PS: Advanced Probability Theory
Sheet 7 Solutions

lucas.teyssier@univie.ac.at

Exercise 1. For every $x \geq 0$,

$$\mathbb{P}\left(\frac{X_n}{n} > x\right) = \mathbb{P}(X_n \geq \lfloor nx \rfloor + 1) = \sum_{k \geq \lfloor nx \rfloor + 1} \left(1 - \frac{1}{n}\right)^{k-1} \frac{1}{n} = \left(1 - \frac{1}{n}\right)^{\lfloor nx \rfloor} \xrightarrow[n\to\infty]{} e^{-x} = \mathbb{P}(\mathrm{Exp}(1) > x).$$

This proves that $X_n/n \xrightarrow[n\to\infty]{(d)} \mathrm{Exp}(1)$.

Exercise 2.

a) (A.J.) The difficulty here is that the supremum is taken over a non countable set, so an argument is needed. If we sort $(X_1, \dots, X_n)$ to get $X_{(1)} \leq \cdots \leq X_{(n)}$, it is not hard to see that

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| = \max_{0 \leq i \leq n} \max \left(\left| \frac{i}{n} - F(X_{(i)}) \right|, \left| \frac{i}{n} - F(X_{(i+1)}) \right| \right)$$

with the convention $X_{(0)} = -\infty$ and $X_{(n+1)} = +\infty$. The $X_{(i)}$'s being measurable (this is left to the reader), it shows that $\sup_{x\in\mathbb{R}} |F_n(x) - F(x)|$ is measurable as well.

b) (A.J.) Let $\varepsilon > 0$. As f is continuous on a compact set, f is uniformly continuous and we can find $x_1 < x_2 < \cdots < x_p$ with x_1 and x_p being the end points of I and such that $\sup_{1\leq i\leq p-1} |f(x_{i+1}) - f(x_i)| \leq \varepsilon/2$. As $f_n \to f$ pointwise and as there is only a finite number of x_i's, there exists $n_0 \geq 1$ s.t. for all $n \geq n_0$, $\sup_{1\leq i\leq p} |f_n(x_i) - f(x_i)| \leq \varepsilon/2$.

Let $n \geq n_0$, $x \in I$ and consider $i \in \{1, \dots, p-1\}$ s.t. $x_i \leq x \leq x_{i+1}$. Since f_n and f are non-decreasing (f is non-decreasing as a pointwise limit of non-decreasing functions), we have

$$f(x) - f_n(x) \leq f(x_{i+1}) - f_n(x_i) = (f(x_{i+1}) - f(x_i)) + (f(x_i) - f_n(x_i)) \leq \varepsilon.$$

Similarly $f(x) - f_n(x) \geq -\varepsilon$ and we have shown that for all $n \geq n_0$, $\sup_{x\in I} |f(x) - f_n(x)| \leq \varepsilon$. It concludes the question.

c) As $\mathbb{Q}$ is countable, it suffices to show that for some $x \in [0,1] \cap \mathbb{Q}$, $F_n(x) \xrightarrow[n\to\infty]{a.s.} F(x)$.

Let $x \in [0,1] \cap \mathbb{Q}$. As the variables $\mathbb{1}_{X_n \leq x}$ are i.i.d and have an expectation, we conclude using the strong law of large numbers:

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{X_i \leq x\}} \xrightarrow[n\to\infty]{a.s.} \mathbb{E}\left[\mathbb{1}_{X_i \leq x}\right] = \mathbb{P}(X_1 \leq x) = F(x).$$

d) In this case, F is continuous. This combined with c) implies that a.s., for all $x \in [0,1]$, $F_n(x) \to F(x)$. As F is continuous and the F_n's are non-decreasing, we deduce from b) that almost surely, $F_n \xrightarrow[n\to\infty]{unif} F$. (There is no problem outside of $[0,1]$.)

e) (A.J.) One can show that for all x, y, $F^{\leftarrow}(y) \leq x$ if an only if $y \leq F(x)$. From this, we see that the cumulative distribution functions of $F^{\leftarrow}(U_i)$ and X_i are the same. They hence must have the same law.

The probability (which makes sense thanks to a))

$$\mathbb{P}\left(\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |F_n(x) - F(x)| = 0\right)$$

depends only on the law of the X_i's. It can thus be computed with X_i replaced by $F^{\leftarrow}(U_i)$ for which we have:

$$\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{F^{\leftarrow}(U_i)\leq x\}} - F(x)\right| = \sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{U_i\leq F(x)\}} - F(x)\right| = \sup_{y\in F(\mathbb{R})}\left|\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{U_i\leq y\}} - y\right|$$
$$\leq \sup_{y\in[0,1]}\left|\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{U_i\leq y\}} - y\right|.$$

By d), this last quantity converges almost surely to 0. It concludes the proof.

Exercice 3.

a) As observed by some students, the wording of the exercise was not clear. X and the X_n's have their values in $\mathbb{N}$. (If not, $X_n \sim \delta_n$ provides a counter example.)

Let $\mu(\cdot) = \mathbb{P}(X \in \cdot)$ and for all n, $\mu_n(\cdot) = \mathbb{P}(X_n \in \cdot)$. As usually, the space $\mathbb{N}$ is endowed with the topology of the euclidian distance, so the Borel sets of $\mathbb{N}$ are all its subsets. By Portmanteau's theorem (point (4)), if $X_n \to X$ in distribution, then for every singleton $\{k\} \subset \mathbb{Z}$, $\mu_n(\{k\}) \xrightarrow[n\to\infty]{} \mu(\{k\})$[1].

Reciprocally, assume that for every $k \in \mathbb{N}$, $\mathbb{P}(X_n = k) \xrightarrow[n\to\infty]{} \mathbb{P}(X = k)$. Then, for every $x \in \mathbb{R}$,

$$F_n(x) = \sum_{k=1}^{\lfloor x\rfloor}\mathbb{P}(X_n = k) \xrightarrow[n\to\infty]{} \sum_{k=1}^{\lfloor x\rfloor}\mathbb{P}(X = k) = F(x),$$

so $X_n \xrightarrow[n\to\infty]{(d)} X$.

b) It follows from a), and that for all $k \in \mathbb{Z}$,

$$\mathbb{P}(X_n = k) = \binom{n}{k}\left(\frac{c}{n}\right)^k\left(1 - \frac{c}{n}\right)^{n-k} \sim_{n\to\infty} \frac{n^k}{k!}\frac{c^k}{n^k}e^{-c} = e^{-c}\frac{c^k}{k!} = \mathbb{P}(\text{Poiss}(c) = k).$$

c) For the typist, the number of mistakes has the law of a Binomial$(1000, c/1000)$, with $c = 1$. So $\mathbb{P}(X_{1000} = 0) = \left(1 - \frac{1}{1000}\right)^{1000} \approx 0.3677$. Approximating by a Poisson law of parameter

[1] We usually write $\mu(k)$ instead of $\mu(\{k\})$ for singletons, to have lighter notations.

1 (using b)), we would have guessed that it is about $\mathbb{P}(\text{Poiss}(1) = 0) = 1/e \approx 0.3679$, which is very close to the real result.
For the emails, let's approximate this directly by a Poisson law of parameter 10. Then $\mathbb{P}(\text{Poiss}(10) = 1) = 10e^{-10} \approx 0.00045$. This is very little! To get more familiar with this type of values, scroll down on the following link and see the two tables. http://villemin.gerard.free.fr/aMaths/Probabil/PoissonT.htm

Exercise 4. (A.J.)

a) By denoting $(\cdot)_+$ and $(\cdot)_-$ the positive and negative parts, we have

$$0 = 1 - 1 = \int f - f_n = \int (f - f_n)_+ - \int (f - f_n)_-.$$

Hence

$$\int |f - f_n| = \int (f - f_n)_+ + \int (f - f_n)_- = 2 \int (f - f_n)_+$$

and this last integral goes to zero by dominated convergence theorem: $(f - f_n)_+$ is dominated by f.

b) The functions f_n, f satisfy the assumptions of the previous question and

$$\int |f - f_n| \xrightarrow[n\to\infty]{} 0.$$

Then for all $x \in \mathbb{R}$, we have by triangle inequality

$$\begin{aligned} |\mathbb{P}(X_n \le x) - \mathbb{P}(X \le x)| &= \left| \int_{-\infty}^{x} f_n(y)dy - \int_{-\infty}^{x} f(y)dy \right| \le \int_{-\infty}^{x} |f_n(y) - f(y)|\, dy \\ &\le \int_{-\infty}^{+\infty} |f_n(y) - f(y)|\, dy \end{aligned}$$

which goes to zero as $n \to \infty$. By definition, it shows that $X_n \to X$ in distribution.

Exercise 5. For the first point, see https://en.wikipedia.org/wiki/Proofs_of_convergence_of_random_variables#Convergence_in_distribution_to_a_constant_implies_convergence_in_probability.
For the second point, see https://en.wikipedia.org/wiki/Proofs_of_convergence_of_random_variables#propB3, and apply the continuous function $(x, y) \mapsto x + y$ to the couple.

Exercise 6. (For the general case, an extra assumption of inner regularity is needed.)
Let $d \in \mathbb{N}^*$ and consider only the case of $\mathbb{R}^d$.

Let $\varepsilon > 0$. For each $0 \le i \le n$, as μ_i is tight, there exists a compact K_i such that $\mu_i(K_i) \ge 1 - \varepsilon$. $K := \bigcup_{i=1}^{n} K_i$ is a finite union of compacts, hence compact, and for all $1 \le i \le n$, $\mu_i(K) \ge 1 - \varepsilon$. This proves that the familly $(\mu_i)_{1 \le i \le n}$ is tight.

Remark. Every probability measure on a complete separable metric space is tight. See Prop 2.7.18 in https://b-ok.cc/book/3308724/702f8d.