PS: Advanced Probability Theory
Sheet 7

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Exercise 1. We say that a random variable X is geometric with parameter $p = 1 - q$ if for all $k \in \mathbb{N}^*$, $\mathbb{P}(X = k) = q^{k-1}p$.
Let X_n be a geometric random variable with parameter $1/n$. Show that $X_n/n \to \text{Exp}(1)$ in distribution.

Exercise 2 (Glivenko-Cantelli)**.** Let $(X_i, i \geq 1)$ be a sequence of i.i.d. real-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by F the cumulative distribution function of X_1. For $n \geq 1$, define the n-th empirical distribution function

$$\forall x \in \mathbb{R}, F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{X_i \leq x\}}.$$

This exercise aims to show that a.s (F_n) converges uniformly to F:

$$\mathbb{P}\left(\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \to 0 \right) = 1. \tag{1}$$

a) Justify that (1) makes sense by showing that for all $n \geq 1$,

$$\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is measurable.

b) (Optional question, Second Dini's theorem) Let $I \subset \mathbb{R}$ be a *compact* interval and $f, f_n : I \to \mathbb{R}, n \geq 1$. Assume that f is continuous and that for all $n \geq 1$, f_n is non-decreasing. If (f_n) converges pointwise to f, then (f_n) converges uniformly to f.

c) Show that almost surely, for all $x \in [0, 1] \cap \mathbb{Q}$, $F_n(x) \to F(x)$.

d) Deduce (1) in the particular case where the X_i's are uniform r.v. on $[0, 1]$.

e) We come back to the general case and we consider the general inverse function of F:

$$\forall y \in [0, 1], F^{\leftarrow}(y) := \inf\{x \in \mathbb{R} : F(x) \geq y\}.$$

Show that if $(U_i, i \geq 1)$ is a sequence of i.i.d. uniform r.v. on $[0, 1]$, then $(X_i, i \geq 1)$ and $(F^{\leftarrow}(U_i), i \geq 1)$ have the same law. Conclude with (1) in the general case.

Exercise 3. a) Let X, X_n be random variables with values in $\mathbb{N}$. Show that X_n converges to X in distribution if and only if $\mathbb{P}(X_n = k) \to \mathbb{P}(X = k)$ for all $k \geq 0$.

b) As an application, recall the Binomial distribution $B(n,p)$ and the Poisson distribution with parameter λ. Show that if X_n is Binomial $(n, c/n)$ then X_n converges in distribution to a Poisson random variable with parameter $\lambda = c$.

This is very useful in practice:

c) Examples (optional): A copyist makes a typo with probability 0.1% for each character. In a page containing 1000 characters, what is (approximately) the probability that there isn't any typo? You receive on average 10 emails in a working day. What is (approximately) the chance you get exactly one email in a given day? (assume that emails arrive independently of one another, with constant frequency).

Exercise 4. a) (Optional question, Scheffé's lemma) Let $f, f_n : \mathbb{R} \to [0, \infty), n \geq 1$, be measurable functions such that for all $n \geq 1$, $\int f_n(x)dx = \int f(x)dx = 1$. If (f_n) converges pointwise to f, then (f_n) converges in L_1 to f, i.e. $\int |f_n(x) - f(x)|\, dx \to 0$.

b) Deduce that if (X_n) is a sequence of random variables having densities (f_n) which satisfy $f_n \to f$ pointwise where f is the density of X, then (X_n) converges in distribution to X.

Exercise 5. Show that if $X_n \to 0$ in distribution then $X_n \to 0$ in probability.
Show that if $X_n \to 0$ in probability and $Y_n \to Y$ in distribution then $X_n + Y_n \to Y$ in distribution. This implication is sometimes called Slutsky's lemma/theorem.

Exercise 6. Let $d \in \mathbb{N}^*$. Show that if $\{\mu_1, \ldots, \mu_n\}$ is a finite collection of probability distributions on $\mathbb{R}^d$, then it is tight.

Optional exercise

Exercise A. (Skorokhod representation theorem)
Let X_n be a sequence of random variables defined on some probability spaces $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$. Suppose that $X_n \to X$ in distribution. Show that there exists $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence X'_n of random variables and a random variable X' defined on this space such that X'_n has the same law as X_n for every n, X' has the same law as X' and X'_n converges a.s to X'.
Hint: use the Lebesgue-Stieltjes construction of a random variable with given law.