PS: Advanced Probability Theory
Sheet 6 Solutions

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Exercise 1. (Moritz Dober)

a) Let $\varepsilon > 0$. Then we have for $n \geq 1$

$$\begin{aligned}\mathbb{P}\left(|(X_n + Y_n) - (X + Y)| > \varepsilon\right) &\leq \mathbb{P}\left(|X_n - X| + |Y_n - Y| > \varepsilon\right)\\ &\leq \mathbb{P}\left(|X_n - X| > \varepsilon/2 \text{ or } |Y_n - Y| > \varepsilon/2\right)\\ &\leq \underbrace{\mathbb{P}\left(|X_n - X| > \varepsilon/2\right)}_{\to 0} + \underbrace{\mathbb{P}\left(|Y_n - Y| > \varepsilon/2\right)}_{\to 0} \to 0,\end{aligned}$$

as n tends to infinity.

b) Let $\varepsilon > 0$. Then using Markov's inequality we get for $n \geq 1$

$$\mathbb{P}\left(|X_n - X| > \varepsilon\right) \leq \frac{\mathbb{E}\left[|X_n - X|\right]}{\varepsilon} \to 0 \text{ as } \to \infty.$$

Now let $(X_n)_{n\geq 1}$ be a sequence of r.v. such that $\mathbb{P}\left(X_n = n\right) = 1/n$ and $\mathbb{P}\left(X_n = 0\right) = 1 - 1/n$. Then we have for $n \geq 1$ and $\varepsilon > 0$

$$\mathbb{P}\left(|X_n| > \varepsilon\right) \leq \mathbb{P}\left(X_n = n\right) = 1/n \to 0 \text{ as } n \to \infty$$

and hence X_n converges to 0 in probability. On the other hand $\mathbb{E}\left[|X_n|\right] = 1$ for all n and thus X_n does not converge to 0 in L^1. Therefore convergence in probability does not imply convergence in L^1.
To see that L^1-convergence does not imply a.s.-convergence let $(X_n)_{n\geq 1}$ be a sequence of independent r.v. such that $\mathbb{P}\left(X_n = 1\right) = 1/n$ and $\mathbb{P}\left(X_n = 0\right) = 1 - 1/n$. Then $X_n \to 0$ in L^1 since $\mathbb{E}\left[|X_n|\right] = 1/n$ for $n \geq 1$. But we have $\sum_{n\geq 1} \mathbb{P}\left(X_n = 1\right) = \sum_{n\geq 1} 1/n = \infty$, hence by the second Borel-Cantelli lemma $X_n = 1$ i.o. almost surely.
To show that the converse is also wrong in general let $(X_n)_{n\geq 1}$ be a sequence of r.v. with $\mathbb{P}\left(X_n = n^2\right) = 1/n^2$ and $\mathbb{P}\left(X_n = 0\right) = 1 - 1/n^2$. Then $X_n \to 0$ a.s. since $\sum_{n\geq 1} \mathbb{P}\left(X_n = n^2\right)$ $= \sum_{n\geq 1} 1/n^2$ converges and hence $X_n = 0$ for n large enough a.s. by the first Borel-Cantelli lemma. On the other hand we have that $\mathbb{E}\left[|X_n|\right] = 1$ for all n and thus $X_n \not\to 0$ in L^1.

c) We have seen in the lecture that if $X_n \to X$ in probability then there exists a subsequence $(n_k)_{k\geq 1}$ such that $X_{n_k} \to X$ a.s.. Therefore we get $|X| = \lim_{k\to\infty} |X_{n_k}| \leq Z$ on the event of convergence and hence $|X| \leq Z$ a.s.. Not let $\varepsilon > 0$, then we have

$$\begin{aligned}\mathbb{E}\left[|X_n - X|\right] &= \mathbb{E}\left[|X_n - X| \mathbb{1}_{\{|X_n - X| > \varepsilon\}}\right] + \mathbb{E}\left[|X_n - X| \mathbb{1}_{\{|X_n - X| \leq \varepsilon\}}\right]\\ &\leq 2\mathbb{E}\left[Z \mathbb{1}_{\{|X_n - X| > \varepsilon\}}\right] + \varepsilon\end{aligned}$$

and $\mathbb{E}\left[Z\mathbb{1}_{\{|X_n-X|>\varepsilon\}}\right]$ converges to 0 as $n\to\infty$ since it holds in general that if $Y\in L^1$ and $A_n\in\mathcal{F}$ for $n\geq 1$ with $\mathbb{P}(A_n)\to 0$ as $n\to\infty$ then $\mathbb{E}\left[Y\mathbb{1}_{A_n}\right]\to 0$ as $n\to\infty$. Indeed, suppose not, then there exists $c>0$ and $(n_k)_{k\geq 1}$ s.t. $\mathbb{E}\left[|Y|\,\mathbb{1}_{A_{n_k}}\right]>c$ for all $k\geq 1$. Since $\mathbb{1}_{A_n}\to 0$ in probability there exists a subsequence $(k_i)_{i\geq 1}$ s.t. $\mathbb{1}_{A_{n_{k_i}}}\to 0$ a.s.. Moreover $|Y|<\infty$ a.s. (since $Y\in L^1$) and so $|Y|\,\mathbb{1}_{A_{n_{k_i}}}\to 0$ a.s.. Therefore by the dominated convergence theorem $\mathbb{E}\left[|Y|\,\mathbb{1}_{A_{n_{k_i}}}\right]\to 0$, contradicting $\mathbb{E}\left[|Y|\,\mathbb{1}_{A_{n_k}}\right]>c$ for all $k\geq 1$.

Exercise 2. (M. D.)

a) Let $\varepsilon>0$. Using the hint and Markov's inequality we get for $n\geq 1$

$$n\mathbb{P}\left(|X_1|\geq\varepsilon\sqrt{n}\right)=n\mathbb{P}\left(|X_1|\,\mathbb{1}_{\{|X_1|\geq\varepsilon\sqrt{n}\}}\geq\varepsilon\sqrt{n}\right)\leq\frac{1}{\varepsilon^2}\mathbb{E}\left[X_1^2\mathbb{1}_{\{|X_1|\geq\varepsilon\sqrt{n}\}}\right].$$

Since X_1 has finite second moment and hence $|X_1|,X_1^2<\infty$ a.s. we have that $X_1^2\mathbb{1}_{\{|X_1|\geq\varepsilon\sqrt{n}\}}$ decreases to $X_1^2\mathbb{1}_{\{|X_1|=\infty\}}=0$ a.s.. We conclude by applying the dominated convergence theorem.

b) By exercise 1 it is enough to prove convergence in L^1. Let $\varepsilon>0$. We have for $n\geq 1$

$$\begin{aligned}\mathbb{E}\left[\max_{1\leq i\leq n}|X_i|\right]&=\mathbb{E}\left[\max_{1\leq i\leq n}|X_i|\,\mathbb{1}_{\{\max_{1\leq i\leq n}|X_i|<\varepsilon\sqrt{n}\}}\right]+\mathbb{E}\left[\max_{1\leq i\leq n}|X_i|\,\mathbb{1}_{\{\max_{1\leq i\leq n}|X_i|\geq\varepsilon\sqrt{n}\}}\right]\\&\leq\varepsilon\sqrt{n}+\mathbb{E}\left[\max_{1\leq i\leq n}X_i^2\right]^{1/2}\mathbb{P}\left(\max_{1\leq i\leq n}|X_i|\geq\varepsilon\sqrt{n}\}\right)^{1/2},\end{aligned}$$

where we used the Cauchy-Schwarz inequality as suggested. Furthermore,

$$\begin{aligned}\mathbb{P}\left(\max_{1\leq i\leq n}|X_i|\geq\varepsilon\sqrt{n}\}\right)&=\mathbb{P}\left(\bigcup_{i=1}^n\{|X_i|\geq\varepsilon\sqrt{n}\}\right)\\&\leq\sum_{i=1}^n\mathbb{P}\left(|X_i|\geq\varepsilon\sqrt{n}\right)=n\mathbb{P}\left(|X_1|\geq\varepsilon\sqrt{n}\right)\end{aligned}$$

and the last expression converges to 0 as $n\to\infty$ by a). We also have

$$\mathbb{E}\left[\max_{1\leq i\leq n}X_i^2\right]\leq\sum_{i=1}^n\mathbb{E}\left[X_i^2\right]=n\mathbb{E}\left[X_1^2\right].$$

Altogether we proved

$$\frac{1}{\sqrt{n}}\mathbb{E}\left[\max_{1\leq i\leq n}|X_i|\right]\leq\varepsilon+\mathbb{E}\left[X_1^2\right]^{1/2}n\mathbb{P}\left(|X_1|\geq\varepsilon\sqrt{n}\right)<2\varepsilon$$

for n large enough since $X_1\in L^2$ by assumption.

Exercise 3. (M. D.)

a) Fix $x \in \mathbb{R}$. For $i \geq 1$ we have that $X_i(x) := \mathbb{1}_{\{U_i \leq x\}}$ is a Bernoulli r.v. with parameter x and the $X_i(x)$ are independent of one another. Hence $S_n(x) = X_1(x) + ... + X_n(x)$ is a binomial r.v. with parameters n and x.
By the weak LLN we also get that $S_n(x)/n$ converges in probability to $\mathbb{E}[X_1(x)] = x$. Furthermore, for $\varepsilon > 0$, we get from Chebyshev's inequality a bound for $\mathbb{P}(|S_n(x)/n - x| > \varepsilon)$ that does not depend on x:

$$\mathbb{P}\left(\left|\frac{S_n(x)}{n} - x\right| > \varepsilon\right) \leq \frac{\mathrm{Var}(S_n(x)/n)}{\varepsilon^2} = \frac{\mathrm{Var}(S_n(x))}{(n\varepsilon)^2} = \frac{nx(1-x)}{(n\varepsilon)^2} \leq \frac{1}{4n\varepsilon^2},$$

since $x(1-x) \leq 1/4$ for all $x \in [0,1]$.

b) We have for $n \geq 1$ and $x \in [0,1]$

$$\mathbb{E}\left[f\left(\frac{S_n(x)}{n}\right)\right] = \sum_{t \in \mathbb{R}} t \cdot \mathbb{P}\left(f\left(\frac{S_n(x)}{n}\right) = t\right) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \underbrace{\mathbb{P}(S_n(x) = k)}_{=\binom{n}{k}x^k(1-x)^{n-k}} = B_n(x).$$

c) Since f is continuous on the compact interval $[0,1]$ we have that f is uniformly continuous and bounded by some $C > 0$. Let $\varepsilon > 0$, then there exists $\delta > 0$ such that for all $x, y \in [0,1]$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Therefore $\mathbb{P}(|f(S_n(x)/n) - f(x)| > \varepsilon) \leq \mathbb{P}(|S_n(x)/n - x| \geq \delta) < \varepsilon/(2C)$ for n large enough, by a).
Now, similar to 1)c) we have

$$\begin{aligned} |B_n(x) - f(x)| &= |\mathbb{E}[f(S_n(x)/n)] - f(x)| \leq \mathbb{E}[|f(S_n(x)/n) - f(x)|] \\ &\leq \mathbb{E}\left[|f(S_n(x)/n) - f(x)| \mathbb{1}_{\{|f(S_n(x)/n) - f(x)| > \varepsilon\}}\right] + \varepsilon \\ &\leq 2C\mathbb{P}(|f(S_n(x)/n) - f(x)| > \varepsilon) + \varepsilon < 2\varepsilon \end{aligned}$$

for n large enough.

Exercise 4. (M. D.)

a) By assumption for all $k \geq 1$ there exists $N_k \geq 1$ such that $\mathbb{P}(|X_n - X_m| > 2^{-k}) < 2^{-k}$ for all $n, m \geq N_k$. Set $n_k := \max\{N_1, ..., N_k\}$. Then clearly the n_k are increasing, $n_k, n_{k+1} \geq N_k$ and hence $\mathbb{P}(|X_{n_{k+1}} - X_{n_k}| > 2^{-k}) < 2^{-k}$ for all k.

b) We have $\sum_{k \geq 1} \mathbb{P}(|X_{n_{k+1}} - X_{n_k}| > 2^{-k}) \leq \sum_{k \geq 1} 2^{-k} < \infty$. Thus, by the first Borel-Cantelli lemma we get that $|X_{n_{k+1}} - X_{n_k}| \leq 2^{-k}$ eventually on some event A of probability one. Let $\varepsilon > 0$ and choose $N \geq 1$ such that $2^{-N} < \varepsilon$. Fix $\omega \in A$ and let $K = K(\omega) \geq 1$ such that $|X_{n_{k+1}}(\omega) - X_{n_k}(\omega)| \leq 2^{-k}$ for all $k \geq K$. Then have for $m > l \geq \max\{K, N+1\}$

$$|X_{n_m}(\omega) - X_{n_l}(\omega)| \leq \sum_{k=l}^{m-1} |X_{n_{k+1}}(\omega) - X_{n_k}(\omega)| \leq \sum_{k \geq l} 2^{-k} = 2^{-(l-1)} \leq 2^{-N} < \varepsilon.$$

We deduce that $(X_{n_k}(\omega))_{k \geq 1}$ is a Cauchy sequence in $\mathbb{R}$ and thus convergent. Therefore $(X_{n_k})_{k \geq 1}$ converges a.s. towards some r.v. X.

c) Let $\varepsilon > 0$, then we have

$$\begin{aligned} \mathbb{P}(|X_n - X| > \varepsilon) &\leq \mathbb{P}(|X_n - X_{n_k}| + |X_{n_k} - X| > \varepsilon) \\ &\leq \mathbb{P}\left(|X_n - X_{n_k}| > \frac{\varepsilon}{2} \text{ or } |X_{n_k} - X| > \frac{\varepsilon}{2}\right) \\ &\leq \mathbb{P}\left(|X_n - X_{n_k}| > \frac{\varepsilon}{2}\right) + \mathbb{P}\left(|X_{n_k} - X| > \frac{\varepsilon}{2}\right), \end{aligned}$$

and both of these terms get arbitrary small when n and n_k are large enough (the first term by assumption and the second one since a.s.-convergence implies convergence in probability).

Exercise 5. (M. D.) Let $(X_n)_{n\geq 1}$ be a sequence of independent r.v. such that $\mathbb{P}\left(X_n = n^2\right) = 1/n$ and $\mathbb{P}(X_n = 0) = 1 - 1/n$. Then X_n converges to 0 in probability. However, if $i \geq \lceil \sqrt{n} \rceil$, then $i^2 \geq n$, and hence

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i < 1\right) \leq \mathbb{P}\left(\bigcap_{i=\lceil\sqrt{n}\rceil}^{n} \{X_i = 0\}\right) = \prod_{i=\lceil\sqrt{n}\rceil}^{n} \mathbb{P}(X_i = 0)$$
$$= \prod_{i=\lceil\sqrt{n}\rceil}^{n} \left(1 - \frac{1}{i}\right) \leq \left(1 - \frac{1}{n}\right)^{n-\sqrt{n}} \to \frac{1}{e} < 1 \text{ as } n \to \infty.$$

We deduce that $(\sum_{i=1}^{n} X_i)/n$ does not converge to 0 in probability.

Exercise A. (M. D.) We have (by taking countable intersections) that $\limsup_{n\to\infty} V_n \geq p_c$ $\mathbb{P}$-a.s. iff $\mathbb{P}(V_n > p \text{ i.o.}) = 1$ for all $p < p_c$ which is equivalent to $\mathbb{P}(V_n \leq p \text{ eventually}) = 0$ for all $p < p_c$.
Fix $p < p_c$ and define $\omega_e = \mathbb{1}_{\{U_e \leq p\}}$ for $e \in \mathbb{E}^d$. Then $(\omega_e)_{e\in\mathbb{E}^d}$ has the law $\mathbb{P}_p$ and $V_n \leq p$ eventually iff $\omega_{e_n} = 1$ eventually. But $\omega_{e_n} = 1$ for n large enough is equivalent to A_n "hitting" an infinite p-open cluster, which does a.s. not exist since $p < p_c$.