PS: Advanced Probability Theory
Sheet 5 Solutions

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Exercise 1.
As for $d \geq 2$, $\mathbb{Z}^2 \subset \mathbb{Z}^d$, we only need to prove the result for $d = 2$.
In the course ("Topological fact" in Lecture 12), we have seen using duality that for all p,

$$1 - \theta(p) = \mathbb{P}_p(|C_0| < \infty) \leq \sum_{n \geq 1} n 4^n (1-p)^n,$$

which tends to 0 as p tends to 1.

Exercise 2. (Antoine Jego) This exercise was hard. Here is the solution written by last year's PS course teacher, Antoine Jego. It uses a result which you can find for example in the following article by Grimmett and Marstrand. https://royalsocietypublishing.org/doi/pdf/10.1098/rspa.1990.0100

We introduce p_c^* the critical percolation probability when the underlying graph is not longer $\mathbb{Z}^d$ but is the half space $\mathbb{N} \times \mathbb{Z}^{d-1}$. In particular, when $q > p_c^*$, with positive probability there exists an infinite path of q-open edges starting from the origin and lying in $\mathbb{N} \times \mathbb{Z}^{d-1}$. We are going to show that $\limsup V_n \leq p_c^*$ a.s. By admitting the non trivial result $p_c = p_c^*$, it will answer the question.

Let $q > p_c^*$. Let $n_1 \geq 1$ large and for $N \geq n_1$ consider the first exit time of the cube $[-N, N]^d$:

$$\tau_N := \inf \left\{ n \geq 1 : A_n \cap \left(\mathbb{R}^d \backslash [-N, N]^d \right) \neq \varnothing \right\}.$$

Let $y \in \mathbb{Z}^d$. With the notation of the statement of the exercise, the event $\{y_{\tau_N - 1} = y\}$ reads as the event that the invasion has broken out at y. Notice that if y is connected to infinity by q-open edges lying outside $[-N, N]^d$, then $\limsup V_n \leq q$ on the event $\{y_{\tau_N - 1} = y\}$. Hence

$$\begin{aligned} &\mathbb{P}\left(\limsup V_n > q \,\middle|\, \sup_{n_1 \leq n \leq \tau_N} V_n > q, y_{\tau_N - 1} = y \right) \\ &\leq \mathbb{P}\left(\begin{array}{c} y \text{ is not connected to } \infty \text{ by} \\ q\text{−open edges lying outside } [-N,N]^d \end{array} \,\middle|\, \sup_{n_1 \leq n \leq \tau_N} V_n > q, y_{\tau_N - 1} = y \right) \end{aligned}$$

Now observe that the event $\{\sup_{n_1 \leq n \leq \tau_N} V_n > q, y_{\tau_N - 1} = y\}$ is independent of U_e for $e \in \mathbb{E}^d$ lying outside $[-N, N]^d$. So we get forget the conditioning in the last probability and

$$\mathbb{P}\left(\limsup V_n > q \,\middle|\, \sup_{n_1 \leq n \leq \tau_N} V_n > q, y_{\tau_N - 1} = y \right) \leq \mathbb{P}\left(\begin{array}{c} y \text{ is not connected to } \infty \text{ by} \\ q\text{−open edges lying outside } [-N,N]^d \end{array} \right).$$

As $q > p_c^*$, there exists $\alpha > 0$ which depends only on q s.t.

$$\mathbb{P}\left(\begin{array}{c} y \text{ is not connected to } \infty \text{ by} \\ q\text{−open edges lying outside } [-N,N]^d \end{array}\right) \leq 1-\alpha.$$

(Make sure you understand this claim! This is here that we need $q > p_c^*$.) Wrapping things up, we have proven that

$$\begin{aligned}
\mathbb{P}\left(\limsup V_n > q\right) &\leq \mathbb{P}\left(\limsup V_n > q,\ \sup_{n_1 \leq n \leq \tau_N} V_n > q\right) \\
&= \sum_{y \in \mathbb{Z}^d} \mathbb{P}\left(\limsup V_n > q,\ \sup_{n_1 \leq n \leq \tau_N} V_n > q, y_{\tau_N - 1} = y\right) \\
&\leq (1-\alpha) \sum_{y \in \mathbb{Z}^d} \mathbb{P}\left(\sup_{n_1 \leq n \leq \tau_N} V_n > q, y_{\tau_N - 1} = y\right) \\
&= (1-\alpha)\mathbb{P}\left(\sup_{n_1 \leq n \leq \tau_N} V_n > q\right).
\end{aligned}$$

As

$$\lim_{n_1 \to \infty} \lim_{N \to \infty} \mathbb{P}\left(\sup_{n_1 \leq n \leq \tau_N} V_n > q\right) = \mathbb{P}\left(\limsup V_n > q\right),$$

it implies that $\mathbb{P}\left(\limsup V_n > q\right) \leq (1-\alpha)\mathbb{P}\left(\limsup V_n > q\right)$ and $\mathbb{P}\left(\limsup V_n > q\right) = 0$. Since this is true for all $q > p_c^*$, it concludes the proof of: $\limsup V_n \leq p_c^*$ a.s.

Exercise 3. For $n \geq 1$ and $0 < p < 1$, let

$$g_n(p) = \mathbb{P}_p\left(0 \leftrightarrow Q(0,n)^c\right),$$

which is increasing. Let also B_n be the set of edges who have at least one endpoint in $Q(0,n)$.

a) Let $n \geq 1$. If $q \geq p$, then

$$\begin{aligned}
|g_n(q) - g_n(p)| &= \mathbb{P}(0 \text{ is } q\text{-connected to } Q(0,n) \text{ but not } p\text{-connected to } Q(0,n)) \\
&\leq \mathbb{P}(\exists e \in B_n,\ \ U_e \in [p,q]) \\
&\leq |q-p|\,\#B_n,
\end{aligned}$$

which tends as $q - p \to 0$, and hence proves the continuity of g_n.

Remark. It is also possible to show that g_n is polynomial in p.

b) Let $x \in [0,1)$ and $\varepsilon > 0$. We want to prove that there exists some $\delta > 0$ such that for all $y \in [x, x+\delta)$, $|f(y) - f(x)| \leq \epsilon$.
As $f_n(x) \to f(x)$, there exists n such that $|f_n(x) - f(x)| \leq \varepsilon$.
As f_n is continuous at x, there exists $\delta > 0$ such that for all y such that $|y - x| \leq \delta$, $|f_n(x) - f_n(y)| \leq \varepsilon$.
For all $y \in [x, x+\delta)$, as by assumption f is non-decreasing and $f(y) \leq f_n(y)$,

$$|f(y) - f(x)| = f(y) - f(x) \leq f_n(y) - f(x) \leq |f_n(y) - f_n(x)| + |f_n(x) - f(x)| \leq 2\varepsilon.$$

This proves that f_n is right continuous.
Replacing f_n by g_n and f by θ proves the last claim.

Exercise 4.

a) We want to show that $\theta(p) - \theta(q) \longrightarrow 0$ as $q \to p$.
For $q < p$, as the event $\{|C_q| = \infty\}$ is included in the event $\{|C_p| = \infty\}$,

$$\theta(p) - \theta(q) = \mathbb{P}(|C_p| = \infty, |C_q| < \infty).$$

Finally, as the events $\{|C_p| = \infty, |C_q| < \infty\}$ are decreasing,

$$\lim_{q\to p} \mathbb{P}(|C_p| = \infty, |C_q| < \infty) = \mathbb{P}\left(\bigcap_{q<p, q\in\mathbb{Q}} \{|C_p| = \infty, |C_q| < \infty\}\right) = \mathbb{P}(|C_p| = \infty, |C_q| < \infty \text{ for all } q < p).$$

b) If $|C_p| = \infty$, then by Burton and Keane, C_p is *the* infinite p-cluster, which contains the infinite r-cluster (by coupling). We exactly said that $\mathbb{P}(I_r \subset C_p \mid |C_p| = \infty) = 1$, which is what was to be proven.

c) Let γ be a finite path of p-open edges from 0 to I_r. Let q such that

$$\max_{e\in\gamma} U_e < q < p.$$

Then $\gamma \subset C_q$ and so, as $r \leq q$, the whole component I_r is in C_q, which proves that C_q is infinite.
To conclude,

$$\begin{aligned}\mathbb{P}\left(|C_p| = \infty, |C_q| < \infty \text{ for all } q < p\right) =& \mathbb{P}\left(|C_p| = \infty, I_r \subset C_p, |C_q| < \infty \text{ for all } q < p\right)\\ \leq& \mathbb{P}\left(|C_q| < \infty \text{ for all } q < p \mid |C_p| = \infty, I_r \subset C_p\right)\\ =& 0 \quad \text{by the first part of c)},\end{aligned}$$

which by a) proves the left-continuity of θ at p.

Exercise 5. (Moritz Dober)

a) Let $x = \sum_{n\geq 1} a_n 2^{-n} \in [0, 1)$, $(a_n)_{n\geq 1} \in \mathcal{S}$, then

$$\bar{T}(x) = (b \circ T \circ b^{-1})(x) = b(T((a_n)_{n\geq 1})) = b((a_{n+1})_{n\geq 1}) = \sum_{n\geq 1} \frac{a_{n+1}}{2^n} = 2\sum_{n\geq 2} \frac{a_n}{2^n}.$$

In the case where $x \in [0, 1/2)$, i.e. $a_1 = 0$, we have $x = \sum_{n\geq 2} a_n 2^{-n}$, and hence $\bar{T}(x) = 2x$. On the other hand, if $x \in [1/2, 1)$, i.e. $a_1 = 1$, then we get $x = 1/2 + \sum_{n\geq 2} a_n 2^{-n}$, and thus $\bar{T}(x) = 2x - 1$.
We conclude that $\bar{T}(x) = 2x \bmod 1$.

b) Let $\mathbb{P} = (b^{-1})_*(\lambda) = \lambda \circ b$ be the pushforward of the Lebesgue measure λ w.r.t. b^{-1}. Consider a uniform r.v. U on $[0, 1)$ and its binary expansion $(X_n)_{n\geq 1}$ as in exercise 3 on sheet 2, X_n i.i.d. $\sim$ Bernoulli(1/2). Then $\mathbb{P}$ is the law of $(X_n)_{n\geq 1}$.

We have seen in the lecture that translation invariant events for two-sided sequences of i.i.d. r.v. are trivial, but almost the same proof works for one-sided sequences. Therefore, events invariant under T are trivial for $\mathbb{P}$.
Now let $B \in \mathcal{B}([0, 1))$ be invariant under $\bar{T}$, i.e. $\bar{T}(B) \subseteq B$. But we have

$$(b \circ T \circ b^{-1})(B) = \bar{T}(B) \subseteq B \Leftrightarrow T(b^{-1}(B)) \subseteq b^{-1}(B),$$

i.e. $b^{-1}(B)$ is invariant under T and hence trivial for $\mathbb{P}$. We conclude

$$\lambda(B) = \lambda(b(b^{-1}(B))) = \mathbb{P}\left(b^{-1}(B)\right) \in \{0, 1\}.$$

Exercise A. (Antoine Jego)

a) This is a classical result called Fekete's Subadditive Lemma. The key of the proof is to think of the Euclidean division. Let $l := \inf_{n\geq 1} a_n/n$. We will assume that $l > -\infty$. If $l = -\infty$ very few arguments need to be changed. Let $\varepsilon > 0$ and take $n_0 \geq 1$ such that

$$l \leq \frac{a_{n_0}}{n_0} \leq l + \varepsilon.$$

If $n \geq n_0$, by writing $n = qn_0 + r$ where $q \geq 1$ and $0 \leq r \leq n_0 - 1$ and with the convention $a_0 = 0$, we have (it can be shown by induction)

$$\frac{a_n}{n} \leq \frac{qa_{n_0} + a_r}{n} \leq \frac{qn_0}{qn_0 + r}\frac{a_{n_0}}{n_0} + \frac{\sup_{1\leq s\leq n_0-1} a_s}{n}.$$

Hence

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_{n_0}}{n_0} \leq l + \varepsilon.$$

As this is true for all $\varepsilon > 0$ it shows that $\limsup a_n/n \leq l$. As we have, by definition of l, $\liminf a_n/n \geq l$, this concludes the proof.

b) Thanks to the previous question, it is enough to show that $(\log(a_n)/n, n \geq 1)$ is subadditive. Take $n, m \geq 1$. We want to show that $a_{n+m} \leq a_n a_m$. If we denote by $\mathcal{A}_n$ the set of self-avoiding paths of length n, it can be checked that the following map is injective:

$$\begin{aligned} \mathcal{A}_{n+m} &\to \mathcal{A}_n \times \mathcal{A}_m \\ (A_0, \dots, A_{n+m}) &\mapsto (A_0, \dots, A_n), (A_n - A_n, A_{n+1} - A_n, \dots, A_{n+m} - A_n). \end{aligned}$$

In words, we cut a self-avoiding path of length $n + m$ in two, both of which are themselves self-avoiding. As this map is injective, the cardinality of $\mathcal{A}_{n+m}$ is not larger than the one of $\mathcal{A}_n \times \mathcal{A}_m$, i.e $a_{n+m} \leq a_n a_m$.

c) Paths that always go north or east are self-avoiding. Hence $a_n \geq 2^n$ and $c \geq 2$. At each step, the walk cannot come back on its previous vertex. In other words, at each step the walk has at most 3 possibilities (except for the very first step). Hence $a_n \leq 4.3^{n-1}$ and $c \leq 3$.

d) To show strict inequalities we have to work a bit more. We are going to show that $c \geq 1 + \sqrt{2} > 2$. The upper bound is left to the reader.

Denote by α_n the number of self-avoiding paths of length n going always west, north or east and denote by $\alpha_n^{\leftarrow}$ (resp. $\alpha_n^{\uparrow}$, $\alpha_n^{\rightarrow}$) the number of such paths which finish with a step going west (resp. north, east) so that $\alpha_n = \alpha_n^{\leftarrow} + \alpha_n^{\uparrow} + \alpha_n^{\rightarrow}$. From such a path of length n, building a self-avoiding path of length $n+1$ is way easier than for general self-avoiding paths: we just need to check local rules. More specifically, we have for all $n \geq 1$,

$$\begin{cases} \alpha_n^{\leftarrow} = \alpha_n^{\rightarrow} & \text{by symmetry} \\ \alpha_{n+1}^{\uparrow} = \alpha_n^{\leftarrow} + \alpha_n^{\uparrow} + \alpha_n^{\rightarrow} & \text{(no restriction)} \\ \alpha_{n+1}^{\leftarrow} = \alpha_n^{\leftarrow} + \alpha_n^{\uparrow} & \text{(east cannot be followed by west)} \end{cases}$$

Hence,

$$\begin{pmatrix} \alpha_{n+1}^{\uparrow} \\ \alpha_{n+1}^{\leftarrow} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_n^{\uparrow} \\ \alpha_n^{\leftarrow} \end{pmatrix}.$$

The eigenvalues of the previous matrix being $1 + \sqrt{2}$ and $1 - \sqrt{2}$, it shows that there exist $\lambda, \mu \in \mathbb{R}$ such that for all $n \geq 1$, $\alpha_n = \lambda(1 + \sqrt{2})^n + \mu(1 - \sqrt{2})^n$. In (iii) we showed that $\alpha_n \geq 2^n$. Therefore λ has to be positive and $c = \lim a_n^{1/n} \geq \lim \alpha_n^{1/n} = 1 + \sqrt{2}$.