PS: Advanced Probability Theory
Sheet 4 Solutions

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Exercise 1.

a)
$$\infty = \mathbb{E}\left[|X_1|\right] = \sum_{n=1}^{\infty} \mathbb{P}(X_1 \geq n) = \sum_{n=1}^{\infty} \mathbb{P}(X_n \geq n).$$

As the X_n are independant, we can apply Borel-Cantelli II and conclude that

$$\mathbb{P}(X_n \geq n \text{ i.o.}) = 1. \tag{1}$$

b) Thanks to the hint, we see that the event $\left\{\frac{S_n}{n} \text{ converges in } \mathbb{R}\right\}$ is included in the event $\left\{\frac{X_n}{n} \xrightarrow[n\to\infty]{} 0\right\}$, which because of a) has probability 0.

Exercise 2. (ignoring integer parts)

a) Clear by independance of the X_n.

b) Let $\varepsilon > 0$. For $n \in \mathbb{N}^*$, using a)

$$\mathbb{P}(\ell_n \geq (1+\varepsilon)\log_2(n)) = \frac{1}{n^{1+\varepsilon}}.$$

As $\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} < \infty$, by Borel-Cantelli I,

$$\mathbb{P}(\ell_n \geq (1+\varepsilon)\log_2(n) \text{ i.o.}) = 0,$$

and deduce immediately from this that for all $\varepsilon > 0$,

$$\mathbb{P}\left(\frac{L_n}{\log_2(n)} < 1+\varepsilon \text{ i.o.}\right) = 1.$$

We conclude taking the (countable) intersection on all $\varepsilon \in \mathbb{Q}_+^*$.

c) Let $n \in \mathbb{N}^*, \varepsilon > 0$, and set $M = M(n,\varepsilon) = (1-\varepsilon)\log_2(n)$. Then,

$$\begin{aligned}
\mathbb{P}(L_n \le M) &= \mathbb{P}(\forall 1 \le i \le n,\ \ell_i \le M) \\
&\le \mathbb{P}(\forall 1 \le j \le \frac{n}{M},\ \ell_{jM} \le M) \\
&= \prod_{j=1}^{n/M} \mathbb{P}(\ell_{jM} \le M) \qquad \text{(by independence)} \\
&= \left(1 - \frac{1}{n^{1-\varepsilon}}\right)^{n/M} \qquad \text{(by Question a))} \\
&\le \exp\left(\frac{-n^{\varepsilon}}{M}\right) \qquad (\text{as } 1 - x \le e^{-x}),
\end{aligned}$$

which is clearly summable. This allows us to apply Borel-Cantelli I and conclude, proceeding as in b).

Exercise 3.

a) Let $x \in \mathbb{R}$. Then

$$\sqrt{2\pi}\mathbb{P}(X_1 > x) = \int_x^\infty e^{-t^2/2}dt = \int_x^\infty \frac{-1}{t}\left(-te^{-t^2/2}\right)dt = \frac{e^{-x^2/2}}{x} - \int_x^\infty \frac{1}{t^2}e^{-t^2/2}dt.$$

which is equivalent to $\frac{e^{-x^2/2}}{x}$ as $x \to \infty$, because

$$\int_x^\infty \frac{1}{t^2}e^{-t^2/2}dt \le \frac{1}{x^2}\int_x^\infty e^{-t^2/2}dt = o_{x\to+\infty}\left(\int_x^\infty e^{-t^2/2}dt\right).$$

b) Let $\varepsilon > 0$. From a), we see that

$$\sum_{n=1}^\infty \mathbb{P}(X_n/\sqrt{2\log(n)} > 1 + \varepsilon) < \infty.$$

By Borel-Cantelli I, and taking a countable intersection as in 2)a), we deduce that a.s.,

$$\limsup_{n\to\infty} X_n/\sqrt{2\log n} \le 1.$$

As $\sum \frac{1}{n\sqrt{\log(n)}}$ diverges,

$$\sum_{n=1}^\infty \mathbb{P}(X_n/\sqrt{2\log(n)} \le 1) = \infty.$$

As the X_i's are independent, we can apply Borel-Cantelli II, to deduce that a.s.

$$\limsup_{n\to\infty} X_n/\sqrt{2\log n} \ge 1.$$

Remark. The fact that $\int_2^\infty \frac{1}{x\log(x)^a}dx = \infty$ iff $a \le 1$ is classical. If you forgot it, please compute the integral making the change of variables $y = \log(x)$.

Exercise 4. (Solution of Moritz Dober)

a) For $\varepsilon \in \{0,1\}$ the event $\{X_n = \varepsilon\}$ is the same as U being in every second of dyadic intervals in (0,1) of length 2^{-n}, hence $\mathbb{P}(X_n = \varepsilon) = 1/2$. Furthermore we have that $X_1 = \varepsilon_1, ..., X_n = \varepsilon_n$ iff $U \in [\sum_{i=1}^n \varepsilon_i 2^{-i}, \sum_{i=1}^n \varepsilon_i 2^{-i} + 2^{-n})$. Therefore $\mathbb{P}(X_1 = \varepsilon_1, ..., X_n = \varepsilon_n) = 2^{-n}$. The general case follows by taking finite unions of these events.

b) Let $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be a bijection and for $n, i \geq 1$ set $X_{n,i} := X_{f(n,i)}$. Then clearly $(X_{n,i})_{n,i\geq 1}$ have the same law as X_1 and since f is injective they are also independent, i.e. the family $(\sigma(X_{n,i}))_{n,i\geq 1}$ is independent. Therefore by regrouping the filtrations $\mathcal{F}_n = \sigma(X_{n,i} : i \geq 1)$ are independent of one another.

c) Set $U_n := \sum_{i\geq 1} X_{n,i} 2^{-i}$. Then the U_n are independent since $\sigma(U_n) \subseteq \mathcal{F}_n$ and we have for $x = \sum_{i\geq 1} \varepsilon_i 2^{-i} \in (0,1)$, $\varepsilon_i \in \{0,1\}$,

$$\begin{aligned}\mathbb{P}(U_n > x) &= \mathbb{P}\left(\bigsqcup_{i\geq 1}\{X_{n,1} = \varepsilon_1, ..., X_{n,i-1} = \varepsilon_{i-1}, X_{n,i} > \varepsilon_i\}\right) \\ &= \sum_{i\geq 1} 2^{-(i-1)} \underbrace{2^{-1}(1-\varepsilon_i)}_{=\mathbb{P}(X_{n,i}>\varepsilon_i)} = 1 - x,\end{aligned}$$

hence $\mathbb{P}(U_n \leq x) = x$, i.e. $U_n \sim U(0,1)$.

d) As in exercise 3 on sheet 3 define $F_n(x) : x \in \mathbb{R} \mapsto \mu_n((-\infty, x]) \in [0,1]$ for $n \geq 1$ and let F_n^{-1} be their càdlàg inverses, i.e.

$$F_n^{-1}(u) := \sup\{x \in \mathbb{R} \mid F_n(x) < u\} \text{ for } u \in [0,1].$$

Then $Y_n := F_n^{-1}(X_n)$ are independent as functions of independent r.v. and $Y_n \sim \mu_n$ since (as in exercise 3 on sheet 3) the law of Y_n coincides with μ_n on $\{(-\infty, x] \mid x \in \mathbb{R}\}$.