PS: Advanced Probability Theory
Sheet 3 Solutions

lucas.teyssier@univie.ac.at

Exercise 1. First of all, we have for $d \in \mathbb{N}$

$$\mathbb{P}(d \mid X) = \mathbb{P}\left(\bigsqcup_{n\geq 1}\{X = dn\}\right) = \sum_{n\geq 1}\mathbb{P}(X = dn) = \sum_{n\geq 1}\frac{(dn)^{-s}}{\zeta(s)} = d^{-s}.$$

Now let $k, n_i \in \mathbb{N}$ for $i = 1, ..., k$ and let $p_1, ..., p_k$ be distinct prime numbers, then

$$\mathbb{P}(p_i^{n_i} \mid X, \text{ for } i = 1, ..., k) = \mathbb{P}(p_1^{n_1} \cdots p_k^{n_k} \mid X) = (p_1^{n_1} \cdots p_k^{n_k})^{-s} = \prod_{i=1}^{k}\mathbb{P}(p_i^{n_i} \mid X),$$

hence the events $\{p^{n_p} \mid X\}_{p \text{ prime}, \, n_p \in \mathbb{N}}$ are independent. Let us deduce Euler's formula from this and continuity of measure. Let $\{p_i\}_{i\geq 1}$ be the set of all prime numbers. We have

$$\begin{aligned}\frac{1}{\zeta(s)} &= \mathbb{P}(X = 1) = \mathbb{P}\left(\bigcap_{i\geq 1}\{p_i \nmid X\}\right) = \lim_{n\to\infty}\mathbb{P}(p_i \nmid X, \text{ for } i = 1, ..., n)\\ &= \lim_{n\to\infty}\prod_{i=1}^{n}\mathbb{P}(p_i \nmid X) = \lim_{n\to\infty}\prod_{i=1}^{n}(1 - p_i^{-s}) = \prod_{p \text{ prime}}(1 - p^{-s}).\end{aligned}$$

Furthermore,

$$\mathbb{P}(X \text{ is square-free}) = \mathbb{P}\left(\bigcap_{i\geq 1}\{p_i^2 \nmid X\}\right) = \lim_{n\to\infty}\prod_{i=1}^{n}\mathbb{P}\left(p_i^2 \nmid X\right) = \prod_{p \text{ prime}}(1 - p^{-2s}) = \frac{1}{\zeta(2s)}.$$

To determine the distribution of $\operatorname{hcf}(X, Y)$ we first consider the case where X and Y are coprime. By independence of different primes dividing X, Y and of X and Y we get

$$\begin{aligned}\mathbb{P}(\operatorname{hcf}(X, Y) = 1) &= \mathbb{P}\left(\bigcap_{i\geq 1}\{p_i \mid X, Y\}^{\complement}\right) = \lim_{n\to\infty}\mathbb{P}\left(\bigcap_{i=1}^{n}\{p_i \mid X, Y\}^{\complement}\right)\\ &= \lim_{n\to\infty}\prod_{i=1}^{n}\mathbb{P}\left(\{p_i \mid X, Y\}^{\complement}\right) = \lim_{n\to\infty}\prod_{i=1}^{n}(1 - \mathbb{P}(p_i \mid X, Y))\\ &= \lim_{n\to\infty}\prod_{i=1}^{n}(1 - \mathbb{P}(p_i \mid X)\,\mathbb{P}(p_i \mid Y)) = \lim_{n\to\infty}\prod_{i=1}^{n}(1 - p_i^{-2s}) = \frac{1}{\zeta(2s)}.\end{aligned}$$

For the general statement we use the fact that $\text{hcf}(a,b)=n$ if and only if there exist $k,l\geq 1$ with $\text{hcf}(k,l)=1$ such that $a=nk$ and $b=nl$. We conclude

$$\begin{aligned}\mathbb{P}\left(\text{hcf}(X,Y)=n\right)&=\mathbb{P}\left(\bigsqcup_{\substack{k,l\geq 1\\ \text{hcf}(k,l)=1}}\{X=nk,Y=nl\}\right)=\sum_{\substack{k,l\geq 1\\ \text{hcf}(k,l)=1}}\mathbb{P}\left(X=nk,Y=nl\right)\\&=\sum_{\substack{k,l\geq 1\\ \text{hcf}(k,l)=1}}n^{-2s}\,\mathbb{P}\left(X=k,Y=l\right)=n^{-2s}\,\mathbb{P}\left(\text{hcf}(X,Y)=1\right)=\frac{n^{-2s}}{\zeta(2s)}.\end{aligned}$$

Exercise 2.

a) Let μ be the law of X. Since X,Y are independent we have by disintegration, for all $t\in\mathbb{R}$,

$$\mathbb{P}\left(X+Y\leq t\right)=\int_{\mathbb{R}}\mathbb{P}\left(x+Y\leq t\right)\,d\mu(x)=\int_{\mathbb{R}}\mathbb{P}\left(Y\leq t-x\right)\,d\mu(x)=\int_{\mathbb{R}}G(t-x)\,d\mu(x).$$

b) If X has density f w.r.t. the Lebesgue measure λ we get by a), for all $t\in\mathbb{R}$,

$$\mathbb{P}\left(X+Y\leq t\right)=\int_{\mathbb{R}}G(t-x)f(x)\,d\lambda(x)=(f*G)(t).$$

The other case where Y has a density g w.r.t. λ follows by symmetry.

c) Using the transformation formula and Tonelli's theorem, we have by b), for all $t\in\mathbb{R}$,

$$\begin{aligned}\mathbb{P}\left(X+Y\leq t\right)&=\int_{\mathbb{R}}G(t-x)f(x)\,d\lambda(x)=\int_{\mathbb{R}}\left(\int_{-\infty}^{t-x}g(s)\,d\lambda(s)\right)f(x)\,d\lambda(x)\\&=\int_{\mathbb{R}}\left(\int_{-\infty}^{t}g(s-x)\,d\lambda(s)\right)f(x)\,d\lambda(x)\\&=\int_{-\infty}^{t}\underbrace{\left(\int_{\mathbb{R}}g(s-x)f(x)\,d\lambda(x)\right)}_{=(f*g)(s)}\,d\lambda(s).\end{aligned}$$

d) Let X,Y be independent exponential r.v. with parameter λ and λ' respectively, i.e. they have densities $f(x)=\mathbb{1}_{[0,\infty)}(x)\lambda e^{-\lambda x}$ and $g(x)=\mathbb{1}_{[0,\infty)}(x)\lambda' e^{-\lambda' x}$ respectively. Then, for $x\geq 0$

$$(f*g)(x)=\int_0^x\lambda e^{-\lambda t}\lambda' e^{-\lambda'(x-t)}\,dt=\lambda\lambda' e^{-\lambda' x}\int_0^x e^{(-\lambda+\lambda')t}\,dt.$$

We clearly have $(f*g)(x)=0$ for $x<0$, hence $X+Y$ has density

$$(f*g)(x)=\begin{cases}\mathbb{1}_{[0,\infty)}(x)\frac{\lambda\lambda'}{\lambda'-\lambda}(e^{-\lambda x}-e^{-\lambda' x}), & \text{if } \lambda\neq\lambda'\\ \mathbb{1}_{[0,\infty)}(x)\lambda^2 xe^{-\lambda x}, & \text{if } \lambda=\lambda'\end{cases}.$$

Now, let X,Y be independent uniform r.v. on $(0,1)$, i.e. they have densities $f=g=\mathbb{1}_{(0,1)}$.

Then $X+Y$ has density

$$(f*g)(x) = \int_{\mathbb{R}} \mathbb{1}_{(0,1)}(t) \underbrace{\mathbb{1}_{(0,1)}(x-t)}_{=\mathbb{1}_{(x-1,x)}(t)} \, dt = \int_0^1 \mathbb{1}_{(x-1,x)}(t) \, dt = \begin{cases} x, & \text{if } x \in (0,1] \\ 2-x, & \text{if } x \in (1,2) \\ 0, & \text{otherwise} \end{cases} .$$

This is the density of a symmetric triangular distribution, a continuous version of e.g. the distribution of the sum of two independent dice.

Exercise 3.

Remark. Unless it is stated explicitly, you can always assume in this course that r.v are real-valued.

a) Clearly X_1, X_2 are measurable. Let $x_1, x_2 \in [0,1]$, then

$$\mathbb{P}(X_1 \le x_1) = \mathbb{P}([0,x_1] \times [0,1]) = x_1$$

and likewise $\mathbb{P}(X_2 \le x_2) = x_2$. Furthermore

$$\mathbb{P}(X_1 \le x_1, X_2 \le x_2) = \mathbb{P}([0,x_1] \times [0,x_2]) = x_1 x_2,$$

hence X_1, X_2 are independent uniform r.v. on [0,1].

b) Let $F(x) = \mu((-\infty, x])$, $G(x) = \nu((-\infty, x])$ and let $F^{-1}, G^{-1} : [0,1] \to \mathbb{R}$ be their càdlàg inverses, i.e. $F^{-1}(u) = \sup\{y \in \mathbb{R} \mid F(y) < u\}$ and analogously for G^{-1}. Then as in the proof of the Lebesgue Stieltjes theorem

$$u \le F(x) \iff F^{-1}(u) \le x \text{ for all } u \in [0,1], \ x \in \mathbb{R}.$$

Now set $X' = F^{-1}(X_1)$, $Y' = G^{-1}(X_2)$, then X', Y' are independent as functions of independent r.v. and we have for $x \in \mathbb{R}$

$$\mathbb{P}(X' \le x) = \mathbb{P}\left(F^{-1}(X_1) \le x\right) = \mathbb{P}(X_1 \le F(x)) = F(x),$$

hence the law of X' coincides with μ and likewise for Y'.

Exercise 4.

a) The function $\tilde{f} : (S \times \Omega, \mathcal{S} \otimes \mathcal{F}) \to [0, \infty)$ given by $\tilde{f}(x, \omega) := f(x, Y(\omega))$ is measurable since f and Y are. Hence (as in Tonelli's theorem) for $x \in S$ the sections $\tilde{f}_x = f(x, Y) : \Omega \to [0, \infty)$ are measurable and so is $x \mapsto \int_\Omega \tilde{f}_x \, d\mathbb{P} = \mathbb{E}[f(x,Y)] = g(x)$.

To show that $\mathbb{E}[g(X)] = \mathbb{E}[f(X,Y)]$ we first assume that f is an indicator function, i.e. $f = \mathbb{1}_Q$ for some $Q \in \mathcal{S} \otimes \mathcal{S}'$. Let $\mu = X_*(\mathbb{P})$ be the law of X, then we have

$$\begin{aligned} \mathbb{E}[g(X)] &= \int_\Omega g(X(x)) \, d\mathbb{P}(x) = \int_\Omega \int_\Omega f(X(x), Y(y)) \, d\mathbb{P}(y) \, d\mathbb{P}(x) \\ &= \int_\Omega \int_\Omega \underbrace{\mathbb{1}_Q(X(x), Y(y))}_{=\mathbb{1}_{\{(X(x),Y) \in Q\}}(y)} \, d\mathbb{P}(y) \, d\mathbb{P}(x) = \int_\Omega \mathbb{P}((X(x), Y) \in Q) \, d\mathbb{P}(x) \\ &\overset{*}{=} \int_S \mathbb{P}((x, Y) \in Q) \, d\mu(x) \overset{**}{=} \mathbb{P}((X,Y) \in Q) = \mathbb{E}[f(X,Y)], \end{aligned}$$

where we used the main property of the pushforward measure in * and disintegration in ** (since X, Y are independent).
Now, the case where f is a non-negative simple function follows immediately by linearity of expectation and the case where f is non-negative and measurable by monotone convergence/definition of the integral.

b) Take $f(x, y) := |x + y|$, then by a) and assumption we have $\mathbb{E}[g(X)] = \mathbb{E}[f(X, Y)] = \mathbb{E}[|X + Y|] < \infty$. Hence $g(X) < \infty$ almost surely. In particular there exists $\omega \in \Omega$ with $g(X(\omega)) < \infty$. Using the hint ($|Y| \leq |x| + |Y + x|$ for all $x \in \mathbb{R}$) we conclude

$$\mathbb{E}[|Y|] \leq |X(\omega)| + \underbrace{\mathbb{E}[|Y + X(\omega)|]}_{=g(X(\omega))} < \infty.$$

By symmetry we also have that X is integrable.

c) Just choose a r.v. X that is not integrable and set $Y = -X$, e.g. $X : x \in (0, 1) \mapsto \frac{1}{x} \in \mathbb{R}$.

Acknoledgements. Many thanks to Moritz Dober for writing those solutions.