PS: Advanced Probability Theory
Sheet 1 Solutions

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Exercise 1 (L.T.)

a) By definition of the expectation and rewriting the integral,

$$\mathbb{E}[X] = \int_\Omega X(\omega)\mathbb{P}(d\omega) = \int_\Omega \int_0^\infty 1_{X(\omega)\geq x} dx \mathbb{P}(d\omega). \tag{1}$$

Then, applying Fubini-Tonelli to the nonnegative function $1_{X(\cdot)\geq \cdot}$ (i.e. $(\omega, x) \mapsto 1_{X(\omega)\geq x}$)[1] gives

$$\mathbb{E}[X] = \int_0^\infty \int_\Omega 1_{X(\omega)\geq x}\mathbb{P}(d\omega)dx = \int_0^\infty \mathbb{P}(X \geq x)dx. \tag{2}$$

If X has support in $\mathbb{N}$, we deduce that

$$\mathbb{E}[X] = \sum_{n=1}^\infty \int_{n-1}^n \mathbb{P}(X \geq x)dx = \sum_{n=1}^\infty \mathbb{P}(X \geq n). \tag{3}$$

b) Let us first observe that for all $n \geq 0$,

$$\mathbb{P}(N \geq n+1) = \mathbb{P}(X_1 = \max_{1\leq i\leq n} X_i), \tag{4}$$

and consequently, by symmetry, that

$$\mathbb{P}(N \geq n+1) \geq \frac{1}{n}. \tag{5}$$

We deduce, using Question a), that

$$\mathbb{E}[N] \geq \sum_{n=1}^\infty \frac{1}{n} = \infty. \tag{6}$$

Remark. You can think about whether the inequality in Equation (5) is always strict.

[1] To be more rigorous, you could write on which space this function lives but this is not necessary.

Exercise 2 (Moritz Dober)

Take $\Omega = I$, $\mathcal{F} = \mathcal{P}(\Omega)$, $\mathbb{P} = \sum_{i\in I} p_i \cdot \delta_i$ (where δ_i is the Dirac measure on $\{i\}$) and $X : \Omega \to \mathbb{R}$ given by $X(i) = x_i$.

Then we have $\mathbb{P}(X = x_i) = \mathbb{P}(\{i\}) = p_i$. Its law is unique since for every such X it holds that $\mathbb{P}(X \in \{x_i \mid i \in I\}) = \sum_{i\in I} p_i = 1$ and hence for $A \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{P}(X \in A) = \mathbb{P}(X \in A \cap \{x_i \mid i \in I\}) = \sum_{i\in I} \mathbb{P}(X \in A \cap \{x_i\}) = \sum_{i\in I} p_i \cdot \delta_{x_i}(A).$$

Exercise 3 (Daniel Bäumer)

a) It holds that $P(-\log U \leq t) = \lambda(-\log U \leq t) = \lambda(U \geq e^{-t}) = 1 - e^{-t}$. Hence, $F(t) = 1 - e^{-t}$, which is the distribution function of an exponentially distributed random variable with parameter 1.

b) In the proof of the Lebesgue-Stieltjes theorem, we applied the cadlag-inverse of a given function with the required properties to a uniformly distributed random variable. The construction in (a) is almost the same, except that the inverse of F is $-\log(1 - U)$, which has the same distribution.

c) Computing the distribution function first, $P(U^2 \leq t) = P(U \leq \sqrt{t}) = \sqrt{t}$. This is seen to have a density w.r.t. the Lebesgue measure, since $F_{U^2}(t) = \int_0^t \frac{1}{2\sqrt{s}} ds$, hence $f_{U^2}(t) = \frac{1}{2\sqrt{t}}$ and the law of U^2 (which is uniquely determined by the distribution function, as seen in Exercise 4) is

$$\mu_{U^2}(B) = \int_B \frac{1}{2\sqrt{x}} dx.$$

Exercise 4, first solution (Moritz Dober)

Claim: open intervals are in $\sigma(\mathcal{A})$

For $a, b, c, d \in \mathbb{R}$ with $a < b$, $c < d$ we have

$$(a, b] = (-\infty, b] \cap (a, \infty) = (-\infty, b] \cap (-\infty, a]^{\complement} \in \sigma(\mathcal{A})$$

and therefore

$$(c, d) = \bigcup_{n\geq 1} \underbrace{(c, d - \frac{d-c}{n}]}_{\in\sigma(\mathcal{A})} \in \sigma(\mathcal{A}).$$

Every open set $O \in \mathcal{O}$ can be represented as union of open intervals with rational endpoints, hence as a countable union of open intervals. This implies $\mathcal{O} \subseteq \sigma(\mathcal{A})$ and hence $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{O}) \subseteq \sigma(\mathcal{A})$.

Clearly $\mathcal{A} \subseteq \sigma(\mathcal{O})$ (and thus $\sigma(\mathcal{A}) \subseteq \sigma(\mathcal{O})$) since $(-\infty, a] = \bigcap_{n\geq 1}(-\infty, a + \frac{1}{n})$.

$\mathcal{A}$ is a π-system since $(-\infty, a] \cap (-\infty, b] = (-\infty, \min\{a, b\}] \in \mathcal{A}$.

Now, let X, Y be two random variables with the same distribution functions $F_X = F_Y$. Then $\mu_X|_{\mathcal{A}} = \mu_Y|_{\mathcal{A}}$ since

$$\mu_X((-\infty, a]) = F_X(a) = F_Y(a) = \mu_Y((-\infty, a]).$$

Set $\mathcal{D} := \{A \in \mathcal{B}(\mathbb{R}) \mid \mu_X(A) = \mu_Y(A)\}$ which is a Dynkin system with $\mathcal{A} \subseteq \mathcal{D}$.

By Dynkin's lemma the σ-algebra generated by $\mathcal{A}$ is the same as the Dynkin system generated by $\mathcal{A}$, i.e. $\sigma(\mathcal{A}) = \delta(\mathcal{A})$. Therefore we conclude

$$\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{O}) = \sigma(\mathcal{A}) = \delta(\mathcal{A}) \subseteq \delta(\mathcal{D}) = \mathcal{D},$$

i.e. $\mu_X \equiv \mu_Y$ on $\mathcal{B}(\mathbb{R})$.

Exercise 4, second solution (Daniel Bäumer)

In order to prove that $\mathcal{A}$ generates the Borel σ-algebra on $\mathbb{R}$, we need to show that $\mathcal{A}$ contains all open sets in $\mathbb{R}$. By taking complements and intersections, it is clear that $\sigma(\mathcal{A})$ contains all intervals of the form $(a, b]$; for a sequence $b_n \uparrow b$, we also have $\bigcup_{n=1}^{\infty}(a, b_n] = (a, b)$, so $\sigma(\mathcal{A})$ contains all open intervals. The fact that every open set in $\mathbb{R}$ is a countable union of open intervals finishes the proof.

The intersection of any two sets in $\mathcal{A}$ is equal to the smaller of the two, so $\mathcal{A}$ is intersection stable. Now, looking at two random variables X_1, X_2 with the same distribution and laws μ_1, μ_2, it immediately follows that $\mu_1((-\infty, a]) = \mu_2((-\infty, a])$ for all $a \in \mathbb{R}$. This just means $\mu_1 = \mu_2$ on $\mathcal{A}$.

Since $\mathcal{D} := \{A \in \mathcal{B}(\mathbb{R}) : \mu_1(A) = \mu_2(A)\}$ is a Dynkin system that contains the π-system $\mathcal{A}$, we can apply Dynkin's lemma: $\mathcal{D} \supset \sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$, so $\mu_1 = \mu_2$.