PS: Advanced Probability Theory
Sheet 10 Solutions

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Exercice 1. (Daniel Bäumer) First, a conditional version of Fatou's lemma:
For a.s. nonnegative random variables Y_n, it holds

$$\liminf_{n\to\infty} \mathbb{E}[Y_n|\mathcal{G}] = \lim_{n\to\infty} \inf_{k\geq n} \mathbb{E}[Y_k|\mathcal{G}] \geq \lim_{n\to\infty} \mathbb{E}[\inf_{k\geq n} Y_k|\mathcal{G}] = \mathbb{E}[\lim_{n\to\infty} \inf_{k\geq n} Y_k|\mathcal{G}] = \mathbb{E}[\liminf_{n\to\infty} Y_n|\mathcal{G}],$$

by conditional MCT. Applying this to $Z - X_n$, we obtain

$$\mathbb{E}[Z - X|\mathcal{G}] = \mathbb{E}[\liminf_{n\to\infty}(Z - X_n)|\mathcal{G}] \leq \liminf_{n\to\infty} \mathbb{E}[Z - X_n|\mathcal{G}] = \mathbb{E}[Z|\mathcal{G}] - \limsup_{n\to\infty} \mathbb{E}[X_n|\mathcal{G}].$$

Similarly,

$$\mathbb{E}[X + Z|\mathcal{G}] = \mathbb{E}[\liminf_{n\to\infty}(X_n + Z)|\mathcal{G}] \leq \liminf_{n\to\infty} \mathbb{E}[X_n + Z|\mathcal{G}] = \liminf_{n\to\infty} \mathbb{E}[X_n|\mathcal{G}] + \mathbb{E}[Z|\mathcal{G}].$$

From the first relation,

$$\limsup_{n\to\infty} \mathbb{E}[X_n|\mathcal{G}] \leq \mathbb{E}[X|\mathcal{G}],$$

from the second

$$\liminf_{n\to\infty} \mathbb{E}[X_n|\mathcal{G}] \geq \mathbb{E}[X|\mathcal{G}],$$

which together imply a.s. convergence. Since

$$|\mathbb{E}[X_n|\mathcal{G}]| \leq \mathbb{E}[|X_n||\mathcal{G}] \leq \mathbb{E}[Z|\mathcal{G}],$$

the sequence $\mathbb{E}[X_n|\mathcal{G}]$ is bounded from above by an integrable random variable; hence, it converges to X in L^1 by classical DCT. □

Exercise 2. (D.B.) If $f = \mathbb{1}_{A\times B}$, $\mathbb{E}[f(X, y)] = \mathbb{P}(X \in A)$ if $y \in B$ and 0 otherwise.
By properties of conditional expectation, we also see that

$$\mathbb{E}[f(X, Y)|\mathcal{G}] = \mathbb{E}[\mathbb{1}_{x\in A}\mathbb{1}_{y\in B}|\mathcal{G}] = \mathbb{1}_{Y\in B}\mathbb{E}[\mathbb{1}_{x\in A}|\mathcal{G}] = \mathbb{1}_{Y\in B}\mathbb{E}[\mathbb{1}_{x\in A}] = \mathbb{P}(X \in A)$$

if $Y \in B$ and 0 otherwise. Therefore, the equality holds for indicator functions.
For linear combinations of indicators, the result follows from linearity of (conditional) expectation.
If $f \geq 0$ is an arbitrary measurable function, there is a sequence f_n of simple functions such that $f_n \uparrow f$. Applying (conditional) MCT to that sequence yields the general result. □

Exercise 3. (Moritz Dober)

a) Let $\mathcal{F}_n = \sigma(X_1, ..., X_n)$, $n \geq 1$. Clearly M_n is measurable w.r.t. $\mathcal{F}_n$ for all n. Moreover M_n is non-negative since cosh is and $M_n \leq \cosh(un)/\cosh(u)^n$ because cosh is an even function, non-decreasing on $\mathbb{R}^+$ and $|S_n| \leq n$. In particular M_n is integrable for all n. To verify $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$ we use that $\cosh(x) = (e^x + e^{-x})/2$ for $x \in \mathbb{R}$. We have for $n \geq 1$

$$\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \frac{1}{2\cosh(u)^{n+1}}\mathbb{E}\left[e^{uS_{n+1}} + e^{-uS_{n+1}}|\mathcal{F}_n\right] \\
&= \frac{1}{2\cosh(u)^{n+1}}\left(\mathbb{E}\left[e^{uS_n}e^{uX_{n+1}}|\mathcal{F}_n\right] + \mathbb{E}\left[e^{-uS_n}e^{-uX_{n+1}}|\mathcal{F}_n\right]\right) \\
&= \frac{1}{2\cosh(u)^{n+1}}(e^{uS_n}\underbrace{\mathbb{E}\left[e^{uX_{n+1}}\right]}_{=\cosh(u)} + e^{-uS_n}\underbrace{\mathbb{E}\left[e^{-uX_{n+1}}\right]}_{=\cosh(u)}) = M_n,
\end{aligned}$$

since S_n is $\mathcal{F}_n$-measurable and X_{n+1} is independent of $\mathcal{F}_n$.

b) We have that $\tau_a < \infty$ a.s.. Indeed, $A_n := \{X_{2an+1} = ... = X_{2a(n+1)} = 1\}$, $n \geq 0$, are independent, $\mathbb{P}(A_n) = (1/2)^{2a} > 0$ and the second Borel-Cantelli lemma gives that a.s. infinitely many A_n occur, but $A_n \subseteq \{\tau_a < \infty\}$. Furthermore $\cosh \geq 1$, $|S_{\tau_a \wedge n}| \leq a$ and hence $|M_{\tau_a \wedge n}| \leq \cosh(ua)$, so $\{M_{\tau_a \wedge n} : n \geq 1\}$ is UI. Under this conditions we can apply the optional stopping theorem to get $\mathbb{E}[M_{\tau_a}] = \mathbb{E}[M_1] = 1$. On the other hand we have

$$\mathbb{E}[M_{\tau_a}] = \mathbb{E}\left[\frac{\cosh(uS_{\tau_a})}{\cosh(u)^{\tau_a}}\right] = \cosh(ua)\mathbb{E}\left[\cosh(u)^{-\tau_a}\right].$$

We deduce that $\mathbb{E}[\cosh(u)^{-\tau_a}] = \cosh(ua)^{-1}$.

Exercise 4. (D.B.)

a) $M_n := \mathbb{E}[X|\mathcal{F}_n]$ is a martingale by the tower property of conditional expectation. It holds for all n

$$|M_n| = |\mathbb{E}[X|\mathcal{F}_n]| \leq \mathbb{E}[|X||\mathcal{F}_n],$$

hence $\mathbb{E}[|M_n|\mathbb{1}_A] \leq \mathbb{E}[|X|\mathbb{1}_A]$ for all $A \in \mathcal{F}_n$. It follows from the characterization in exercise 9.4 that M_n is UI.
Now, Doob's convergence theorem can be applied: There is a random variable M such that $M_n \to M$ a.s.; by uniform integrability, also $M_n \to M$ in L^1. We only need to show that $M = \mathbb{E}[X|\mathcal{F}_\infty]$.
Looking at the Dynkin-system $\{A : \mathbb{E}[M\mathbb{1}_A] = \mathbb{E}[X\mathbb{1}_A]\}$, this contains all $\mathcal{F}_n$, therefore also $\bigcup_{n\geq 1} \mathcal{F}_n = \mathcal{F}_\infty$. Since M is measurable with respect to $\mathcal{F}_\infty$, it must be $M = \mathbb{E}[X|\mathcal{F}_\infty]$.

b) $M_n := \mathbb{E}[X|\mathcal{F}_n]$ is a martingale by the tower property of conditional expectation. It holds for all n

$$|M_n| = |\mathbb{E}[X|\mathcal{F}_n]| \leq \mathbb{E}[|X||\mathcal{F}_n],$$

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Looking at the Dynkin-system $\{A : \mathbb{E}[M\mathbb{1}_A] = \mathbb{E}[X\mathbb{1}_A]\}$, this contains all $\mathcal{F}_n$, therefore also $\bigcup_{n\geq 1} \mathcal{F}_n = \mathcal{F}_\infty$. Since M is measurable with respect to $\mathcal{F}_\infty$, it must be $M = \mathbb{E}[X|\mathcal{F}_\infty]$.

Exercise 5. (M.D.)

a) We have that $S_n = \sum_{i=1}^n X_i$ is a martingale. Furthermore using the Cauchy–Schwarz inequality and that the X_i are independent we get

$$\mathbb{E}\left[|S_n|\right]^2 \leq \mathbb{E}\left[S_n^2\right] = \mathrm{Var}(S_n) = \sum_{i=1}^n \mathrm{Var}(X_i) \leq \sum_{i\geq 1} \mathrm{Var}(X_i) < \infty,$$

so (S_n) is bounded in L^1. We conclude from the martingale convergence theorem that S_n converges a.s..

b) Set $Y_n = X_n/a_n$, $n \geq 2$, then the Y_n are still independent, $\mathbb{E}\left[Y_n\right] = 0$ and

$$\sum_{n\geq 2} \mathrm{Var}(Y_n) = \sum_{n\geq 2} \frac{\sigma^2}{a_n^2} = \sigma^2 \sum_{n\geq 2} \frac{1}{n \log(n)^{1+2\varepsilon}} < \infty.$$

Using a) we deduce that $\sum_{n\geq 2} Y_n$ converges with probability 1.

c) We have for sequences $(b_k)_{k\geq 1}$, $(c_k)_{k\geq 1}$ and $n \geq 2$

$$\sum_{k=1}^n b_k c_k = B_n c_n - \sum_{k=1}^{n-1} B_k \Delta c_k,$$

where $B_k = b_1 + ... + b_k$ and $\Delta c_k = c_{k+1} - c_k$.
Set $S'_n = \sum_{i=2}^n Y_i$ and let S' be its a.s.-limit. Fix ω in the set of convergence and consider $X_i = X_i(\omega)$, $Y_i = Y_i(\omega)$... Let $\varepsilon > 0$, then there exists $N \geq 1$ such that $|S'_n - S'| < \varepsilon$ for all $n \geq N$. Using the above version of discrete summation by parts we get for $n > N$

$$\begin{aligned}\frac{S_n}{a_n} &= \frac{1}{a_n}\sum_{i=2}^n X_i = \frac{1}{a_n}\sum_{i=2}^n Y_i a_i = \frac{1}{a_n}\left(S'_n a_n - \sum_{i=2}^{n-1} S'_i \Delta a_i\right) \\ &= \underbrace{S'_n}_{\to S'} - \underbrace{\frac{1}{a_n}\sum_{i=2}^{N-1} S'_i \Delta a_i}_{\to 0} - \underbrace{\frac{1}{a_n}\sum_{i=N}^{n-1} S'_i \Delta a_i}_{\in \frac{a_n - a_N}{a_n}[S'-\varepsilon, S'+\varepsilon]} \quad .\end{aligned}$$

Letting $n \to \infty$ we obtain that $\limsup_{n\to\infty} |S_n/a_n| < \varepsilon$. Since ε and ω were arbitrary we conclude that a.s. $\limsup_{n\to\infty} S_n/a_n = 0$.