PS: Advanced Probability Theory
Sheet 10

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Exercise 1 (Dominated convergence theorem for conditional expectation)**.** Let $X, Z, X_n, n \geq 1$ be random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. Assume that a.s. $X_n \to X$ and $\sup_n |X_n| \leq Z$ with $\mathbb{E}[Z] < \infty$. Show that $\mathbb{E}[X_n|\mathcal{G}] \to \mathbb{E}[X|\mathcal{G}]$ a.s. and in L^1.

Exercise 2. Suppose $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra and X, Y are random variables. We suppose that Y is $\mathcal{G}$-measurable and X is independent from $\mathcal{G}$. Let f be a measurable Borel function from $\mathbb{R}^2$ to $\mathbb{R}^+$ and $\Phi : y \mapsto \mathbb{E}(f(X, y)) \in \mathbb{R}^+$. Show that almost surely,

$$\mathbb{E}(f(X, Y)|\mathcal{G}) = \Phi(Y).$$

Hint. Start proving the result for f of the form $1_{A \times B}$ where A, B are Borel sets.

Exercise 3. Let $X_i, i \geq 1$, be i.i.d. variables with $\mathbb{P}(X_1 = -1) = \mathbb{P}(X_1 = 1) = 1/2$. Consider the random walk $S_n = X_1 + \cdots + X_n$ and for $a \geq 1$, let τ_a be the exit time of $[-(a-1), a-1]$:

$$\tau_a = \inf\{n \geq 1 : S_n = a \text{ or } -a\}.$$

a) Show that for all $u > 0$, $M_n := \frac{\cosh(uS_n)}{\cosh(u)^n}$ is a martingale.

b) Deduce an expression for $u \in \mathbb{R} \mapsto \mathbb{E}[\cosh(u)^{-\tau_a}]$.

Exercise 4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_n, n \geq 1)$.

a) Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Show that $M_n := \mathbb{E}[X|\mathcal{F}_n]$ is a martingale which converges to $\mathbb{E}[X|\mathcal{F}_\infty]$ in L^1 where $\mathcal{F}_\infty = \sigma(\mathcal{F}_k, k \geq 1)$.

b) Conversely, let $(M_n, \mathcal{F}_n)$ be a martingale bounded in L^p for some $p > 1$. Show that there exists a random variable $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $M_n = \mathbb{E}[X|\mathcal{F}_n]$.

Exercise 5.

a) Suppose X_i are independent with $\mathbb{E}(X_i) = 0$. Suppose also that $\sum_i \mathrm{Var}(X_i) < \infty$. Show that $\sum_i X_i$ converges almost surely.

b) Now suppose that $X_i, i \geq 2$, are i.i.d. with $\mathbb{E}(X_i) = 0$ and $\mathrm{Var}(X_i) = \sigma^2 < \infty$. Show that $\sum_n (X_n/a_n)$ converges a.s., where $a_n = n^{1/2}(\log n)^{1/2+\varepsilon}$ (and $\varepsilon > 0$ is fixed) for $n \geq 2$.

c) Deduce that $\limsup_{n\to\infty} S_n/a_n = 0$ a.s.
Hint. $\left(\frac{1}{a_n}\right)$ is decreasing. Apply a discrete summation by parts.
(This step is deterministic and known as Kronecker's lemma.)

In fact, a result due to Kolmogorov (the so called iterated law of logarithm) shows that

$$\limsup_{n\to\infty} \frac{S_n}{n^{1/2}(\log\log n)^{1/2}} = \sigma\sqrt{2}, a.s.$$

Optional exercise.

Exercise A.

a) Let $(W_i^{(j)}, i \geq 1, j = 1 \ldots 2^{i-1})$ be i.i.d. random variables with $W > 0$ and $\mathbb{E}[W] = 1$. For $i \geq 1$ and $x \in [0,1]$, we write $W_i(x) = W_i^{(j)}$ where j is so that $x \in [(j-1)2^{-i+1}, j2^{-i+1})$. For any Borel subset I of $[0,1]$, we finally define

$$M_n(I) = \int_I \prod_{i=1}^n W_i(x)dx.$$

Show that $M_n(I)$ is a nonnegative martingale for the filtration $\mathcal{F}_n = \sigma(W_i^{(j)}, i = 1 \ldots n, j = 1 \ldots 2^{i-1})$.

b) Let $M_\infty(I)$ be the almost sure limit of $(M_n(I))$ and assume that $\mathbb{E}\left[W^2\right] < 2$. Show that $(M_n(I))$ is bounded in L^2. Deduce that if I is a non trivial interval of $[0,1]$ then $M_\infty(I)$ is positive with positive probability.

One can actually show that the sequence of random measures (M_n) converges in probability (for a suitable metric on the set of measures) towards a random measure M_∞. Moreover M_∞ is trivial iif $\mathbb{E}\left[W^2\right] \geq 4$.