PS: Advanced Probability Theory
Sheet 1

Dear students, this course is until further notice an online course. Please send me an email at lucas.teyssier@univie.ac.at so that I know who is taking the course and wants to have credits for it. Waiting for more information you can start to solve the exercises.

Exercise 1.

a) Let X be a non-negative random variable. Show that

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq x)\, dx.$$

Deduce that if X is a discrete random variable with support included in $\mathbb{N}$, then

$$\mathbb{E}[X] = \sum_{n \geq 1} \mathbb{P}(X \geq n).$$

b) Consider a sequence $(X_n, n \geq 1)$ of i.i.d. real-valued random variables (independent and identically distributed). Define $N = \inf\{n > 1 : X_n > X_1\}$. Show that it is better to accept the first offer than to wait for a higher one: $\mathbb{E}[N] = \infty$.

Exercise 2. Let $\{x_i\}_{i \in I}$ be a countable set of points in $\mathbb{R}$ (i.e., the index set I is countable or finite, and for each $i \in I$, $x_i \in \mathbb{R}$). Let $(p_i)_{i \in I}$ be such that $p_i \geq 0$ and $\sum_{i \in I} p_i = 1$. Show that there exists a random variable X on some probability space such that $\mathbb{P}(X = x_i) = p_i$. Show that its law is unique.

Exercise 3. Let U be a uniform random variable on $[0, 1]$.

a) Show that $-\log U$ is an exponential random variable with parameter 1.

b) What is the connection between the above fact and the the Lebesgue-Stieltjes theorem?

c) Let U be a uniform random variable on $[0, 1]$ and let $X = U^2$. Compute its law: does it have a density with respect to Lebesgue measure? If so what is it? What is its distribution function?

Exercise 4. (This uses concepts from measure theory, so you are encouraged to take a look at notes from this course).
Consider the collection $\mathcal{A}$ of semi-infinite closed intervals of the form $(-\infty, a]$ for $a \in \mathbb{R}$. Show that these generate the Borel σ-algebra on $\mathbb{R}$. Show that they form a π-system (i.e., if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$). Use Dynkin's lemma to show that two random variables with the same distribution function have the same law.

Feel free to ask any question at lucas.teyssier@univie.ac.at

Other exercises (optional)

Exercise A. A full deck of 52 cards is divided in half at random. Find an expression for the probability that each half contains the same number of red and black cards. Evaluate this expression as a decimal expansion. Use Stirling's formula

$$n! \sim_{n\to\infty} \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

to find an approximation for the same probability and evaluate this approximation as a decimal expansion.

Exercise B. Let X, Y and Z be real-valued random variables on a same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

a) Assume that $X = Y$ almost surely ($\mathbb{P}$-almost everywhere). Show that the law of X and Y are the same. What can you say about the converse?

b) Assume now that X and Y have the same law. Show that for any measurable function $f : \mathbb{R} \to \mathbb{R}$, $f(X)$ and $f(Y)$ have the same law. Show that XZ and YZ do *not* have necessarily the same law.