Invariance of the Equilibrium Set of Games with an Endogenous Sharing Rule*

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Abstract

We consider games with an endogenous sharing rule and provide conditions for the invariance of the equilibrium set, i.e., for the existence of a common equilibrium set for the games defined by each possible sharing rule. Applications of our results include Bertrand competition with convex costs, electoral competition, and contests.

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1 Introduction

Consider the classical problem of a Bertrand duopoly, i.e., two firms set prices to compete for customers. A difficulty in modelling this situation is that when the firms set the same price, customers are indifferent with respect to where to buy, so that it is unclear how to specify the firms' profits (payoffs). Games with an endogenous sharing rule, introduced by Simon and Zame (1990), avoid this difficulty by specifying players' payoffs by a correspondence rather than by a function, thus taking a broad stand on how to specify players' payoffs.

When analyzing a game with an endogenous sharing rule, one may be interested in obtaining the existence of a strategy profile that is an equilibrium for each possible sharing rule, henceforth, an invariant equilibrium. Indeed, whenever such invariant equilibrium exists, the choice of the sharing rule becomes immaterial. Specifically, the predictions provided by such invariant equilibrium are robust to the actual sharing rule that happens to occur, e.g. the actual choice made by consumers regarding which of the two duopolists to buy from.

An appealing scenario occurs when each strategy that is an equilibrium for some sharing rule is an invariant equilibrium, henceforth, when the equilibrium set is invariant. In this case, any equilibrium is robust to the actual sharing rule that happens to occur and, based on this robustness notion, there is no need to select amongst the set of equilibria. Furthermore, as pointed out by Lebrun (1996) and Jackson and Swinkels (2005), the invariance of the equilibrium set is also important from a practical viewpoint. Indeed, it allows us to analyze the equilibrium set of the game defined by a sharing rule we may be interest in by analyzing the equilibrium set of the game defined by any other (simpler, easier to analyze) sharing rule. In particular, in the presence of incomplete information, it is often easier to establish the existence of equilibrium for some type-independent sharing rule by first showing that some type-dependent sharing rule has an equilibrium and then appeal to the invariance of the equilibrium set.

In this paper, we establish results concerning the invariance of the equilibrium set for general games with an endogenous sharing rule. Our first key condition, called “virtual continuity,” roughly says that each player can, with a probability close to one, avoid points at which the payoff correspondence is multi-valued while virtually...
achieving the same payoff given the strategies of the other players, regardless of the particular sharing rule which is in force. We show that, under this condition, the equilibrium set coincides with the set of invariant equilibrium in the class of games defined by efficient sharing rules. More precisely, any strategy that is an equilibrium of the game defined by some sharing rule is also an equilibrium in the game defined by any efficient sharing rule. This means that for equilibrium analysis of virtually continuous games with an endogenous sharing rule, one may focus on efficient sharing rules. In this light, our result has the interesting interpretation that, in equilibrium, indeterminacies are resolved efficiently. Moreover, as we illustrate using simple Bertrand examples, this result is also useful to compute the equilibrium set of games with an endogenous sharing rule.

Our second key condition, called “strong indeterminacy,” roughly requires that indeterminacies are not fully eliminated by focusing on efficient sharing rules. More precisely, if a player has, at some action profile, more than one possible payoff, then there are at least two efficient payoff profiles at that action profile given different payoffs to that player. Our main result states that each game with an endogenous sharing rule satisfying virtual continuity and strong indeterminacy has an invariant equilibrium set.

Jackson and Swinkels (2005) have established the invariance of the equilibrium set for a specific setting of private-value auctions. Our contribution is to extend this conclusion to a general framework. The generality of our approach allows us to obtain new equilibrium invariance results for Bertrand competition with convex costs (along the lines of Dasgupta and Maskin (1986) and Maskin (1986)), for electoral competition (as in Duggan (2007)) and contests (as in Moldovanu and Sela (2001)).

The paper is organized as follows. In Section 2 we consider a simple motivating example to illustrate our results. In Section 3 we present our main results. Section 4 contains applications of our results. In Section 5 we present extensions of our results, in particular to Bayesian contexts.
2 A motivating example

Consider a standard Bertrand duopoly with zero costs and one commodity whose demand is \( d(x) = 1 - x \) where \( x \) is the lowest price in the market. Each of the two firms sets a price in the unit interval to attract customers. If prices are different (i.e. \( x_1 \neq x_2 \)), then the firm setting the lowest price, firm \( i \) say, receives a profit of \( (1 - x_i)x_i \) whereas the other firm receives a profit of zero. If \( x_1 = x_2 \), profits are indeterminate; if \( \theta \) denotes the fraction of the demand allocated to firm 1, then \( \theta(1 - x_1)x_1 \) is the profit of firm 1, and \( (1 - \theta)(1 - x_1)x_1 \) is that of firm 2; \( \theta \) is allowed to take any value in \([0, 1]\). The situation can thus be described by an endogenous sharing rule with two players (the two firms), each having as its action set the interval \([0, 1]\), and with a payoff correspondence \( Q : [0, 1]^2 \to \mathbb{R}^2 \) defined by setting, for each \( x \in [0, 1]^2 \),

\[
Q(x) = \begin{cases} 
(x_1(1 - x_1), 0) & \text{if } x_1 < x_2, \\
(0, x_2(1 - x_2)) & \text{if } x_1 > x_2, \\
\{(x_1\theta(1 - x_1), x_1(1 - \theta)(1 - x_1)) : \theta \in [0, 1]\} & \text{if } x_1 = x_2.
\end{cases}
\]

It turns out that the discussion of how to specify payoff in this Bertrand duopoly problem is immaterial for equilibrium analysis. Indeed, a particular way of specifying payoffs amounts to choosing a (measurable) selection of the payoff correspondence and, for any such choice, the resulting normal-form game has a unique equilibrium (both in pure and mixed strategies) where both firms set a price of zero. This is well-known for the case of the equal-sharing rule defined by setting \( \theta = 1/2 \) independently of the price vector \( x \) (see, e.g. Kaplan and Wettstein (2000)) and it will follow from our invariance result in the general case.

3 Invariance of the equilibrium set

3.1 Preliminaries

A game \( \Gamma = (N, (X_i)_{i \in N}, Q) \) with an endogenous sharing rule is defined by a finite set \( N \) of players, a compact metric space \( X_i \) of actions for each \( i \in N \), and an upper hemicontinuous (uhc in the sequel) payoff correspondence \( Q : X \to \mathbb{R}^N \) with non-empty compact values, writing \( X = \prod_{i \in N} X_i \).
We write $M_i$ for the set of Borel probability measures on $X_i$, and $M = \prod_{i \in N} M_i$. Given $\sigma = (\sigma_1, \ldots, \sigma_n) \in M$, we write $\tau_\sigma$ for the corresponding product measure on $X$. For each $i \in N$ and $x_i \in X_i$, $\delta_{x_i}$ denotes the Dirac measure at $x_i$.

Let $\Gamma = (N, (X_i)_{i \in N}, Q)$ be an endogenous sharing rule game. We write $S_Q$ for the set of all measurable selections of $Q$. For each $q \in S_Q$, $G_q = (X_i, q_i)_{i \in N}$ is a normal-form game. For each $i \in N$ and $\sigma \in M$, let $\bar{q}_i : M \to \mathbb{R}$ be defined by setting $\bar{q}_i(\sigma) = \int_X q_i d\tau_\sigma$. The mixed extension of $G_q$ is the normal-form game $G_\bar{q} = (M_i, \bar{q}_i)_{i \in N}$. A mixed strategy Nash equilibrium of $G_q = (X_i, q_i)_{i \in N}$ is a pure strategy Nash equilibrium of $G_\bar{q}$. Let $E(G_q)$ denote the set of mixed strategy Nash equilibria of $G_q$. From now on, we abuse notation and write $q_i(\sigma)$ instead of $\bar{q}_i(\sigma)$ for each $i \in N$ and $\sigma \in M$.

We say that $\sigma \in M$ is an equilibrium of $\Gamma$ if $\sigma \in \bigcup_{q \in S_Q} E(G_q)$. We write $E(\Gamma)$ for the set of all equilibria of $\Gamma$. For $\sigma \in E(\Gamma)$, we say that $\sigma$ is an invariant equilibrium of $\Gamma$ if $\sigma \in \bigcap_{q \in S_Q} E(G_q)$. We write $I(\Gamma)$ for the set of invariant equilibria of $\Gamma$.

Given $x \in X$, we say that $r \in Q(x)$ is efficient if $r' \in Q(x)$ and $r' \geq r$ implies $r' = r$. Define a correspondence $Q_{\text{eff}} : X \to \mathbb{R}^n$ by setting

$$Q_{\text{eff}}(x) = \{ r \in Q(x) : r \text{ is efficient} \}$$

for each $x \in X$, and write $S_{Q_{\text{eff}}}$ for the set of all selections of $Q_{\text{eff}}$. Of course, $S_{Q_{\text{eff}}} \neq \emptyset$, as $Q$ assumes non-empty compact values. For $\sigma \in E(\Gamma)$, we say that $\sigma$ is a $Q_{\text{eff}}$-invariant equilibrium of $\Gamma$ if $\sigma \in \bigcap_{q \in S_{Q_{\text{eff}}}} E(G_q)$. Let $I_{\text{eff}}(\Gamma)$ be the set of $Q_{\text{eff}}$-invariant equilibria.

For each $i \in N$, write $\pi_i$ for the projection of $\mathbb{R}^n$ onto the $i$-th copy of $\mathbb{R}$. Let $Q_i = \pi_i \circ Q$ and note that $Q_i$ is uhc with nonempty and compact values. Let $D_i$ be the set of action profiles at which $Q_i$ is multi-valued, i.e.,

$$D_i = \{ x \in X : \#(Q_i(x)) > 1 \}.$$

We say that $\Gamma = (N, (X_i)_{i \in N}, Q)$ is strongly indeterminate if $\#(\pi_i \circ Q_{\text{eff}}(x)) > 1$ for each $i \in N$ and $x \in D_i$. Equivalently, $\Gamma$ is strongly indeterminate if $D_i = D_{i\text{eff}}$ for each $i \in N$, where

$$D_{i\text{eff}} = \{ x \in X : \#(\pi_i \circ Q_{\text{eff}}(x)) > 1 \}.$$

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Footnote 1: The set $\{(q, \sigma) \in S_Q \times M : \sigma \in E(G_q)\}$ is the set of solutions of $\Gamma$ (see Simon and Zame (1990)); the projection of this set in $M$ equals $E(\Gamma)$.
Lemma 1. \(D_i\) is measurable for each \(i \in N\).

Proof. Because \(Q\) is uhc with non-empty and compact values, there is a sequence \(\langle q_k \rangle_{k \in \mathbb{N}}\) of measurable selections of \(Q\) such that \(\{q_k(x): k \in \mathbb{N}\}\) is dense in \(Q(x)\) for each \(x \in X\) (see Castaing and Valadier (1977, Corollary III.3, p. 63 and Theorem III.6, p. 65)). Now, for each \(i \in N\), \(X \setminus D_i = \{x \in X: q_{k,i}(x) = q_{0,i}(x)\text{ for all } k \in \mathbb{N}\}\).

For convenience of later reference, we record the following fact in form of a lemma; the lemma implies in particular that \(S_{Q_{\text{eff}}} \neq \emptyset\) (consider \(f(r) = \sum_{i \in N} r_i\)).

Lemma 2. If \(f: \mathbb{R}^n \to \mathbb{R}\) is continuous, then there is a \(q \in S_Q\) such that, for each \(x \in X\), \(f(q(x)) = \max_{r \in Q(x)} f(r)\)


For each \(q \in S_Q\) and \(i \in N\), player \(i\)'s value function is the function \(v_{q_i} : M \to \mathbb{R}\) defined by setting \(v_{q_i}(\sigma) = \sup_{\sigma_i' \in M_i} q_i(\sigma_i', \sigma_{-i})\) for each \(\sigma \in M\).

Recall that the \(\sigma\)-algebra of the universally measurable subsets of \(X\) equals \(\bigcap_{\mu} B_{\mu}\), where the intersection is over all the Borel probability measures \(\mu\) on \(X\) and \(B_{\mu}\) denotes the \(\mu\)-completion of the Borel \(\sigma\)-algebra.\(^2\) Each Borel probability measures \(\mu\) on \(X\) has an unique extension to the universally measurable subsets of \(X\), which we also denote by \(\mu\).

Lemma 3. \(D_{i_{\text{eff}}}^i\) is universally measurable for each \(i \in N\).

Proof. (a) By Lemma 2, \(Q_{\text{eff}}\) has nonempty values; we now show that \(\text{graph}(Q_{\text{eff}})\) is a Borel subset of \(X \times \mathbb{R}^N\). To see this, set \(A = \mathbb{R}^N\) and \(B = \mathbb{R}^N\), and for each \(m \in \mathbb{N}\setminus\{0\}\) let

\[
G_m = \{(x, a, b) \in X \times A \times B: a, b \in Q(x), b \geq a, b_i \geq a_i + 1/m\text{ for some } i \in N\}.
\]

Let \(H_m\) be the projection of \(G_m\) onto \(X \times A\). Because \(Q\) is uhc with compact values, \(G_m\) is compact, and hence so is \(H_m\). Now \(\text{graph}(Q_{\text{eff}}) = \text{graph}(Q) \setminus \bigcup_{m \in \mathbb{N}\setminus\{0\}} H_m\).

(b) By Castaing and Valadier (1977, Theorem III.22, p. 74), it follows from (a) that there is a sequence \(\langle q_k \rangle_{k \in \mathbb{N}}\) of universally measurable selections of \(Q\) such that the set \(\{q_k(x): k \in \mathbb{N}\}\) is dense in \(Q(x)\) for each \(x \in X\). Now, for each \(i \in N\), \(X \setminus D_{i_{\text{eff}}}^i = \{x \in X: q_{k,i}(x) = q_{0,i}(x)\text{ for all } k \in \mathbb{N}\}\).

\(^2\)It is easy to see that this definition is equivalent to Castaing and Valadier (1977, Definition 21, p. 73) where the intersection is over all the Borel positive bounded measures \(\mu\) on \(X\).
3.2 Virtual continuity

Our definition of virtual continuity is as follows. We say that an endogenous sharing rule game \( \Gamma = (N, (X_i)_{i \in N}, Q) \) is \textit{virtually continuous} if for each \( q \in S_Q, i \in N, \varepsilon > 0 \) and \( \sigma \in M \) there is a \( \mu_i \in M_i \) such that

\[
\tau_{(\mu_i, \sigma_{-i})}(D_i) < \varepsilon \quad \text{and} \quad q_i(\mu_i, \sigma_{-i}) > q_i(\sigma) - \varepsilon.
\]

Example 1 below shows in particular that the game considered in Section 2 is virtually continuous.

\textbf{Example 1.} Consider an endogenous sharing rule game \( \Gamma = (N, (X_i)_{i \in N}, Q) \) where \( N = \{1, 2\}, X_1 = X_2 = [0, 1], \) and \( D_1, D_2 \subseteq \Delta \), writing \( \Delta = \{(t, t) \colon t \in [0, 1]\} \). For \( x_1 \in X_1, x_2 \in X_2, \) and \( F \subseteq X \), write \( F_{x_1}, F_{x_2} \) for the sections of \( F \) at \( x_1 \) and \( x_2 \) respectively. Consider the following hypothesis. For each \( i = 1, 2 \), each \( \varepsilon > 0 \), and each \( \bar{x} \in D_i \), there is a set \( C_i(\bar{x}) \subseteq [0, 1] \) such that (i) \( \bar{x} \) is a condensation point of \( C_i(\bar{x}) \), (ii) \( Q_i(x) > \max Q_i(\bar{x}) - \varepsilon \) for each \( x \in (C_i(\bar{x}) \times \{\bar{x}\}) \setminus \{\bar{x}\} \), where \( j \neq i \) (writing \( (x_2, x_1) \) instead of \((x_1, x_2)\) if \( i = 2 \); note also that \( \#(Q_i(x)) = 1 \) for \( x \in X \setminus D_i \), so the inequality in (ii) is defined). Then virtual continuity holds. To see this, consider any \( q \in S_Q, \sigma \in M, \varepsilon > 0 \), and \( i \in \{1, 2\} \). Without loss of generality, take \( i = 1 \). Using Fubini’s theorem, we can find an \( \bar{x}_1 \in [0, 1] \) such that \( \int_{[0, 1]} q_1(\bar{x}_1, x_2) \, d\sigma_2(x_2) \geq q_1(\sigma) \). If \( \sigma_2(D_{1, \bar{x}_1}) = 0 \), then \( \tau_{(\delta_{\bar{x}_1}, \sigma_2)}(D_1) = 0 \), and we are done by taking \( \delta_{\bar{x}_1} \) for \( \mu_1 \). Otherwise, let \( \bar{x} \) be the unique point in \( D_1 \) determined by \( \bar{x}_1 \), and choose \( C_1(\bar{x}) \) corresponding to \( \varepsilon \) and \( \bar{x} \) according to the hypotheses above. By (i) there is a sequence \( \langle x_{1,k} \rangle \) in \( C_1(\bar{x}) \) with \( x_{1,k} \to \bar{x}_1 \) such that \( \sigma_2(D_{1, x_{1,k}}) = 0 \) for each \( k \), because \( D_1 \subseteq \Delta \). Now \( \tau_{(\delta_{x_{1,k}}, \sigma_2)}(D_1) = 0 \) for all \( k \). Moreover, because each \( x \in X \setminus D_1 \) is a continuity point of \( q_1 \), (ii) and Fatou’s lemma ensure that \( q_1(\delta_{x_{1,k}}, \sigma_2) > q_1(\sigma) - \varepsilon \) if \( k \) is large.

\textbf{Remark 1.} It should be obvious that in Example 1 one may take any compact metric spaces (which need not be the same) for \( X_1 \) and \( X_2 \), and for \( \Delta \) any subset of \( X_1 \times X_2 \) such that all sections \( \Delta_{x_1} \) and \( \Delta_{x_2} \) \( x_1 \in X_1, x_2 \in X_2 \), are empty or singletons.

Verifying virtual continuity in a particular game with an endogenous sharing rule is potentially daunting as one needs to consider all possible selections of the payoff correspondence. The next result may be helpful in this regard. Define \( m_i : X \to \mathbb{R}, i \in N, \) by setting \( m_i(x) = \max_{q \in Q(x)} q_i \) for each \( x \in X \), which is possible because \( Q \).
takes non-empty compact values, and note that because $Q$ is also uhc, $m_i$ is upper semicontinuous, therefore measurable.

**Lemma 4.** Let $\Gamma = (N, (X_i)_{i \in N}, Q)$ be a game with an endogenous sharing rule. Then $\Gamma$ is virtually continuous if and only if, for each $i \in N$, $\varepsilon > 0$ and $\sigma \in M$ there is a $\mu_i \in M_i$ such that

$$\tau_{(\mu_i, \sigma_{-i})}(D_i) < \varepsilon \text{ and } m_i(\mu_i, \sigma_{-i}) > m_i(\sigma) - \varepsilon.$$  

**Proof.** The “only if” part is true because for each $i \in N$ there is a $q \in S_Q$ (which may depend on $i$) such that $q_i(x) = m_i(x)$ for all $x \in X$ (see Lemma 2). For the “if” part, let $i \in N$, $q \in S_Q$, $\varepsilon > 0$ and $\sigma \in M$ be given. Since $X$ is compact and $Q$ is uhc and takes compact values, there is a number $B$ such that $||y|| \leq B$ for all $y \in Q(x)$ and $x \in X$. Let $\eta > 0$ be such that $(1 + 2B)\eta < \varepsilon$. By hypothesis, there is a $\mu_i \in M_i$ such that $m_i(\sigma) < m_i(\mu_i, \sigma_{-i}) + \eta$ and $\tau_{(\mu_i, \sigma_{-i})}(D_i) < \eta$. Now

$$m_i(\mu_i, \sigma_{-i}) - q_i(\mu_i, \sigma_{-i}) = \int_{D_i} (m_i - q_i) d\tau_{(\mu_i, \sigma_{-i})} < 2B\eta,$$  

so

$$q_i(\sigma) \leq m_i(\sigma) < m_i(\mu_i, \sigma_{-i}) + \eta < q_i(\mu_i, \sigma_{-i}) + (1 + 2B)\eta < q_i(\mu_i, \sigma_{-i}) + \varepsilon.$$  

Also, $\tau_{(\mu_i, \sigma_{-i})}(D_i) < \varepsilon$. Thus the “if” part follows. 

It is important to note that if an endogenous sharing rule game $\Gamma = (N, (X_i)_{i \in N}, Q)$ is virtually continuous, then, for every player, the value functions defined from the elements of $S_Q$ all agree.

**Lemma 5.** If $\Gamma = (N, (X_i)_{i \in N}, Q)$ is virtually continuous, then $v_{q_i} = v_{q'_i}$ for each $q, q' \in S_Q$ and $i \in N$; in particular, if $\sigma \in E(G_q)$, then $\sigma \in E(G_{q'})$ if and only if $q_i(\sigma) = q'_i(\sigma)$ for each $i \in N$.

**Proof.** Fix $i \in N$ and $\sigma \in M$. Let $\varepsilon > 0$ and $H = \{\mu_i \in M_i : \tau_{(\mu_i, \sigma_{-i})}(D_i) < \varepsilon\}$. Consider any $q, q' \in S_Q$. Then, by virtual continuity, since $q, q'$ agree on $X \setminus D_i,$

$$|v_{q_i}(\sigma) - v_{q'_i}(\sigma)| = \left| \sup_{\mu_i \in H} q_i(\mu_i, \sigma_{-i}) - \sup_{\mu_i \in H} q'_i(\mu_i, \sigma_{-i}) \right| \leq \varepsilon 2B$$  

where $B$ as in the proof of Lemma 4. As $\varepsilon$ is arbitrary, $v_{q_i}(\sigma) = v_{q'_i}(\sigma)$. 

**Remark 2.** Virtual continuity also implies that if $M$ is given the narrow topology, then the value functions are continuous; see Lemma 10 in Appendix A.
3.3 Structure of the equilibrium set

The first result in this section provides a characterization of equilibria in virtually continuous games.

**Theorem 1.** If $\Gamma = (N, (X_i)_{i \in N}, Q)$ is virtually continuous, then $E(\Gamma) = I_{\text{eff}}(\Gamma)$.

*Proof.* Clearly $I_{\text{eff}}(\Gamma) \subseteq E(\Gamma)$. For the reverse inclusion, let $q \in S_Q$, $\sigma \in E(G_q)$ and $\check{q} \in S_{Q_{\text{eff}}}$. Then $\tau_{\sigma}(\{x \in X: \hat{q}_i(x) > q_i(x)\}) = 0$ for all $i \in N$. Indeed, pick any $i \in N$ and write $F = \{x \in X: \hat{q}_i(x) > q_i(x)\}$. Suppose that $\tau_{\sigma}(F) > 0$. Set $\check{q} = \check{q}_{1F} + q_{1X_F}$.

Then $\check{q} \in S_Q$ and $\hat{q}_i(\sigma) > q_i(\sigma)$. Let $B$ be as in the proof of Lemma 4, and let $\varepsilon > 0$ be such that $\check{q}_i(\sigma) - (2B + 1)\varepsilon > q_i(\sigma)$. By virtual continuity, there is a $\sigma'_i \in M_i$ such that $\check{q}_i(\sigma'_i, \sigma_{-i}) > \check{q}_i(\sigma) - \varepsilon$ and $\tau_{(\sigma'_i, \sigma_{-i})}(D_i) < \varepsilon$. Hence

$$q_i(\sigma'_i, \sigma_{-i}) - \check{q}_i(\sigma'_i, \sigma_{-i}) - 2B\varepsilon > q_i(\sigma) - (2B + 1)\varepsilon > q_i(\sigma),$$

contradicting the assumption that $\sigma$ is an equilibrium for $q$. Thus $\tau_{\sigma}(F) = 0$.

Suppose now that there is an $i \in N$ such that $\tau_{\sigma}(\{x \in X: \hat{q}_i(x) < q_i(x)\}) > 0$. As $\check{q} \in S_{Q_{\text{eff}}}$, there must then be a $j \in N$ such that $\tau_{\sigma}(\{x \in X: \hat{q}_j(x) > q_j(x)\}) > 0$. But this is impossible by what has been shown in the previous paragraph.

It follows that for all $i \in N$, $\tau_{\sigma}(\{x \in X: \hat{q}_i(x) \neq q_i(x)\}) = 0$. By Lemma 5, we conclude that $\sigma$ is an equilibrium for $\check{q}$. $\square$

A simple application of Theorem 1 can be made in the context of the equilibrium existence result for endogenous sharing rule games established by Simon and Zame (1990). Recall that in this latter result, the payoff correspondence is required to take convex values, and that an equilibrium needs to exist just for some selection which cannot be specified. In both of these aspects, Theorem 1 can be used to get an improvement if virtual continuity holds.

**Theorem 2.** If $\Gamma = (N, (X_i)_{i \in N}, Q)$ is virtually continuous, then $E(\Gamma) \neq \emptyset$; in fact, $E(G_q) \neq \emptyset$ for every $q \in S_{Q_{\text{eff}}}$.

*Proof.* Define an endogenous sharing rule game $\Gamma' = (N, (X_i)_{i \in N}, Q')$ by letting $Q'(x)$ be the convex hull of $Q(x)$ for each $x \in X$. (By Aliprantis and Border (2006, Theorem 17.35, p. 573), $Q'$ is uhc and takes non-empty compact values, as required by our definition of a game with an endogenous sharing rule.)
Using Lemma 4, we can see that virtual continuity of $\Gamma$ implies virtual continuity of $\Gamma'$, because $\max_{r \in Q(x)} r_i = \max_{r \in Q'(x)} r_i$. Also we have $S_{Q,\text{eff}} \cap S_{Q',\text{eff}} \neq \emptyset$ (consider a $q \in S_Q$ such that $q(x)$ solves $\max_{r \in Q(x)} \sum_{i \in N} r_i$ for each $x \in X$).

Now by Simon and Zame (1990), $E(\Gamma') \neq \emptyset$, so by the previous paragraph and Theorem 1, $E(\Gamma) \neq \emptyset$, and in particular, $E(G_q) \neq \emptyset$ for every $q \in S_{Q,\text{eff}}$.

**Remark 3.** Instead of appealing to Simon and Zame (1990) and our invariance result Theorem 1, Theorem 2 can also be proved by using Reny (1999, Corollary 5.2). Indeed, give $M$ the narrow topology. Then virtual continuity implies that, for every $q \in S_Q$, the mixed extension of $G_q$ is payoff secure; see Lemma 9 in the Appendix. Moreover, if $q \in S_Q$ is such that $q(x)$ solves $\max_{r \in Q(x)} \sum_{i \in N} r_i$ for each $x \in X$ (see Lemma 2), then the mixed extension of $G_q$ is reciprocal upper semicontinuous; see Reny (1999, Proposition 5.1). Thus Reny (1999, Corollary 5.2) applies to such a $q$.

**Remark 4.** None of our results requires the payoff correspondence to take convex values. Actually, under our condition of virtual continuity, convexifying payoffs does not alter the equilibrium set, and also not the invariant equilibrium set. Indeed, let $\Gamma = (N, (X_i)_{i \in N}, Q)$ be a virtually continuous game and define $\Gamma' = (N, (X_i)_{i \in N}, Q')$ by letting $Q'(x)$ be the convex hull of $Q(x)$ for each $x \in X$. As shown in the proof of Theorem 2, $\Gamma'$ is virtually continuous and $S_{Q,\text{eff}} \cap S_{Q',\text{eff}} \neq \emptyset$; this, together with Theorem 1, implies that $E(\Gamma) = E(\Gamma')$. Since $S_Q \subseteq S_{Q'}$, it follows that $I(\Gamma') \subseteq I(\Gamma)$.

For the converse, let $\sigma \in I(\Gamma)$. Let $q \in S_{Q'}$ and $i \in N$. Let $q_s \in S_Q \cap S_{Q'}$ be such that $q_s(x) = \min_{r \in Q(x)} r_i = \min_{r \in Q'(x)} r_i$ for each $x \in X$. Then $q_i(\sigma) \geq q_s(\sigma)$. By Lemma 5, applied in $\Gamma'$, $v_{q_i}(\sigma) = v_{q_s}(\sigma)$. As $\sigma \in I(\Gamma)$, $v_{q_i}(\sigma) = q_i(\sigma)$. It follows that $q_i(\sigma) = v_{q_i}(\sigma)$. As $i \in N$ is arbitrary, $\sigma \in E(G_q)$. As $q \in S_{Q'}$ is arbitrary, $\sigma \in I(\Gamma')$.

The next result provides a characterization of invariant equilibria.

**Theorem 3.** If $\Gamma = (N, (X_i)_{i \in N}, Q)$ is virtually continuous, then

$$I(\Gamma) = \{ \sigma \in E(\Gamma) : \tau_\sigma(D_i) = 0 \text{ for all } i \in N \}.$$

**Proof.** That $\{ \sigma \in E(\Gamma) : \tau_\sigma(D_i) = 0 \text{ for all } i \in N \} \subseteq I(\Gamma)$ is immediate from Lemma 5.

For the reverse inclusion, consider any $i \in N$. There are $q, \bar{q} \in S_Q$ such that for every $x \in X$, $\bar{q}_i(x) = \max_{r \in Q(x)} r_i$ and $q_i(x) = \min_{r \in Q(x)} r_i$ (see Lemma 2). Now if $\sigma \in M$ is such that $\tau_\sigma(D_i) > 0$, then $\bar{q}_i(\sigma) > q_i(\sigma)$, so by Lemma 5, $\sigma \notin I(\Gamma)$.

\[ \square \]
We now address the question of when $E(\Gamma) = I(\Gamma)$. Having this equality is of interest because it implies that any selection of the payoff correspondence can be used to find an equilibrium for any other selection. A sufficient condition for having $E(\Gamma) = I(\Gamma)$ is that $\Gamma$ be strongly indeterminate in addition to being virtual continuous.

**Theorem 4.** Let $\Gamma = (N, (X_i)_{i \in N}, Q)$ be virtually continuous. Then the following holds:

1. If $\sigma \in E(\Gamma)$ then $\tau_\sigma(D_i^{\text{eff}}) = 0$ for all $i \in N$.

2. If $\Gamma$ is strongly indeterminate, then $I(\Gamma) = E(\Gamma)$.

**Proof.** (i) Given $\sigma \in M$ and $i \in N$, if $\tau_\sigma(D_i^{\text{eff}}) > 0$, then there are $q, \tilde{q} \in S_{Q_{\text{eff}}}$ such that $q_i(\sigma) \neq \tilde{q}_i(\sigma)$. Indeed, as shown in the proof of Lemma 3, $Q_{\text{eff}}$ has a measurable graph and nonempty values, so by Castaing and Valadier (1977, Theorem III.22) that there is a sequence $\langle h_k \rangle_{k \in \mathbb{N}}$ of (Borel) measurable functions $h_k : X \to \mathbb{R}^n$ and a Borel set $Y \subseteq X$ with $\tau_\sigma(Y) = 0$ such that $\{h_k(x) : k \in \mathbb{N}\}$ is a dense subset of $Q_{\text{eff}}(x)$ for all $x \in X \setminus Y$. Modify each $h_k$ on $Y$ so that it becomes a member of $S_{Q_{\text{eff}}}$ (by making it equal to $q$ on $Y$ for some $q \in S_{Q_{\text{eff}}}$; recall that $S_{Q_{\text{eff}}} \neq \emptyset$). Because $D_i^{\text{eff}}$ belongs to the $\tau_\sigma$-completion of the Borel $\sigma$-algebra of $X$, if $\tau_\sigma(D_i^{\text{eff}}) > 0$ then there a Borel set $H \subseteq D_i^{\text{eff}}$ with $\tau_\sigma(H) > 0$. Set $H' = H \setminus Y$. Then also $\sigma(H') > 0$.

Note that $\{h_k,i(x) : k \in \mathbb{N}\}$ is a dense subset of $Q_{\text{eff},i}(x)$ for each $x \in H'$. Hence, $\#(\{h_k,i(x) : k \in \mathbb{N}\}) > 1$ for all $x \in H'$. Thus for some $k \in \mathbb{N} \setminus \{0\}$ and some Borel set $B \subseteq H'$ with $\tau_\sigma(B) > 0$ we have $h_{k,i}(x) > h_{0,i}(x)$ for all $x \in B$, or for some $k \in \mathbb{N} \setminus \{0\}$ and some Borel set $B \subseteq H'$ with $\tau_\sigma(B) > 0$ we have $h_{k,i}(x) < h_{0,i}(x)$ for all $x \in B$. In either case, set $q = h_0$ and $\tilde{q} = 1_B h_k + 1_{X \setminus B} h_0$.

(ii) Virtual continuity, Theorem 1, and Lemma 5 combine to say that whenever $\sigma \in E(\Gamma)$ and $q, \tilde{q} \in S_{Q_{\text{eff}}}$, then $q_i(\sigma) = \tilde{q}_i(\sigma)$. Thus (ii) yields part 1.

(iii) If $D_i = D_i^{\text{eff}}$ for each $i \in N$, then part 1 and Theorem 3 imply that $I(\Gamma) = E(\Gamma)$. \hfill \Box

**Example 2.** If $\Gamma = (N, (X_i)_{i \in N}, Q)$ is a constant-sum endogenous sharing rule game (i.e., for some $c \in \mathbb{R}$, $\sum_{i \in N} r_i = c$ for all $r \in Q(x)$ and all $x \in X$) then $Q = Q_{\text{eff}}$ and thus $\Gamma$ is strongly indeterminate. Consequently, if such a $\Gamma$ is virtually continuous, then $I(\Gamma) = E(\Gamma)$. Thus, in particular, if $\Gamma$ is a two-person, constant-sum endogenous sharing rule game which satisfies the hypothesis in Example 1, then $I(\Gamma) = E(\Gamma)$. 

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Remark 5. In this remark we take a brief look at quasi-concave games. Recall that a game $\Gamma = (N, (X_i)_{i \in N}, Q)$ is quasi-concave if, for each $i \in N$, $X_i$ is a convex subset of a locally convex topological vector space and there is a $q \in S_Q$ such that, for each $i \in N$ and $x_{-i} \in X_{-i}$, the function $x_i \mapsto q_i(x_i, x_{-i})$ is quasi-concave. Identify, for each $i \in N$, $X_i$ with the set of Dirac measures at the points of this space. Using Bich and Laraki (2017, Theorem 3.4) it follows that if $\Gamma$ is quasi-concave, then $X \cap E(\Gamma) \neq \emptyset$. Moreover, if $\Gamma$ is virtually continuous and strongly indeterminate, then $E(\Gamma) = I(\Gamma)$ by Theorem 4 and, therefore, $X \cap E(\Gamma) = X \cap I(\Gamma)$. In words, invariance in mixed strategies implies invariance in pure strategies. However, as it can be easily shown, the latter holds when virtually continuity is required to hold only for pure strategies, in the following sense: For each $q \in S_Q$, $i \in N$, $\varepsilon > 0$ and $x \in X$, there is an $\bar{x}_i \in X_i$ such that $(\bar{x}_i, x_{-i}) \not\in D_i$ and $q_i(\bar{x}_i, x_{-i}) > q_i(x) - \varepsilon$.

Example 3. Modify the example in Section 2 by taking $X_2 = [c, 1]$ with $0 < c < 1/2$ (i.e., firm 2 has constant marginal cost $c$ with $0 < c < 1/2$), so that for each $x \in X$,

$$Q(x) = \begin{cases} (x_1(1 - x_1), 0) & \text{if } x_1 < x_2, \\ (0, (x_2 - c)(1 - x_2)) & \text{if } x_1 > x_2, \\ \{(x_1 \theta(1 - x_1), (x_1 - c)(1 - \theta)(1 - x_1)) : \theta \in [0, 1]\} & \text{if } x_1 = x_2. \end{cases}$$

Note that $D_1 = \{x \in X : x_1 = x_2\}$.

It follows by Example 1 that $\Gamma$ is virtually continuous. However, $E(\Gamma) \neq I(\Gamma)$. Indeed, if $q \in S_Q$ is such that $\theta(c, c) = 1$, then $(\delta_c, \delta_c) \in E(G_q) \subseteq E(\Gamma)$. But $(\delta_c, \delta_c) \not\in E(G_q)$ if $q \in S_Q$ is such that $\theta(c, c) < 1$. Thus $(\delta_c, \delta_c) \not\in I(\Gamma)$.

This shows that virtual continuity is not enough to ensure $E(\Gamma) = I(\Gamma)$, and thus highlights the role of strong indeterminacy as a condition in Theorem 4. Indeed this latter condition fails at $(c, c)$, because $(c, c) \in D_1$ but $\#(\pi_1 \circ Q_{\text{eff}}(x)) = 1$.

Example 4. The following example (whose setup is due to Simon and Zame (1990)) illustrates why our notion of virtual continuity is required to hold for all selections of the payoff correspondence, rather than just for one. Let $N = \{1, 2\}$, $X_1 = [0, 3]$, $X_2 = [3, 4]$, and for each $x \in X$,

$$Q(x) = \begin{cases} \{(\frac{x_1 + 3}{2}, 4 - \frac{x_1 + 3}{2})\} & \text{if } x \neq (3, 3), \\ \{(\alpha, 4 - \alpha) : \alpha \in [0, 4]\} & \text{if } x = (3, 3) \end{cases}$$
Note that if $\tilde{q} \in S_Q$ is such that $\tilde{q}(3,3) = (3,1)$, then $E(G_{\tilde{q}}) = \{(\delta_3, \delta_3)\}$.

The function $\tilde{q}$ just specified is continuous, so the requirements of virtual continuity hold for $\tilde{q}$. Also, strong indeterminacy holds; in fact $Q = Q_{\text{eff}}$. But $I(\Gamma) = \emptyset$. Indeed, suppose $(\sigma_1, \sigma_2) \in I(\Gamma)$. Then, in particular, $(\sigma_1, \sigma_2) \in E(G_{\tilde{q}})$, which implies that $(\sigma_1, \sigma_2) = (\delta_3, \delta_3)$. But $(\delta_3, \delta_3) \notin E(G_{\tilde{q}})$ if $\tilde{q} \in S_Q$ is such that $\tilde{q}(3,3) = (4,0)$. This contradiction shows that $I(\Gamma) = \emptyset$. Finally, note that $\hat{q}_1(x,3) < 3$ whenever $x_1 < 3$, i.e., whenever $(x_1,3) \in X_1 \setminus D_1$. Thus the requirements of virtual continuity are not satisfied for $\hat{q}$.

4 Applications

4.1 Bertrand competition with convex costs

In this section we consider a Bertrand oligopoly with convex costs. The standard formalization of Bertrand competition, according to which the firm posting the lowest price serves the entire demand, leads to difficulties in this setting. This is so because firms may prefer to tie to reduce the quantity produced. But this desire of a firm to tie is rather artificial and a consequence of the assumption that the firm posting the lowest price must serve the entire demand. In other words, the standard formalization is not appropriate for the case of convex costs; rather, it is more appropriate to allow firms to choose the quantity they want to supply. We allow for this by allowing each firm to choose a price and the maximum production level it is willing to produce. Our formalization is analogous to that of Dasgupta and Maskin (1986, Section 2.2) where each firm has an exogenously given capacity; here, in contrast, we assume that the capacity of each firm is endogenous, i.e. it is chosen by the firm. Our formalization is also analogous to that of Maskin (1986) where firms choose both prices and quantities, and firms produce to order, i.e., produce only after the entire price profile has been observed.

As in Dasgupta and Maskin (1986), there is a market for a single commodity with a continuum of consumers represented by the unit interval $[0,1]$. Consumers are identical, and the representative consumer’s demand for the commodity is a continuous and decreasing function $d : \mathbb{R}_+ \to \mathbb{R}_+$ such that there exists $\bar{p} > 0$ satisfying $d(p) > 0$ for all $p < \bar{p}$ and $d(p) = 0$ for all $p \geq \bar{p}$. There is a continuous, increasing and strictly
convex cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with $c(0) = 0$.

There are $n \in \mathbb{N}$ firms. Each firm $i \in N = \{1, \ldots, n\}$ chooses a price $p_i$ and a capacity $s_i$, the latter being the maximum amount the firm is willing to produce. Let $P = [0, \bar{p}]$ and $S = [0, d(0)]$.

To specify how the demand is allocated to firms, it is convenient to consider first the case of two firms. In this case, if one firm offers a price $p$ lower than the price $p'$ offered by the other firm, it serves the entire market up to its capacity $s$. A fraction $\frac{(d(p)-s)^+}{d(p)} = \max\{d(p)-s,0\} / d(p)$ of consumers is not served and each of these consumers demands $d(p')$ from the firm offering the highest price. When both firms set the same price, then the demand at the common price is split by each firm up to its capacity.

Formally, the quantities produced by firms are described by the correspondence $\Phi : (P \times S)^2 \to \mathbb{R}^2_+$ defined by setting, for each $(p, s) \in (P \times S)^2$,

$$\Phi(p, s) = \begin{cases} 
\{ \min\{d(p_1), s_1\}, \min\{\frac{(d(p_1)-s_1)^+}{d(p_1)}d(p_2), s_2\} \} & \text{if } p_1 < p_2, \\
\{ \phi \in \mathbb{R}^2_+ : \phi_1 + \phi_2 \leq d(p_1), \phi_i \leq s_i \text{ and } \}
\{ d(p_1) - \phi_1 - \phi_2|[s_i - \phi_i] = 0 \text{ for each } i = 1, 2 \} & \text{if } p_1 = p_2, \\
\{ \min\{\frac{(d(p_2)-s_2)^+}{d(p_2)}d(p_1), s_1\}, \min\{d(p_2), s_2\} \} & \text{if } p_1 > p_2.
\end{cases}$$

When prices are different, these quantities are the same as in both Dasgupta and Maskin (1986) and Maskin (1986) (with the proportional rationing rule in the latter). There are, however, differences between our formalization and theirs when prices are equal. First, we allow for indeterminacy, whereas they do not. Second, we rule out the possibility that a firm produces less than its capacity when there is unfulfilled demand (i.e. $\phi_i < s_i$ and $\phi_1 + \phi_2 < d(p_1)$ for some $i$ is not possible); in contrast this is allowed in both Dasgupta and Maskin (1986) and Maskin (1986).

We now return to the general case of $n$ firms. We start by defining the following correspondence $\Phi : (P \times S)^n \to \mathbb{R}^n_+$. Fix any $(p, s) \in (P \times S)^n$. Order the elements of the set $\{p_1, \ldots, p_n\}$ so that $p^{(1)} < \cdots < p^{(L(p))}$. Set $N^{(l)} = \{i \in N : p_i = p^{(l)}\}$ for each $l = 1, \ldots, L(p)$. Define numbers $D^{(l)}(p, s), l = 1, \ldots, L(p)$, recursively in the following way. Set $D^{(1)}(p, s) = d(p^{(1)})$; given that $D^{(l')}(p, s)$ has been specified for all $l'$ with $1 \leq l' \leq l - 1 < L(p)$, set

$$D^{(l)}(p, s) = \frac{D^{(l-1)}(p, s) - \min\{D^{(l-1)}(p, s), \sum_{j \in N^{(l-1)}} s_j\}}{d(p^{(l-1)})}d(p^{(l)}).$$
Now set
\[ \Phi(p, s) = \left\{ \phi \in \mathbb{R}_+^n : \phi_i \leq s_i, \ i = 1, \ldots, n, \right\} \]
\[ \sum_{j \in N(l)} \phi_j = \min \left\{ D^{(l)}(p, s), \sum_{j \in N(l)} s_j \right\}, \ l = 1, \ldots, L(p) \} . \]

The correspondence \( \Phi \) is closed. To see this, let \((p_k, s_k)\) be a sequence in \((P \times S)^n\) with \((p_k, s_k) \to (p, s) \in (P \times S)^n\), and \((\phi_k)\) a sequence in \(\mathbb{R}^n\) with \(\phi_k \in \Phi(p_k, s_k)\) for each \(k\) and \(\phi_k \to \phi\). We may assume that \(L(p_k)\) is constant along the sequence \((p_k)\), say \(L(p_k) = K\) for all \(k\). Then for each \(k\), we have \(L(p) \leq L(p_k) = K\), and we can group the elements of \(\{1, \ldots, K\}\) into non-empty disjoint sets \(A^{(1)}, \ldots, A^{(L(p))}\) so that for each \(1 \leq l \leq L(p)\) and any choice of \(p_{k,i}\) with \(p_{k,i} = p_{k}^{(h)}\) for some \(h \in A^{(l)}\), \(k \in \mathbb{N}\), we have \(p_{k,i} \to p^{(l)}\). It is straightforward to check, using induction, together with continuity of \(d\) and continuity of taking minima, that for each \(l = 1, \ldots, L(p)\),
\[ \sum_{h \in A^{(l)}} \sum_{i \in N(h)} \phi_{k,i} \to \min \left\{ D^{(l)}(p, s), \sum_{j \in N(l)} s_j \right\} \] (just recalculate the limits of the sums \(\sum_{h \in A^{(l)}} \sum_{i \in N(h)} \phi_{k,i}\)). Consequently \(\phi \in \Phi(p, s)\).

Since the values taken by \(\Phi\) are included in a common compact set, the fact that \(\Phi\) is closed implies that \(\Phi\) is uhc and takes compact values. Clearly \(\Phi\) takes non-empty values.

The above specification of \(\Phi\) allows firms to choose any capacity. However some choices are easily seen to be redundant. In fact, for each \(i \in N\), it suffices to consider capacity choices that are solutions of the problem
\[ \max_{0 \leq s \leq d(p_i)} ps - c(s). \]

Note that because \(c\) is strictly convex, a solution of this problem is unique. Let \(s^* : P \to \mathbb{R}\) describe this solution as a function on \(P\). Note that \(s^*\) is continuous, with \(s^*(0) = s^* (\bar{p}) = 0\), and that, given \(p_i \in P\), if \(z \geq 0\) is a number with \(z \leq s^*(p_i)\), then \(z' = z\) is the profit maximizing quantity choice of \(i\) subject to \(z' \in [0, z]\).

This discussion leads to consider the following game \(\Gamma\) with an endogenous sharing rule. For each \(i \in N\) let the action set be \(P\) and define the payoff correspondence \(Q : P^n \to \mathbb{R}^n\) by setting, for each \(p \in P^n\),
\[ Q(p) = \{(p_1 \phi_1 - c(\phi_1), \ldots, p_n \phi_n - c(\phi_n)) : \phi \in \Phi(p, \bar{s}(p))\} , \]
writing \( s(p) = (s^*(p_1), \ldots, s^*(p_n)) \). Then \( Q \) takes non-empty values. By the facts that the map \( c \) is continuous and the correspondence \( \Phi \) takes compact values, we see that \( Q \) takes compact values, and in addition, using the facts that \( \Phi \) is uhc, the maps \( d \) and \( s^* \) are continuous, and that \( s^*(\bar{p}) = 0 \), we can see that \( Q \) is uhc.

To check strong indeterminacy, fix \( p \in P^n \) and consider any \( i \in N \). Let \( p^{(l)} \) be the element of the order \( p^{(1)} < \cdots < p^{(L(p))} \) such that \( p_i = p^{(l)} \). Suppose \( p \in D_i \). Then \( p_i > 0 \), \( s^*(p_i) > 0 \), \( \#(N^{(l)}) > 1 \), and \( \sum_{j \in N^{(l)}} s^*(p_j) > D^{(l)}(p, s(p)) = 0 \). These facts together imply strong indeterminacy, because (see above) payoffs are strictly increasing on \([0, s^*(p_j)]\).

As for virtual continuity, fix \( i \in N \), \( p_i \in P \) with \( p_i > 0 \), and \( \varepsilon > 0 \). Note that given any \( 0 \leq p_i' < p_i \) and \( p_{-i} \in P^{n_i} \), and given any \( q \in S_Q \), for some numbers \( 0 \leq \alpha \leq \beta \leq 1 \) we have \( q_i(p_i, p_{-i}) = p_i \min\{ad(p_i), s^*(p_i)\} - c(\min\{od(p_i), s^*(p_i)\}) \) and \( q_i(p_i', p_{-i}) = p_i' \min\{\beta d(p_i'), s^*(p_i')\} - c(\min\{\beta d(p_i'), s^*(p_i')\}) \). Continuity of \( s^* \), \( c \), and \( d \), together with compactness of \([0, 1] \), imply that there is a \( \delta > 0 \) such that whenever \( p_i - \delta < p_i' < p_i \) and \( \alpha \in [0, 1] \), then

\[
p_i' \min\{\alpha d(p_i'), s^*(p_i')\} - c(\min\{\alpha d(p_i'), s^*(p_i')\}) > p_i \min\{\alpha d(p_i), s^*(p_i)\} - c(\min\{\alpha d(p_i), s^*(p_i)\}) - \varepsilon.
\]

As payoffs are non-decreasing on \([0, s^*(p_i)]\), it follows that \( q_i(p_i', p_{-i}) > q_i(p_i, p_{-i}) - \varepsilon \) for all \( p_{-i} \in P^{n_i} \) whenever \( p_i - \delta < p_i' < p_i \). Consequently the hypotheses of Lemma 8 are satisfied. Thus virtual continuity holds.

By Theorems 2 and 4, we conclude that \( I(\Gamma) = E(\Gamma) \neq \emptyset \).

### 4.2 Electoral competition

We consider a location/voting model as in Duggan (2007, Section 6). The setting is as follows. There are 2 players \( i = 1, 2 \) (e.g. political candidates), choosing locations \( x_1, x_2 \), respectively, in a compact and convex subset \( A \) of \( \mathbb{R}^m \), \( m > 0 \), with nonempty interior. When these location differ, then, for each \( i = 1, 2 \), the payoff is given by

\[
u_i(x_1, x_2) = \nu(\{ \alpha \in A : ||\alpha - x_i|| < ||\alpha - x_j||, j \neq i \}),
\]

where \( || \cdot || \) denotes the Euclidean norm and \( \nu \) is a measure on \( A \) which is absolutely continuous with respect to \((m\text{-dimensional}) \) Lebesgue measure. The interpretation is
that there is a set of individuals whose location in $A$ is distributed according to $\nu$ and that each individual is attracted to the player located closest to him. (Note that as long as $x_1 \neq x_2$, absolute continuity of $\nu$ with respect to Lebesgue measure implies that points $a \in A$ with $\|a - x_1\| = \|a - x_2\|$ don’t matter for the payoffs of the two players.) Now if $x_1 = x_2$, there is no canonical way to determine payoffs; in fact, perturbing such a situation can lead to different payoff sharings in the limit when the perturbations vanish; see the example given in Section 2. It is therefore natural to analyze this situation using a game with an endogenous sharing rule, rather than to make an ad hoc specification of payoffs as in Duggan (2007), where it is assumed that whenever players choose the same locations, payoffs are distributed in equal shares.

Without loss of generality, we assume that $\nu(A) = 1$. Let $S = \{p \in \mathbb{R}^m : \|p\| = 1\}$. For each $p \in S$ and each $z \in A$, let $\theta(p, z) = \nu(\{a \in A : pa < pz\})$. Note that for $x_1, x_2 \in A$ with $x_1 \neq x_2$, we can write $u_1(x_1, x_2) = \theta\left(\frac{x_2 - x_1}{\|x_2 - x_1\|}, \frac{1}{2}(x_1 + x_2)\right)$ and $u_2(x_1, x_2) = \theta\left(\frac{x_1 - x_2}{\|x_1 - x_2\|}, \frac{1}{2}(x_1 + x_2)\right)$. If $x_1, x_2 \in A$ with $x_1 = x_2 = z$, let

$$Q(x_1, x_2) = \{(r_1, r_2) : r_2 = 1 - r_1, \ r_1 = \theta(p, z) \text{ for some } p \in S\}.$$  

If $x_1, x_2 \in A$ with $x_1 \neq x_2$, let $Q(x_1, x_2) = \{u_1(x_1, x_2), u_2(x_1, x_2)\}$.

**Lemma 6.** (a) The correspondence $Q$ is closed. (b) Virtual continuity is satisfied.

**Proof.** (i) Given $z \in \text{bd}(A)$, there is a $p \in S$ such that $pz \geq pa$ for all $a \in A$ and such that $z - \lambda p \in A$ for all $\lambda > 0$ sufficiently small. Indeed, let $C$ be the set of all $p \in \mathbb{R}^m$ such that $pz \geq pa$ for all $a \in A$. Then $C$ convex, with $0 \in C$. We must have $(z - C) \cap \text{int}(A) \neq \emptyset$. Otherwise, as $\text{int}(A) \neq \emptyset$, there would be a non-zero $v \in \mathbb{R}^m$ such that $va \leq v(z - p)$ for all $a \in A$ and $p \in C$, by the separation theorem. The fact that $0 \in C$ implies that $v \in C$, and the fact that $z \in A$ implies that $vp \leq 0$ for all $p \in C$. But these implications contradict each other because $v \neq 0$ means $vv > 0$.

(ii) The map $\theta$ is continuous. To see this, suppose $p_k \to p$ in $S$ and $z_k \to z$ in $A$. Set $B_k = \{a \in A : p_k a < p_k z_k\}$ and $B = \{a \in A : pa < pz\}$. Observe that

$$\bigcap_{m=0}^{\infty} \bigcup_{k \geq m} B \Delta B_k \subseteq \{a \in A : pa = pz\}.$$  

Because $\nu$ is absolutely continuous with respect to Lebesgue measure, it follows that $\nu(B \Delta B_k) \to 0$, and therefore that $\nu(B_k) \to \nu(B)$. 

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(iii) Using (ii) we see that $Q$ is closed. As for virtual continuity, wlog consider player 1. Suppose that $x_1 = x_2 = z \in \text{int}(A)$. Let $(r_1, r_2) \in Q(x_1, x_2)$, and let $p \in S$ be such that $r_1 = \theta(p, z)$. As $x_1 \in \text{int}(A)$, we have $x_\lambda = x_1 - \lambda p \in A$ for all sufficiently small $\lambda > 0$. Now, for such $\lambda$,

$$u_1(x_\lambda, x_2) = \theta \left( \frac{x_2 - x_1 + \lambda p}{\|x_2 - x_1 + \lambda p\|}, \frac{1}{2}(x_1 + \lambda p + x_2) \right) = \theta(p, z + \frac{1}{2} \lambda p),$$

and by (ii), $\theta(p, z + \frac{1}{2} \lambda p) \to \theta(p, z)$ as $\lambda \to 0$, so $u_1(x_\lambda, x_2) \to r_1$ as $\lambda \to 0$. This holds, in particular, if $r_1 = \max\{r'_1: (r'_1, r'_2) \in Q(x_1, x_2)\}$.

Suppose next that $x_1 = x_2 = z \in \text{bd}(A)$. Choose $p \in S$ with respect to $z$ according to (i). Then, setting $r_1 = \theta(p, z)$, we have

$$r_1 = \theta(p, z) = 1 = \max\{r'_1: (r'_1, r'_2) \in Q(x_1, x_2)\},$$

and because $z - \lambda p \in A$ for all sufficiently small $\lambda > 0$, we can again choose $x_\lambda$ for player 1 to get $u_1(x_\lambda, x_2) \to r_1$. In view of Example 1 and Remark 1 it follows that virtual continuity is satisfied.

Obviously, the game $\Gamma$ we have discussed is a constant-sum game and thus satisfies strong indeterminacy. It therefore follows from Lemma 6 and Theorems 2 and 4 that $I(\Gamma) = E(\Gamma) \neq \emptyset$.

**Remark 6.** (a) Contrary to the case $m = 1$, if $m > 1$ then (because $S$ is connected if $m > 1$) continuity of $\theta$ implies that the correspondence $Q$ takes convex values. In particular, because $\max_{p \in S} \theta(p, z) \geq 1/2 \geq \min_{p \in S} \theta(p, z)$ for each $z \in A$, equal sharing is allowed when payoffs are indeterminate and $m > 1$. Equal sharing is also allowed when $m = 1$ by convexifying payoffs as in Remark 4.

(b) From (iii) in the proof of Lemma 6 we see that on $(A \times A) \backslash (D \cap (\text{bd}(A) \times \text{bd}(A)))$, $Q$ is the smallest closed correspondence which includes the map $(u_1, u_2)$ (in the sense of set inclusion of the graphs), writing $D$ for the diagonal in $A \times A$. Now if $x_1 = x_2 = z \in \text{bd}(A)$ and $p \in S$, consider any $z_0 \in \text{int}(A)$. Then by (ii) in the proof of Lemma 6, $\theta(p, \lambda z_0 + (1 - \lambda) z) \to \theta(p, z)$ as $\lambda \to 0$, and it follows that on the entire domain $A \times A$, $Q$ is the smallest closed correspondence which includes the map $(u_1, u_2)$.

**Remark 7.** The analysis of this section does not extend to the case of three or more players. Indeed, suppose $N = \{1, 2, 3\}$, let $X_i = [-1, 1]$ for each $i \in N$, write $\nu$ for
Lebesgue measure, and let \( Q : X \to \mathbb{R}^3 \) be the smallest closed correspondence which includes the map \( u : X' \to \mathbb{R}^3 \), where \( X' = \{ x \in X : x_i \neq x_j \text{ for each } i \neq j \} \) and

\[
u_i(x) = \nu(\{ a \in [-1, 1] : |a - x_i| < |a - x_j| \text{ for all } j \neq i \}).
\]

Consider \( i = 1 \). There is a \( q \in S_Q \) such that \( q_1(0, 0, 1) = q_1(0, 0, -1) = 1 \). Let \( \sigma = (\delta_0, \delta_0, \frac{\delta_1 + \delta_{-1}}{2}) \). Then \( q_1(\sigma) = 1 \), and a simple calculation shows that for each \( x_1 \not\in \{-1, 0, 1\} \), \( q_1(\delta x_1, \sigma_{-1}) \leq 3/4 \). In light of Lemma 7 in Appendix A.1, it follows that \( \Gamma \) is not virtually continuous.

\section{5 Extension: Incomplete Information}

In this section we extend our results to the case of incomplete information. Specifically, we consider incomplete information games with indeterminate outcomes, as introduced by Jackson, Simon, Swinkels, and Zame (2002), i.e., games where the assignment of payoffs to type/action profiles factors through a correspondence to some space of possible outcomes. We will present two results. The first one can be interpreted as assuming that the auctioneer knows the realizations of players’ types and can use this information when implementing tie breaking rules. In Jackson and Swinkels (2005), this case is called that of an “omniscient auctioneer.” In the second one, which is a corollary of the first, we turn to the more realistic case where the auctioneer does not have any information about players’ types.

A game with indeterminate outcomes is described as follows. There is a finite set \( N = \{1, \ldots, n\} \) of players. For each \( i \in N \), there is a compact metric action space \( A_i \) and a compact metric type space \( T_i \). Write \( A = \prod_{i \in N} A_i \) and \( T = \prod_{i \in N} T_i \). Type profiles, i.e., elements of \( T \), are chosen according to a (Borel) probability measure \( \lambda \) on \( T \); write \( \lambda_i \) for the marginal measure on \( T_i, i \in N \). As usual in the context of Bayesian games, it is assumed that \( \lambda \) is absolutely continuous with respect to the product \( \lambda_1 \times \cdots \times \lambda_n \). There is an outcome space \( \Omega \), assumed to be a compact metric space. Players’ actions determine a set of possible outcomes via an uhc correspondence \( \Theta : A \to \Omega \) with nonempty and compact values. The payoffs (or utilities) of players are determined by a continuous function \( u : T \times \text{graph}(\Theta) \to \mathbb{R}^N \).
The payoff correspondence $Q : T \times A \to \mathbb{R}^N$ is now defined by setting

$$Q(t, a) = \{ u(t, a, \omega) : \omega \in \Theta(a) \}$$

for each $(t, a) \in T \times A$. Note that $Q$ is uhc and has nonempty and compact values.

Additional notation is as before: given $i \in N$, $\pi_i$ denotes the projection of $\mathbb{R}^N$ on the $i$th coordinate, and we write $Q_i = \pi_i \circ Q$, $q_i = \pi_i \circ q$ for $q \in S_Q$, and $D_i = \{(t, a) \in T \times A : \#(Q_i(t, a)) > 1 \}$.

Following Balder (1988), we describe a mixed strategy $\sigma_i$ of player $i$ by a Young measure from $T_i$ to $A_i$, i.e., a map from $T_i$ to the space $M(A_i)$ of probability measures on $A_i$ such that the map $t_i \mapsto \sigma_i(t_i)(B)$ is Borel-measurable for each Borel set $B$ in $A_i$. As in Section 3, $M_i$ is the set of mixed strategies available for player $i$, now with the interpretation as a space of Young measures. Again, we write $M = \prod_{i \in N} M_i$ for the set of all profiles of mixed strategies.

As for payoffs, consider any $\sigma = (\sigma_1, \ldots, \sigma_n) \in M$. For every $t = (t_1, \ldots, t_n) \in T$ write $\sigma(t)$ for the Borel measure on $A$ defined by setting $\sigma(t) = \sigma_1(t_1) \times \cdots \times \sigma_n(t_n)$. Then the map $t \mapsto \sigma(t)$ is a Young measure from $T$ to $M(A)$ (to see that $t \mapsto \sigma(t)(B)$ is measurable for each $B \in \mathcal{B}(A)$, observe that this is true if $B$ is a product of Borel subsets $B_i$ of $T_i$, $i = 1, \ldots, n$, and use the monotone class theorem). By Neveu (1965, Proposition III.2.1) it follows that there is a uniquely determined probability measure $\tau_{\sigma}$ on $T \times A$ such that $\tau_{\sigma}(E \times B) = \int_E \sigma(t)(B) \, d\lambda(t)$ for each $E \in \mathcal{B}(T)$ and $B \in \mathcal{B}(A)$.

Now, for any $q \in S_Q$ and $i \in N$, the integral $\int_{T \times A} q_i(t, a) \, d\tau_{\sigma}(t, a)$ is defined, because $q_i$ is bounded and measurable, and by the generalized version of Fubini’s theorem (see again Neveu (1965, Proposition III.2.1)) we have

$$\int_{T \times A} q_i(t, a) \, d\tau_{\sigma}(t, a) = \int_T \int_A q_i(t, a) \, d\sigma(t)(a) \, d\lambda(t).$$

Because, given any realization $t \in T$ of possible type profiles, the payoff of player $i$ is $\int_A q_i(t, a) \, d\sigma(t)(a)$ (exactly as in the deterministic framework of the previous sections), we see that in the Bayesian framework considered now, the payoff of this player can be written as $\int_{T \times A} q_i(t, a) \, d\tau_{\sigma}(t, a)$. Again, we use the expression $q_i(\sigma)$ as abbreviation.

---

3In Milgrom and Weber (1985) such notion of a mixed strategy is called a behavioral strategy and, as they note, is equivalent to the notion of a distributional strategy that they consider.

4We use $\mathcal{B}(T)$ and $\mathcal{B}(A)$ to denote the Borel $\sigma$-algebra of $T$ and $A$, respectively.
As in Section 4.1 on Bertrand competition, one may restrict players’ choices of strategies so as that these have certain dominance properties (see Example 6 below). This can be done by specifying, for each \( i \in N \), an uhc correspondence \( \Phi_i: T_i \to A_i \), with non-empty closed values, and considering only strategy profiles \( \sigma \) such that for each \( i \in N \), \( \sigma_i(t_i)(\Phi_i(t_i)) = 1 \) for \( \lambda_i \)-a.e. \( t_i \in T_i \). Write \( W_i \) for the set of all \( \sigma_i \in M_i \) satisfying this restriction, and let \( W = \prod_{i \in N} W_i \).

Given such correspondences \( \Phi_i \), it might be of interest to get an invariance result in \( W \). For this the following notion of virtual continuity is appropriate (see Example 6). Writing \( \Phi \) for the list \( (\Phi_1, \ldots, \Phi_n) \), we say that a game is \( \Phi \)-\textit{virtually continuous} if for any \( q \in S_Q, i \in N, \sigma \in M_i \times W_{-i} \), and \( \varepsilon > 0 \), there is a \( \mu_i \in W_i \) such that \( \tau_{(\mu_i, \sigma_{-i})}(D_i) < \varepsilon \) and \( q_i(\mu_i, \sigma_{-i}) > q_i(\sigma) - \varepsilon \).

Finally, given \( \Phi \), some generality can be gained by relaxing strong indeterminacy into the requirement that there be a Borel set \( K \subseteq T \times A \) such that both \( \tau_\sigma(K) = 0 \) for each \( \sigma \in W \) and \( \#(\pi_i \circ Q_{\text{eff}}(t, a)) \geq 1 \) for each \( i \in N \) and each \( (t, a) \in D_i \setminus K \) (see Example 6). We will call this notion \( \Phi \)-\textit{strong indeterminacy}.

**Theorem 5.** Let \( \Gamma = (N, (T_i, A_i, \Phi_i, u_i)_{i \in N}, \lambda, \Theta) \) be a game with indeterminate outcomes. Suppose that \( \Gamma \) is \( \Phi \)-virtually continuous and \( \Phi \)-strongly indeterminate. Then \( E(\Gamma) \cap W = I(\Gamma) \cap W \neq \emptyset \).

**Proof.** The proof of the equality \( E(\Gamma) \cap W = I(\Gamma) \cap W \) (points (a)-(e) below) amounts, in essence, to a reinterpretation of the proofs of Lemma 5 and Theorems 1, 3, and 4, with \( T \times A \) in place of \( X \); note that for the arguments in the proofs of those results it does not matter whether or not the \( \tau_\sigma \)'s appearing there are product measures.

(a) For each \( q \in S_Q \) and \( i \in N \), define value functions \( v_{q_i}: M \to \mathbb{R} \) in the same way as in Section 3. Then, provided that \( \sigma \in W \), \( \Phi \)-virtual continuity implies that \( v_{q_i}(\sigma) = v_{q'_i}(\sigma) \) for any \( q, q' \in S_Q \) and any \( i \in N \). This follows as in the proof of Lemma 5, just replace \( M_i \) by \( W_i \) in the definition of the set \( H \) there. Consequently, for any \( q, q' \in S_Q \), if \( \sigma \in W \) then \( \sigma \in E(G_q) \) implies \( \sigma \in E(G_{q'}) \) if and only if \( q_i(\sigma) = q'_i(\sigma) \) for each \( i \in N \).

(b) From (a) we see that \( E(\Gamma) \cap W = I_{\text{eff}}(\Gamma) \cap W \), arguing as in the proof of Theorem 1 (replacing virtual continuity by \( \Phi \)-virtual continuity).

(c) Next note that \( I(\Gamma) \cap W = \{ \sigma \in E(\Gamma) \cap W : \tau_\sigma(D_i) = 0 \text{ for all } i \in N \} \); see the proof of Theorem 3.

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(d) Putting (a) and (b) together we see that if \( \sigma \in E(\Gamma) \cap W \) and \( q, \tilde{q} \in S_{Q_{\text{eff}}} \), then \( q_i(\sigma) = \tilde{q}_i(\sigma) \). It follows from this by arguments as in (i) of the proof of Theorem 4 that if \( \sigma \in E(\Gamma) \cap W \), then \( \tau_\sigma(D_i^{\text{eff}}) = 0 \) for each \( i \in N \).

(e) \( \Phi \)-strong indeterminacy means that if \( \sigma \in W \), then \( \tau_\sigma(D_i^{\text{eff}}) = 0 \) implies \( \tau_\sigma(D_i) = 0 \). From (c) and (d) we therefore conclude that \( E(\Gamma) \cap W = I(\Gamma) \cap W \).

(f) It remains to see that \( E(\Gamma) \cap W \neq \emptyset \). To this end, let \( \tilde{q} \in S_Q \) be such that \( \sum_{i \in N} \tilde{q}_i(t, a) = \max_{r \in Q(t, a)} \sum_{i \in N} r_i \) for each \( (t, a) \in T \times A \). Note that since \( Q \) is uhc with nonempty compact values, \( (t, a) \mapsto \sum_{i \in N} \tilde{q}_i(t, a) \) is bounded and usc. Taking \( \lambda_i \) as control measure for \( W_i \), give each \( W_i \) the narrow topology for Young measures (see Appendix A.3). Then (by Theorem 9 in Appendix A.3) each \( W_i \) becomes a non-empty compact convex subset of a locally convex topological vector space. Consider the normal form game \( \bar{G} = (W_i, \tilde{q}_i)_{i \in N} \), where payoffs are specified as above. Suppose temporarily that \( \bar{G} \) has a Nash equilibrium, say \( \bar{\sigma} = (\bar{\sigma}_1, \ldots, \bar{\sigma}_n) \). Thus, for each \( i \in N \), \( \bar{\sigma}_i \in W_i \) and \( \tilde{q}_i(\bar{\sigma}) \geq \tilde{q}_i(\sigma_i, \bar{\sigma}_{-i}) \) for all \( \sigma_i \in W_i \). Pick any \( i \in N \) and suppose there is a \( \mu_i \in M_i \) with \( \tilde{q}_i(\mu_i, \bar{\sigma}_{-i}) > \tilde{q}_i(\bar{\sigma}) \). Then, given \( \varepsilon > 0 \), \( \Phi \)-virtual continuity implies that there is a \( \sigma_i \in W_i \) such that \( \tilde{q}_i(\sigma_i, \bar{\sigma}_{-i}) > \tilde{q}_i(\mu_i, \bar{\sigma}_{-i}) - \varepsilon \), which implies \( \tilde{q}_i(\sigma_i', \bar{\sigma}_{-i}) > \tilde{q}_i(\bar{\sigma}) \) if \( \varepsilon \) is small enough. But this contradicts the fact that \( \bar{\sigma} \) is a Nash equilibrium of \( \bar{G} \) and we conclude that \( \bar{\sigma} \in E(\Gamma) \cap W \). Now by Reny (1999, Theorem 3.1), \( G \) has a Nash equilibrium if \( G \) is quasi-concave, payoff secure, and \( \sigma \mapsto \sum_{i \in N} \tilde{q}_i(\sigma) \) is usc on \( W \). Quasi-concavity is clear. The other facts are established in what follows.

(g) Give \( W = \prod_{i \in N} W_i \) the product topology defined from the \( W_i \)'s. Let \( \tilde{W} \) be the set of all Young measures from \( T \) to \( M(A) \). Take \( \lambda \) as control measure for \( \tilde{W} \) and give \( \tilde{W} \) the corresponding narrow topology for Young measures. As noted above, given \( \sigma \in W \), the map \( t \mapsto \sigma_1(t_1) \times \cdots \times \sigma_n(t_n) \) is a Young measure from \( T \to M(A) \). We may therefore define a map \( f: W \to \tilde{W} \) by setting

\[
f(\sigma)(t) = \sigma_1(t_1) \times \cdots \times \sigma_n(t_n), \quad t \in T, \quad \sigma \in W.
\]

It follows from Lemmata 12 and 13 in Appendix A.3 that \( f \) is continuous. Now let \( \rho: T \times A \to \mathbb{R} \) be a bounded and usc. By Balder (1988, Theorem 2.2), the map \( \bar{\sigma} \mapsto \int_T \int_A \rho(t, a) d\bar{\sigma}(t) d\lambda(t) : \tilde{W} \to \mathbb{R} \) is usc. Consequently, as \( f \) is continuous, the map \( \sigma \mapsto \int_T \int_A \rho(t, a) d\sigma(t) d\lambda(t) : W \to \mathbb{R} \) is usc. By the definition of \( f \), it follows that the map \( \sigma \mapsto \int_{T \times A} \rho(t, a) d\tau_\sigma(t, a) : W \to \mathbb{R} \) is usc.
(h) As noted above, $(t, a) \mapsto \sum_{i \in N} \tilde{q}_i(t, a)$ is bounded and usc. Consequently, in view of (g), the map

$$
\sigma \mapsto \sum_{i \in N} \tilde{q}_i(\sigma) = \sum_{i \in N} \int_{T \times A} \tilde{q}_i(t, a) \, d\tau_{\sigma}(t, a) = \int_{T \times A} \sum_{i \in N} \tilde{q}_i(t, a) \, d\tau_{\sigma}(t, a)
$$

is usc on $W$.

(i) Combining Lemmata 12–14 in Appendix A.3 shows that if $i \in N$, $\mu_i \in W_i$, and $\langle \sigma_k \rangle$ is a sequence in $W$ with $\sigma_k \to \sigma$, then $\tau_{(\mu_i, \sigma_{k-1})} \to \tau_{(\mu_i, \sigma_{-1})}$. From this and the argument in the proof of Lemma 9 we can see that $\tilde{G}$ is payoff secure.

Theorem 5 implies an invariance and existence result for selections of the payoff correspondence which are determined by selections of the outcome correspondence, i.e., for elements $q$ of $S_Q$ which can be written in the form $q(t, a) = u(t, a, \theta(a))$ for some measurable selection $\theta$ of $\Theta$. Write $S_Q'$ for the set of all $q \in S_Q$ which can be written in this form. Let $E^*(\Gamma) = \bigcup_{q \in S_Q'} E(G_q)$ and $I^*(\Gamma) = \bigcap_{q \in S_Q'} E(G_q)$.

**Theorem 6.** Let $\Gamma = (N, (T_i, A_i, \Phi_i, u_i)_{i \in N}, \lambda, \Theta)$ be a game with indeterminate outcomes. Suppose that $\Gamma$ is $\Phi$-virtually continuous and $\Phi$-strongly indeterminate. Then $E^*(\Gamma) \cap W = I^*(\Gamma) \cap W \neq \emptyset$.

**Proof.** Note that $I(\Gamma) \cap W \subseteq I^*(\Gamma) \cap W \subseteq E^*(\Gamma) \cap W \subseteq E(\Gamma) \cap W$, just by definition, and apply Theorem 5. \qed

Such a result does not hold in general as shown by Jackson, Simon, Swinkels, and Zame (2002). Thus the conditions of virtual continuity and strong indeterminacy are important in Theorem 6.

Note, however, that, in contrast with both Jackson, Simon, Swinkels, and Zame (2002) and Jackson and Swinkels (2005), Theorem 6 does not require players’ payoff function to be affine in the outcome. Such feature is important as it allows one to cover applications such as Bayesian version of the Bertrand competition setting of Section 4.1.

We remark that Theorem 6 is important because the type of a player may be his own private information, and because with selections of the payoff correspondence that are obtained via selections of the outcome correspondence no issues concerning type revelation arise.
We illustrate Theorem 5 with two examples. In both of them the next theorem, which provides a way to show that virtual continuity holds in a wide class of games with indeterminate outcomes, is used.

**Theorem 7.** Fix \( \ell \in \mathbb{N} \setminus \{0\} \) and let \( \Gamma = (N, (T_i, A_i, \Phi_i, u_i)_{i \in N}, \lambda, \Theta) \) be a game with indeterminate outcomes such that \( A_i \subseteq \mathbb{R}^\ell \) for all \( i \in N \). For each \( i \in N \), write

\[
\Delta_i = \left\{ a \in \prod_{j \in N} A_j: a_{i,h} = a_{j,h'} \text{ for some } j \in N \setminus \{i\} \text{ and some } 0 \leq h, h' \leq \ell \right\}.
\]

Suppose the following:

1. \( A_i \) is convex and has a non-empty interior for each \( i \in N \).
2. For each \( q \in S_Q, i \in N \), and \( \sigma \in M_i \times W_{-i} \) there is a \( \sigma_i' \in W_i \) such that \( q_i(\sigma_i', \sigma_{-i}) \geq q_i(\sigma) \).
3. \( D_i \subseteq T \times \Delta_i \) for each \( i \in N \).
4. For each \( i \in N \) and each \( \varepsilon > 0 \), there is a \( \lambda_i \)-null set \( C_i \subseteq T_i \), a measurable map \( f_i: \text{graph}(\Phi_i) \to A_i \) and a correspondence \( \Lambda_i: \text{graph}(\Phi_i) \to A_i \), with measurable graph, such that for each \( (t_i, a_i) \in \text{graph}(\Phi_i) \cap ((T_i \setminus C_i) \times f_i(\text{graph}(\Phi_i))) \) the following hold: (i) \( f_i(t_i, a_i) \in \Phi_i(t_i) \) and \( q_i(t_i, t_{-i}, f_i(t_i, a_i), a_{-i}) \geq q_i(t_i, t_{-i}, a_i, a_{-i}) \) for each \( q \in S_Q \) and each \( (t_{-i}, a_{-i}) \in T_{-i} \times A_{-i} \), (ii) \( \Lambda_i(t_i, a_i) \subseteq \Phi_i(t_i) \), (iii) \( \Lambda_i(t_i, a_i) \) is open, (iv) \( a_i \) is a cluster point of \( \Lambda_i(t_i, a_i) \), and (v) for any \( a_i' \in \Lambda_i(t_i, a_i) \) and any \( (t_{-i}, a_{-i}) \in T_{-i} \times A_{-i} \), if \( (t_i, t_{-i}, a_i, a_{-i}) \in D_i \), and \( (t_i, t_{-i}, a_i', a_{-i}) \notin D_i \), then \( Q_i(t_i, t_{-i}, a_i', a_{-i}) > \max Q_i(t_i, t_{-i}, a_i, a_{-i}) - \varepsilon \) (recall: \( \#Q_i(t_i, t_{-i}, a_i', a_{-i}) = 1 \) if \( (t_i, t_{-i}, a_i', a_{-i}) \notin D_i \)).

Then \( \Gamma \) is \( \Phi \)-virtually continuous.

**Remark 8.** This remark clarifies what is intended with condition (4)(i). Ignoring this condition momentarily, the restriction that (4) imposes is the requirement that (ii)--(iv) be satisfied at the same time as (v). Condition (4)(i) helps in this regard as it allows to reduce the set of points at which conditions (ii)-(v) need hold. Note, in particular, that (i) does not impose any restriction in addition to those imposed by (ii)-(v), because one can always set \( f_i(t_i, a_i) = a_i \) for each \( i \in N \) and \( (t_i, a_i) \in \text{graph}(\Phi_i) \).
Proof of Theorem 7. Without loss of generality, consider $i = 1$. Fix any $q \in S_Q$, $\sigma \in M_1 \times W_1$, and $\varepsilon > 0$. Use (2) and Lemma 11 to find a $h' \in S_{\Phi_1}$ such that $q_1(\delta_{h'}, \sigma_{-1}) \geq q_1(\sigma)$. Define $h : T_1 \to A_1$ by setting $h(t_1) = f_1(t_1, h'(t_1))$ for $t_1 \in T_1$. By (4)(i), $h \in S_{\Phi_1}$. Also by (4)(ii), $q_1(t_1, t_{-1}, h(t_1), a_{-1}) \geq q_1(t_1, t_{-1}, h'(t_1), a_{-1})$ for each $t_1 \in T_1$ and each $(t_{-1}, a_{-1}) \in T_{-1} \times A_{-1}$; thus, $q_1(\delta_{h}, \sigma_{-1}) \geq q_1(\delta_{h'}, \sigma_{-1}) \geq q_1(\sigma)$.

We claim that for each $k \in \mathbb{N} \setminus \{0\}$ there is a $g_k \in S_{\Phi_1}$ such that

(a) $\|h(t_1) - g_k(t_1)\| < 1/k$ for all $t_1 \in T_1$;

(b) $\tau(\delta, \sigma_{-1})(D_1) = 0$;

(c) $g_k(t_1) \in \Lambda_1(t_1, h(t_1))$ for $\lambda_1$-a.e. $t_1 \in T_1$.

To see this, let $\lambda_0^{(x)}$ be the product measure on $T_{-1}$ defined from the measures $\lambda_2, \ldots, \lambda_n$ and let $\tau^{(x)}_{-1}$ be the uniquely determined probability measure on $T_{-1} \times A_{-1}$ such that $\tau^{(x)}_{-1}(C \times B) = \int_C \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(B) \, d\lambda_0^{(x)}(t)$ for each $C \in \mathcal{B}(T_{-1})$ and $B \in \mathcal{B}(A_{-1})$. We claim that the set $E = \{a_1 \in A_1 : \tau^{(x)}_{-1}(T_{-1} \times A_{1,a_1}) = 0\}$ is a dense $G_\delta$-set, writing $A_{1,a_1}$ for the section of $A_1$ at $a_1$. Indeed, for each $i \in \mathbb{N} \setminus \{1\}$ and each $0 \leq h \leq \ell$, the set $R_{i,h} = \{r \in \mathbb{R} : \tau^{(x)}_{-1}((t_{-1}, a_{-1}) \in T_{-1} \times A_{-1} : a_{i,h} = r) > 0\}$ is countable. Observe that if $a_1 \notin E$, then there must be an $i \in \mathbb{N} \setminus \{1\}$ and $0 \leq h, h' \leq \ell$ such that $\tau^{(x)}_{-1}((t_{-1}, a_{-1}) \in T_{-1} \times A_{-1} : a_{i,h} = a_{1,h'}) > 0$. We must therefore have $A_1 \setminus E = \bigcup_{h} \bigcup_{h' \neq 1} \bigcup_{r \in R_{i,h}}\{a_1 \in A_1 : a_{1,h'} = r\}$. Because $A_1$ is convex and has non-empty interior, the set $\{a_1 \in A_1 : a_{1,h'} = r\}$ is closed and nowhere dense in $A_1$ for each $r \in \mathbb{R}$, and the claim about $E$ follows by Baire’s category theorem.

Now, for each $k \in \mathbb{N} \setminus \{0\}$, define a correspondence $F_k : T_1 \setminus C_1 \to A_1$ by setting

$$F_k(t_1) = \{a_1 \in A_1 : \|h(t_1) - a_1\| < 1/k\} \cap \Lambda_1(t_1, h(t_1)) \cap E$$

for each $t_1 \in T_1 \setminus C_1$. Then $F_k$ has a measurable graph. As $(t_1, h(t_1)) \in \text{graph}(\Phi_1)$ and $h(t_1) \in f_1(\text{graph}(\Phi_1))$ for each $t_1 \in T_1 \setminus C_1$, it follows by (4)(iv) and the properties of $E$ that $F_k$ has non-empty values. Consequently, by Castaing and Valadier (1977, Theorem III.22, p. 74), $F_k$ has a universally measurable selection $g_k : T_1 \setminus C_1 \to A_1$. Choosing a suitable extension to all of $T_1$, and making modifications on a $\lambda_1$-negligible set, if necessary, we obtain a $g_k \in S_{\Phi_1}$ such that (a) and (c) hold.
As for (b), observe that, for each \( k \in \mathbb{N} \setminus \{0\}, \)
\[
\int_T \delta_{g_k}(t_1) \times \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(\Delta_1) \, d(\lambda_1 \times \ldots \times \lambda_n)(t)
\]
\[
= \int_T \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(\Delta_{1,g_k(t_1)}) \, d(\lambda_1 \times \ldots \times \lambda_n)(t)
\]
\[
= \int_{T_1} \int_{T_{-1}} \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(\Delta_{1,g_k(t_1)}) \, d\lambda_1^\times(t_{-1}) \, d\lambda_1(t_1)
\]
\[
= \int_{T_1} \tau_{-1}^\times(T_{-1} \times \Delta_{1,g_k(t_1)}) \, d\lambda_1(t) = 0, \text{ because } g_k(t_1) \in E \text{ for } \lambda_1\text{-a.e. } t_1 \in T_1.
\]

We must therefore have \( \delta_{g_k}(t_1) \times \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(\Delta_1) = 0 \) for \( \lambda_1 \times \ldots \times \lambda_n\text{-a.e. } t \in T, \) hence also for \( \lambda\text{-a.e. } t \in T, \) because \( \lambda \) is absolutely continuous with respect to \( \lambda_1 \times \ldots \times \lambda_n. \) Consequently \( \tau_{(\delta_{g_k},\sigma_1)}(T \times \Delta_1) = 0, \) and thus (3) implies (b).

As the countable union of null sets is a null set, there must be a \( \lambda\)-null set \( H \subseteq T \) such that \( \delta_{g_k}(t_1) \times \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(\Delta_1) = 0 \) for all \( k \in \mathbb{N} \setminus \{0\} \) and all \( t \in T \setminus H. \) Let \( H_1 \subseteq T_1 \) be the exceptional set from (c) and let \( H' = H \cup (H_1 \times T_{-1}) \cup (C_1 \times T_{-1}), \) so that \( H' \) is a \( \lambda\)-null set in \( T. \) Fix any \( t \in T \setminus H'. \) Using Fubini’s theorem and the fact that the countable union of null sets is a null set, we see that for \( \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)\text{-a.e. } a_{-1} \in A_{-1} \) we have \( \delta_{g_k(t_1)}(\Delta_{1,a_{-1}}) = 0 \) for all \( k \in \mathbb{N} \setminus \{0\}, \) i.e., \( (g_k(t_1), a_{-1}) \notin \Delta_1 \) and thus \( (t_1, t_{-1}, g_k(t_1), a_{-1}) \notin D_1 \) (as \( D_1 \subseteq \Delta_1). \) Combining this with (c), (4)((iv)), and the fact that \( (t_1, t_{-1}, h(t_1), a_{-1}) \) is a continuity point of \( q_1 \) if \( (t_1, t_{-1}, h(t_1), a_{-1}) \notin D_1, \) we see that
\[
\lim_{k \to \infty} q_1(t_1, t_{-1}, g_k(t_1), a_{-1}) \geq q_1(t_1, t_{-1}, h(t_1), a_{-1}) - \varepsilon
\]
for \( \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)\text{-a.e. } a_{-1} \in A_{-1}. \) Hence, by Fatou’s lemma,
\[
\lim_{k \to \infty} \int_{A_{-1}} q_1(t_1, t_{-1}, g_k(t_1), a_{-1}) \, d\sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(a_{-1}) 
\]
\[
\geq \int_{A_{-1}} q_1(t_1, t_{-1}, h(t_1), a_{-1}) \, d\sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(a_{-1}) - \varepsilon,
\]
or, in other words,
\[
\lim_{k \to \infty} \int_A q_1(t_1, t_{-1}, a_{-1}) \, d\delta_{g_k}(t_1) \times \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(a_{-1}) 
\]
\[
\geq \int_A q_1(t_1, t_{-1}, a_{-1}) \, d\delta_{h}(t_1) \times \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(a_{-1}) - \varepsilon.
\]
Since this is true for \( \lambda \text{-a.e.} \in T \), we now see, again using Fatou’s lemma, that

\[
\lim_{k \to \infty} q_1(\delta_{g_k}, \sigma_{-i}) = \lim_{k \to \infty} \int_T \int_A q_1(t, a) \, d(\delta_{g_k}(t_1) \times \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)) \, d\lambda(t)
\]

\[
\geq \int_T \lim_{k \to \infty} \int_A q_1(t, a) \, d(\delta_{g_k}(t_1) \times \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)) \, d\lambda(t)
\]

\[
\geq \int_T \int_A q_1(t, a) \, d(\delta_h(t_1) \times \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)) \, d\lambda(t) - \varepsilon
\]

\[
= q_1(\delta_h, \sigma_{-1}) - \varepsilon \geq q_1(\sigma) - \varepsilon, \text{ by the choice of } \delta_h.
\]

Thus, as \( q \in S_Q, \sigma \in M_1 \times W_{-1}, \) and \( \varepsilon > 0 \) are arbitrary, the requirements of \( \Phi \)-virtual continuity are satisfied for player 1. As the consideration of player 1 does not imply any loss of generality, \( \Gamma \) is \( \Phi \)-virtually continuous.

As a first application of Theorem 5, we consider a general contest with incomplete information.

**Example 5.** Consider the following game with indeterminate outcomes which models a contest as formalized in Moldovanu and Sela (2001). There are \( n \) contestants \( i = 1, \ldots, n \) who compete for one of \( n \) prizes with values \( V_1 \geq V_2 \geq \cdots \geq V_n \geq 0 \).\(^5\) The allocation of prizes is determined by the contestants’ effort. For example, contestant can be firms investing in R& D (their “effort”) and prizes be their share of total demand.

Contestants simultaneously choose an effort level. Each contestant suffers a disutility \( c(t_i, a_i) \) from his own effort \( a_i \), where \( t_i \in \bar{T} \) denotes his ability, \( \bar{T} \) is a nonempty compact subset of \( \mathbb{R}_+ \), and \( c : \bar{T} \times \mathbb{R}_+ \to \mathbb{R}_+ \) is continuous and satisfies \( c(t_i, 0) = 0 \) for each \( t_i \in \bar{T} \). Abilities are drawn according to a probability measure \( \lambda \) on \( \bar{T}^n \), which is absolutely continuous with respect to the product of its marginals, and each contestant’s ability is his own private information. We assume that there is \( \bar{a} > 0 \) such that \( V_1 < c(t, \bar{a}) \) for each \( t \in \bar{T} \).\(^6\)

\(^5\)The assumption that the number of prizes equals the number of contestants is without loss of generality. Indeed, the case where there are \( p < n \) prizes, which is allowed in Moldovanu and Sela (2001), is identified with \( V_j = 0 \) for all \( j = p + 1, \ldots, n \).

\(^6\)Moldovanu and Sela (2001) assume that \( \bar{T} = [m, 1] \) for some \( 0 < m < 1 \), \( c(t_i, a_i) = t_i \gamma(a_i) \) where \( \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \) is strictly increasing and differentiable and satisfies \( \gamma(0) = 0 \). The existence of \( \bar{a} \) then follows when \( \gamma \) is linear or convex. Note also that, unlike Moldovanu and Sela (2001), we do not assume that types are independent with a continuous and strictly positive density.
For each $i \in N$, let $T_i = \bar{T}$ and $A_i = [0, \bar{a}]$. When all contestants choose different effort levels, then the first prize goes to the player with the highest effort, the second prize goes to the player with the second highest effort and so on. In case of ties in effort levels, randomization is used to determine the allocation of prizes. For example, if players 1, 2 and 4 choose the highest effort level, then the first three prizes are randomly allocated to players 1, 2 and 4. We let $H$ be the set of allocations, i.e. the set of 1-1 functions from $N$ (players) to $N$ (prizes). The outcome space $\Omega$ is the set of probability measures on $H$. Some notation is needed to define the outcome correspondence. Given $a \in A$, order the elements of the set $\{a_1, \ldots, a_n\}$ so that $a^{(1)} > \cdots > a^{(L(a))}$. For each $l = 1, \ldots, L(a)$, set $N_a^{(l)} = \{i \in N : a_i = a^{(l)}\}$ and $n_a^{(l)} = \#(N_a^{(l)})$; furthermore, define $J_a^{(1)} = \{1, \ldots, n_a^{(1)}\}$ and, for each $1 < l \leq L(a)$, $J_a^{(l)} = \{n_a^{(l-1)} + 1, \ldots, n_a^{(l-1)} + n_a^{(l)}\}$. Given $a \in A$, the set of feasible allocations is denoted by $H(a)$ and consists of those $h \in H$ with the property that, for each $i \in N$, if $1 \leq l \leq L(a)$ is such that $a_i = a^{(l)}$ then $h_i \in J_a^{(l)}$. We then define $\Theta : A \rightarrow \Omega$ in this context by setting, for each $a \in A$,

$$\Theta(a) = \{\omega \in \Omega : \omega_h = 0 \text{ for each } h \not\in H(a)\}.$$  

The payoff of contestant $i \in N$ equals the expected value of the prize received minus his disutility of effort; thus contestant $i$’s payoff function $u_i : T \times A \times \Omega \rightarrow \mathbb{R}$ is given by $u_i(t, a, \omega) = \sum_{h \in H(a)} \omega_h V_{h_i} - c(t_i, a_i)$.

Evidently, we have

$$\sum_{i \in N} u_i(t, a, \omega) = \sum_{j=1}^n V_j - \sum_{i=1}^n c(t_i, a_i)$$

for each $t \in T$, $a \in A$, and $\omega \in \Theta(a)$. This implies easily that the game $\Gamma$ just defined is strongly indeterminate.

We next show that $\Gamma$ is virtually continuous by using Theorem 7 with $\Phi_i(t_i) = A_i$ for each $i \in N$ and $t_i \in T_i$ (so that $W_i = M_i$). It is clear that conditions (1)–(3) in Theorem 7 hold. As for condition (4), fix $i \in N$ and $\varepsilon > 0$. Let $C_i = \emptyset$ for each $i \in N$. Let $\eta \in (0, \bar{a})$ be such that $V_i < c(t, a)$ for each $a > \bar{a} - \eta$, and define $f_i$ by setting, for each $(t_i, a_i) \in T_i \times A_i$,

$$f_i(t_i, a_i) = \begin{cases} 
    a_i & \text{if } a_i \leq \bar{a} - \eta; \\
    0 & \text{otherwise.}
\end{cases}$$
Then (i) of condition (4) in Theorem 7 holds. Note that \( f_i(T_i \times A_i) \subseteq [0, \bar{a} - \eta] \). Let \( \delta \in (0, \eta) \) be such that \( |c(t, a) - c(t, a')| < \varepsilon \) whenever \( |a - a'| < \delta \), \( a, a' \in [0, \bar{a}] \) and \( t \in \bar{T} \), and define \( \Lambda_i \) by setting

\[
\Lambda_i(t_i, a_i) = \begin{cases} 
(a_i, a_i + \delta) & \text{if } a_i \leq \bar{a} - \eta, \\
A_i & \text{otherwise}
\end{cases}
\]

for each \( (t_i, a_i) \in T_i \times A_i \). Clearly \( \Lambda_i \) has a measurable graph and (ii)–(iv) of condition (4) in Theorem 7 are satisfied. As for (v) of that condition, let \( t_i \in T_i \), \( a_i \in f_i(T_i \times A_i) \), \( a'_i \in \Lambda_i(t_i, a_i) \), \( (t_i, a'_i, \overline{a}_i) \in T_{-i} \times A_{-i} \) and suppose \( (t_i, t_{-i}, a_i, \overline{a}_i) \in D_i \) and \( (t_i, t_{-i}, a'_i, \overline{a}_i) \notin D_i \). Let \( l \in \{1, \ldots, L(a)\} \) be such that \( a_i = a^{(l)} \) and let \( l' \in \{1, \ldots, L(a'_i, \overline{a}_i)\} \) be such that \( a'_i = (a'_i, \overline{a}_i)^{(l')} \). Since \( a'_i > a_i \), we must have \( \min J_{(a'_i, \overline{a}_i)}^{(l')} \geq \max J_{a}^{(l)} \). Therefore

\[
Q_i(t_i, t_{-i}, a'_i, \overline{a}_i) - \max Q_i(t_i, t_{-i}, a_i, \overline{a}_i)
= V_{\min J_{(a'_i, \overline{a}_i)}^{(l')}} - V_{\max J_{a}^{(l)}} - c(t_i, a'_i) + c(t_i, a_i) > -\varepsilon,
\]

as desired.

By Theorems 5 we conclude that \( E(\Gamma) = I(\Gamma) \neq \emptyset \).

The next theorem provides a way to see that strong indeterminacy holds in a wide class of games with indeterminate outcomes.

**Theorem 8.** Let \( \Gamma = (N, (T_i, A_i, \Phi_i, u_i)_{i \in N}, \lambda, \Theta) \) be a game with indeterminate outcomes. Suppose that, for some \( \ell \in \mathbb{N} \setminus \{0\} \), \( A_i, T_i \subseteq \mathbb{R}^\ell \) for each \( i \in N \). Let \( \Delta_i \) be as in the statement of Theorem 7 and suppose (2) of Theorem 7 is satisfied. For each \( i \in N \) and each \( 1 \leq h \leq \ell \), let \( f_{i,h}: A_{i,h} \rightarrow \mathbb{R} \) be a measurable function such that \( f_{i,h}(a_{i,h}) \geq t_{i,h} \) whenever \( a_i \in \Phi_i(t_i) \). For each \( i \in N \) and each \( 1 \leq h \leq \ell \), write

\[
\Delta_{i,h} = \{(t, a) \in T \times A: a_{i,h} = a_{j,h}, j \neq i, 0 \leq h' \leq \ell\}.
\]

Suppose the following:

(a) \( Q(t, a) = Q_{eB}(t, a) \) if \( (t, a) \in T \times A \) is such that \( a_i \in \Phi_i(t_i) \) for each \( i \in N \) and such that \( (t, a) \in \Delta_{i,h} \) implies \( f_{i,h}(a_{i,h}) > t_{i,h}, i \in N, 1 \leq h \leq \ell \).

(b) \( \lambda_i(\{t_{i,h}\} \times T_{i,-h}) = 0 \) for each \( i \in N, t_i \in T_i \), and \( 0 \leq h \leq \ell \).
Then $\Gamma$ is $\Phi$-strongly indeterminate.

**Proof.** For each $i, j \in N$, $i \neq j$, and each $0 \leq h, h' \leq \ell$, let

$$K_{i,0} = (T \times A) \setminus (\text{graph}(\Phi_i) \times T_{-i} \times A_{-i})$$

and

$$K_{i,j,h,h'} = \{(t, a) \in T \times A: f_{i,h}(a_{i,h}) = t_{i,h} \text{ and } a_{i,h} = a_{j,h'}\}.$$

Let

$$K = \bigcup_{i \in N} K_{i,0} \cup \bigcup_{i \in N} \bigcup_{j \in N, j \neq i} \bigcup_{h=1}^{\ell} \bigcup_{h'=1}^{\ell} K_{i,j,h,h'}.$$ 

By hypothesis, if $(t, a) \in D_i \setminus K$, then $Q(t, a) = Q_{\text{eff}}(t, a)$. We therefore need to show that $\tau_\sigma(K) = 0$ whenever $\sigma \in W$. Thus fix any $\sigma \in W$. By the definition of $W$, $\tau_\sigma(K_{i,0}) = 0$ for each $i \in I$. Consider any $K_{i,j,h,h'}$. Then, by the fact that $i \neq j$,

$$\tau_\sigma(K_{i,j,h,h'}) = \int_T \sigma(t)\{(a \in A: f_{i,h}(a_{i,h}) = t_{i,h}, a_{i,h} = a_{j,h'})\} \, d\lambda(t)$$

$$\leq \int_T \sigma(t)\{(a \in A: f_{i,h}(a_{j,h'}) = t_{i,h})\} \, d\lambda(t)$$

$$= \int_T \left( \int_{A_i} \sigma_i(t_{-i})(\{a_{-i} \in A_{-i}: f_{i,h}(a_{j,h'}) = t_{i,h}\}) \, d\sigma_i(t_{-i})(a_{-i}) \right) \, d\lambda(t)$$

$$= \int_T \sigma_{-i}(t_{-i})(\{a_{-i} \in A_{-i}: f_{i,h}(a_{j,h'}) = t_{i,h}\}) \, d\lambda(t)$$

$$= \int_T \sigma_{-i}(t_{-i})(\{a_{-i} \in A_{-i}: f_{i,h}(a_{j,h'}) = t_{i,h}\}) \, \rho(t) \, d\lambda^{(\times)}(t)$$

$$= \int_{T_{-i}} \left( \int_{T_i} \sigma_{-i}(t_{-i})(\{a_{-i} \in A_{-i}: f_{i,h}(a_{j,h'}) = t_{i,h}\}) \, \rho(t) \, d\lambda_i(t_i) \right) \, d\lambda^{(\times)}_{-i}(t_{-i})$$

$$= 0,$$

because for each $t_{-i} \in T_{-i}$, by Fubini’s theorem,

$$\lambda_i \times \sigma_{-i}(t_{-i})(\{(t_i, a_{-i}) \in T_i \times A_{-i}: f_{i,h}(a_{j,h'}) = t_{i,h}\})$$

$$= \int_{T_i} \sigma_{-i}(t_{-i})(\{a_{-i} \in A_{-i}: f_{i,h}(a_{j,h'}) = t_{i,h}\}) \, d\lambda_i(t_i)$$

$$= \int_{A_{-i}} \lambda_i(\{t_i \in T_i: f_{i,h}(a_{j,h'}) = t_{i,h}\}) \, d\sigma_{-i}(t_{-i})(a_{-i}) = 0$$

by hypothesis (b), so $\sigma_{-i}(t_{-i})(\{a_{-i} \in A_{-i}: f_{i,h}(a_{j,h'}) = t_{i,h}\}) = 0$ for $\lambda_i$-a.e. $t_i \in T_i$.

Since a finite union of null sets is a null set, it follows that $\tau_\sigma(K) = 0$. \hfill \Box
As a second application of Theorem 5, we consider a Bertrand oligopoly with incomplete information.

Example 6. Consider a Bertrand oligopoly with linear cost functions and one commodity whose demand is \(d(x)\) where \(x\) is the lowest price in the market and \(d : \mathbb{R}_+ \to \mathbb{R}_+\) is a continuous function. There is incomplete information regarding firms’ marginal costs. For some \(\bar{a} > 0\) and each firm \(i \in N\), let \(A_i = [0, \bar{a}]\) and \(T_i\) be a compact subset of \(\mathbb{R}_+\). Let \(\lambda\) be a probability measure on \(T\) which is absolutely continuous with respect to the product of its marginals and such that \(\lambda_i\) is atomless for each \(i \in N\). The outcomes of the game are the fractions of total demand that each firm gets. Thus, we let the outcome space \(\Omega\) be \(\{\omega \in [0,1]^n : \sum_{i \in N} \omega_i = 1\}\), with the interpretation that \(\omega_i \in [0,1]\) is the fraction of total demand that firm \(i\) satisfies, \(i \in N\). The natural choice of the outcome correspondence \(\Theta : A \to \Omega\) in the context is given by setting, for each \(a \in A\),

\[
\Theta(a) = \left\{ \omega \in \Omega : \omega_i > 0 \text{ only if } a_i = \min_{j \in N} a_j \text{ for each } i \in N \right\}.
\]

The payoff functions \(u_i : T \times A \times \Omega \to \mathbb{R}\) are then given by

\[
u_i(t, a, \omega) = \omega_i d\left(\min_{j \in N} a_j\right) (a_i - t_i)
\]

for each \(i \in N\). Finally, let \(\Phi_i(t_i) = \{a_i \in A_i : a_i \geq t_i\}\), i.e. \(\Phi_i(t_i)\) is the set of prices above marginal cost, which we assume to be nonempty.

To see that the game \(\Gamma\) just defined is virtually continuous, we check that the hypotheses of Theorem 7 are satisfied. Clearly conditions (1) and (3) of that theorem are satisfied. As for condition (2), without loss of generality consider player 1. Fix \(q \in S_Q\) and \(\sigma \in M\). By Lemma 11 there is a measurable map \(g : T_1 \to A_1\) such that \(q_1(\delta_g, \sigma_i) \geq q_1(\sigma)\). Define \(h : T_1 \to A_1\) by setting \(h(t_1) = \max\{g(t_1), t_1\}\). Then \(\delta_h \in W_1\). Moreover, we have \(q_1(t_1, t_{-1} , h(t_1), a_{-1}) \geq q_1(t_1, t_{-1} , g(t_1), a_{-1})\) for each
$(t_1, t_{-1}, a_{-1}) \in T_1 \times T_{-1} \times A_{-1}$, so

$$q_1(\delta_h, \sigma_{-1}) = \int_{T_1} \int_{A_{-1}} q_1(t_1, t_{-1}, a_1, a_{-1}) \, d\delta_h(t_1) \times \sigma_{-1}(t_{-1}) \, d\lambda(t)$$

$$= \int_{T_1} \int_{A_{-1}} q_1(t_1, t_{-1}, h(t_1), a_{-1}) \, d\sigma_{-1}(t_{-1}) \, d\lambda(t)$$

$$\geq \int_{T_1} \int_{A_{-1}} q_1(t_1, t_{-1}, g(t_1), a_{-1}) \, d\sigma_{-1}(t_{-1}) \, d\lambda(t)$$

$$= \int_{T_1} \int_{A} q_1(t_1, t_{-1}, a_1, a_{-1}) \, d\delta_g(t_1) \times \sigma_{-1}(t_{-1}) \, d\lambda(t)$$

$$= \sigma_{-1} \geq q_1(\sigma_1, \sigma_{-1}).$$

Thus condition (2) of Theorem 7 holds. To see that condition (4) of that theorem holds, fix $\varepsilon > 0$ and $i \in N$. Let $\eta > 0$ be such that $|d(a'_i)(a'_i - t_i) - d(a_i)(a_i - t_i)| < \varepsilon$ whenever $t_i \in T_i$ and $a_i, a'_i \in A_i$ are such that $|a'_i - a_i| < \eta$. Let $C_i = \{a_i \cap T_i\}$, and note that $\lambda_i(C_i) = 0$ because $\lambda_i$ is atomless. Let $f_i(t_i, a_i) = a_i$ for each $i \in N$ and $(t_i, a_i) \in \text{graph}(\Phi_i)$. Define the correspondence $\Lambda_i: \text{graph}(\Phi_i) \to A_i$ by setting

$$\Lambda_i(t_i, a_i) = \begin{cases} 
\max\{t_i, a_i - \eta\}, a_i \text{ if } a_i > t_i \\
(a_i, \bar{a}) \text{ if } a_i = t_i < \bar{a} \\
\{\bar{a}\} \text{ if } a_i = t_i = \bar{a}.
\end{cases}$$

Then $f_i$ and $\Lambda_i$ satisfy the requirements in (4) of Theorem 7 for the given $i$ and $\varepsilon$. As $i \in N$ and $\varepsilon > 0$ are arbitrary, (4) of Theorem 7 is satisfied.

Thus, by Theorem 7, $\Gamma$ is $\Phi$-virtually continuous. For each $i \in N$, define a map $h_i: A_i \to \mathbb{R}$ by setting $h_i(a_i) = a_i$. Using Theorem 8, with $\ell = 1$ and $f_{i,1} = h_i$ for each $i \in N$, we see that $\Gamma$ is $\Phi$-strongly indeterminate. Now by Theorem 5 we can conclude that $I(\Gamma) \cap W = E(\Gamma) \cap W \neq \emptyset$.

**Remark 9.** It is straightforward to generalize the above example to the case of more than one commodity by using the full generality of Theorem 7 with $\ell > 1$. It is also interesting to contrast the conclusion of the above Bertrand example with Example 3. In both examples, the two firms have asymmetric costs with probability one; however, invariance of the equilibrium set holds in Example 6 but not in Example 3. The difference is that the possible types of each firm are distributed atomlessly in Example 6, so that Theorem 8 applies to yield strong indeterminacy, which is not the case in the other example.
A Appendix

A.1 Virtual continuous games

Let $\Gamma = (N, (X_i)_{i \in N}, Q)$ be a game with an endogenous sharing rule. One aspect of Example 1 is that Dirac measures (i.e., pure strategies) are taken for the $\mu_i$’s of the definition of virtual continuity. In Lemma 7 below it is shown that this is always possible.

**Lemma 7.** $\Gamma$ is virtually continuous if and only if for each $q \in S_Q$, $i \in N$, $\varepsilon > 0$ and $\sigma \in M$, there exists $\bar{x}_i \in X_i$ such that $\tau(\delta_{\bar{x}_i, \sigma_{-i}})(D_i) < \varepsilon$ and $q_i(\bar{x}_i, \sigma_{-i}) > q_i(\sigma) - \varepsilon$.

**Proof.** The sufficiency part is obvious. For the necessity part, fix $0 < \varepsilon < 1$ and $i \in N$. Choose $a \geq 0$ so that $q_i(x) + a > 0$ for all $x \in X$. Set $k = \max\{1, \sup\{q_i(x) : x \in X\}\}$ and $\varepsilon' = \varepsilon^2/(2(k + a))$. Note that $\varepsilon' \leq \varepsilon/2$.

Virtual continuity gives a $\mu_i$ so that $\tau(\mu_i, \sigma_{-i})(D_i) < \varepsilon'$ and $q_i(\mu_i, \sigma_{-i}) > q_i(\sigma) - \varepsilon'$.

Let $E = \{x_i \in X_i : \sigma_{-i}(D_i x_i) \geq \varepsilon\}$ where $D_i x_i \subseteq X_{-i}$ is the section of $D_i$ at $x_i$. By Fubini’s theorem, $\mu_i(E) < \varepsilon'/\varepsilon$. Thus $\mu_i(E) < \varepsilon/(2(k + a))$ by the choice of $\varepsilon$ and $\varepsilon'$.

Now, again using Fubini’s theorem,

$$\int_{X_i \setminus E} \int_{X_{-i}} (q_i(x_i, x_{-i}) + a) d\sigma_{-i}(x_{-i}) d\mu_i(x_i)$$

$$> q_i(\sigma) + a - \varepsilon' - \int_{E} \int_{X_{-i}} (q_i(x_i, x_{-i}) + a) d\sigma_{-i}(x_{-i}) d\mu_i(x_i)$$

$$\geq q_i(\sigma) + a - \varepsilon' - \mu_i(E)(k + a)$$

$$\geq q_i(\sigma) + a - \varepsilon/2 - \varepsilon/2$$

$$= q_i(\sigma) + a - \varepsilon.$$

There must therefore be an $\bar{x}_i \in X_i \setminus E$ such that

$$\mu_i(X_i \setminus E) \int_{X_{-i}} (q_i(\bar{x}_i, x_{-i}) + a) d\sigma_{-i}(x_{-i}) > q_i(\sigma) + a - \varepsilon.$$

Because $\int_{X_{-i}} (q_i(\bar{x}_i, x_{-i}) + a) d\sigma_{-i}(x_{-i}) \geq 0$ by the choice of $a$, it follows that

$$\int_{X_{-i}} (q_i(\bar{x}_i, x_{-i}) + a) d\sigma_{-i}(x_{-i}) > q_i(\sigma) + a - \varepsilon$$

and hence that $\int_{X_{-i}} q_i(\bar{x}_i, x_{-i}) d\sigma_{-i}(x_{-i}) > q_i(\sigma) - \varepsilon$. Finally, we have $\tau(\delta_{\bar{x}_i, \sigma_{-i}})(D_i) < \varepsilon$ because $\bar{x}_i \in X_i \setminus E$. \qed
Lemma 7 implies that virtual continuity in mixed strategies implies virtual continuity in pure strategies (see Remark 5 for the definition of the latter). However, the converse does not hold, as the following example shows.

Example 7. Suppose $N = \{1, 2\}$ and $X_1 = X_2 = [0, 1]$. Define a correspondence $Q_1 : X \to \mathbb{R}$ by setting

$$Q_1(x_1, x_2) = \begin{cases} [0, 1] & \text{if } -1/4 + x_1 \leq x_2 \leq 1/4 + x_1 \\ \{1\} & \text{otherwise.} \end{cases}$$

Define a correspondence $Q_2 : X \to \mathbb{R}$ by setting $Q_2(x) = \{0\}$ for all $x \in X$. Let $Q = Q_1 \times Q_2$. Evidently virtual continuity holds for pure strategies. But if $\sigma_2$ is the restriction of Lebesgue measure to the Borel sets of $[0, 1]$, then $\tau_\sigma(D_1) \geq 1/4$ for all $\sigma_1 \in M_1$, so virtual continuity fails for mixed strategies.

The next result is used in our treatment of Bertrand competition.

Lemma 8. Let $\Gamma = (N, (X_i)_{i \in N}, Q)$ be such that $X_i = A$ for each $i \in N$, where $A$ is a perfect compact subset of $\mathbb{R}^m$, $m \geq 1$. For each $i \in N$, write

$$\Delta_i = \left\{ x \in X : x_i = x_j \text{ for some } j \in N \setminus \{i\} \right\}.$$ 

Suppose the following:

1. $D_i \subseteq \Delta_i$ for each $i \in N$.

2. Given $\varepsilon > 0$, $i \in N$ and $x_i \in A$, there is an open subset $\Lambda_i(x_i)$ of $A$ such that
   (i) $x_i$ is a cluster point of $\Lambda_i(x_i)$,
   (ii) whenever $a_i \in \Lambda_i(x_i)$ and $x_{-i} \in A^{n-1}$ are such that $(x_i, x_{-i}) \in D_i$ and $(a_i, x_{-i}) \notin D_i$, then $Q_i(a_i, x_{-i}) > \max Q_i(x_i, x_{-i}) - \varepsilon$ (recall that $\#(Q_i(a_i, x_{-i})) = 1$ if $(a_i, x_{-i}) \notin D_i$).

Then virtual continuity holds.

Proof. To see this, consider any $q \in S_Q$, $\sigma \in M$, $\varepsilon > 0$, and $i \in N$. Without loss of generality, take $i = 1$ and let $x_1 \in A$ be such that

$$q_1(\delta_{x_1}, \sigma_{-1}) \geq q_1(\sigma).$$

(i) For each $k \in \mathbb{N} \setminus \{0\}$ there is $a_k \in A$ such that

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(a) \( \tau(\delta_{a_k}, \sigma_{-1})(D_1) = 0 \) and

(b) \( ||a_k - x_1|| < 1/k \).

To see this, first observe that the set \( E = \{ a_1 \in A : \tau_{\sigma_{-1}}(\Delta_{1,a_1}) = 0 \} \) is dense, writing \( \Delta_{1,a_1} \) for the section of \( \Delta_1 \) at \( a_1 \). Indeed, for each \( i \in \mathbb{N}\setminus\{1\} \), the set of all \( r \in \mathbb{R}^m \) such that \( \tau_{\sigma_{-1}}(\{x_{-1} \in A^{n-1} : x_i = r \}) > 0 \) is countable. Now if \( a_1 \notin E \) there must be \( i \in \mathbb{N}\setminus\{1\} \) such that \( \tau_{\sigma_{-1}}(\{x_{-1} \in A^{n-1} : x_i = a_1 \}) > 0 \). Thus \( A \setminus E \) is countable. Because \( A \) is perfect, every point of \( A \) is a condensation point for \( A \), and the claim about \( E \) follows.

Now for each \( k \in \mathbb{N}\setminus\{0\} \), choose \( a_k \in \Lambda_1(x_1) \cap E \) such that (b) holds; such \( a_k \) exist by condition (2)(i), because \( E \) is dense and \( \Lambda_1(x_1) \) is open. As for (a), note that we have

\[
\tau(\delta_{a_k}, \sigma_{-1})(D_1) \leq \tau(\delta_{a_k}, \sigma_{-1})(\Delta_1) = \tau_{\sigma_{-1}}(\Delta_{1,a_k}) = 0
\]

for each \( k \), because \( a_k \in E \).

(ii) Using Fubini’s theorem and the fact that the countable union of null sets is a null set, we see from (b) that for \( \tau_{\sigma_{-1}} \)-a.e. \( x_{-1} \in A_{-1} \) we have \( \delta_{a_k}(D_1, x_{-1}) = 0 \) for the \( x_{-1} \)-section of \( D_1 \), i.e., \( (a_k, x_{-1}) \notin D_1 \), for all \( k \in \mathbb{N}\setminus\{0\} \). Hence, from (2)(ii), and because \( (x_1, x_{-1}) \) is a continuity point of \( q_1 \) whenever \( (x_1, x_{-1}) \notin D_1 \), we must have

\[
\lim_{k \to \infty} q_1(a_k, x_{-1}) \geq q_1(x_1, x_{-1}) - \varepsilon
\]

for \( \tau_{\sigma_{-1}} \)-a.e. \( x_{-1} \in A^{n-1} \). It follows that

\[
\lim_{k \to \infty} q_1(\delta_{a_k}, \sigma_{-1}) = \lim_{k \to \infty} \int_{A^{n-1}} q_1(a_k, x_{-1}) \, d\tau_{\sigma_{-1}}(x_{-1}) \\
\geq \int_{A^{n-1}} q_1(x_1, x_{-1}) \, d\tau_{\sigma_{-1}}(x_{-1}) - \varepsilon = q_1(\sigma) - \varepsilon,
\]

by Fatou’s lemma. Thus, by (i)(a), and as \( \varepsilon > 0 \) is arbitrary, the requirements of virtual continuity are satisfied for player 1.

Returning to the definition of virtual continuity, it would of course be more intuitive, and for typical applications probably also sufficient (this is actually the case for the applications we will consider in this paper), to require \( \tau(\mu, \sigma_{-1})(D_i) = 0 \) in the definition of virtual continuity, rather than just \( \tau(\mu, \sigma_{-1})(D_i) < \varepsilon \) for \( \varepsilon > 0 \). However, the “\( \tau(\mu, \sigma_{-1})(D_i) < \varepsilon \)”-clause adds some generality.
**Example 8.** Let $N = \{1, 2\}$, and $X_1 = X_2 = [0, 1]$. Define a correspondence $Q_1 : X \to \mathbb{R}$ by setting

$$Q_1(x_1, x_2) = \begin{cases} 
[0, 1] & \text{if } x_1 \leq x_2 \leq x_1/2 + 1/2 \text{ or } x_1 = 1 \\
\{1\} & \text{otherwise.}
\end{cases}$$

Define a correspondence $Q_2 : X \to \mathbb{R}$ by setting $Q_2(x) = \{0\}, x \in X$. Let $Q = Q_1 \times Q_2$. Then $Q$ is uhc with non-empty compact values, and

$$D_1 = \{x \in X : x_1 \leq x_2 \leq x_1/2 + 1/2\} \cup \{x \in X : x_1 = 1\}.$$

Pick any $q \in S_Q$. Of course, the requirements of virtual continuity are satisfied for $i = 2$. Consider $i = 1$. Pick any $\sigma_2 \in M_2$. Let $x_{1,k}, k \in \mathbb{N}$, be such that $x_{1,k} < 1$ for all $k$ but $x_{1,k} \to 1$. Then, by the choice of $Q$, $\tau(\delta_{x_{1,k}}, \sigma_2)(D_1) \to 0$ and $q_1(\delta_{x_{1,k}}, \sigma_2) \to 1$, from which we can see that the requirements of virtual continuity are satisfied for $i = 1$. However, if $\sigma_2$ has full support, then $\tau(\sigma_1, \sigma_2)(D_1) > 0$ for all $\sigma_1 \in M_1$.

**Remark 10.** Example 8 shows, in particular, that our definition of virtual continuity allows the sets $D_i$ to be quite large. In fact, in that example, $D_1$ has a non-empty interior.

**Lemma 9.** Give $M$ the narrow topology. Then virtual continuity implies that each $q \in S_Q$ is mixed strategy payoff secure, i.e., for all $\sigma \in M$, $i \in N$, and $\varepsilon > 0$ there is a $\mu_i \in M_i$ and a neighborhood $V$ of $\sigma$ such that $q_i(\mu_i, \sigma'_{-i}) > q_i(\sigma) - \varepsilon$ for all $\sigma' \in V$.

**Proof.** Fix $q \in S_Q$, $\sigma \in M$, $i \in N$, and $\varepsilon > 0$. Let $\varepsilon' > 0$ be such that $B\varepsilon' < \varepsilon$ where $B$ is as in the proof of Lemma 4. By virtual continuity there is a $\mu_i \in M_i$ such that $\tau(\mu_i, \sigma_{-i})(D_i) < \varepsilon'/2$ and $q_i(\mu_i, \sigma_{-i}) > q_i(\sigma) - \varepsilon'/2$. We need to show that whenever $\langle \sigma_k \rangle$ is a sequence in $M$ with $\sigma_k \to \sigma$ then $\lim_k q_i(\mu_i, \sigma_{k,-i}) > q_i(\mu_i, \sigma_{-i}) - \varepsilon$.

Given such a sequence $\langle \sigma_k \rangle$, note that $\tau(\mu_i, \sigma_{k,-i}) \to \tau(\mu_i, \sigma_{-i})$ and use Skorokhod’s theorem to find measurable maps $h$ and $h_k, k \in \mathbb{N}$, from $[0, 1]$ to $X$ such that $\lambda \circ h^{-1} = \tau(\mu_i, \sigma_{-i})$, $\lambda \circ h_k^{-1} = \tau(\mu_i, \sigma_{k,-i})$ for all $k$, and $h_k(t) \to h(t)$ $\lambda$-a.e. on $[0, 1]$, writing $\lambda$ for Lebesgue measure on $[0, 1]$. Set $E = h^{-1}(D_i)$ and $F = [0, 1] \setminus E = h^{-1}(X \setminus D_i)$.

Because each $x \in X \setminus D_i$ is a continuity point of $q_i$, we have $q_i(h_k(t)) \to q_i(h(t))$.
a.e. on $F$. Consequently

$$\lim_{k} q_i(\mu_i, \sigma_{k,-i}) = \lim_{k} \int_{[0,1]} q_i \circ h_k \, d\lambda \geq \lim_{k} \int_{E} q_i \circ h_k \, d\lambda + \lim_{k} \int_{F} q_i \circ h_k \, d\lambda$$

$$= \lim_{k} \int_{E} q_i \circ h_k \, d\lambda + \int_{F} q_i \circ h \, d\lambda.$$

Now, by the choice of $B$, we have $\int_{E} q_i \circ h_k \, d\lambda \geq -B\lambda(E) > -B\varepsilon'/2$ for each $n$; also

$$\int_{F} q_i \circ h \, d\lambda = q_i(\mu_i, \sigma_{-i}) = \int_{E} q_i \circ h \, d\lambda \geq q_i(\mu_i, \sigma_{-i}) - B\lambda(E) > q_i(\mu_i, \sigma_{-i}) - B\varepsilon'/2$$

(because $q_i(\mu_i, \sigma_{-i}) = \int_{[0,1]} q_i \circ h \, d\lambda$). Putting all these together, proves the claim. \qed

**Lemma 10.** Let $M$ be given the narrow topology. Then virtual continuity implies that $v_{q_i}$ is continuous for each $q \in S_Q$ and each $i \in N$.

**Proof.** Fix $q \in S_Q$ and $i \in N$. By Lemma 9 we can see that $v_{q_i}$ is lower semicontinuous. By Lemma 2 there is a $q' \in S_q$ such that $q'$ is upper semicontinuous and, thus, $v_{q'_i}$ is also upper semicontinuous. By Lemma 5, $v_{q_i}$ is upper semicontinuous. \qed

### A.2 Young measures

In this section we establish a result on Young measures that is needed for our main results.

**Lemma 11.** In the context and notation of Section 5, let $\sigma = (\sigma_1, \ldots, \sigma_n) \in M$ and $q: T \times A \to \mathbb{R}$ be a bounded measurable function. Then, for any $i \in N$, there is a measurable map $g: T_i \to A_i$ such that $g(t_i) \in \text{supp}(\sigma_i(t_i))$ for $\lambda_i$-a.e. in $T_i$ and $q(\delta_g, \sigma_{-i}) \geq q(\sigma_i, \sigma_{-i})$.

**Proof.** Without loss of generality consider $i = 1$. Assume first that $q(t, a) \geq 0$ for all $(t, a) \in T \times A$. For each $t_{-1} \in T_{-1}$ write $\sigma_{-1}^{(x)}(t_{-1})$ for the product measure on $A_{-1}$ defined from the measures $\sigma_2(t_2), \ldots, \sigma_n(t_n)$. Write $\lambda_{-1}^{(x)}$ for the product measure on $T_{-1}$ defined from the measures $\lambda_2, \ldots, \lambda_n$, and $\tau_{-1}^{(x)}$ for the uniquely determined probability measure on $T_{-1} \times A_{-1}$ such that $\tau_{-1}^{(x)}(C \times B) = \int_C \sigma_{-1}^{(x)}(t_{-1})(B) \, d\lambda_{-1}^{(x)}(t_{-1})$ for each $C \in \mathcal{B}(T_{-1})$ and $B \in \mathcal{B}(A_{-1})$. Let $\rho: T \to \mathbb{R}_+$ be a Radon-Nikodym derivative of $\lambda$ with respect to $\lambda_1 \times \ldots \times \lambda_n$. Define $\tilde{q}: T \times A \to \mathbb{R}_+$ by setting $\tilde{q}(t, a) = \rho(t)q(t, a)$ for each $(t, a) \in T \times A$. In the sequel of this proof, Tonelli’s theorem (in its ordinary
and generalized version; see Neveu (1965, Proposition III.2.1)) is used repeatedly and invoked without explicit reference.

Note that whenever \( O \subseteq A_1 \) is open, then for any \( t_1 \in T_1 \), \( \text{supp}(\sigma_1(t_1)) \cap O \neq \emptyset \) if and only if \( \sigma_1(t_1)(O) > 0 \). Therefore, by the definition of Young measure, for such sets \( O \), the set \( \{t_1 \in T_1 : \text{supp}(\sigma_1(t_1)) \cap O \neq \emptyset \} \) is measurable. Using Castaing and Valadier (1977, Proposition III.13, p. 69), this implies that the correspondence \( t_1 \mapsto \text{supp}(\sigma_1(t_1)) : T_1 \rightarrow A_1 \) has a measurable graph. Next note that the maps \( h: T_1 \rightarrow \mathbb{R}_+ \cup \{+\infty\} \) and \( h_1: T_1 \times A_1 \rightarrow \mathbb{R}_+ \cup \{+\infty\} \), defined by setting

\[
h(t_1) = \int_{A_1} \int_{T_{-1}} \int_{A_{-1}} \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, d\sigma^{(x)}_{-1}(t_{-1})(a_{-1}) \, d\lambda^{(x)}_{-1}(t_{-1}) \, d\sigma_1(t_1)(a_1)
\]

and

\[
h_1(t_1, a_1) = \int_{T_{-1}} \int_{A_{-1}} \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, d\sigma^{(x)}_{-1}(t_{-1})(a_{-1}) \, d\lambda^{(x)}_{-1}(t_{-1})
\]

respectively, are measurable. It follows that the correspondence \( F: T_1 \rightarrow A_1 \), defined by setting

\[
F(t_1) = \begin{cases} 
\text{supp}(\sigma_1(t_1)) \cap \{a_1 \in A_1 : h_1(t_1, a_1) \geq 0\} & \text{if } h_1(t_1) < \infty \\
A_1 & \text{otherwise},
\end{cases}
\]

has a measurable graph. Clearly \( F(t_1) \neq \emptyset \) for all \( t_1 \in T_1 \). Using Castaing and Valadier (1977, III.22, p. 74), there is a universally measurable map \( g' \) such that \( g'(t_1) \in F(t_1) \) for all \( t_1 \in T_1 \). Observe that the set \( \{t_1 \in T_1 : \int_{T_{-1}} \rho(t_1, t_{-1}) \, d\lambda^{(x)}_{-1}(t_{-1}) = +\infty\} \) is a \( \lambda_1 \)-null set, so the same must be true of the set \( \{t_1 \in T_1 : h(t_1) = +\infty\} \) because \( q \) is bounded. Modifying \( g' \) on a \( \lambda_1 \)-null set, if necessary, we can therefore find a measurable map \( g: T_1 \rightarrow A_1 \) such that \( \int_{T_1} h_1(t_1, g(t_1)) \, d\lambda_1(t_1) \geq \int_{T_1} h(t_1) \, d\lambda_1(t_1) \) and \( g(t_1) \in \text{supp}(\sigma_1(t_1)) \) for \( \lambda_1 \)-a.e. \( t_1 \in T_1 \).

Now, for each fixed \( t_1 \in T_1 \),

\[
\int_{A_1 \times (T_{-1} \times A_{-1})} \tilde{q}(t_1, \cdot) \, d\sigma_1(t_1) \times \tau^{(x)}_{-1}
= \int_{A_1} \int_{T_{-1} \times A_{-1}} \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, d\tau^{(x)}_{-1}(t_{-1}, a_{-1}) \, d\sigma_1(t_1)(a_1)
= \int_{T_{-1} \times A_{-1}} \int_{A_1} \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, d\sigma_1(t_1)(a_1) \, d\tau^{(x)}_{-1}(t_{-1}, a_{-1}).
\]
Consequently,

\[ q(\sigma_1, \sigma_{-1}) \]
\[ = \int_{T \times T_{-1}} \int_A \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, d\sigma_1(t_1) \times \sigma_{-1}^{(x)}(t_{-1})(a_1, a_{-1}) \, d\lambda_1 \times \lambda_{-1}^{(x)}(t_1, t_{-1}) \]
\[ = \int_{T_1} \int_{T_{-1}} \int_A \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, d\sigma_1(t_1) \times \sigma_{-1}^{(x)}(t_{-1})(a_1, a_{-1}) \, d\lambda_{-1}^{(x)}(t_{-1}) \, d\lambda_1(t_1) \]
\[ = \int_{T_1} \int_{T_{-1}} \int_A \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, d\sigma_1(t_1)(a_1) \, d\sigma_{-1}^{(x)}(t_{-1})(a_{-1}) \, d\lambda_{-1}^{(x)}(t_{-1}) \, d\lambda_1(t_1) \]
\[ = \int_{T_1} \int_{A_1} \int_{T_{-1} \times A_{-1}} \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, d\tau_{-1}^{(x)}(t_{-1}, a_{-1}) \, d\sigma_1(t_1)(a_1) \, d\lambda_1(t_1) \]
\[ = \int_{T_1} \int_{A_1} \int_{T_{-1}} \int_{A_{-1}} \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, d\sigma_{-1}^{(x)}(t_{-1})(a_{-1}) \, d\lambda_{-1}^{(x)}(t_{-1}) \, d\sigma_1(t_1)(a_1) \, d\lambda_1(t_1) \]
\[ = \int_{T_1} h(t_1) \, d\lambda_1(t_1) \]
\[ \leq \int_{T_1} h_1(t_1, g(t_1)) \, d\lambda_1(t_1) \]
\[ = \int_{T_1} \int_{T_{-1}} \int_{A_{-1}} \tilde{q}(t_1, t_{-1}, g(t), a_{-1}) \, d\sigma_{-1}^{(x)}(t_{-1})(a_{-1}) \, d\lambda_{-1}^{(x)}(t_{-1}) \, d\lambda_1(t_1) \]
\[ = \int_{T_1} \int_{T_{-1}} \int_A \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, d\delta_g(t_1) \times \sigma_{-1}^{(x)}(t_{-1})(a_1, a_{-1}) \, d\lambda_{-1}^{(x)}(t_{-1}) \, d\lambda_1(t_1) \]
\[ = \int_{T_1 \times T_{-1}} \int_A \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, d\delta_g(t_1) \times \sigma_{-1}^{(x)}(t_{-1})(a_1, a_{-1}) \, d\lambda_1 \times \lambda_{-1}^{(x)}(t_1, t_{-1}) \]
\[ = q(\delta_g, \sigma_{-1}). \]

Thus the lemma is true whenever \( q \) is non-negative. But this implies that the lemma is true for any bounded \( q \), by the fact that if \( f : T \times A \to \mathbb{R} \) is constant-valued, then \( f(\sigma_1, \sigma_{-1}) = f(\sigma'_1, \sigma_{-1}) \) for any \( \sigma'_1 \in M_1 \). \( \square \)

### A.3 Spaces of Young measures

Fix a probability space \((T, \Sigma, \nu)\) and a Polish space \( X \), and let \( \mathcal{R} \) denote the set of all Young measures from \( T \) to \( X \).

A Carathéodory integrand on \( T \times X \), with control measure \( \nu \), is a measurable function \( q : T \times X \to \mathbb{R} \) such that \( q(t, \cdot) \) is continuous for each \( t \in T \) and such that for some \( \nu \)-integrable \( \theta_q : T \to \mathbb{R}_+ \), \( \sup\{|q(t, x)| : x \in X\} \leq \theta_q(t) \) for each \( t \in T \). Write
for the set of all such functions. Now the narrow topology for Young measures on $\mathcal{R}$, with $\nu$ as control measure, is the coarsest topology on $\mathcal{R}$ such that for each $q \in \mathcal{G}_\nu$ the functional

$$
\gamma \mapsto \int_T \int_X q(t, x) d\gamma(t)(x) d\nu(t) : \mathcal{R} \to \mathbb{R}
$$

is continuous. With this topology, $\mathcal{R}$ becomes a subset of a locally convex topological vector space (see Balder (2002, Step 2, p. 462)). It should be noted that, in general, the narrow topology for Young measures is not a Hausdorff topology.

If $\kappa: T \to X$ is a correspondence, then $\mathcal{R}_\kappa$ denotes the subset of $\mathcal{R}$ defined by setting

$$
\mathcal{R}_\kappa = \{ \gamma \in \mathcal{R} : \text{supp}(\gamma(t)) \subseteq \kappa(t) \text{ for almost all } t \in T \}.
$$

The following theorem gathers several properties of the space $\mathcal{R}_\kappa$ (see Carmona and Podczeck (2014, Theorem 10) for a proof).

**Theorem 9.** Let $\kappa: T \to X$ be a correspondence with measurable graph such that $\kappa(t)$ is non-empty and compact for all $t \in T$. Give $\mathcal{R}$ the narrow topology for Young measures, with $\nu$ as control measure. Then the subset $\mathcal{R}_\kappa$ of $\mathcal{R}$ is non-empty, convex, closed, compact, and sequentially compact.

Now for each $i = 1, \ldots, n$, let $(T_i, \Sigma_i, \lambda_i)$ be a probability measure, $X_i$ a Polish space, and write $\mathcal{R}_i$ for the set of all Young measures from $T_i$ to $X_i$. Set $X = \prod_{i=1}^n X_i$, $T = \prod_{i=1}^n T_i$, and $\Sigma = \bigotimes_{i=1}^n \Sigma_i$. Let $\mathcal{R}$ be the set of all Young measures from $T$ to $X$, and $g: \prod_{i \in \mathbb{N}} \mathcal{R}_i \to \mathcal{R}$ the map defined by setting

$$
g(\sigma_1, \ldots, \sigma_n)(t) = \sigma_1(t_1) \times \cdots \times \sigma_n(t_n), \ t \in T,
$$

for all $(\sigma_1, \ldots, \sigma_n) \in \prod_{i \in \mathbb{N}} \mathcal{R}_i$. Write $\lambda^{(\times)}$ for the product measure defined from the measures $\lambda_i, \ i = 1, \ldots, n$, and let $\lambda$ be any probability measure on $(T, \Sigma)$. For each $i = 1, \ldots, n$, give $\mathcal{R}_i$ the narrow topology for Young measures, with $\lambda_i$ as control measure. Write $\mathcal{R}_\lambda$ for $\mathcal{R}$ endowed with the narrow topology for Young measures, taking $\lambda$ as control measure, and $\mathcal{R}_{\lambda^{(\times)}}$ for $\mathcal{R}$ endowed with the narrow topology for Young measures, taking $\lambda^{(\times)}$ as control measure. Then the following three lemmata are true.

**Lemma 12.** The map $g$ is continuous.
Proof. Using induction, this follows from Balder (1988, Theorem 2.5).

Lemma 13. If $\lambda$ is absolutely continuous with respect to $\lambda^{(x)}$, then the identity from $\mathcal{R}_{\lambda^{(x)}}$ to $\mathcal{R}_{\lambda}$ is continuous.

Proof. Let $h$ be a Radon-Nikodym derivative of $\lambda$ with respect to $\lambda^{(x)}$. Note that if $q$ is a Carathéodory integrand on $T \times X$ with control measure $\lambda$, then $q \times h$ is a Carathéodory integrand on $T \times X$ with control measure $\lambda^{(x)}$.

Lemma 14. Suppose that $T$ is a Polish space, and that $\Sigma$ is the Borel $\sigma$-algebra. Give $M(T \times X)$ the narrow topology. Then the function $\sigma \mapsto \tau_{\sigma}$ from $\mathcal{R}_{\lambda}$ to $M(T \times X)$ is continuous.

Proof. Let $c : T \times X \to \mathbb{R}$ be bounded and continuous. Then $c$ is a Carathéodory integrand on $T \times X$. Since

$$\int c \mathrm{d}\tau_{\sigma} = \int_T \left( \int_A c(t,a) \mathrm{d}\sigma(a) \right) \mathrm{d}\lambda(t),$$

the result follows.

References


