Strict pure strategy Nash equilibrium in large finite-player games when the action set is a manifold

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Abstract

We present results on the relationship between non-atomic games (in distributional form) and approximating games with a large but finite number of players. Specifically, in a setting with differentiable payoff functions, we show that: (1) The set of all non-atomic games has an open dense subset such that any finite-player game that is sufficiently close (in terms of distributions of players' characteristics) to a game in this subset and has sufficiently many players has a strict pure strategy Nash equilibrium (Theorem 1), and (2) any equilibrium distribution of any non-atomic game is the limit of equilibrium distributions defined from strict pure strategy Nash equilibria of finite-player games (Theorem 2). This supplements our paper Carmona and Podczeck (2020b) "Strict Pure Strategy Nash Equilibria in Large Finite-Player Games" [Theoretical Economics, forthcoming], where analogous results are established for the case where the action set of players is a subset of some Euclidean space, with non-empty interior, and payoff functions are such that equilibrium actions are in the interior of the action set. The goal of the present paper is to remove these assumptions.

Keywords: Large games, pure strategy, Nash equilibrium, generic property, differentiable manifold.

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1 Introduction

In a seminal paper, Schmeidler (1973, Theorem 2) proved the existence of pure strategy Nash equilibria in games with a continuum of players and finite action spaces. This was done by Schmeidler in the context of anonymous games (i.e., games where the payoff of a player is, apart from his/her own action, determined by the distribution of the actions made by the other players). After all, of course, the continuum specification is an idealization of situations with a large but finite number of players, and the question arose whether Nash equilibrium existence results for games with a continuum of players are reflected in large finite-player games. In terms of ϵ-equilibria, this question was answered positively first by Rashid (1983). Since then, an extensive literature on this topic emerged.[1]

In two recent papers (Carmona and Podczeck, 2020a,b) we have addressed the question to which extent pure strategy Nash equilibrium existence results for anonymous games with a continuum of players (for short, non-atomic games in the sequel) carry over into pure strategy Nash equilibrium existence results for large finite-player games. In Carmona and Podczeck (2020a) we have shown that, with finite action sets, there is a generic set of non-atomic games such that sufficiently large finite-player games which are close (in terms of distributions of players' characteristics) to a game in this set have indeed Nash equilibria in pure strategies. Based on this result we have shown in addition that, in fact, any equilibrium distribution of any non-atomic game can be approximated by equilibrium distributions defined from pure strategy equilibria of large finite-player games (provided that action sets are finite).

In Carmona and Podczeck (2020b) we have dropped the requirement that action sets be finite and have proved analogous results in a setting with differentiable payoff functions. Actually, the results in this latter paper were formulated in a framework where the action set of players has non-empty interior in some Euclidean space and payoff functions satisfy a boundary assumption guaranteeing that equilibrium strategies are always in the interior of the action set. These assumptions were made to be in a convenient position to employ differentiability assumptions on payoff functions.

The purpose of the present note is to obtain results analogous to those in Carmona and Podczeck (2020b) for the case where the boundary assumption on payoff functions made there is not met, or the action set of players does not have non-empty interior in any Euclidean space. This makes it necessary to impose some smooth structure on the boundary of action sets. To achieve this, we find it convenient to work with local coordinates, assuming that every point in the action set has a neighborhood that look like a certain piece of the non-positive orthant of some Euclidean space. Technically speaking, we assume that the action set of players is a compact differentiable manifold with corners; see the next section for a definition. We emphasize that even though the general structure of the proofs of the results of the present note is similar to that in Carmona and Podczeck (2020b), numerous adjustments in the details become necessary to accommodate the proofs to the setting of action sets studied now (cf.

[1]For an overview of this literature, we refer to Carmona and Podczeck (2020a).

Section 5.1 and Remark 2 in Section 3).

To illustrate the usefulness of dispensing with the assumption that players' action sets have non-empty interior in some Euclidean space, and thus the usefulness of the present results to economic applications, consider a large population of individuals who live, or are considering living, in a given city. Following Salop (1979), it is convenient to model the city as the unit-circumference of a circle. The optimal choice of any individual on where in the city to live depends on the choice of all others through their influence, in particular, on how popular each neighborhood is. Thus we may specify the payoff of each individual as a function of his choice and of summary statistics of the distribution of the choice of all other players, for instance its first m (non-central) moments. If players' payoff functions are C^2, then our results apply to yield a strict equilibrium in all sufficiently large finite-player games that are close to a generic non-atomic game.

To illustrate the usefulness of removing the assumption that equilibrium actions are in the interior of the action set, we present as a special case of our results an example on existence of Cournot equilibria for generic distributions of cost functions if the number of firms is large. In this application, any inverse demand function and any cost function are allowed provided that they are C^2. This is in contrast with what we considered in the working paper version of Carmona and Podczeck (2020b) where some constraints on inverse demand and cost functions were imposed to guarantee that firms' optimal actions are strictly positive and strictly below their capacities.

Technically, a circle is an example of a manifold without boundary, which is a special case of manifolds with corners. An example of a genuine manifold with corners is a compact interval in $\mathbb{R}^n$. Many other examples of compact differentiable manifolds with corners are possible. These include finite sets, disks, simplices, and stripes in any Euclidean space, or the set of solutions to some systems of equations. In fact, any subset of a Euclidean space which is homeomorphic to a differentiable manifold with corners can be given a structure such that it becomes a differentiable manifold with corners, too; e.g., a square in $\mathbb{R}^2$, being homeomorphic to a circle in $\mathbb{R}^2$, can be given a smooth structure.

It is true that our specification of action sets is not a must. An alternative would be to define action sets globally as the sets of those points x in $\mathbb{R}^n$ satisfying conditions of the form $g_i(x) = 0$ and $g_j(x) \leq 0$ where the g_i and g_j are finitely many differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that constraint qualification holds. In practically all applications in economics, action sets which are covered as manifolds in our paper are defined in this way. The examples from above can be written in this form. One could then use the famous Kuhn-Tucker theorem to characterize maxima of payoff functions. However, we find it more convenient to state first and second order conditions for maxima of payoff functions in local coordinates. For this reason, and for sake of generality,[2] we prefer to use the notion of differentiable manifold to specify action sets.

[2]Not every differentiable manifold with corners can be described as a set of the form $\{x \in \mathbb{R}^n : g_i(x) = 0, g_j(x) \leq 0\}$, where the functions g_i and g_j are as above; e.g., the Möbius strip cannot be defined this way. See Guillemin and Pollack (1974, p. 106).

The rest of this note is organized as follows. In the next section some general notation is settled. In Section 3 the model and the main results are presented. This is followed by a section with the example on Cournot oligopoly. Proofs can be found in Section 5. An appendix contains some auxiliary material.

2 Notation and terminology

For a subset A of a topological space X, $\operatorname{int} A$ denotes the interior of A, $\operatorname{cl} A$ the closure of A, and $A \setminus B$ set-theoretic subtraction. If $A \subseteq \mathbb{R}^n$, then $\operatorname{co} A$ denotes the convex hull of A. If $X \times Y$ is the product of any sets X and Y, proj_X denotes the projection of $X \times Y$ onto X. We write $\operatorname{dom} f$ for the domain of a map f, and $f \restriction X$ for the restriction of f to a subset X of its domain.

If X is a metric space, we write δ_x for the Dirac measure at $x \in X$. If μ is a Borel measure on a separable metric space X, we write $\operatorname{supp}(\mu)$ for the support of μ, i.e., the smallest closed subset of X with full measure. If μ is a Borel measure on a product $X \times Y$ of metric spaces, μ_X and μ_Y denote the marginal measures on X and Y respectively.

Euclidean spaces are regarded as being equipped with the Euclidean norm. For any point x in such a space, and any number $r > 0$, we write $B(x, r)$ for the open ball of center x and radius r, and $\bar{B}(x, r)$ for the closed ball of center x and radius r.

If $O \subseteq \mathbb{R}^n$ is a non-empty open set and $f \colon O \to \mathbb{R}^m$ a C^1 map, we write $Df(x)$ for the derivative of f at $x \in X$; if f is C^2, we write $D^2 f(x)$ for the second order derivative of f at x.

By $\mathbb{R}^n_-$ we denote the set $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_i \leq 0,\ i = 1, \ldots, n\}$. Let U be a (relatively) open subset of $\mathbb{R}^n_-$. A map $f \colon U \to \mathbb{R}^m$ is called C^k if there is an open set $\tilde{U}$ in $\mathbb{R}^n$ and a C^k-map $\tilde{f} \colon \tilde{U} \to \mathbb{R}^m$ (C^k in the usual sense of calculus) such that $\tilde{f} \restriction U = f$. Note that the interior of $\mathbb{R}^n_-$ is dense in $\mathbb{R}^n_-$, so, by continuity, the derivative of $\tilde{f}$ at points belonging to the topological boundary of $\mathbb{R}^n_-$ is uniquely defined.

A subset M of some $\mathbb{R}^q$ is an n-dimensional C^k-manifold with corners, if there is a C^k-structure with corners on M, i.e., a family $\langle (O_i, \theta_i) \rangle_{i \in I}$ such that (i) $\langle O_i \rangle_{i \in I}$ is an open cover of M, (ii) for each $i \in I$, θ_i is a homeomorphism from O_i onto a (relatively) open subset U_i of $\mathbb{R}^n_-$, (iii) for any $i, j \in I$, the map $\theta_j \circ \theta_i^{-1} \colon \theta_i(O_i \cap O_j) \to \theta_j(O_i \cap O_j)$ is C^k (in the sense of the previous paragraph), (iv) the family $\langle (O_i, \theta_i) \rangle_{i \in I}$ is maximal in the sense that it is not a proper subfamily of any family $\langle (O_i, \theta_i) \rangle_{i \in J}$ for which (i)-(iii) hold. Any subfamily $\langle (O_\lambda, \theta_\lambda) \rangle_{\lambda \in \Lambda}$ of $\langle (O_i, \theta_i) \rangle_{i \in I}$ which covers M is called an atlas on M, and any member $(O_\lambda, \theta_\lambda)$ of such a subfamily is called a chart on M. Because it is possible that $\theta_i(O_i)$ is actually open in $\mathbb{R}^n$ for each $i \in I$, a C^k-manifold without boundary is a special case of a C^k-manifold with corners.[3]

[3] We refer to Hirsch (1976) and Lee (2013) for material (including definitions and examples) concerning differentiable manifolds without boundary, and in particular to Lee (2013, pp. 415–417) for differentiable manifolds with corners. We note that it is possible to define an n-dimensional C^k-manifold with corners more abstractly as a second countable Hausdorff space M together with a C^k-structure with corners on M.

Let N and M be C^k-manifolds with corners, of dimensions n and m respectively. Let $U \subseteq N$ be open and $f\colon U \to M$ a continuous map. The continuous map f is said to be C^k if for every pair (O, θ) and (W, ψ) of charts in N and M with $O \cap U \neq \emptyset$ and $f(O \cap U) \subseteq W$ the map $\psi \circ f \circ \theta^{-1} \restriction \theta(O \cap U)$ is C^k. Recall that if a continuous f is C^k for some pair of charts as above, then it is C^k for any such pair of charts.

Let M be a C^k-manifold with corners and N a manifold without boundary. Then the product $N \times M$ is a C^k-manifold with corners, of dimension $n+m$; a typical chart is given by $(O \times W, \theta \times \psi)$ where (O, θ) is a chart on N and (W, ψ) a chart on M.

If N is open in an affine subspace L of some Euclidean space, we treat N as having a C^k-structure that allows to work with the single chart consisting of the set N itself together with the restriction to N of the canonical identification of L with some $\mathbb{R}^k$; in particular, if $N = \mathbb{R}$, we can take the identity on $\mathbb{R}$ for the latter map. In such cases, we most of the time suppress explicitly mentioning charts.

3 The model and the results

In the games we consider, all players have the same action set A, which is assumed to be an n-dimensional compact C^2-manifold with corners. By what was noted in the previous section, the case where A is a C^2-manifold without boundary is a special case; in this case, A necessarily has empty interior in any ambient Euclidean space.

The payoff of a player depends on his action a and some externality e which is determined by the distribution of the actions of the other players. We assume that there is a C^1-map $g\colon A \to \mathbb{R}^m$ such that for any player and any distribution τ_A on A induced by the actions of the other players in a game, $e = e(\tau_A) = \int g(a)\, d\tau_A(a)$. Set $E = \{\int g\, d\tau_A\colon \tau_A \text{ is a probability measure on } A\}$. Note that E is a convex and compact subset of $\mathbb{R}^m$, and that E is just the convex hull of the compact set $g(A)$.

We assume that g is such that E has non-empty interior. This does not impose any loss of generality, because E is convex in any case and therefore has non-empty interior relative to the smallest affine subspace of $\mathbb{R}^m$ including it. Indeed, write Z for the smallest affine subspace of $\mathbb{R}^m$ including E and let k be its dimension. Let $h'\colon \mathbb{R}^m \to \mathbb{R}^m$ be an affine bijection identifying Z with a coordinate subspace of $\mathbb{R}^m$. Write h for the composition of h' with the projection of this coordinate subspace onto the canonically associated $\mathbb{R}^k$. Then $h \circ g$ is continuously differentiable and $h(E) = h(\operatorname{co} g(A)) = \operatorname{co} h(g(A))$. Thus, replacing g by $h \circ g$, the assumption under discussion is satisfied.

In a non-atomic game, for any player, the set of potential externalities is E and a payoff function is a real-valued function with domain $A \times E$. In a finite-player game, with a set I of players, $\#(I) \geq 2$, the externalities any player can potentially observe are elements of the set

$$E_{\#(I)-1} = \left\{ e \in E\colon e = \frac{1}{\#(I)-1} \sum_{j \in I \setminus \{i\}} g(a_j),\ a_j \in A \right\}.$$

This set is, in general, a proper subset of E, so the domains of the payoff functions in a game with a finite set I of players have to be taken to be $A \times E_{\#(I)-1}$.

Write E_l for $\frac{1}{l}\sum_{i=1}^{l} g(A)$ if $l \geq 1$. Then, in any of the games we consider, a payoff function is a real-valued function u with $\operatorname{dom} u = A \times E$ or $\operatorname{dom} u = A \times E_l$ for $l \geq 1$. We write $\varphi(u, e)$ for the best reply set of a player with payoff function u if he faces e as externality. Thus

$$\varphi(u, e) = \big\{a \in A \colon u(a, e) = \max_{a' \in A} u(a', e)\big\}.$$

We assume that a payoff function is C^2 in the sense that it can be extended to a C^2-function defined on $A \times \tilde{E}$, where $\tilde{E} \supseteq E$ or $\tilde{E} \supseteq E_l$ is open in $\mathbb{R}^m$ (considering $A \times \tilde{E}$ as a C^2-manifold; see Section 2). We write $\widehat{\mathcal{U}}$ for the set of all payoff functions, and $\mathcal{U}$ for the subset consisting of the elements whose domain is $A \times E$. The following lemma prepares the choice of a suitable topology on the set $\widehat{\mathcal{U}}$.

Lemma 1. *Every $u \in \widehat{\mathcal{U}}$ can be extended to a map $\bar{u}$ in $\mathcal{U}$.*

Proof. Consider any $u \in \widehat{\mathcal{U}}$ with $\operatorname{dom} u = A \times E_l$. Let $(O_h, \theta_h)_{h\in\Lambda}$ be an atlas on A. Let $(\lambda_h)_{h\in\Lambda}$ be a subordinated C^2-partition of unity. For each h set $F_h = \theta_h(\operatorname{supp}(\lambda_h))$. Fix any h. Set $u_h = u \circ (\theta_h, id_E)^{-1} \restriction F_h \times E_l$. By the Whitney extension theorem, u_h extends to a C^2-function $\bar{u}_h \colon F_h \times E \to \mathbb{R}$. Do this construction for each $h \in \Lambda$, and then define $\bar{u} \in \mathcal{U}$ by setting $\bar{u}(a, e) = \sum_{h\in\Lambda} \lambda_h(a)\bar{u}_h(\theta_h(a), e)$ for each $(a, e) \in A \times E$. □

In view of Lemma 1, we can give $\widehat{\mathcal{U}}$ the topology generated by all sets of the form

$$\begin{aligned}
&\mathcal{N}^r(u; (O, \theta), K, \epsilon)\\
&= \Big\{u' \in \widehat{\mathcal{U}} \colon \inf_{f\in Z(u), g\in Z(u')} \|D^k(f \circ (\theta \times id_E)^{-1})(x) - D^k(g \circ (\theta \times id_E)^{-1})(x)\|\\
&\qquad\qquad\qquad + \rho_E\big(\operatorname{proj}_E(\operatorname{dom} u), \operatorname{proj}_E(\operatorname{dom} u')\big) < \epsilon\Big\}
\end{aligned}$$

where $u \in \widehat{\mathcal{U}}$, (O, θ) is a chart on A, K is a compact subset of O, $\epsilon > 0$ is a real number, $Z(u)$ and $Z(u')$ are the sets of all extensions of u and u', respectively, to elements of $\mathcal{U}$, k runs over $\{0, 1, 2\}$, x runs over $\theta(K) \times E$, and ρ_E is the Hausdorff metric on the non-empty compact subsets of E. Then, since A and E are compact, $\widehat{\mathcal{U}}$ is a Polish space; moreover, the subspace $\mathcal{U}$ of $\widehat{\mathcal{U}}$ can be given a norm making it a separable Banach space (cf. Hirsch, 1976, p. 64, Exercise 6).

Remark 1. Had we taken, in an attempt for simplicity, the space of payoff functions to be $\mathcal{U}$ also for finite-player games, it would be possible that we had two sequences $\langle u_l\rangle_{l\in\mathbb{N}}$ and $\langle u'_l\rangle_{l\in\mathbb{N}}$ of payoff functions, one of them being convergent and the other not, even though for all l the restrictions of u_l and u'_l would agree on the relevant set $A \times E_l$. With the space $\widehat{\mathcal{U}}$ and its topology, this problem does not appear. In fact, in the present setup, whether or not a sequence $\langle u_l\rangle$ of elements of $\widehat{\mathcal{U}}$ is convergent depends only on the sequence of sets $\langle Z_{u_l}\rangle$ of all extensions of the elements u_l to elements of $\mathcal{U}$, and each set Z_{u_l} in turn depends only on u_l. In this sense, whether or not a sequence of payoff functions is convergent depends only on players' payoffs at their own actions and the externalities they can potentially observe in actual games.

Remark 2. In Carmona and Podczeck (2020b), a topology on the set of payoff functions was defined solely in terms of the Hausdorff metric of the graphs of these functions. This approach worked by an interplay of the assumptions that (1) the action set of players has non-empty interior, (2) payoff functions satisfy a boundary condition under which equilibrium actions are in the interior of the action set, and (3) the externality mag g is an open map, which guarantees by (1) and (2) that equilibrium externalities in non-atomic games are, generically, in the interior of E. In the present note, none of these assumptions is made.

Now a finite-player game is given by a pair (I, G) where I is a finite set of players, with $\#(I) \geq 2$, and G is a map from I to $\widehat{\mathcal{U}}$ such that $\operatorname{dom} G(i) = A \times E_{\#(I)-1}$ for each $i \in I$. Every finite-player game (I, G) defines a distribution on $\widehat{\mathcal{U}}$, i.e., a distribution of payoff functions. We write ν_G for such a distribution; thus, for any Borel set B in $\widehat{\mathcal{U}}$, $\nu_G(B) = \#(\{i \in I \colon G(i) \in B\})/\#(I)$.

A *strategy profile* in a finite-player game (I, G) is a map $f \colon I \to A$. Given any strategy profile f, we write $e_{f,i}$ for the externality faced by player i; that is,

$$e_{f,i} = \sum_{j \in I \setminus \{i\}} g(f(j))/(\#(I) - 1),$$

or, in other words, $e_{f,i} = \int g(a)\, d\tau_{A,f,i}(a)$, where $\tau_{A,f,i}$ is the distribution of the actions chosen by the players $j \in I \setminus \{i\}$; thus, for any Borel set $B \subseteq A$,

$$\tau_{A,f,i}(B) = \#(\{j \in I \setminus \{i\} \colon f(j) \in B\})/(\#(I) - 1).$$

A strategy profile $f \colon I \to A$ is a *pure strategy Nash equilibrium* if $f(i) \in \varphi(G(i), e_{f,i})$ for each $i \in I$. A pure strategy Nash equilibrium is called *strict* if $\#(\varphi(G(i), e_{f,i})) = 1$ for each $i \in I$.

A non-atomic game is specified as a Borel probability measure ν on $\mathcal{U}$ with compact support. Thus, as in Mas-Colell (1984), there is no explicit set of agents. We add the compact support condition to the formalization in Mas-Colell (1984); this condition requires that players' characteristics in a non-atomic game are not too dispersed. We write $\mathcal{G}$ for the set of all non-atomic games. A Borel probability measure τ on $\mathcal{U} \times A$ is an *equilibrium distribution* of a non-atomic game $\nu \in \mathcal{G}$ if $\tau_{\mathcal{U}} = \nu$ and $\operatorname{supp}(\tau) \subseteq \{(u, a) \in \mathcal{U} \times A \colon a \in \varphi(u, e(\tau_A))\}$. By a standard fact, every $\nu \in \mathcal{G}$ has an equilibrium distribution.

We write $\mathcal{M}$ for the set of all Borel probability measures on $\widehat{\mathcal{U}}$ (or, in other words, for the set of all distributions of payoff functions) with compact support and give $\mathcal{M}$ the topology such that $\nu_n \to \nu$ in $\mathcal{M}$ if both $\nu_n \to \nu$ in the narrow topology and $\operatorname{supp}(\nu_n) \to \operatorname{supp}(\nu)$ in the Hausdorff metric topology on the set of non-empty compact subsets of $\widehat{\mathcal{U}}$.[4] The subset $\mathcal{G}$ of $\mathcal{M}$ consisting of the non-atomic games is given the subspace topology. We can now state the following definition, which is central in our results.

Definition 1. A sequence $\langle (G_n, I_n) \rangle_{n\in\mathbb{N}}$ of finite-player games converges to a non-atomic game $\nu \in \mathcal{G}$ if $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$ in the topology of $\mathcal{M}$.

The interpretation of the topology on $\mathcal{M}$ is as in Carmona and Podczeck (2020a,b): The narrow topology on $\mathcal{M}$ means that two games are "close" if they involve similar players' characteristics with similar frequencies; the Hausdorff metric topology on the non-empty compact subsets of $\mathcal{M}$ means that two games are close only if they involve similar players' characteristics. The latter requires, by the compact support assumption on non-atomic games, that for a sequence of finite-player games to converge to a non-atomic game, players' characteristics must not become too diverse if the number of players grows to infinity. We emphasize that by what was noted in Remark 1, our model is such that whether or not a sequence $\langle (I_n, G_n) \rangle_{n\in\mathbb{N}}$ of finite-player game converges to a non-atomic game depends, besides of the requirement that $\#(I_n) \to \infty$, only on the sequence of constellations of payoff functions on the relevant sets $A \times E_{\#(I_n)-1}$, $n \in \mathbb{N}$.

Remark 3. Given $\nu \in \mathcal{G}$, sequences $\langle (I_n, G_n) \rangle_{n\in\mathbb{N}}$ of finite-player games which converge to ν do exist. Indeed, let $\nu \in \mathcal{G}$ be given. Write $X = \text{supp}(\nu)$, and $\nu^{\mathbb{N}}$ for the product measure on $X^{\mathbb{N}}$ defined from $\mathbb{N}$ copies of ν. By the Glivenko-Cantelli lemma, for $\nu^{\mathbb{N}}$-almost every sequence $\langle u_n \rangle$ of elements of X, the sequence $\langle \nu_n \rangle$ of probability measures, defined by setting $\nu_n = 1/(n+1) \sum_{i=0}^{n} \delta_{u_i}$ for each $n \in \mathbb{N}$, converges to ν narrowly. Fix such a sequence $\langle u_n \rangle$. Then $\text{supp}(\nu_n) \subseteq X$ for each n, so narrow convergence of $\langle \nu_n \rangle$ to ν implies that we also have $\text{supp}(\nu_n) \to \text{supp}(\nu)$ in the Hausdorff metric topology on the non-empty compact subsets of $\mathcal{U}$. Thus $\nu_n \to \nu$ in the topology of $\mathcal{M}$. For each $n \in \mathbb{N}$, set $I_n = \{0, \dots n+1\}$ and define $G_n \colon I_n \to \widehat{\mathcal{U}}$ by $G_n(i) = u_i \restriction (A \times E_{\#(I_n)-1})$ for each $i \in I_n$. Evidently the sequence $\langle E_{\#(I_n)-1} \rangle_{n\in\mathbb{N}}$ converges to E in the Hausdorff metric topology on the non-empty compact subsets of E. Consequently, for any sequence $\langle i_n \rangle_{n\in\mathbb{N}}$ with $i_n \leq n$ for each n, we have $u_{i_n} - G_n(i) \to 0$, by the choice of the topology of $\widehat{\mathcal{U}}$. It is now plain that the sequence (I_n, G_n) converges to ν.

In our example of Cournot oligopoly considered in Section 4, the admissible payoff functions will be of a special form, which is captured in the next definition. Together with the second assertion of Theorem 1 below, this definition prepares our proof of Theorem 3 on existence of Cournot equilibrium.

[4] Recall that the narrow topology on the set of Borel measures on a metrizable topological space is the topology of pointwise convergence on the bounded continuous functions defined on this space, evaluation being given by integration. Recall also that on the set of all non-empty *compact* subsets of a metric space Z, the Hausdorff metric induces a topology which depends only on the topology of Z, but not on the metric inducing the latter topology. Thus, if Z is any metrizable topological space, we may speak simply of the Hausdorff metric topology on the (set of) non-empty compact subsets of Z, without referring to any particular metric inducing the topology of Z.

Definition 2. Given any $u \in \mathcal{U}$, $\mathcal{U}_u$ is the set of all $u' \in \mathcal{U}$ which can be written in the form $u' = u - v$ where v is an element of $\mathcal{U}$ which does not depend on e; $\mathcal{G}_u$ is the set of all $\nu \in \mathcal{G}$ with $\text{supp}(\nu) \subseteq \mathcal{U}_u$.

Here is our main result. It translates Carmona and Podczeck (2020b, Theorem 1) into a setting of differentiable manifolds.

Theorem 1. *There is an open and dense subset $\mathcal{G}^*$ of $\mathcal{G}$ such that if $\nu \in \mathcal{G}^*$ and $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ is a sequence of finite-player games converging to ν, then there is an $N \in \mathbb{N}$ such that (I_n, G_n) has a strict pure strategy Nash equilibrium if $\#(I_n) \geq N$. Further, for any $u \in \mathcal{U}$, $\mathcal{G}^* \cap \mathcal{G}_u$ is dense in $\mathcal{G}_u$ where $\mathcal{G}_u$ corresponds to u according to Definition 2.*

(For the proof, see Section 5.2.) The next definition provides a notion of an equilibrium distribution of a non-atomic game not to be an artifact of having continuum many players.

Definition 3. Let $\nu \in \mathcal{G}$ and τ an equilibrium distribution for ν. A sequence $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ of finite-player games converging to ν is said to *asymptotically implement* (ν, τ) if for all n larger than some $N \in \mathbb{N}$, (I_n, G_n) has a strict pure strategy Nash equilibrium f_n such that the sequence of distributions of the maps $G_n \times f_n$ converges to τ narrowly. We say that (ν, τ) is *asymptotically implementable* if it can be asymptotically implemented by some sequence $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ of finite-player games.

Based on our main result, we have the following.

Theorem 2. *Every (ν, τ), where $\nu \in \mathcal{G}$ is a non-atomic game and τ is an equilibrium distribution for ν, is asymptotically implementable.*

(For the proof, see Section 5.3.) As in Carmona and Podczeck (2020a,b), we stress the fact that every non-atomic game can be taken in Theorem 2, not just one from a generic set.

4 An application

In some applications, one is interested in a class of games that forms a proper subset of $\mathcal{G}$. An example, which we consider in this section, is a model of Cournot oligopoly where an inverse demand function is given and the payoff function of different players (i.e., firms) can differ only if their cost functions differ. That is, only payoff functions that can be written as the difference between some cost function and the revenue function defined from the given inverse demand function are relevant. In this case, it is desirable to obtain a generic subset of distributions of cost functions having certain properties, not a generic subset of distributions of all possible payoff functions.[5]

[5] Another setting where the approach of this section might be useful is that of monopolistic competition considered e.g. in Chipman (1970), Hart (1979), or Páscoa (1993).

We assume that the set of possible outputs that can be produced by an individual firm is the interval $[0, m]$, where $m > 0$; the number $m > 0$ can be interpreted as a capacity constraint. Let X be the set of all twice continuously differentiable functions on $[0, m]$, equipped with the topology of C^2-uniform convergence. The inverse demand function is given in terms of output per firm (independently of the actual number of firms in an oligopoly) and specified by an element p of X.[6] Let Z be the subset of X consisting of the elements v of X with $v(0) = 0$ and $Dv(a) > 0$ for all $a \in [0, m]$. Give Z the subspace topology induced by the topology of X and let $\mathcal{K}$ be a non-empty open subset of Z. The elements of $\mathcal{K}$ are the possible cost functions. A Cournot oligopoly is a pair (I, G) where I is a finite set of firms, with $\#(I) \geq 2$, and $G\colon I \to \mathcal{K}$ is a map assigning cost functions to firms. A strategy profile $f\colon I \to [0, m]$ is a Cournot equilibrium of (I, G) if

$$p\left(\tfrac{1}{\#(I)} f(i) + \tfrac{1}{\#(I)} \sum_{j \in I \setminus \{i\}} f(j)\right) f(i) - G(i)(f(i))$$
$$\geq p\left(\tfrac{1}{\#(I)} a + \tfrac{1}{\#(I)} \sum_{j \in I \setminus \{i\}} f(j)\right) a - G(i)(a)$$

for each $i \in I$ and each $a \in [0, m]$. Let $\mathcal{M}_{\mathcal{K}}$ be the set of all Borel probability measures on $\mathcal{K}$, with compact support. Give $\mathcal{M}_{\mathcal{K}}$ the topology analogous to that of $\mathcal{M}$.

Theorem 3. *There is an open dense subset $\mathcal{M}^*_{\mathcal{K}}$ of $\mathcal{M}_{\mathcal{K}}$ such that if $\nu \in \mathcal{M}^*_{\mathcal{K}}$ and $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ is a sequence of Cournot oligopolies such that $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$, then there is an $N \in \mathbb{N}$ such that (I_n, G_n) has a Cournot equilibrium if $\#(I_n) \geq N$.*

(The proof is in Section 5.4.)

5 Proofs

5.1 Comments on the proof of Theorem 1

The structure of the proof of our main theorem is similar to that of Theorem 1 in Carmona and Podczeck (2020b). After some preliminaries—points (a) and (b) of the present proof—we define in (c) a set $\mathcal{G}^*$ of games as the set of those non-atomic games for which, at some equilibrium distribution, (i) best replies are given locally as a function, which is continuous in payoff functions and externalities, and actually C^1 in the externalities, and (ii) the function, which is then locally defined, sending externalities to the externalities determined by the composition of best reply function and the externality function g has a derivative satisfying some rank condition. In (d) we show that $\mathcal{G}^*$ is open in the set of all non-atomic games, and in (e) that $\mathcal{G}^*$ is dense in this latter set. Finally, in (f), we establish existence of pure strategy Nash equilibria

[6] If desired, one can add additional economically meaningful assumptions such as $p(a) > 0$ for all $a \in [0, m)$ and $p(m) \geq 0$ and/or p is decreasing.

for finite-player games with a large number of players and a distribution of payoff functions near a non-atomic game belonging $\mathcal{G}^*$. The idea in this step of the proof is as follows. Fix a non-atomic game, ν say, belonging to $\mathcal{G}^*$, and an equilibrium distribution τ of ν, witnessing that ν belongs $\mathcal{G}^*$. Let $e(\tau_A)$ denote the externality induced by τ. "Large number of players" in finite-player games means that the differences between the mean externality of the actions of all players and those of all but one player are small. A consequence will be that, for such games, if the distribution of payoff functions is close to ν, then, in a neighborhood of $e(\tau_A)$, best replies can also be described by a continuous function, because $\mathcal{G}^*$ is open, so that equilibrium existence can be proved by a fixed point theorem.

The main difference of the present paper to Carmona and Podczeck (2020b) comes from the fact that in this latter paper it was assumed that the action set of players has non-empty interior in some $\mathbb{R}^n$ and that payoff functions are such that equilibrium actions are in the interior of the action set. In the present paper, however, it is allowed, in conformity with standard game theory, that equilibrium actions of player are boundary points of their action set. For this reason, it needs to be shown that if a best reply action of a player is on the boundary of the action set, then, generically, best replies remain there if the determinants of the player's payoff which are out of his/her control, i.e., payoff function or externality, change slightly. At this point, smoothness of the boundary of the action set becomes essential.

Moreover, in the present paper, it is not possible to derive that, generically, equilibrium externalities are in the interior of the externalities set. As a consequence, the rank condition in the definition of the generic set of non-atomic games has to be imposed relative to some linear subspace of the ambient of the externalities set.

5.2 Proof of Theorem 1

(a) We start by introducing some additional notation used in this proof. If f is a C^1 function defined on an open set X in some Euclidean space, then $D_{(i)}f(x)$ denotes the (partial) derivative of f with respect to the ith coordinate of the argument x at the point $x \in X$. If f is defined on a product $X \times Y$, then, if Y is any set and X is as above, $D_1 f(x,y)$ means the derivative of $f(\cdot,y)$ at $(x,y) \in X \times Y$ if $f(\cdot,y)$ is C^1, and if X is any set and Y is an open set in an Euclidean space, we write $D_2 f(x,y)$ for the derivative of f with respect to y at $(x,y) \in X \times Y$ if $f(x,\cdot)$ is C^1; in particular, $D_{1(i)}f(x,y)$ stands for the derivative with respect to the ith coordinate of x at $(x,y) \in X \times Y$. We write $D_1^2 f(x,y)$ for the second order derivative of $f(\cdot,y)$ at $(x,y) \in X \times Y$ if $f(\cdot,y)$ is C^2. Finally, for $x \in \mathbb{R}^n_-$, $I(x)$ is the set of coordinates $I(x) = \{i = 1,\ldots,n \colon x_i = 0\}$ and $H(x)$ is the linear subspace of $\mathbb{R}^n$ given as $H(x) = \{x \in \mathbb{R}^n \colon x_i = 0 \text{ for all } i \in I(x)\}$.

(b) As noted in Section 3, every $\nu \in \mathcal{G}$ has an equilibrium distribution. Write $\mathcal{G}_1$ for the subset of $\mathcal{G}$ consisting of those ν such that for some equilibrium distribution τ of ν,

(i) $\#(\varphi(u, e(\tau_A))) = 1$ for each $u \in \operatorname{supp} \nu$;

(ii) for each $u \in \operatorname{supp} \nu$, writing a_u for the unique element of $\varphi(u, e(\tau_A))$, and taking any chart (O, θ) on A with $a_u \in O$, $D_{1(i)}(u \circ (\theta \times id_E)^{-1}))(\theta(a_u), e(\tau_A)) > 0$ for all $i \in I(\theta(a_u))$, and $D_1^2(u \circ (\theta \times id_E)^{-1})(\theta(a_u), e(\tau_A))$ is negative definite on $H(\theta(a_u))$;

(iii) there are an affine subspace L in $\mathbb{R}^m$, open neighborhoods V' of $\operatorname{supp}(\nu)$ in $\mathcal{U}$ and W' of $e(\tau_A)$ in $E \cap L$ such that $(*)$ $g(\varphi(u, e)) \subseteq L$ for each $(u, e) \in V' \times W'$. (If $\dim L = 0$, then $(*)$ means that $g(\varphi(u, e)) = \{e(\tau_A)\}$ for each $(u, e) \in V' \times W'$.)

We claim that given any $\nu \in \mathcal{G}_1$ and any equilibrium distribution τ of ν such that (i)–(iii) are satisfied, there are an open neighborhoods V of $\operatorname{supp} \nu$ in $\mathcal{U}$ and W of $e(\tau_A)$ in $E \cap L$ such that, on $V \times W$, the best reply of u against e can be described by a continuous map $h \colon V \times W \to A$ such that (1) $h(u, \cdot)$, is C^1 for each $u \in V$, (2) the map $D_2(g \circ h)$ (in other words, the map $(u, e) \mapsto D_2(g \circ h)(u, e))$ is continuous on $V \times W$,[7] and (3) $g(h(u, e)) \in L$ for each $(u, e) \in V \times W$.

To see this, pick any $u \in \operatorname{supp} \nu$. As above, let $a_u \in A$ be the unique element of $\varphi(u, e(\tau_A))$. Let (O, θ) be a chart on A with $a_u \in O$. Then there is a compact neighborhood U_{a_u} of a_u in A, with $U_{a_u} \subseteq O$, such that $D_1^2(u \circ (\theta \times id_E)^{-1})(\theta(a), e(\tau_A))$ is negative definite on $H(\theta(a_u))$ and $D_{1(i)}(u \circ (\theta \times id_E)^{-1})(\theta(a), e(\tau_A)) > 0$ for each $i \in I(\theta(a_u))$ and each $a \in U_{a_u}$. We can find numbers r_1, r_2 such that $u(a_u, e(\tau_A)) > r_1 > r_2 > u(a, e(\tau_A))$ for each $a \in A \backslash U_{a_u}$, and in particular, $r_1 > u(a, e(\tau_A))$ for each $a \in \operatorname{cl}(A \backslash U_{a_u})$. Using the choice of the topology on $\mathcal{U}$, together with the fact that the action set A is compact, we see that there are open neighborhoods V_u of u in $\mathcal{U}$ and W_u of $e(\tau_A)$ in E such that $u'(a_u, e) > r_1 > u'(a, e)$ for each $u' \in V_u$, $e \in W_u$, and $a \in A_{u'} \backslash U_{a_u}$, and such that $D_1^2(u' \circ (\theta \times id_E)^{-1})(\theta(a), e)$ is negative definite on $H(\theta(a_u))$ and $D_{1(i)}(u \circ (\theta \times id_E)^{-1})(\theta(a), e) > 0$ for each $i \in I(\theta(a_u))$, $u' \in V_u$, $e \in W_u$, and $a \in U_{a_u}$. We can assume that U_{a_u} is such that $\theta(U_{a_u})$ is convex. Then for each $u' \in V_u$ and $e \in W_u$, $u'(\cdot, e) \circ \theta^{-1}$ is strictly concave on $\theta(U_{a_u}) \cap H(\theta(a_u))$. Consequently, for each $u' \in V_u$ and $e \in W_u$, the best reply of u' against e is unique. By the fact that $D_{1(i)}(u' \circ (\theta \times id_E)^{-1})(\theta(a), e) > 0$ for each $i \in I(\theta(a_u))$, $u' \in V_u$, $e \in W_u$, and $a \in U_{a_u}$, it follows that, on $V_u \times W_u$, the best reply correspondence φ can be identified with a function h_u such that $\theta \circ h_u$ takes values in $H(\theta(a_u))$. As A is compact, h_u is continuous. Note that $D_{1(i)}(u' \circ (\theta \times id_E)^{-1})(\theta(h_u(u', e)), e) = 0$ for each $i \in \{1, \dots n\} \backslash I(\theta(a_u))$, $u' \in V_u$ and $e \in W_u$. Thus, since $\theta \circ h_u$ takes values in $H(\theta(a_u))$ and $D_1^2 u' \circ (\theta \times id_E)^{-1})(\theta(h_u(u', e)), e)$ is non-singular on $H(\theta(a_u))$, the implicit function theorem applied on $H(\theta(a_u))$ shows that $\theta \circ h(u', \cdot)$ is C^1 for each $u' \in V_u$.

Do this construction for each $u \in \operatorname{supp} \nu$. By compactness of $\operatorname{supp} \nu$ there are points $u_1, \dots, u_k \in \operatorname{supp} \nu$ such that $\operatorname{supp} \nu \subseteq V = \bigcup_{i=1}^k V_{u_i}$. Set $W = \bigcap_{i=1}^k W_{u_i}$. Let

[7] In view of the notation introduced in (a), recall from Section 2 that if a set is (relatively) open in some affine subspace L of $\mathbb{R}^m$, then this set can be viewed as an open set in some $\mathbb{R}^k$.

$h\colon V \times W \to A$ be the uniquely determined function such that $h(u,\cdot)$ is the restriction to W of some $u' \in V$. Then h is continuous and (1) holds. As for (2), pick any chart (O,θ) on A with $h^{-1}(O) \neq \emptyset$. Note that on $h^{-1}(O)$, the map $D_2(g \circ h)$ can be written in the form $D_2(g \circ h) = D_2((g \circ \theta^{-1}) \circ (\theta \circ h))$. Thus, for each $(u,e) \in h^{-1}(O)$, $D_2(g \circ h)(u,e) = D(g \circ \theta^{-1})(\theta(h(u,e)))D_2(\theta \circ h)(u,e)$. Since $g \circ \theta^{-1}$ is C^1 on $\theta(O)$ and h is continuous on $h^{-1}(O)$, $(u,e) \mapsto D(g \circ \theta^{-1})(\theta(h(u,e)))$ is continuous on the latter set, and by the choice of the topology on $\mathcal{U}$, so is $(u,e) \mapsto D_2(\theta \circ h)(u,e)$. Thus $D_2(g \circ h)$ is continuous on $h^{-1}(O)$. As (O,θ) is arbitrary (modulo that $h^{-1}(O) \neq \emptyset$), it follows that (2) is true. Finally, shrinking V and W, if necessary, we get (3).

Let $\nu \in \mathcal{G}_1$ and τ an equilibrium distribution for ν such that (i)–(iii) are satisfied. Affinely changing coordinates, if necessary, we can assume that $L = \{0\} \subseteq \mathbb{R}^m$ if $\dim L = 0$, and otherwise, if $\dim L = k > 0$, that $L = \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^m$, where $\{0\} \subseteq \mathbb{R}^{m-k}$. In the second case, identifying $\mathbb{R}^k \times \{0\}$ with $\mathbb{R}^k$, we can then find an open neighborhood W_L of $e(\tau_A)$ in $\mathbb{R}^k$ so that we can define a function $\xi_\tau\colon W_L \to \mathbb{R}^k$ by setting

$$\xi_\tau(e) = \int g(h(u,e))\, d\nu(u) - e$$

for each $e \in W_L$; then by the generalized version of Leibniz' rule in Schwartz (1967, Chap IV.11, Theorem 115), ξ_τ is C^1 on W_L, with $D\xi_\tau(e) = \int D_2(g \circ h)(u,e)\, d\nu(u) - I$ for each $e \in W_L$, where I is the $(k \times k)$-identity matrix. In the first case, we take for ξ_τ and $D\xi_\tau$ the identity on $\{0\} \subseteq \mathbb{R}^m$. (If ξ_τ is the identity on $\{0\}$, then ξ_τ is linear, so it makes sense to take for $D\xi_\tau$ the identity on $\{0\}$, too.)

(c) Let $\mathcal{G}^*$ be the subset of $\mathcal{G}$ consisting of those $\nu \in \mathcal{G}$ such that for some equilibrium distribution τ of ν, (i)–(iii) of (b) are satisfied and $D\xi_\tau(e(\tau_A))$ has full rank, where ξ_τ is associated with τ as above; note that the full rank condition holds automatically if $\dim L = 0$. (Note also that while the choice of the neighborhood W_L of $e(\tau_A)$, i.e., the domain of ξ_τ, involves some arbitrariness if $\dim L > 0$, $D\xi_\tau(e(\tau_A))$ is uniquely determined.) In (d) below we will show that $\mathcal{G}^*$ is open in $\mathcal{G}$, and in (e) prove the denseness part.

(d) Fix $\nu \in \mathcal{G}^*$. Let τ be an equilibrium distribution for ν, witnessing that $\nu \in \mathcal{G}^*$. Let L, W, V, h, and ξ_τ be corresponding to τ as in (b).

We need to show that ν has a neighborhood U in $\mathcal{G}$ such that $U \subseteq \mathcal{G}^*$. To start, fix an open neighborhood U of ν, with $\operatorname{supp}(\nu') \subseteq V$ for each $\nu' \in U$. If $\dim L = 0$, then, in view of (3) in (b), every $\nu' \in U$ trivially belongs to $\mathcal{G}^*$. We may therefore assume in the following that $\dim L > 0$.

(i) Choose a compact neighborhood W_1 of $e(\tau_A)$ in L with $W_1 \subseteq W$. Then there is a $k \in \mathbb{N}$ and neighborhood $V_1 \subseteq V$ of $\operatorname{supp}(\nu)$ such that $\|D_2(g \circ h)(u,e)\| \leq k$ for each $(u,e) \in V_1 \times W_1$. Indeed, otherwise, for each $k \in \mathbb{N}\setminus\{0\}$, we can find points $e_k \in W_1$ and $u_k \in V$ such that $\|D_2(g \circ h)(u_k,e_k)\| > k$ but $\operatorname{dist}(u_k, \operatorname{supp}(\nu)) < 1/k$. Since W_1 and $\operatorname{supp}(\nu)$ are compact we may assume that $(u_k,e_k) \to (u,e)$ for some $(u,e) \in \operatorname{supp}(\nu) \times W_1$. Now $D_2(g \circ h)(u_k,e_k) \to D_2(g \circ h)(u,e)$ because $D_2(g \circ h)$ is continuous, and we get a contradiction.

(ii) Write W_1' for the interior of W_1 in E. As above, make an affine change of coordinates so that L can be identified with $\mathbb{R}^k$. Let W_L be associated with ξ_τ as above. Let $W_2 = W_L \cap \pi(W_1')$ where π is the coordinate projection of $\mathbb{R}^m$ onto $\mathbb{R}^k$. Then W_2 is an open neighborhood of $e(\tau_A)$ in $\mathbb{R}^k$. We can now define a map $\xi_U \colon U \times W_2 \to \mathbb{R}^k$ by setting

$$\xi_U(\nu', e) = \int g(h(u, e))\, d\nu'(u) - e.$$

As above, appealing to Leibniz' rule, we see that for each fixed $\nu' \in U$, $\xi_U(\nu', \cdot)$ is C^1 on W_2, with $D_2\xi_U(\nu', e) = \int D_2(g \circ h)(u, e)\, d\nu'(u) - I$ for each $e \in W_2$ where I is the $(k \times k)$-unit matrix. Now ξ_U is continuous and $D_2\xi_U(\nu', e)$ depends continuously on (ν', e). Indeed, suppose that $e_k \to e$ in W_1 and $u_k \to u$ in V_1. Then $h(u_k, e_k) \to h(u, e)$, because the map h is continuous, and because the map $D_2(g \circ h)$ is continuous, also $D_2(g \circ h)(u_k, e_k) \to D_2(g \circ h)(u, e)$. That is, uniformly on compact subsets of V_1, we have both $h(\cdot, e_k) \to h(\cdot, e)$ and $D_2(g \circ h)(\cdot, e_k) \to D_2(g \circ h)(\cdot, e)$. Since h takes values in the compact set A, the sequence $\langle h(\cdot, e_k)\rangle$ is (coordinate-wise) uniformly integrable, and by (i), so is the sequence $\langle (g \circ h)(\cdot, e_k)\rangle$. Using Billingsley (1968, Theorems 5.4, 5.5 and the remarks stated after the proof of the latter theorem), it follows that if $\nu_k \to \nu'$ in U, then $\xi_U(\nu_k, e_k) \to \xi_U(\nu', e)$ as well as $D_2\xi_U(\nu_k, e_k) \to D_2\xi_U(\nu', e)$. Thus, on $U \times W_2$, ξ_U is continuous and $D_2\xi_U(\nu', e)$ depends continuously on (ν', e), as claimed.

Now as τ is an equilibrium distribution for ν, we have $\xi_U(\nu, e(\tau_A)) = 0$, and since $\nu \in \mathcal{G}^*$, $D_2\xi_U(\nu, e(\tau_A)) \equiv D\xi_\tau(e(\tau_A))$ has full rank. Hence, by a version of the implicit function theorem (see Schwartz, 1967, Chap. III.8, Theorem 25, or Mas-Colell, 1985, Chap. 1, C.3.3), there is an open neighborhood U_1 of ν in $\mathcal{G}$, with $U_1 \subseteq U$, and a continuous map $\nu' \mapsto e(\nu') \colon U_1 \to W_2$ such that for each $\nu' \in U_1$, $\xi_U(\nu', e(\nu')) = 0$. Also, since $D_2\xi_U(\nu', e)$ depends continuously on (ν', e), the derivative of $\xi_U(\nu', \cdot)$ with respect to e at $e(\nu')$ has full rank for each $\nu' \in U_1$, shrinking U_1 if necessary.

Fix $\nu' \in U_1$ and set $\tau' = \nu' \circ \big(id \times h(\cdot, e(\nu'))\big)^{-1}$. Then

$$\operatorname{supp}(\tau') \subseteq \{(u, a) \in \mathcal{U} \times A \colon a \in \varphi(u, e(\nu'))\},$$

by the choice of h, and

$$e(\tau_A') = \int g(h(u, e(\nu')))\, d\nu'(u) = \xi_U(\nu', e(\nu')) + e(\nu') = e(\nu').$$

Thus τ' is an equilibrium distribution for ν'. By the choices of V and W, and since $e(\nu') \in W_2$ for each $\nu' \in U_1$, (i)–(iii) of (b) are true for τ'. Finally, note that we must have $\xi_{\tau'} = \xi_U(\nu', \cdot)$ on some neighborhood of $e(\tau_A') = e(\nu')$, so $D\xi_{\tau'}(e(\tau_A'))$ has maximal rank. Thus every ν' in the neighborhood U_1 of ν belongs to $\mathcal{G}^*$. As $\nu \in \mathcal{G}^*$ is arbitrary, $\mathcal{G}^*$ is open.

(e) We next show that $\mathcal{G}^*$ is dense in $\mathcal{G}$. Since $\mathcal{U}$ is a separable metric space the set $\mathcal{G}'$ of $\nu \in \mathcal{G}$ such that $\operatorname{supp}(\nu)$ is finite is dense in $\mathcal{G}$. Let $\nu \in \mathcal{G}'$; thus $\nu = \sum_{j=1}^{k} \alpha_j \delta_{u_j}$

with $\alpha_j > 0$, $j = 1, \ldots, k$, and $\sum_{j=1}^k \alpha_j = 1$. Let τ be an equilibrium distribution for ν. Then $\tau_A = \sum_{j=1}^k \alpha_j \tau_{A,j}$ for probability measures $\tau_{A,j}$ on A such that for each j and each $a \in \operatorname{supp} \tau_{A,j}$ we have $a \in \varphi(u_j, e(\tau_A))$. Note that

$$e(\tau_A) \equiv \int g(a)\, d\tau_A(a) = \sum_{j=1}^k \alpha_j \int g(a)\, d\tau_{A,j}(a)\,.$$

Now, for each $j = 1, \ldots, k$,

$$\int g(a)\, d\tau_{A,j}(a) = \int_{\operatorname{supp}(\tau_{A,j})} g d\tau_{A,j} \in \operatorname{co}(g(\operatorname{supp}(\tau_{A,j}))),$$

so by Caratheodory's theorem there are points $a_{j,h} \in \operatorname{supp} \tau_{A,j}$ and numbers $\alpha_{j,h} > 0$, $h = 1, \ldots, h_j$, with $\sum_{h=1}^{h_j} \alpha_{j,h} = 1$, such that $\int g(a)\, d\tau_{A,j}(a) = \sum_{h=1}^{h_j} \alpha_{j,h} g(a_{j,h})$.

For each $j = 1, \ldots, k$ and $h = 1, \ldots, h_j$, choose a C^2-function $\rho_{j,h} \colon A \to \mathbb{R}$, taking a unique global maximum at $a_{j,h}$, such that, for some chart $(O_{j,h}, \theta_{j,h})$ on A with $a_{j,h} \in O_{j,h}$, $D^2(\rho_{j,h} \circ \theta_{j,h}^{-1})(\theta(a_{j,h}))$ is negative definite on $H(\theta(a_{j,h}))$ and if $i \in I(\theta(a_{j,h}))$ then $D_{(i)}(\rho_{j,h} \circ \theta_{j,h}^{-1})(\theta(a_{j,h})) > 0$. For each $0 < \lambda < 1$ and each (j,h), define a payoff function $u_{j,h,\lambda} \in \mathcal{U}$ by setting

$$u_{j,h,\lambda}(a,e) = u_j(a,e) + \lambda \rho_{j,h}(a)$$

for each $(a,e) \in A \times E$. Note that for each (j,h,λ), the fact that $a_{j,h} \in \varphi(u_j, e(\tau_A))$, meaning in particular that $D_1^2 u_j \circ (\theta_{j,h} \times id_E)^{-1})(\theta_{j,h}(a_{i,h}), e(\tau_A))$ is negative semi-definite on $H(\theta(a_{j,h}))$ and that $D_{1(i)}(u_j \circ (\theta_{j,h} \times id_E)^{-1})(\theta_{j,h}(a_{j,h}), e(\tau_A)) \geq 0$ for each $i \in I(\theta(a_{j,h}))$, implies by the choice of $\rho_{j,h}$ that $\varphi(u_{j,h,\lambda}) = \{a_{j,h}\}$, that

$$\begin{aligned} &D_{1(i)}(u_{j,h,\lambda} \circ (\theta_{j,h} \times id_E)^{-1})(\theta_{j,h}(a_{j,h}), e(\tau_A)) \\ &\quad = D_{1(i)}(u_j \circ (\theta_{j,h} \times id_E)^{-1})(\theta_{j,h}(a_{j,h}), e(\tau_A)) + D_{(i)}(\rho_{j,h} \circ \theta_{j,h}^{-1})(\theta(a_{j,h})) > 0 \end{aligned}$$

for each $i \in I(\theta(a_{j,h}))$, and that $D_1^2 u_j \circ (\theta_{j,h} \times id_E)^{-1})(\theta_{j,h}(a_{j,h}), e(\tau_A))$ is negative definite on $H(\theta(a_{j,h}))$. Now for each $0 < \lambda < 1$, let $\nu_\lambda \in \mathcal{G}'$ be defined by setting $\nu_\lambda = \sum_{j=1}^k \alpha_j \sum_{h=1}^{h_j} \alpha_{j,h} \delta_{u_{j,h,\lambda}}$ and set $\tau_\lambda = \sum_{j=1}^k \alpha_j \sum_{h=j}^{h_j} \alpha_{j,h} \delta_{(u_{j,h,\lambda}, a_{j,h})}$. Note that $\int g(a)\, d\tau_{\lambda,A}(a) = \int g(a)\, d\tau_A(a)$, or, in other words, that $e(\tau_{\lambda,A}) = e(\tau_A)$. It follows that for each $0 < \lambda < 1$, τ_λ is an equilibrium distribution for ν_λ such that (i) and (ii) from (a) are satisfied. Making λ small, we get ν_λ as close to ν as we please. Thus, setting

$$\mathcal{G}_2 = \{\nu' \in \mathcal{G}' \colon \nu \text{ has an equilibrium distribution satisfying (i) and (ii) of (b)}\},$$

then $\mathcal{G}_2$ is dense in $\mathcal{G}'$, therefore dense in $\mathcal{G}$.

Let $\nu \in \mathcal{G}_2$ and let τ be an equilibrium distribution for ν such that (i) and (ii) of (b) are satisfied. We can write $\nu = \sum_{i=1}^k \alpha_i \delta_{u_i}$ and $\tau = \sum_{i=1}^k \alpha_i \delta_{(u_i, a_i)}$, where $\alpha_i > 0$ for each $i = 1, \ldots, k$. Let L be an affine subspace of $\mathbb{R}^m$ such that L is of minimal dimension subject to the condition that there are neighborhoods V of

$\text{supp}(\nu)$ in $\mathcal{U}$ and W of $e(\tau_A)$ in E with $g(\varphi(u,e)) \subseteq L$ for each $(u,e) \in V \times W$. Suppose this dimension is ℓ where $0 \le \ell \le m$. Then there are $\ell+1$ sequences $\langle (u_{j,n}, e_{j,n}) \rangle_{n \in \mathbb{N}}$ in $\mathcal{U} \times W$, $j = 1, \ldots, \ell+1$, together with points $a_{j,n} \in \varphi(u_{j,n}, e_{j,n})$, such that (1), for each $j = 1, \ldots, \ell+1$, $e_{j,n} \to e(\tau_A)$ and $u_{j,n} \to u_i$ for some $i = 1, \ldots, k$, and (2), for each n, L is the smallest affine subspace of $\mathbb{R}^m$ that includes each set $\{g(a_{j,n})\colon j = 1, \ldots, \ell+1\}$. Because φ is upper hemi-continuous and $\varphi(u_i, e(\tau_A)) = \{a_i\}$ for each i, (1) implies that for each $j = 1, \ldots, \ell+1$, $a_{j,n} \to a_i$ for some $i = 1, \ldots, k$, and in particular, by the choice of τ, that (3) if $a_i \in \partial A$ and $a_{j,n} \to a_i$, then $a_{j,n} \in \partial A$ if n is sufficiently large. For each $i = 1, \ldots, k$, let $J(i) = \{j\colon a_{j,n} \to a_i\}$ if there is a j such that $a_{j,n} \to a_i$, and let $J(i) = \{i\}$ otherwise.

Set $a_{j,n} = a_i$ for each n if there is no j such that $a_{j,n} \to a_i$. For each $i = 1, \ldots, k$ and $j \in J(i)$, set $\beta_j = \frac{\alpha_i}{\#(J(i))}$. Then $\beta_j > 0$ for each $j \in J = \bigcup_{i=1}^k J(i)$, and $\sum_{j \in J} \beta_j = 1$, so from (2) in the previous paragraph we see that $e_n = \sum_{j \in J} \beta_j g(a_{j,n})$ belongs to the relative interior of $E \cap L$ for each n. Moreover, $e_n \to \sum_{j \in J} \beta_j g(a_j) = e(\tau_a)$, by choice of the numbers β_j and since g is continuous.

For each $i = 1, \ldots, k$, choose a chart (O_i, θ_i) on A, with $a_i \in O_i$, and a number $r_1 > 0$, so that $\bar{B}(\theta_i(a_i), r_1) \cap \mathbb{R}^n_- \subseteq \theta_i(O_i)$. Let $r_2 > 0$ be a number with $r_2 < r_1$. For each $i = 1, \ldots, k$, let $\rho_i\colon \mathbb{R}^n \to [0,1] \subseteq \mathbb{R}$ be a C^2-map such that $\rho_i(x) = 1$ if $x \in B(\theta_i(a_i), r_2)$ and $\rho_i(x) = 0$ if $x \notin B(\theta_i(a_i), r_1)$. For each $i = 1, \ldots, k$ and $j \in J(i)$, since $a_{j,n} \to a_i$, we can assume that $a_{j,n} \in O_i$ for all n, and then that $\theta_i(a_{j,n}) \in B(\theta_i(a_i), r_2)$ for all n.

For each $i = 1, \ldots, k$, $j \in J(i)$, and n, define a map $v_{i,j,n}\colon A \to \mathbb{R}$ by setting, for each $a \in A$,

$$v_{i,j,n}(a) = \begin{cases} \rho_i(\theta_i(a))\big(D_1(u_i \circ (\theta_i \times id_E)^{-1})(\theta_i(a_{j,n}), e_n) \\ \qquad\qquad - D_1(u_i \circ (\theta_i \times id_E)^{-1})(\theta_i(a_i), e(\tau_A))\big)\theta_i(a) \\ \quad \text{if } \theta_i(a) \in \theta_i(O_i), \\ 0 \quad \text{otherwise.} \end{cases}$$

Then each $v_{i,j,n}$ is C^2, so for each $i = 1, \ldots, k$, $j \in J(i)$, and n, we can define a payoff function $u_{i,j,n} \in \mathcal{U}$ by setting $u_{i,j,n}(a,e) = u_i(a,e) - v_{i,j,n}(a)$ for each $(a,e) \in A \times E$.

Evidently $u_{i,j,n} \to u_i$ for each $j \in J(i)$ and each $i = 1, \ldots, k$. Therefore, setting $\nu_n = \sum_{j \in J} \beta_j \delta_{u_{i,j,n}}$, we have $\nu_n \to \nu$. Set $\tau_n = \sum_{j \in J} \beta_j \delta_{(u_{i,j,n}, a_{j,n})}$, so that we also have $\tau_n \to \tau$. Using (3) above and the fact that τ is equilibrium distribution for ν satisfying (i) and (ii) of (b), we see that if n is large enough, then τ_n is an equilibrium distribution for ν_n, also satisfying (i) and (ii) of (b). Finally, since $e(\tau_{n,A}) = e_n$ for each n, we also have (iii) of (b), by the choices of L and e_n, if n is large enough. It follows that $\mathcal{G}_1 \cap \mathcal{G}_2$ is dense in $\mathcal{G}_2$, therefore dense in $\mathcal{G}$.

Pick any $\nu \in \mathcal{G}_1 \cap \mathcal{G}_2$ and let τ be an equilibrium distribution of ν such that (i)–(iii) from (b) are satisfied. Write $\nu = \sum_{i=1}^h \alpha_i \delta_{u_i}$, let $\bar{a}_1, \ldots, \bar{a}_h$ be the optimal actions, and write $\bar{e} = e(\tau_A)$. As above, let the affine subspace L of $\mathbb{R}^m$ associated with τ according to (iii) of (a) be identified with $\mathbb{R}^k$; we may assume $k > 0$, for otherwise ν trivially belongs to $\mathcal{G}^*$. Choose charts $(O_1, \theta_1), \ldots, (O_h, \theta_h)$ with $\bar{a}_i \in O_i$ for each $i = 1, \ldots, h$, so that $D\xi_\tau(\bar{e})$ can be written in the form

$$\sum_{i=1}^{h} A_i B_i^{-1} C_i - I$$

where I is the $k \times k$-unit matrix, and for each i, A_i is an $k \times n$-matrix, B_i an $n \times n$-matrix, and C_i an $n \times k$-matrix. If $\det(\sum_{i=1}^{h} A_i B_i^{-1} C_i - I) \neq 0$, then $\nu \in \mathcal{G}^*$. Otherwise, pick any number $\lambda > 0$. For each $i = 1, \ldots, h$ replace u_i by $u_{i,\lambda}$, defined by setting $u_{i,\lambda}(a, e) = u_i(a, e) + \lambda u_i(a, \bar{e})$ for each $(a, e) \in A \times E$, so that $u_{i,\lambda} \in \mathcal{U}$. Then $\nu_\lambda = \sum_{i=1}^{h} \alpha_i \delta_{u_{i,\lambda}}$ has an equilibrium distribution τ_λ with the same optimal actions $\bar{a}_1, \ldots, \bar{a}_h$ and with $e(\tau_{\lambda,A}) = \bar{e}$. In particular, (i) and (ii) of (b) are satisfied for τ_λ. Moreover, if $\lambda \to 0$, then $\mathrm{supp}(\nu_\lambda)$ becomes close to $\mathrm{supp}(\nu)$ in the Hausdorff metric topology, so also $(*)$ of (iii) of (b) is satisfied for τ_λ if λ is sufficiently close to 0. Thus $\nu_\lambda \in \mathcal{G}_1 \cap \mathcal{G}'$ if λ is sufficiently close to 0. Now, for the same charts on A as before, $D\xi_{\tau_\lambda}(\bar{e})$ equals

$$\frac{1}{(1+\lambda)} \sum_{i=1}^{h} A_i B_i^{-1} C_i - I,$$

with the same matrices A_i, B_i, C_i as above. Using the fact that the characteristic polynomial of the matrix $\sum_{i=1}^{h} A_i B_i^{-1} C_i$ can have only finitely may zeros, we see that $\det\big(\frac{1}{(1+\lambda)} \sum_{i=1}^{h} A_i B_i^{-1} C_i - I\big) \neq 0$ for all sufficiently small $0 < \lambda < 1$. Thus we have $\nu_\lambda \in \mathcal{G}^*$ for all sufficiently small $0 < \lambda < 1$. Moreover, making $0 < \lambda < 1$ small, we get ν_λ as close to ν as we please. Thus $\mathcal{G}^* \cap \mathcal{G}_1 \cap \mathcal{G}_2$ is dense in $\mathcal{G}_1 \cap \mathcal{G}_2$, therefore dense in $\mathcal{G}$ by the previous paragraph.

(f) Let $\nu \in \mathcal{G}^*$ and let $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ be a sequence of finite-player games converging to ν. By the choices of the topologies on $\widehat{\mathcal{U}}$ and $\mathcal{M}$, for each n there are extensions $u_{n,i}$ of the payoff functions $G_n(i)$, $i = 1, \ldots, k_n$, to elements of $\mathcal{U}$ such that the sequence of sets $\{u_{n,i} \colon u \in \mathrm{supp}(\nu_{G_n})\}$, $n \in \mathbb{N}$, converges to $\mathrm{supp}(\nu)$ in the Hausdorff metric topology on the non-empty compact subsets of $\mathcal{U}$. Write $\bar{\nu}_{G_n} = \frac{1}{\# I_n} \sum_{i \in I_n} \delta_{u_{n,i}}$. Then, since $\nu_{G_n} \to \nu$ in the topology of $\mathcal{M}$, also $\bar{\nu}_{G_n} \to \nu$ in the topology of $\mathcal{M}$, and hence in that of $\mathcal{G}$.

By the definition of "manifold with corners" as stated in Section 2, for some closed ball K in some Euclidean space, $A \subseteq K$. Let $\bar{g}$ be a continuous extension of g to K. For any function $f \colon I_n \to K$, we write $\tau_{K,f}$ for the distribution of f, i.e., $\tau_{K,f}(B) = \#\{i \in I_n \colon f(i) \in B\}/\#(I_n)$ for any Borel set $B \subseteq K$, and similarly, given any $i \in I_n \backslash \{i\}$, $\tau_{k,f,i}$ for the distribution of the restriction of f to $I \backslash \{i\}$. If f takes values in A, we also write $\tau_{A,f}$ in place of $\tau_{K,f}$, and $\tau_{A,f,i}$ in place of $\tau_{K,f,i}$. Write $\|\cdot\|_V$ for the variation norm on the space $M(K)$ of all signed Borel measures on K. Note that for any $n \in \mathbb{N}$ and any $f \colon I_n \to K$, $\|\tau_{K,f,i} - \tau_{K,f}\|_V \leq 2/\#(I_n)$ for each $i \in I_n$. As $\bar{g}$ is bounded on K, it follows that for any $\epsilon > 0$ there is an $N_\epsilon \in \mathbb{N}$ such that $\big\| \int \bar{g}(a)\, d\tau_{K,f,i}(a) - \int \bar{g}(a)\, d\tau_{K,f}(a) \big\| < \epsilon$ for each $n \geq N_\epsilon$, $f \colon I_n \to K$, and $i \in I_n$.

Let τ be an equilibrium distribution for ν, witnessing that $\nu \in \mathcal{G}^*$. Let L, V, W, and h be as in the paragraph after the statement of (i)–(iii) in (b). Assume first that

$\dim L > 0$. In addition, assume as earlier that L is of the form $L = \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^m$, so that L can be identified with $\mathbb{R}^k$ whenever this is convenient. Let W_L be an open neighborhood of $e(\tau_A)$ in $\mathbb{R}^k$, with $W_L \times \{0\} \subseteq W$, such that the map ξ_τ associated with τ is defined on W_L; here and elsewhere below, concerning arguments involving the map h, we use the representation of L as $\mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^m$.

As $\nu \in \mathcal{G}^*$, the derivative of ξ_τ at $e(\tau_A)$ has full rank, which implies that on some convex compact neighborhood W_1 of $e(\tau_A)$ in $\mathbb{R}^k$, with $W_1 \subseteq W_L$, $\xi_\tau(e) = 0$ if and only if $e = e(\tau_A)$. Let W_2 be a convex compact neighborhood of $e(\tau_A)$ in $\mathbb{R}^k$ such that $W_2 \subseteq \operatorname{int} W_1$.

Now because $\bar{\nu}_{G_n} \to \nu$, and therefore $\operatorname{supp}(\bar{\nu}_{G_n}) \to \operatorname{supp}(\nu)$ in the Hausdorff metric topology, and by what was noted in the second paragraph of this part of the proof, there is an $N \in \mathbb{N}$ such that for $n \geq N$, $\operatorname{supp}(\bar{\nu}_{G_n}) \subseteq V$ and for each $f \in K^{I_n}$ and $i = 1 \dots, k_n$,

$$\pi\Big(\int \bar{g}(a)\, d\tau_{K,f,i}(a)\Big) - \pi\Big(\int \bar{g}(a)\, d\tau_{K,f}(a)\Big) + e \in W_1$$

whenever $e \in W_2$, writing π for the coordinate projection of $\mathbb{R}^m$ onto $\mathbb{R}^k$. For $n \geq N$, consider the function $\Lambda\colon K^{I_n} \times W_2 \to K^{I_n} \times \mathbb{R}^k$ defined by setting

$$\begin{aligned} &\Lambda(f, e) \\ &= \Bigg(h_1\bigg(\pi\Big(\int \bar{g}(a)\, d\tau_{K,f,1}(a)\Big) - \pi\Big(\int \bar{g}(a)\, d\tau_{K,f}(a)\Big) + e\bigg), \dots, \\ &\qquad h_{k_n}\bigg(\pi\Big(\int \bar{g}(a)\, d\tau_{K,f,k_n}(a)\Big) - \pi\Big(\int \bar{g}(a)\, d\tau_{K,f}(a)\Big) + e\bigg), \pi\Big(\int \bar{g}(a)\, d\tau_{K,f}(a)\Big)\Bigg), \end{aligned}$$

writing $h_i(\cdot)$ in place of $h(u_{n,i}, \cdot)$ for each $i \in \{1, \dots, k_n\}$. Then a fixed point of Λ gives a pure strategy Nash equilibrium of (I_n, G_n). Indeed, fix $n \geq N$ and suppose (f^*, e^*) is a fixed point of Λ. Evidently $f^*(i) = h_i\Big(\pi(\int \bar{g}(a)\, d\tau_{K,f^*,i}(a))\Big)$ for each $i \in I_n$. In particular, $f^*(i) \in A$ for each $i \in I_n$, so $\int \bar{g}(a)\, d\tau_{K,f^*}(a) = \int g(a)\, d\tau_{A,f^*}(a)$ as well as $\int \bar{g}(a)\, d\tau_{K,f^*,i}(a) = \int g(a)\, d\tau_{A,f^*,i}(a)$ for each $i \in I_n$. Fix any $i \in I_n$. For $j \in I_n \backslash \{i\}$, write $z_j = \pi(\int g(a)\, d\tau_{A,f^*,j}(a)) - \pi(\int g(a)\, d\tau_{K,f^*}(a))$, so that $f^*(j) = h_j(z_j + e^*)$ and thus $g(f^*(j)) = g(h_j(z_j + e^*))$. Now, for any $j \in I_n \backslash \{i\}$, $g(h_j(z_j + e^*)) \in \mathbb{R}^k$, by the choice of V and W, because $W_1 \subseteq W$ and $G(i) \in V$ for each $i \in I_n$. Consequently

$$\int g(a)\, d\tau_{A,f^*,i}(a)) = \sum_{j \in I_n \backslash \{i\}} \frac{1}{\#(I_n \backslash \{i\})} g(f^*(j)) \in \mathbb{R}^k,$$

so $\int g(a)\, d\tau_{A,f^*,i}(a)) = \pi(\int g(a)\, d\tau_{A,f^*,i}(a))$, and thus $f^*(i) = h_i(\int g(a)\, d\tau_{A,f^*,i}(a))$. Because $\int g(a)\, d\tau_{A,f^*,i}(a) \in E_{I_n}$ for each $i \in I_n$, and $G(i)$ is just the restriction of u_i to $A \times E_{I_n}$, it follows that f^* is a strict pure strategy Nash equilibrium of (I_n, G_n).

We claim that there is an $N_1 \geq N$ such that for $n \geq N_1$ the fixed point theorem stated in the appendix as Theorem 4 applies to Λ. Clearly Λ is continuous, and for $X = K^{I_n}$ and $Y = W_2$ the requirements of Theorem 4 on X and Y are satisfied.

With the map ξ_τ it is also clear that we have (a) of Theorem 4. Let $\gamma > 0$ be such that $\|\xi_\tau(e)\| \geq \gamma$ for each $e \in \partial W_2$; here and in the sequel, we write ∂W_2 for the topological boundary of W_2 in $\mathbb{R}^k$. We need to show that for some $N_1 \geq N$ also (b) of that theorem is satisfied for Λ and ξ_τ if $n \geq N_1$.

To this end, fix $n \geq N$ and suppose that $f \in K^{I_n}$ and $e \in \partial W_2$ are such that

$$f = \Big(h_1\Big(\pi\Big(\int \bar{g}(a)\, d\tau_{K,f,1}(a)\Big) - \pi\Big(\int \bar{g}(a)\, d\tau_{K,f}(a)\Big) + e\Big), \\ \dots, h_{k_n}\Big(\pi\Big(\int \bar{g}(a)\, d\tau_{K,f,k_n}(a)\Big) - \pi\Big(\int \bar{g}(a)\, d\tau_{K,f}(a)\Big) + e\Big)\Big).$$

Then we must actually have $f \in A^{I_n}$, so we can equivalently write

$$f = \Big(h_1\Big(\pi\Big(\int g(a)\, d\tau_{A,f,1}(a)\Big) - \pi\Big(\int g(a)\, d\tau_{A,f}(a)\Big) + e\Big), \\ \dots, h_{k_n}\Big(\pi\Big(\int g(a)\, d\tau_{A,f,k_n}(a)\Big) - \pi\Big(\int g(a)\, d\tau_{A,f}(a)\Big) + e\Big)\Big).$$

Note that

$$\begin{aligned}
&\Big\|\pi\Big(\frac{1}{k_n}\sum_{i=1}^{k_n} g(f(i))\Big) - e - \xi_\tau(e)\Big\| \\
&\qquad = \Big\|\pi\Big(\frac{1}{k_n}\sum_{i=1}^{k_n} g(f(i))\Big) - e - \Big(\int g(h(u,e))\, d\nu(u) - e\Big)\Big\| \\
&\qquad = \Big\|\pi\Big(\frac{1}{n_k}\sum_{i=1}^{n_k} g(f(i))\Big) - \int g(h(u,e))\, d\nu(u)\Big\| \\
&\qquad \leq \Big\|\pi\Big(\frac{1}{k_n}\sum_{i=1}^{k_n} g(f(i))\Big) - \pi\Big(\frac{1}{k_n}\sum_{i=1}^{k_n} g(h_i(e))\Big)\Big\| \\
&\qquad\qquad + \Big\|\pi\Big(\frac{1}{k_n}\sum_{i=1}^{k_n} g(h_i(e))\Big) - \int g(h(u,e))\, d\nu(u)\Big\| \\
&\qquad = \Big\|\pi\Big(\frac{1}{k_n}\sum_{i=1}^{k_n} g(f(i)) - \frac{1}{k_n}\sum_{i=1}^{k_n} g(h_i(e))\Big)\Big\| \\
&\qquad\qquad + \Big\|\pi\Big(\int g(h(u,e))\, d\bar{\nu}_{G_n}(u)\Big) - \int g(h(u,e))\, d\nu(u)\Big\| \\
&\leq \Big\|\frac{1}{k_n}\sum_{i=1}^{k_n} g(f(i)) - \frac{1}{k_n}\sum_{i=1}^{k_n} g(h_i(e))\Big\| + \Big\|\int g(h(u,e))\, d\bar{\nu}_{G_n}(u) - \int g(h(u,e))\, d\nu(u)\Big\|,
\end{aligned}$$

the last inequality following since we may assume that $\|\cdot\|$ is such that $\|\pi(z)\| \leq \|z\|$ for every $z \in \mathbb{R}^m$, and because $g(h(U \times W)) \subseteq \mathbb{R}^k$ by the choice of V and W, and $\operatorname{supp}(\bar{\nu}_{G_n}) \subseteq V$ if $n \geq N$. Since $\operatorname{supp}\nu$ and ∂W_2 are compact, there are a neighborhood

$V_1 \subseteq V$ of $\operatorname{supp} \nu$ and a $\delta > 0$ such that for $z \in \mathbb{R}^m$ and $e \in \partial W_2$, with $(e, 0) + z \in W$, we have $\|g(h(u, e+z)) - g(h(u, e))\| < \gamma/2$ whenever $u \in V_1$ and $\|z\| < \delta$. Also, by what was noted earlier, we must have $\left\| \int g(a)\, d\tau_{A,f,i}(a) - \int g(a)\, d\tau_{A,f}(a) \right\| < \delta$ for each $i \in I_n$ if n is large. Thus, as $\operatorname{supp}(\bar{\nu}_{G_n}) \to \operatorname{supp}(\nu)$ in the Hausdorff metric topology, we must have $\left\| \frac{1}{k_n} \sum_{i=1}^{k_n} g(f(i)) - \frac{1}{k_n} \sum_{i=1}^{k_n} g(h_i(e)) \right\| < \gamma/2$ for large n if $f \in A^{I_n}$ and $e \in \partial W_2$ are as above. On the other hand, as $\bar{\nu}_{G_n} \to \nu$ narrowly and ∂W_2 is compact, for large n we have $\left\| \int g(h(u, e))\, d\bar{\nu}_{G_n}(u) - \int g(h(u, e))\, d\nu(u) \right\| < \gamma/2$ for each $e \in \partial W_2$. It follows that there is an $N_1 \geq N$ such that $\left\| \pi(\frac{1}{k_n} \sum_{i=1}^{k_n} g(f(i))) - e - \xi_\tau(e) \right\| < \gamma$ for $n \geq N_1$ whenever $f \in A^{I_n}$ and $e \in \partial W_2$ are as above. Consequently, because $\pi(\frac{1}{k_n} \sum_{i=1}^{k_n} g(f(i))) = \pi(\int g(a)\, d\tau_{A,f}(a))$, (b) of Theorem 4 is satisfied if $n \geq N_1$.

We can conclude that for $n \geq N_1$, Λ has a fixed point and thus (I_n, G_n) has a strict pure strategy Nash equilibrium.

Finally, assume $\dim L = 0$. As above, there is an $N \in \mathbb{N}$ such that $\operatorname{supp}(\bar{\nu}_{G_n}) \subseteq V$ if $n \geq N$. For each n and $i \in I_n$, write $\bar{\nu}_{G_n,i} = \frac{1}{\#I_n - 1} \sum_{j \in I_n \setminus \{i\}} \delta_{u_{n,j}}$. Then we have $\operatorname{supp}(\bar{\nu}_{G_n,i}) \subseteq \operatorname{supp}(\bar{\nu}_{G_n})$, therefore also $\operatorname{supp}(\bar{\nu}_{G_n,i}) \subseteq V$ for each $i \in I_n$ if $n \geq N$. For $n \geq N$, we can define $f_n \colon I_n \to A$ by setting $f_n(i) = h(u_{n,i}, e(\tau_A))$ for $i \in I_n$. By (iii) of (b) we see that, for $n \geq N$,

$$e_{f_n,i} = \int g(a)\, d\tau_{A,f_n,i}(a) = \int g(h(u_{n,i}, e(\tau_A)))\, d\nu_{G_n,i} = e(\tau_A)$$

for each $i \in I_n$, because $\operatorname{supp}(\bar{\nu}_{G_n,i}) \subseteq V$. Thus, for each $n \geq N$, f_n is a strict pure strategy Nash equilibrium of (I_n, G_n).

(g) As for the second assertion of the theorem, fix any $u \in \mathcal{U}$. Inspecting part (e) of this proof reveals that if the perturbations of payoff functions made in this part of the proof are applied to payoff functions belonging to the set $\mathcal{U}_u$, then the resulting payoff functions are again elements of $\mathcal{U}_u$. □

5.3 Proof of Theorem 2

(a) Note first that if $\nu \in \mathcal{G}$ and τ is an equilibrium distribution for ν, then there exists a sequence $\langle \nu_n \rangle$ in $\mathcal{G}$, along with a sequence $\langle \tau_n \rangle$ of corresponding equilibrium distributions, such that (i) $\operatorname{supp}(\tau_n)$, and hence also $\operatorname{supp}(\nu_n)$, is finite for each n, (ii) $\tau_n \to \tau$ narrowly, and (iii) $\nu_n \to \nu$ in the topology of $\mathcal{G}$.

To see this, fix $\nu \in \mathcal{G}$ and let τ be an equilibrium distribution for ν. By the definition of $\mathcal{G}$, $\operatorname{supp}(\nu)$ is compact, and hence so is $\operatorname{supp}(\tau)$. We may therefore appeal to Lemma 2 in the appendix, with $\operatorname{supp}(\tau)$ for X, and $g \circ \operatorname{proj}_A$ for h, to find a sequence $\langle \tau_n \rangle$ of probability measures on $\mathcal{U} \times A$ such that $\tau_n \to \tau$ narrowly and for each n, $e(\tau_{n,A}) = e(\tau_A)$ and $\operatorname{supp} \tau_n$ is a finite subset of $\operatorname{supp}(\tau)$. For each n, let ν_n be the marginal measure of τ_n on $\mathcal{U}$. Then $\operatorname{supp}(\nu_n)$ is finite subset of $\operatorname{supp}(\nu)$ for each n, and $\nu_n \to \nu$ narrowly. Together these two facts imply that $\operatorname{supp}(\nu_n) \to \operatorname{supp}(\nu)$ in the Hausdorff metric topology on the non-empty compact subsets of $\mathcal{U}$. Thus $\nu_n \to \nu$ in the topology of $\mathcal{G}$. Finally, for any n, it is clear that because τ is an equilibrium distribution for ν, and because $\text{supp}(\tau_n) \subseteq \text{supp}(\tau)$ and $e(\tau_{n,A}) = e(\tau)$, τ_n is an equilibrium distribution for ν_n.

(b) Let $\mathcal{G}^*$ be defined as in the proof of Theorem 1. Inspecting (e) in that proof we see that whenever $\nu \in \mathcal{G}$ has finite support and τ is an equilibrium distribution for ν such that $\text{supp}(\tau)$ is also finite, then there is a sequence $\langle \nu_n \rangle$ in $\mathcal{G}^*$, with $\nu_n \to \nu$, and a sequence $\langle \tau_n \rangle$ of corresponding equilibrium distributions such that $\tau_n \to \tau$ narrowly and each τ_n satisfies the requirements listed in (c) of the proof of Theorem 1 (note that if $\text{supp}(\tau)$ is finite, then, in (e) of the proof of Theorem 1, the step involving Caratheodory's theorem can be omitted). Putting this fact together with (a) shows that for any $\nu \in \mathcal{G}$ and any equilibrium distribution τ for ν there is a sequence $\langle \nu_n \rangle$ in $\mathcal{G}^*$, with $\nu_n \to \nu$, and a sequence $\langle \tau_n \rangle$ of corresponding equilibrium distributions such that $\tau_n \to \tau$ narrowly and each τ_n satisfies the requirements in (c) of the proof of Theorem 1.

(c) Fix any $\nu \in \mathcal{G}^*$ and let τ be an equilibrium distribution for ν, witnessing that $\nu \in \mathcal{G}^*$. Suppose $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ is a sequence of finite-player games converging to ν. As in (f) of the proof of Theorem 1, for each n let $u_{n,i}$, $i = 1, \ldots, k_n$, be extensions of the payoff functions $G_n(i)$ to elements of $\mathcal{U}$ such that the sets $\{u_{n,i} \colon u \in \text{supp}(\nu_{G_n})\}$, $n \in \mathbb{N}$, converge to $\text{supp}(\nu)$ in the Hausdorff metric topology on the non-empty compact subsets of $\mathcal{U}$. For each n, write $\bar{\nu}_{G_n} = \frac{1}{\#I_n} \sum_{i \in I_n} \delta_{u_{n,i}}$, and $\widehat{G}_n$ for the map sending i to $u_{n,i}$, $i \in I_n$.

Let L, V, W, W_1, and $h \colon V \times W \to A$ be as in (f) of the proof of Theorem 1. Observe that $\tau = \nu \circ (id_V \times h(\cdot, e(\tau_A)))^{-1}$. Let $\langle W_{2,k} \rangle$ be a non-increasing sequence of compact neighborhoods of $e(\tau_A)$ in $\text{int}\, W_1$ such that $\bigcap_{k=0}^{\infty} W_{2,k} = \{e(\tau_A)\}$. Instead with a fixed W_2, the argument in (f) of the proof of Theorem 1 can be applied with each member of the sequence $\langle W_{2,k} \rangle$ to yield an increasing sequence $\langle n_k \rangle$ in $\mathbb{N}$, and for each $k \in \mathbb{N}$ an equilibrium f_k of the game (I_{n_k}, G_{n_k}) such that f_k can be written in the form $f_k(i) = h(\widehat{G}_{n_k}(i), e_k + z_k(i))$, $i \in I_{n_k}$, where $e_k \in W_{2,k}$ and $\|z_k(i)\| \leq \epsilon_k$ for each $i \in I_{n_k}$, and $\epsilon_k \to 0$ as $k \to \infty$.[8]

For each k, let $f'_k \colon I_{n_k} \to A$ be the map defined by setting $f'_k(i) = h(\widehat{G}_{n_k}(i), e_k)$ for each $i \in I_{n_k}$. Let τ_k be the distribution of $\widehat{G}_{n_k} \times f_k$, and τ'_k that of $\widehat{G}_{n_k} \times f'_k$. As $\bar{\nu}_{G_{n_k}} \to \nu$, we can assume, considering only large k, if necessary, that $\text{supp}(\bar{\nu}_{G_{n_k}}) \subseteq V$ for all k, and therefore write $\tau'_k = \bar{\nu}_{G_{n_k}} \circ (id_V \times h(\cdot, e_k))^{-1}$ for all k. Now because $e_k \in W_{2,k}$ for all k and $\bigcap_{k=0}^{\infty} W_{2,k} = \{e(\tau_A)\}$, we have $e_k \to e(\tau_A)$. From this we see that $id_V \times h(\cdot, e_k) \to id_V \times h(\cdot, e(\tau_A))$, uniformly on compact subsets of V, because h is continuous. Consequently

$$\tau'_k = \bar{\nu}_{G_{n_k}} \circ (id_V \times h(\cdot, e_k))^{-1} \to \nu \circ (id_V \times h(\cdot, e(\tau_A)))^{-1} = \tau,$$

i.e., the sequence $\langle \tau'_k \rangle$ of distributions of the maps $\widehat{G}_{n_k} \times f'_k$ converges to τ narrowly. Now note that if $\langle u'_k \rangle$ is any sequence in $\text{supp}(\nu)$, and $\langle z_k \rangle$ a sequence in $\mathbb{R}$ such that

[8] If $\dim L = 0$ then, of course, for all k, $e_k = e(\tau_A)$ and $z_k(i) = 0$ all $i \in I_{n_k}$, and the argument in the sequel becomes much simpler.

$h(u'_k, e_k + z_k)$ is defined and $\|z_k\| \to 0$, then

$$\|h(u'_k, e_k + z_k) - h(u'_k, e_k)\| \to 0$$

because $\mathrm{supp}(\nu)$ is compact, h continuous, and $e_k \to e(\tau_A)$. As $\mathrm{supp}(\bar{\nu}_{G_{n_k}}) \to \mathrm{supp}(\nu)$ in the Hausdorff metric topology, it follows that for every $\epsilon' > 0$ there is a $k_{\epsilon'} \in \mathbb{N}$ such that whenever $k \geq k_{\epsilon'}$, then

$$\|h(\widehat{G}_{n_k}(i), e_k + z_k(i)) - h(\widehat{G}_{n_k}(i), e_k)\| \leq \epsilon'$$

for all $i \in I_{n_k}$, i.e., $\|f_k(i) - f'_k(i)\| \leq \epsilon'$ for all $i \in I_{n_k}$, and thus, for some standard product metric $\tilde{\rho}$ on $\mathcal{U} \times A$,

$$\tilde{\rho}\big((\widehat{G}_{n_k}(i), f_k(i)), (\widehat{G}_{n_k}(i), f'_k(i))\big) \leq \epsilon'$$

for all $i \in I_{n_k}$. In view of this, we can conclude, using Billingsley (1968, Theorem 4.1), that the fact that the sequence $\langle \tau'_k \rangle$ of distributions of the maps $\widehat{G}_{n_k} \times f'_k$ converges narrowly to τ implies that the sequence $\langle \tau_k \rangle$ of distributions of the maps $\widehat{G}_{n_k} \times f_k$ converges narrowly to τ, too.

(d) By Remark 3, for any $\nu \in \mathcal{G}$ a sequence $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ of finite-player games converging to ν does exist. Putting this fact together with (b) and (c), proves the theorem. □

5.4 Proof of Theorem 3

With $A = [0, m]$ and g being the restriction to A of the identity on $\mathbb{R}$, let $\mathcal{U}$, $\mathcal{M}$, and $\mathcal{G}$ be defined as in the general model. Note that $E = g(A) = [0, m]$.

To each $v \in \mathcal{K}$ associate $u_v \in \mathcal{U}$ by setting $u_\nu(a, e) = p(e)a - v(a)$ for $a \in A$ and $e \in E$. Define a map $\kappa \colon \mathcal{K} \to \mathcal{U}$ by setting $\kappa(v) = u_v$ for each $v \in X$. Note that κ is a homeomorphism from $\mathcal{K}$ onto $\kappa(\mathcal{K})$. Let $\tilde{\kappa} \colon \mathcal{M}_\mathcal{K} \to \mathcal{G}$ be defined by setting $\tilde{\kappa}(\nu) = \nu \circ \kappa^{-1}$ for each $\nu \in \mathcal{M}_\mathcal{K}$. As κ is a homeomorphism from $\mathcal{K}$ onto $\kappa(\mathcal{K})$, $\tilde{\kappa}$ is a homeomorphism from $\mathcal{M}_\mathcal{K}$ onto $\tilde{\kappa}(\mathcal{M}_\mathcal{K})$.

Let $\mathcal{G}^* \subseteq \mathcal{G}$ be as guaranteed by Theorem 1. Then $\mathcal{M}^*_\mathcal{K} = \tilde{\kappa}^{-1}(\mathcal{G}^*)$ is open in $\mathcal{M}_\mathcal{K}$, because $\tilde{\kappa}$ is continuous. Let $\nu \in \mathcal{M}^*_\mathcal{K}$ and let $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ be a sequence of Cournot oligopolies such that $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$. By continuity of $\tilde{\kappa}$, we have $\tilde{\kappa}(\nu_{G_n}) \to \tilde{\kappa}(\nu)$. For each n and each $i \in I_n$, define $u_{n,i} \in \mathcal{U}$ by setting $u_{i,n}(a, e) = p\big(\frac{1}{\#(I_n)}a + \frac{\#(I_n - 1)}{\#(I_n)}e\big)a - G_n(i)(a)$. For each n, let $\tilde{\nu}_n = \sum_{i \in I_n} \frac{1}{\#(I_n)} \delta_{u_{i,n}}$, so that $\tilde{\nu}_n \in \mathcal{G}$. Note that since $\tilde{\kappa}(\nu_{G_n}) \to \tilde{\kappa}(\nu)$ and $\#(I_n) \to \infty$, we also have $\tilde{\nu}_n \to \tilde{\kappa}(\nu)$ (use Billingsley (1968, Theorem 4.1)). Because $\tilde{\kappa}(\nu) \in \mathcal{G}^*$, it follows from Theorem 1 (and the definition of Nash equilibrium for finite-player games as stated in Section 3) that (I_n, G_n) has a Cournot equilibrium if n is large.

It remain to show that $\mathcal{M}^*_\mathcal{K}$ is dense in $\mathcal{M}_\mathcal{K}$. As $\tilde{\kappa}$ is a homeomorphism, this means to show that $\mathcal{G}^* \cap \tilde{\kappa}(\mathcal{M}_\mathcal{K})$ is dense in $\tilde{\kappa}(\mathcal{M}_\mathcal{K})$. For this, we can argue as follows. Let $u \in \mathcal{U}$ be given by setting $u(a, e) = p(e)a$ for every $(a, e) \in A \times E$. By the second assertion of Theorem 1, $\mathcal{G}^* \cap \mathcal{G}_u$ is dense in G_u. Let $\mathcal{U}_{u,1}$ be the subset of $\mathcal{U}_u$ consisting of the elements that can be written in the form $u'(a,e) = u(a,e) - v(a)$ with $v(0) = 0$, and let $\mathcal{G}_{u,1}$ be the subset of $\mathcal{G}_u$ consisting of the elements with support included in $\mathcal{U}_{u,1}$. Let $\kappa_1 \colon \mathcal{U}_u \to \mathcal{U}_{u,1}$ be the map given by setting $\kappa_1(u') = u' + v(0)1_A$ for $u' \in \mathcal{U}_u$, and $\tilde{\kappa}_1 \colon \mathcal{G}_u \to \mathcal{G}_{u,1}$ the map defined by setting $\tilde{\kappa}(\nu) = \nu \circ \kappa_1^{-1}$ for $\nu \in \mathcal{G}_u$. Evidently κ_1 is continuous and is the identity on $\mathcal{U}_{u,1}$. Hence $\tilde{\kappa}_1$ has is continuous and is the identity on $\mathcal{G}_{u,1}$; in particular $\tilde{\kappa}_1$ is surjective. As $\mathcal{G}^* \cap \mathcal{G}_u$ is dense in $\mathcal{G}_u$, it follows that $\tilde{\kappa}_1(\mathcal{G}^* \cap \mathcal{G}_{u)}$ is dense in $\mathcal{G}_{u,1}$. Inspecting the definition of $\mathcal{G}^*$ in the proof of Theorem 1 shows that $\tilde{\kappa}_1(\mathcal{G}^* \cap \mathcal{G}_u) \subseteq \mathcal{G}^* \cap \mathcal{G}_{u,1}$. Thus $\mathcal{G}^* \cap \mathcal{G}_{u,1}$ is dense in $\mathcal{G}_{u,1}$. Now by hypothesis, $\kappa(\mathcal{K})$ is open in $\mathcal{U}_{u,1}$ and hence $\tilde{\kappa}(\mathcal{M}_{\mathcal{K}})$ is open in $\mathcal{G}_{u,1}$. We conclude that $\mathcal{G}^* \cap \tilde{\kappa}(\mathcal{M}_{\mathcal{K}})$ is dense in $\tilde{\kappa}(\mathcal{M}_{\mathcal{K}})$. □

6 Appendix

The following theorem is a special version of a result due to Mas-Colell (1983).

Theorem 4. *Let $X \subseteq \mathbb{R}^\ell$ and $Y \subseteq \mathbb{R}^m$ be compact convex sets with non-empty interior. Let $\Lambda \colon X \times Y \to X \times \mathbb{R}^m$ be a continuous function; write Λ_X for $\operatorname{proj}_X \circ \Lambda$ and Λ_Y for $\operatorname{proj}_{\mathbb{R}^m} \circ \Lambda$. Suppose there is an open set $U \subseteq \mathbb{R}^m$, with $Y \subseteq U$, and a C^1-map $\zeta \colon U \to \mathbb{R}^m$ such that, setting $\gamma = \min\{\|\zeta(y)\| : y \in \partial Y\}$ where ∂Y denotes the topological boundary of Y in $\mathbb{R}^m$,*

(a) for some $y^* \in \operatorname{int} Y$, $D\zeta(y^*)$ has full rank and $\zeta(y) = 0$ if and only if $y = y^*$ (so that, in particular, $\gamma > 0$);

(b) if $y \in \partial Y$ and $x = \Lambda_X(x,y)$, then $\|\Lambda_Y(x,y) - y - \zeta(y)\| < \gamma$.

Then Λ has a fixed point, i.e., there is an $(x,y) \in X \times Y$ such that $\Lambda(x,y) = (x,y)$.

The lemma stated next is used in the proof of Theorem 2.

Lemma 2. *Let X be a compact metric space, and μ a probability measure on X with $\operatorname{supp}(\mu) = X$. Let $h \colon X \to \mathbb{R}^m$ be a continuous function. Then there is a sequence $\langle \mu_n \rangle$ of probability measure on X such that $\mu_n \to \mu$ narrowly, $\operatorname{supp}(\mu_n)$ is finite for each n, and $\int h \, d\mu_n = \int h \, d\mu$ for each n.*

Proof. For each $n \in \mathbb{N}$ let $\mathcal{P}_n$ be a finite partition of X into Borel sets $B_{n,i}$ such that $\operatorname{diam}(B_{n,i}) \leq 1/(1+n)$ for each $B_{n,i} \in \mathcal{P}_n$. Fix any n. For each $B_{n,i} \in \mathcal{P}_n$ let $\tilde{\mu}_{n,i}$ be the Borel measure given by setting $\tilde{\mu}_{n,i}(B) = \mu(B \cap B_{n,i})$ for each Borel set B in X. Note that $\tilde{\mu}_{n,i}(B_{n,i}) = \mu(B_{n,i})$ for each $B_{n,i}$, and that $\sum_{i=1}^{k_n} \tilde{\mu}_{n,i} = \mu$, writing $k_n = \#(\mathcal{P}_n)$. Now if $\tilde{\mu}_{n,i}(B_{n,i}) > 0$ set $\hat{\mu}_{n,i} = \frac{1}{\tilde{\mu}_{n,i}(B_{n,i})} \tilde{\mu}_{n,i}$, so that $\hat{\mu}_{n,i}$ is a probability measure on X; if $\tilde{\mu}_{n,i}(B_{n,i}) = 0$, let $\hat{\mu}_{n,i}$ be the zero measure on X. Fix $B_{n,i}$ such that $\tilde{\mu}_{n,i}(B_{n,i}) > 0$. Write $e_{n,i} = \int h \, d\hat{\mu}_{n,i}$. Note that $e_{n,i} \in \operatorname{co}(h(\bar{B}_{n,i}))$. Using Caratheodory's theorem, we can therefore find a probability measure $\mu'_{n,i}$ on X such that $\operatorname{supp}(\mu'_{n,i})$ is a finite subset of $\bar{B}_{n,i}$ and $\int h \, d\mu'_{n,i} = e_{n,i}$. Set $\mu_n = \sum_{i=1}^{k_n} \tilde{\mu}_{n,i}(B_{n,i}) \mu'_{n,i}$. Then μ_n is a probability measure on X such that $\mathrm{supp}(\mu_n)$ finite and

$$\int h\,d\mu_n = \sum_{i=1}^{k_n}\Big(\tilde{\mu}_{n,i}(B_{n,i})\int h\,d\mu'_{n,i}\Big)$$
$$= \sum_{i=1}^{k_n}\Big(\tilde{\mu}_{n,i}(B_{n,i})\int h\,d\hat{\mu}_{n,i}\Big) = \sum_{i=1}^{k_n}\int h\,d\tilde{\mu}_{n,i} = \int h\,d\mu\,,$$

because $\tilde{\mu}_{n,i}(B_{n,i})\hat{\mu}_{n,i} = \tilde{\mu}_{n,i}$ and $\sum_{i=1}^{k_n}\tilde{\mu}_{n,i} = \mu$.

Now to show that $\mu_n \to \mu$ narrowly, fix any continuous map $p\colon X \to \mathbb{R}$. Fix $\epsilon > 0$. Note that since X is compact, p is uniformly continuous. Consequently there is an n_ϵ such that whenever $n \geq n_\epsilon$, then $|p(x) - p(x')| \leq \epsilon$ for all x, $x' \in B_{n,i}$ and all $B_{n,i} \in \mathcal{P}_n$. As p is continuous, it follows that whenever $n \geq n_\epsilon$, then for each $B_{n,i} \in \mathcal{P}_n$ we actually have $|p(x) - p(x')| \leq \epsilon$ for all x, $x' \in \bar{B}_{n,i}$. Thus, for any $n \geq n_\epsilon$, using the facts that $\mu = \sum_{i=1}^{k_n}\tilde{\mu}_{n,i} = \sum_{i=1}^{k_n}\tilde{\mu}_{n,i}(B_{n,i})\hat{\mu}_{n,i}$ and $\tilde{\mu}_{n,i}(B_{n,i}) = \mu(B_{n,i})$ for each $B_{n,i}$, together with the definition of μ_n, we see that

$$\left|\int p\,d\mu - \int p\,d\mu_n\right| \leq \sum_{i=1}^{n_k}\left(\tilde{\mu}_{n,i}(B_{n,i})\left|\int_{\bar{B}_{n,i}} p\,d\hat{\mu}_{n,i} - \int_{\bar{B}_{n,i}} p\,d\mu'_{n,i}\right|\right)$$
$$\leq \sum_{i=1}^{n_k}\big(\tilde{\mu}_{n,i}(B_{n,i})\epsilon\big) = \epsilon\sum_{i=1}^{n_k}\tilde{\mu}(B_{n,i}) = \epsilon\sum_{i=1}^{n_k}\mu(B_{n,i}) = \epsilon.$$

As $\epsilon > 0$ is arbitrary, it follows that $\int p\,d\mu_n \to \int p\,d\mu$. As p is an arbitrary continuous map from X to $\mathbb{R}$, we conclude that $\mu_n \to \mu$. □

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