# Strict pure strategy Nash equilibria in large finite-player games*

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**Abstract**

In the context of anonymous games (i.e., games where the payoff of a player is, apart from his/her own action, determined by the distribution of the actions made by the other players) we present a model in which, generically (in a precise sense), finite-player games have strict pure strategy Nash equilibria if the number of agents is large. A key feature of our model is that payoff functions have differentiability properties. A consequence of our existence result is that, in our model, equilibrium distributions of non-atomic games are asymptotically implementable by pure strategy Nash equilibria of large finite-player games.

**Keywords:** Large games, pure strategy, Nash equilibrium, generic property.

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# 1 Introduction

It is now well-established in economics to address problems of strategic interaction among many negligible individuals by models of anonymous games. In such games, the impact on the payoff of a player by the actions chosen by the other players factors through the distribution of these actions.[1] A particular and convenient class of models of anonymous games is formed by continuum games (games with a "continuum of players" or, more precisely, "non-atomic games"). In continuum games, there is no distinction anymore between the "distribution of the actions chosen by the other players" and the "distribution of the actions chosen by all players," so that, concerning equilibrium existence, these games are rather easy to analyze. In fact, there are several results on existence of pure strategy Nash equilibrium for such games, the pioneering ones provided by Schmeidler (1973) and Mas-Colell (1984). In these results, no linear structure is imposed on players' action sets. After all, of course, continuum games are idealizations of situations with a large but finite number of players, and in this respect the following questions naturally arise:

(1) To what extent do pure strategy Nash equilibrium existence results for continuum games carry over to pure strategy Nash equilibrium existence results for games with a large but finite number of players?

(2) Are equilibria of continuum games artifacts of having a continuum of players or are they realizable as "limits" of pure strategy Nash equilibria of large finite-player games? That is, given a continuum game and an equilibrium of this game, are there "close" large finite-player games having "similar" pure strategy Nash equilibria?

In other words, are continuum games, and equilibria of such games, good idealizations of situations with a large but finite number of players?

The "to what extent" clause has been incorporated in the first question because it is well-known that finite-player games may fail to have pure strategy Nash equilibria (unless action sets are convex and quasi-concavity assumptions on payoff functions are made). In fact, this may be the case for a fixed distribution of payoff functions, regardless of the number of players in such games (see the example after the statement of Theorem 1 in Section 3.5). Thus the best one can hope for in regard to the first of the above questions is to get positive results in terms of genericity analysis. As made clear by the literature on competitive equilibrium in exchange economies, a suitable and convenient setting for genericity analysis is a setting where agents' characteristics have differentiability properties. In this paper, we develop such a setting for anonymous

[1] We follow Kalai (2004) in defining the notion of "anonymous game" this way. Actually, Kalai (2004) speaks of "semi-anonymous" games to indicate that, in the incomplete information setting he considers, a player's identity is encoded in his type. Since we do not consider incomplete information, the prefix "semi" has been dropped. We note that in Khan and Sun (1999) the notion of "anonymous game" is reserved for continuum games specified solely by distributions of players' characteristics.

games, so that there is a generic set of continuum games such that finite-player games forming a sequence with an increasing number of players and a "limit" in this set have pure strategy Nash equilibria if the number of players is large enough.

Of course, a differentiability assumption on payoff functions necessarily implies that the domains of theses functions are subsets of a linear space (for which we take a Euclidean space). In particular, we will have a linear structure on the action sets of players. This is a restriction compared with the analysis in Mas-Colell (1984) where the action sets of players (which in Mas-Colell (1984) are the same for all players in a game) can be any compact metric space.[2]

Based on our existence result, we deal with the second of the above questions. We show that, in our model, any equilibrium of any continuum game is asymptotically implementable in the sense that there exists a sequence of finite-player games, with an increasing number of players, and a corresponding sequence of strict pure strategy Nash equilibria such that the given continuum game and its equilibrium arise as limit (in an appropriate sense).[3] This bears some surprise: It is well known that there is often an "explosion" of Nash equilibria in the limit, which suggests that limits of strict equilibria in sequences of finite-player games "converging" to a given continuum game could form a proper subset of the equilibrium set of the limit continuum game. As our result shows, this intuition can be incorrect.

An important aspect of our analysis is that the equilibria that we obtain for large finite-player games are strict, i.e., best reply sets in an equilibrium are singletons; thus there is no issue of specifying certain actions in the best reply sets as equilibrium actions. This makes our results immune against an objection as formulated by Mas-Colell (1977), which addresses general equilibrium theory, but applies to game theoretic contexts as well; to quote Mas-Colell (1977): "Important as those results are, the notion of equilibrium they deal with has some unattractive features. In particular, knowledge by the consumers of the equilibrium price system (plus the preference maximization hypothesis) does not determine the equilibrium; one needs, in addition, a possibly very careful specification of each consumer's commodity bundles. This makes the equilibrium a 'decentralized' one only in some weak sense."

To note some details of our model, action sets are compact subsets of some Euclidean space, with dense interior, and for any player, the externality, i.e., the channel through which his/her payoff is affected by the actions of the other players, is given,

[2]In fact, action sets in our model will have non-empty interior in some Euclidean space. We remark in this regard that pure strategy Nash equilibria with actions belonging to the interior of action sets do not have a mixed strategy interpretation unless payoff function are linear in the own actions of players. Models of games with action sets having non-empty interior in Euclidean spaces arise in several applications. We mention models of Cournot competition where firms can vary outputs continuously and models of auctions where bids can vary in products of intervals.

[3]This notion of "asymptotic implementation" is more specific than that in Khan and Sun (1996), where "asymptotic implementation" signifies, in the sense of nonstandard analysis, a "transfer" of any results for games with a hyperfinite Loeb spaces of players to games with a finite number of players (see Loeb and Wolff, 2015, p. 355).

as in Balder (2002), Rauh (2003), or Yu and Zhu (2005), by finitely many summary statistics (e.g., the first non-central moments) of the distribution of these actions. From the viewpoint of applications, this is not a big restriction; in fact, in many applications of anonymous games, e.g., Cournot oligopoly games, it is just the mean action of the other players which determines a player's payoff in addition to his/her own action. Payoff functions, in our model, are twice continuously differentiable. The main costs of our results, compared with standard game theory, are an assumption that implies that the best replies of a player against the distributions of the actions of the other players are always in the interior of his/her action set. This assumption is needed to be in a position in which differentiability assumptions on payoff functions can be conveniently exploited. On the set of players' characteristics (i.e., payoff functions) we define a suitable topology; because the actual definition requires some technical constructions, we refer to Section 3.4 and here say only that this topology is defined in terms of graphs of payoff functions to accommodate for the fact that action sets may differ across players. A continuum game is specified as a Borel probability measure with compact support on the space of players' characteristics. The compact support condition requires that players' characteristics in a continuum game are not too dissimilar. Equilibria of continuum games are described by equilibrium distributions, as in Mas-Colell (1984).

"Generic" in the set of continuum games is formally expressed as "open and dense" in the topology that treats two such games as close if they are close in the narrow topology and if their supports are close in the Hausdorff metric topology. "Close" for two continuum games in the former topology means that they involve similar players' characteristics with similar frequencies; the extra requirement of being "close" in the latter topology means that they are close only if they involve similar players' characteristics. In the notion of a generic set of continuum games, "open" means stability against perturbations, and "dense" that every continuum game can be approximated by continuum games belonging to the generic set.

We remark that the generic set of continuum games we identify in the proof of our main result (Theorem 1) is defined intrinsically in the sense that no reference to the particular problem of equilibrium existence in large finite-player games is made. Roughly, this set consists of those continuum games $\nu$ which have an equilibrium distribution such that the corresponding externality (which is the same for all players in a continuum game) has a neighborhood on which (a) the correspondence that sends externalities to the externalities determined by the best replies of the players with characteristics in the support of $\nu$ can be identified with a differentiable function, and (b) at each point, the derivative of this function minus the identity matrix has maximal rank.

The organization of the paper is as follows. In the next section we mention some of the related literature. In Section 3, the model is set up and the results are stated. Proofs can be found in Section 4. In an appendix, some auxiliary lemmata, which combine some more or less well-known mathematical facts, are stated and proved.

## 2 Related literature

Models with a continuum of agents were introduced into economics by Aumann (1964). These models are convenient from a mathematical point of view, and a way of expressing the idea, central to many economic applications, that an individual alone cannot influence aggregate statistics such as the price level or the distributions of actions taken by all participants in a game.

Almost immediately after the publication of Aumann's (1964) contribution, the question arose of what is the meaning of a model with a continuum of agents for situation of a large but finite number of them. To quote Hildenbrand (1970) on this matter: "...as an economist, our interest in these 'ideal economies' is proportional to how much new information can be derived for large but finite economies. In other words, the relevance of the ideal case to the finite case has to be established." Since then, much work has been done responding to questions like this; for instance, Hildenbrand (1970) and Brown and Robinson (1972) in general equilibrium theory, and Khan and Sun (1996) and Khan and Sun (2002) in game theory.

Results related to ours can be found in Rashid (1983), Khan and Sun (1999), Kalai (2004), Carmona and Podczeck (2009), and Carmona and Podczeck (2012b), who have established that sufficiently large finite-player games posses pure strategy approximate equilibria. Here "pure strategy approximate equilibrium" means a strategy profile such that for some numbers $\epsilon > 0$ and $0 \leq \eta < 1$, players which make up a fraction of at least $1 - \eta$ cannot deviate so that payoffs increase more than $\epsilon$,[4] and "sufficiently large" means that these numbers can be chosen arbitrarily small if one takes the number of players large enough.

Relative to that literature, the contribution of the present paper consists in presenting a setting of games which allows to drop the "approximate" qualifier generically (see Theorems 1 and 4 below). This is important because approximate equilibria are not always appealing[5]; thus, by dispensing with this qualification, this is something we do not have to worry about in our results. In fact, we obtain strict equilibria (i.e., players have unique best-replies) generically, so in an equilibrium there is no indeterminacy regarding which actions players will choose. For these reasons, our analysis provides an important supplement to the above mentioned results.

Asymptotic implementation of equilibria of continuum games, as defined in the introduction, was considered in Housman (1988) and Carmona and Podczeck (2012a). The corresponding results in those papers were stated in terms of approximate equilibria of finite-player games. As above, we can drop the "approximate" qualifier. In fact, in our model, every equilibrium of every continuum game, not just an equilibrium of a "generic" continuum game, can be asymptotically implemented in terms of

[4]In Kalai (2004) and some of the results of Carmona and Podczeck (2009) the number $\eta$ is zero.

[5]Consider a finite-player game where each player can choose one number in the interval $[0, 1]$, and each player's payoff is the average choice of the entire population (including his own choice). If this game is played by $n$ players, then the pure strategy profile where each player chooses 0 is an $\varepsilon$-equilibrium with $\varepsilon = 1/n$; on the other hand, for each player, 1 is a strictly dominant strategy.

strict pure strategy Nash equilibria of finite-player games (see Theorems 2 and 4).

In Carmona and Podczeck (2009) we have shown, among other things, that, roughly speaking, existence of equilibria in continuum games and existence of approximate equilibria in large finite-player games are in fact equivalent (under some conditions). In the class of games that can arise in the present model, this equivalence can be augmented by saying that the following statements are equivalent: (i) Every continuum game has an equilibrium distribution; (ii) there is a generic, in particular, dense set of continuum games such that along every sequence of finite player games with a "limit" in this set one has a strict pure strategy Nash equilibrium if the number of players is large enough. That (i) implies (ii) follows from Theorem 1. The other direction is an easy consequence of Lemma 7 in Section 4.2 and the fact that all distributions of payoff functions and distributions of actions that can arise in our model have compact support. We will not give a formal proof of this implication.

In our companion paper Carmona and Podczeck (2019), we deal with questions as in the present paper, but without differentiability assumptions on payoff functions. For finite action spaces, we obtain results analogous to that of the present paper. These results correspond to Schmeidler (1973), where pure strategies are identified with the vertices of a finite-dimensional simplex. In the general case where, as in Mas-Colell (1984), action sets may infinite, so that players may vary actions continuously, we show in Carmona and Podczeck (2019) that for generic continuum games, approximating finite-player games have mixed strategy equilibria such that the diameter of the supports of the mixed strategies tends to zero if the number of agents increases towards infinity; in this sense, mixed strategy equilibria become "nearly pure." Despite this weakening of the conclusion, this result is interesting as it considers the standard setting of large games, where actions spaces are general compact metric spaces and for any player the payoff function is a general continuous function of his/her action and the distribution of the actions chosen by the other players; in particular, no differentiability assumptions on payoff functions are made.

In the present paper, the goal is to allow players' actions to vary continuously and a the same time to obtain strict pure strategy Nash equilibria for large finite-player games, so that, in particular, mixed strategies can be completely dispensed with. For this, we adopt a setting in which payoff functions have differentiability properties. The main idea of our analysis is that, with such payoff functions, there is a generic set of continuum games which have equilibria for which the following are true: (i) at the equilibrium actions, payoff functions are locally strictly concave in the own actions of players, so that best reply sets are singletons locally; (ii) this implies that best reply sets are singletons globally; (iii) this property remains true in the product of some neighborhood of a game belonging to the generic set and some neighborhood of the externality arising in the equilibrium under consideration; (iv) we can use (iii) to get existence of strict pure strategy Nash equilibrium along sequences of finite-player games with a limit in the generic set of continuum games if the number of players is large enough.[6]

A similar idea underlies the analysis of Mas-Colell (1977, Theorem 2), who considers non-convex differentiable preferences in a general equilibrium framework of large economies. Concerning (i) above, see also Trockel (1984, p. 10). For an analysis of large economies where agents' preferences are convex, see Dierker (1975).

Apart from this similarity, the arguments in our proofs and in those of Mas-Colell (1977) differ to a large extent. For instance, in Mas-Colell (1977) existence of equilibria in large finite economies close to a generic continuum economies can be established by just applying an implicit function theorem to a (locally defined) excess demand function depending on prices and distributions of preferences (actually, Mas-Colell (1977) refers to Dierker (1975) at this point). In our setting, we cannot proceed this way, because, in finite-player games, different players may face different summary statistics of the actions of the respective other players, so that the dimension of domain of the problem increases with the number of players; as a consequence, we need a fixed point result to get pure strategy Nash equilibria. We also remark that in Mas-Colell (1977) there is no counterpart of the treatment of asymptotic implementability of any equilibrium of any continuum game.

Finally, we mention that in the literature there are models of continuum games where, different from the distributional approach adopted in the present paper to describe such games and their equilibria, an explicitly specified atomless measure space of players and a measurable map from this space to some space of players' characteristics are taken as given, and equilibria are described directly by strategy profiles (see, e.g., Balder (2002) or the more recent treatment by Khan et al. (2017)). However, in the present paper, the focus is on the connection between continuum games and large finite-player games. Since there is no reasonable topology on sets of players, such a connection cannot be made in terms of graphs of maps, but only in terms of distributions of maps, and for this, the distributional approach to describe continuum games and their equilibria provides all the information needed.[7]

# 3 The model and the results

## 3.1 General notation and terminology

If $X$ is any metric space, we write $\rho_H$ for the Hausdorff metric on the set of non-empty compact subsets of $X$.[8] Recall that on the set of non-empty compact subsets

[6]We can only be sketchy here. We just note that even though we do not assume action sets to be convex, (i) makes sense because we may assume them to have dense interior, and open sets in a Euclidean space include convex neighborhoods of each of their points.

[7]Of course, given an explicit atomless measure space of players and a map from this space to a space of players' characteristics, the joint distribution of this map and an equilibrium strategy profile is an equilibrium distribution.

[8]That is, $\rho_H(A, B) = \max\{\max_{a \in A} d(a, B), \max_{b \in B} d(b, A)\}$ for any two non-empty compact sets $A$, $B \subseteq X$, writing $d$ for the metric of $X$.

of a metric space $X$, the topology defined from the Hausdorff metric depends only on the topology of $X$, not on the particular metric.

For a subset $A$ of a topological space $X$, $\operatorname{int} A$ denotes the interior of $A$, $\partial A$ the boundary of $A$, $\operatorname{cl} A$ the closure of $A$, and $A \backslash B$ set-theoretic subtraction. If $A \subseteq \mathbb{R}^n$, then $\operatorname{co} A$ denotes the convex hull of $A$.

If $\mu$ is a Borel measure on a separable metrizable topological space $X$, we write $\operatorname{supp}(\mu)$ for the support of $\mu$, i.e., the smallest closed subset of $X$ with full measure. (Recall that every Borel measure on a separable metrizable topological space has a support.) If $\mu$ is a Borel measure on a product $X \times Y$ of topological spaces, $\mu_X$ and $\mu_Y$ denote the marginal measures on $X$ and $Y$ respectively.

Euclidean spaces are regarded as being equipped with the Euclidean norm. For any point $x$ in such a space, and any number $r > 0$, we write $B(x, r)$ for the open ball of center $x$ and radius $r$, and $\bar{B}(x, r)$ for the closed ball of center $x$ and radius $r$.

Let $X \subseteq \mathbb{R}^k$ be such that $\operatorname{int} X$ is dense in $X$ (which is true, in particular, if $X$ is convex and $\operatorname{int} X \neq \emptyset$). We say that a function $f \colon X \to \mathbb{R}^\ell$ is continuously differentiable if there is an open $\tilde{X} \subseteq \mathbb{R}^k$ including $X$ such that $f$ can be extended to a function $\tilde{f} \colon \tilde{X} \to \mathbb{R}^\ell$ which is continuously differentiable in the usual sense; the derivatives of $f$ at non-interior points of $X$ are defined to be those of $\tilde{f}$ (note that these derivatives do not depend on the particular choice of the extension $\tilde{f}$ if $\operatorname{int} X$ is dense in $X$.) In this case, we write $Df(x)$ for the derivative of $f$ at $x \in X$, and $Df$ for the map $x \mapsto Df(x)$. If $Df$ happens again to be continuously differentiable in the above sense, we say that $f$ is twice continuously differentiable; in this case, $D^2 f(x)$ stands for $DDf(x)$, and $D^2 f$ for the map $x \mapsto D^2 f(x)$. If $f$ is defined on a product $X \times Y$, where $Y$ is any set and $X$ is as above, then $D_x f(x, y)$ means the derivative of $f(\cdot, y)$ at $x \in X$ if $f(\cdot, y)$ is continuously differentiable; if $f(\cdot, y)$ is twice continuously differentiable, $D_x^2 f(x, y)$ stands for $D_x D_x f(x, y)$.

## 3.2 Actions and externality

We will consider games with a large number of players (as a particular case, with a continuum of players), where the payoff of a player is determined by his/her own action and an externality which is given by finitely many summary statistics of the distribution of the actions of the other players, as in Balder (2002) or Rauh (2003), and payoff functions have differentiability properties. We start setting up the model by fixing an ambient space so as to get suitable sets for the domains of payoff functions.

There is a universe $A$ of possible actions, which is a non-empty and compact subset of $\mathbb{R}^n$, with dense interior. Action sets of games will be included in $A$ and also have dense interior.

Summary statistics of a distribution $\tau_A$ of actions in $A$ are given by the formulae $\int g_1(a)\, d\tau_A(a), \ldots, \int g_m(a)\, d\tau_A(a)$ where $g_i$, $i = 1, \ldots, m$, is a continuously differentiable function from $A$ to $\mathbb{R}$ (given independently of $\tau_A$). We write $g$ for the vector $(g_1, \ldots, g_m)$, and $\int g(a)\, d\tau_A(a)$ for the vector $\left(\int g_1(a)\, d\tau_A(a), \ldots, \int g_m(a)\, d\tau_A(a)\right)$.

A natural example is obtained by setting

$$g(a) = (a_{(1)}, a^2_{(1)} \dots a^k_{(1)}, a_{(2)}, a^2_{(2)} \dots a^k_{(2)}, \dots, a_{(n)}, \dots, a^k_{(n)})$$

for each $a \in A$, where superscripts are exponents and the subscript $(h)$ means the $h$th coordinate of $a$, $h = 1, \dots, n$; in this case, $m = kn$ and $\int g(a)\, d\tau_A(a)$ is the vector of the first $k$ non-central moments of the coordinate-distributions determined by $\tau_A$; see Rauh (2003). A special case of this example is given if $m = n$ and $g$ is the restriction to $A$ of the identity on $\mathbb{R}^n$, so that $\int g(a)\, d\tau_A(a)$ is the "mean action" corresponding to the distribution $\tau_A$ on $A$.

For any player and any distribution $\tau_A$ on $A$ induced by the actions of the other players in a game, the externality is $e(\tau_A) = \int g(a)\, d\tau_A(a)$. Set

$$E = \left\{ \int g\, d\tau_A \colon \tau_A \text{ is a probability measure on } A \right\}.$$

Note that $E$ is a convex and compact subset of $\mathbb{R}^m$, and that $E$ is just the convex hull of the compact set $g(A)$. Every point of $E$ can arise as externality for a player in some continuum game. Thus the set $E$ can be seen as the universe of possible externalities.

In finite-player games, the set of externalities an individual player could actually face is of the form $\frac{1}{l} \sum_{j=1}^{l} g(A_j)$. To ensure that in games with sufficiently many players such sets have dense interior, i.e., are appropriate for differentiability assumptions on payoff functions, we make the following assumption on the map $g$: whenever $O$ is an open set in $\mathbb{R}^n$ with $O \subseteq A$, then $g(O)$ affinely spans $\mathbb{R}^m$ (in other words, the smallest affine subspace in $\mathbb{R}^m$ including $g(O)$ is $\mathbb{R}^m$ itself); see Lemma 10 in the appendix for the desired conclusion. This assumption simply imposes some kind of homogeneity property on the map $g$. We remark that the assumption is satisfied if $g$ is such that $\int g\, d\tau_A$ is, as in the example above, the vector of the first $k$ non-central moments of the coordinate-distributions determined by $\tau_A$; see Lemma 11 in the appendix.[9]

## 3.3 Payoff functions

A payoff function is a real-valued function $u$ with $\operatorname{dom} u = A_u \times E_u$ where $A_u$ and $E_u$ are subsets of the actions universe $A$ and the externalities universe $E$ respectively. The set $A_u$ is the action set of a player with payoff function $u$. (We thus specify actions sets of players by components of the domains of payoff functions; this is for notational efficiency.) The set $E_u$ is referred to as the externalities factor in $\operatorname{dom} u$.

This formalization of payoff functions will be convenient because it gives an easy way to setup a space of payoff functions so that, in actual games, the domain of the payoff function of any player does not depend on action profiles that this player cannot observe. E.g., in a finite-player game, the set of externalities relevant for the

[9]For an example where this assumption fails, put $A = [0, 1]$ and let $g \colon A \to \mathbb{R}$ be a continuously differentiable function with a non-zero derivative at some point of $A$, but such that the derivative vanishes on some non-degenerate subinterval of $A$.

payoff of an individual player is of the form $\frac{1}{l}\sum_{j=1}^{l} g(A_j)$, where the $A_j$'s are the action sets of the other players, so this set is the natural choice of the externalities factor in the domain of the payoff function of this player. (See also Remark 4 below).

We write $\varphi(u,e)$ for the best reply set of a player with payoff function $u$ if he faces $e \in E_u$ as externality. Thus

$$\varphi(u,e) = \big\{a \in A_u \colon u(a,e) = \max_{a' \in A_u} u(a',e)\big\}.$$

We assume that for a payoff function $u$, and the associated sets $A_u$ and $E_u$, the following are satisfied:

(U1) $A_u$ and $E_u$ are non-empty compact subsets of $A$ and $E$, respectively, such that both $A_u$ and $E_u$ have dense interior;

(U2) $u$ is twice continuously differentiable;

(U3) if $(a,e) \in \partial A_u \times E_u$, then there is an $a' \in A_u$ such that $u(a',e) > u(a,e)$.

Note that (U1) and (U2) imply that $\varphi(u,e)$ is non-empty for each $e \in E_u$, and that (U3) implies that $\varphi(u,e) \subseteq \operatorname{int} A_u$ for each $e \in E_u$.

**Remark 1.** Payoff functions can be constructed by perturbations of any real-valued twice continuously differentiable function defined on a set $\tilde{A} \times \tilde{E} \subseteq A \times E$ such that $\tilde{A}$ and $\tilde{E}$ satisfy the requirements in (U1). Indeed, let $\tilde{A} \subseteq A$ and $\tilde{E} \subseteq E$ be any such sets. Pick any $\bar{a} \in \operatorname{int} \tilde{A}$ and a number $r > 0$ such that $B(\bar{a}, r) \subseteq \tilde{A}$. Let $\tilde{u}$ any twice continuously differentiable function from $\tilde{A} \times \tilde{E}$ to $\mathbb{R}$. Set

$$m_1 = \max\{\tilde{u}(a,e) \colon a \in \partial \tilde{A},\ e \in \tilde{E}\}$$

and

$$m_2 = \min\{\tilde{u}(\bar{a},e) \colon e \in \tilde{E}\}$$

Choose a number $k$ with $k + m_2 > m_1$ and a twice continuously differentiable function $\zeta \colon A \to \mathbb{R}_+$ such that $\zeta(a) = k$ for $a \in B(\bar{a}, r/2)$ and $\zeta(a) = 0$ for $a \in A \setminus B(\bar{a}, r)$. Now set $u = \tilde{u} + \zeta$, $A_u = \tilde{A}$, and $E_u = \tilde{E}$. Then $u$ satisfies conditions (U1)–(U3).

### 3.4 Space of payoff functions

The set of payoff functions is denoted by $\mathcal{U}$. In Section 4.2 we show that there is unique metrizable topology on $\mathcal{U}$ such that a sequence $\langle u_k \rangle$ in $\mathcal{U}$ converges to some $u \in \mathcal{U}$ if and only if

(a) $\rho_H(\operatorname{dom} u, \operatorname{dom} u_k) \to 0$ and $\rho_H(\partial A_u, \partial A_{u_k}) \to 0$; (b) if $(a,e) \in \operatorname{dom} u$ and $(a_k, e_k) \in \operatorname{dom} u_k$, $k \in \mathbb{N}$, are such that $(a_k, e_k) \to (a,e)$, then $u_k(a_k,e_k) \to u(a,e)$, $Du_k(a_k,e_k) \to Du(e,a)$, and $D^2 u_k(a_k,e_k) \to D^2 u(e,a)$.[10]

[10] Concerning (a), note that by what has been remarked in Section 3.1, convergence for the Hausdorff metric on the family of non-empty compact subsets of a Euclidean space $X$ is topological, i.e., does not depend on the particular metric inducing the topology of $X$.

In the rest of the paper, $\mathcal{U}$ is regarded as being equipped with this topology, and for definiteness we fix any metric $\rho$ inducing this topology, so that $\mathcal{U}$ can be regarded as a metric space if necessary. None of our results depend on any specific metric inducing the topology of $\mathcal{U}$; in particular, none of our them depend on the choice of the metric $\rho$.

Evidently on subsets of $\mathcal{U}$ consisting of functions with a common domain, the topology of $\mathcal{U}$ is just the topology of $C^2$-uniform convergence. We also note that $\mathcal{U}$ is a separable topological space (see Lemma 2 in Section 4.2).

Concerning the condition "$\rho_H(\partial A_u, \partial A_{u_k}) \to 0$" in (a) above, this condition is needed in addition to the condition "$\rho_H(\operatorname{dom} u, \operatorname{dom} u_k) \to 0$" to have a notion of closeness of action sets such that an interior point of an action set is also an interior point of nearby action sets. This is central for our results, but unfortunately is not implied by the condition "$\rho_H(\operatorname{dom} u, \operatorname{dom} u_k) \to 0$" alone, unless action sets are assumed to be convex. In fact, for convex action sets, "$\rho_H(\operatorname{dom} u, \operatorname{dom} u_k) \to 0$" implies "$\rho_H(\partial A_u, \partial A_{u_k}) \to 0$" (see Wills, 2007). Of course, in scenarios where all players have the same action set (as considered in Theorem 4 below), the condition $\rho_H(\partial A_u, \partial A_{u_k}) \to 0$ can be dropped from (a).

## 3.5 Finite-player games

Recall that $m$ is the dimension of the ambient Euclidean space of the externalities universe $E$. We consider finite-player games given by pairs $(I, G)$ where $I$ is a finite set of players, with $\#(I) \geq \max\{2, m+1\}$, and $G$ is a map from $I$ to $\mathcal{U}$ such that $E_i = \sum_{j \in I \setminus \{i\}} g(A_j)/((\#I) - 1)$ for each $i \in I$, writing $E_i$ for $E_{G(i)}$ and $A_j$ for $A_{G(j)}$. Note that for a player $i$ in a finite-player game $(I, G)$, any distribution of actions chosen by the other players is of the form $\sum_{j \in I \setminus \{i\}} \delta_{a_j}/((\#I) - 1)$ where $\delta_{a_j}$ denotes Dirac measure at point $a_j$ in the action set $A_j$ of $j \in I \setminus \{i\}$. Thus the equality $E_i = \sum_{j \in I \setminus \{i\}} g(A_j)/((\#I) - 1)$ means that the externalities factor in the domain of the payoff function of a player $i$ is exactly the set of externalities the player could actually face in the game $(I, G)$. If $\#(I) > m$, then by Lemma 10, $\operatorname{int} \sum_{j \in I \setminus \{i\}} g(A_j)/((\#I) - 1)$ is dense in $\sum_{j \in I \setminus \{i\}} g(A_j)/((\#I) - 1)$, so the conditions $E_i = \sum_{j \in I \setminus \{i\}} g(A_j)/((\#I) - 1)$ are consistent with (U1) in the assumptions on payoff functions and therefore with the definition of $\mathcal{U}$. Because the focus of our paper is on large games, there is no problem with a condition based on the imagination that a game have sufficiently many players.

A strategy profile in a finite-player game $(I, G)$ is a map $f \colon I \to A$ such that $f(i) \in A_i$ for each $i \in I$. Given any strategy profile $f$, we write $e_{f,i}$ for the externality faced by player $i$; that is, $e_{f,i} = \sum_{j \in I \setminus \{i\}} g(f(j))/(\#(I) - 1)$, or, in other words, $e_{f,i} = \int g(a)\, d\tau_{A,f,i}(a)$, where $\tau_{A,f,i}$ is the distribution of the actions chosen by the players $j \in I \setminus \{i\}$; thus, for any Borel set $B \subseteq A$,

$$\tau_{A,f,i}(B) = \#(\{j \in I \setminus \{i\} \colon f(j) \in B\})/(\#(I) - 1).$$

A strategy profile $f \colon I \to A$ is a pure strategy Nash equilibrium if $f(i) \in \varphi(G(i), e_{f,i})$ for each $i \in I$. A pure strategy Nash equilibrium is called strict if $\#(\varphi(G(i), e_{f,i})) = 1$ for each $i \in I$.

Every finite-player game $(I, G)$ defines a distribution on $\mathcal{U}$, i.e., a distribution of payoff functions. We write $\nu_G$ for such a distribution; thus, for any Borel set $B$ in $\mathcal{U}$, $\nu_G(B) = \#(\{i \in I\colon G(i) \in B\})/\#(I)$.

### 3.6 Continuum games

Recall from Section 3.4 that $\mathcal{U}$ can be regarded as a metric space. Let $\mathcal{M}$ be the set of all Borel probability measures on $\mathcal{U}$ with compact support. (By Lemma 2 in Section 4.2, $\mathcal{U}$ is separable, so any Borel measure on $\mathcal{U}$ has a support). We regard $\mathcal{M}$ as being given the topology such that $\nu_n \to \nu$ if both $\nu_n \to \nu$ in the narrow topology[11] and $\rho_H(\mathrm{supp}(\nu_n), \mathrm{supp}(\nu)) \to 0$, i.e., $\mathrm{supp}(\nu_n) \to \mathrm{supp}(\nu)$ in the Hausdorff metric topology. Note that for any finite-player game, $\nu_G$ as defined in the previous section belongs to $\mathcal{M}$. Given $\nu \in \mathcal{M}$, let

$$E(\nu) = \Big\{ \int g \, d\tau_A : \tau \text{ is a probability measure on } \mathcal{U} \times A \text{ such that } \tau_{\mathcal{U}} = \nu \text{ and } (u, a) \in \mathrm{supp}(\tau) \text{ implies } a \in A_u \Big\}.$$

Following Mas-Colell (1984), we define a continuum game as a distribution on the space of players' characteristics. We add the assumption that the support of the distribution of players' characteristics in a continuum game is compact, i.e., that players' characteristics in a continuum game are not too dissimilar. With a continuum of players, every player is negligible in the strict mathematical sense, so that there is no distinction between the "distribution of the actions chosen by all players" and the "distribution of the actions chosen by all but one player." Thus, because in any game the externalities factor in the domains of payoff functions must be equal to the set of externalities players could actually face, in our model a continuum game is an element $\nu \in \mathcal{M}$ such that $E_u = E(\nu)$ for each $u \in \mathrm{supp}(\nu)$. By Lemma 5(a), requiring these equalities is consistent with the definition of $\mathcal{U}$. We write $\mathcal{G}$ for the set of continuum games and give $\mathcal{G}$ the subspace topology induced by the topology of $\mathcal{M}$ (see the introduction for the meaning of this topology).

Pure strategy Nash equilibria of continuum games are specified in our model in terms of equilibrium distributions, as in Mas-Colell (1984). In our notation, an equilibrium distribution of a continuum game $\nu \in \mathcal{G}$ is a Borel probability measure $\tau$ on $\mathcal{U} \times A$ such that $\tau_{\mathcal{U}} = \nu$ and $\mathrm{supp}(\tau) \subseteq \{(u, a) \in \mathcal{U} \times A\colon a \in \varphi(u, e(\tau_A))\}$. By Mas-Colell (1984), every continuum game $\nu \in \mathcal{G}$ has an equilibrium distribution.

**Remark 2.** To get an example of a continuum game $\nu \in \mathcal{G}$, let $A_1, \dots, A_k$ be nonempty closed subsets of $A$, all with dense interior, and let $\alpha_1, \dots, a_k$ be positive real numbers such that $\sum_{i=1}^k \alpha_i = 1$. Set $\tilde{E} = \sum_{i=1}^k \alpha_i \mathrm{co}\, g(A_i)$. For each $i = 1, \dots, k$,

[11] Recall that the narrow topology on the set of Borel measures on a metrizable topological space is the topology of pointwise convergence on the bounded continuous functions defined on this space, evaluation being given by integration.

choose a $u_i \in \mathcal{U}$ with $A_{u_i} = A_i$ and $E_{u_i} = \tilde{E}$; cf. Remark 1. Set $\nu = \sum_{i=1}^{k} \delta_{u_i}$. Note that we must have $E(\nu) = \tilde{E}$, by the definition of $E(\nu)$. Consequently $\nu \in \mathcal{G}$.

More interesting are examples of continuum games satisfying the following condition. Let us say that a continuum game $\nu \in \mathcal{G}$ has "essentially ordinally non-equivalent payoff functions" if there is a partition of $\mathrm{supp}(\nu)$ into $\nu$-null sets such that two payoff functions belonging to different members of the partition are ordinally non-equivalent ("ordinally non-equivalent" for two payoff functions $u$, $u' \in \mathcal{U}$ meaning that there is no increasing function $h\colon u(A_u \times E(\nu)) \to \mathbb{R}$ such that $u' = h \circ u$.) Note that if a $\nu \in \mathcal{G}$ satisfies this condition, then $\nu$ must be zero on singletons, therefore atomless since $\mathcal{U}$ is a separable metrizable space. In Lemma 8 in Section 4.2 we show that the set of continuum games $\nu \in \mathcal{G}$ with ordinally non-equivalent payoff functions is dense in $\mathcal{G}$.

**Remark 3.** Given $\nu \in \mathcal{G}$, there are plenty of sequences $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ of finite-player games such that $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$ in $\mathcal{M}$. Indeed, in Lemma 7 in Section 4.2 we show, based on the law of large numbers and the Shapley-Folkman theorem, that such sequences do exist. Of course, given such a sequence, there are uncountably many (ordinally non-equivalent) modifications of the countably many payoff functions involved such that the resulting sequences of finite-player games still converge to $\nu$.

**Remark 4.** In our setup, whether or not a sequence of finite-player games converges to some continuum game depends only on players' payoffs at actions in their own actions sets and on the externalities they can potentially observe in an actual game. This would not be the case had we taken $\mathcal{U}$ to be a space of functions defined on the product of the entire actions universe and the entire externalities universe.

## 3.7 Results

Our first result gives precision to the idea that, generically, pure strategy Nash equilibrium existence results for continuum games carry over to large finite-player games in a setting with differentiable payoff functions. In the context of this result, the compact support condition on the distributions of players' characteristics in continuum games means that, along sequences of finite-player games, players' characteristics must not become too dissimilar if the number of players increases towards infinity.

**Theorem 1.** *There is an open and dense subset $\mathcal{G}^*$ of $\mathcal{G}$ such that whenever $\nu \in \mathcal{G}^*$ and $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ is a sequence of finite-player games such that $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$, then there is an $N \in \mathbb{N}$ such that $(I_n, G_n)$ has a strict pure strategy Nash equilibrium if $\#(I_n) \geq N$.*

(See Section 4.3 for the proof.)

The idea of the proof is first to identify an open and dense set $\mathcal{G}^* \subseteq \mathcal{G}$ such that if $\nu \in \mathcal{G}^*$, then $\nu$ has an equilibrium distribution $\tau$ such that, for some neighborhood $V$ of $\mathrm{supp}(\nu)$ and some neighborhood $W$ of $e(\tau_A)$, best replies on $V \times W$ are given by a (continuously differentiable) function. Given such a $\nu$, if $\langle (I_n, G_n) \rangle$ is sequence of finite-player games such that $\nu_{G_n} \to \nu$, then $\text{supp}(\nu_{G_n})$ must be in $V$ eventually, and at this point the idea is to get strict pure strategy Nash equilibria for the games $(I_n, G_n)$ if $n$ is large enough by a fixed point argument, using the fact that best replies on $V \times W$ are given by a function, but taking care of the fact that in finite-player games different players may face different externalities.

The following example shows that, in the context of Theorem 1, we cannot do better than to obtain a result for generic distributions of players' characteristics, regardless of the number of players, i.e., of the size of $I$.

**Example.** Let $A = [-1/2, 3/2]$, and $v\colon A \to \mathbb{R}_+$ a twice continuously differentiable function, with $Dv(1/2) = D^2v(1/2) = v(1/2) = 0$, assuming a global maximum, equal to 1, exactly at the points 0 and 1. Let $g\colon A \to \mathbb{R}$ be the restriction to $A$ of the identity on $\mathbb{R}$. Then $E = [-1/2, 3/2]$, and for each $f\colon I \to A$ and each $i \in I$, the externality $e_{f,i} \in E$ faced by $i$ is the mean of the actions of the players different from $i$. Let $\#(I)$ be even, with $\#(I) \geq 4$. Partition $I$ into sets $H$ and $J$ of equal size. For $i \in H$ the payoff function is $u_H\colon A \times E \to \mathbb{R}$ defined by setting

$$u_H(a, e) = v(a)(3/2 - e) \text{ if } a < 1/2 \text{ and } u_H(a, e) = v(a) \text{ if } a \geq 1/2,$$

and for $i \in J$ the payoff function is $u_J\colon A \times E \to \mathbb{R}$ defined by setting

$$u_J(a, e) = v(a)(e + 1/2) \text{ if } a < 1/2 \text{ and } u_J(a, e) = v(a) \text{ if } a \geq 1/2.$$

Note that for all $i \in I$ and all values of $e_{f,i}$, the best reply sets are included in $\{0, 1\}$, and that if $f\colon I \to A$ is a strategy profile such that $f(i) \in \{0, 1\}$ for all $i \in I$, then $e_{f,i} = \#\{j \in I \backslash \{i\} \colon a_j = 1\}/\#(I - 1)$ for each $i \in I$.

Now suppose $f\colon I \to A$ is a pure strategy Nash equilibrium. Consider $i$, $i' \in H$ and suppose $f(i) = 0$ and $f(i') = 1$. Then, by optimal choice of actions, $e_{f,i} \leq 1/2$ and $e_{f,i'} \geq 1/2$. On the other hand, calculating frequencies, we see that $f(i) = 0$ and $f(i') = 1$ together imply that $e_{f,i} > e_{f,i'}$, and from this contradiction it follows that all members of $H$ must choose the same action, say 0. But then, because $\#(I)$ is even, $e_{f,i} < 1/2$ for all members $i$ of $J$, so they all must play 1, by optimal choice of actions. This, however, means that $e_{f,i} > 1/2$ for all members of $H$, again because $\#(I)$ is even, so their optimal actions are also equal to 1, and this contradiction shows that no pure strategy Nash equilibrium exists.

**Remark 5.** On the level of continuum games, one might be interested in the set $\mathcal{G}_O$ of elements of $\mathcal{G}$ with ordinally non-equivalent payoff functions (see Remark 2), so it is worth mentioning explicitly that, in the context of Theorem 1, $\mathcal{G}_O \cap \mathcal{G}^*$ is open and dense in $\mathcal{G}_O$. This can be seen from Lemma 8 and the general fact that if $X$ is a topological space, $A \subseteq X$ is dense in $X$, and $B \subseteq X$ is open and dense in $X$, then $A \cap B$ is open and dense in $A$.

In this regard, one may ask whether Theorem 1 can be fully placed into an ordinal setting, i.e., a setting of differentiable preference relations, in order to get a result that is on the level of classical genericity analysis of exchange economies. In the latter context, however, the space of preference relations is topologized under the assumption that they be monotone, an assumption that cannot be made in the game theoretic context of our paper, because action sets and externality sets are assumed to be compact. The main problem of placing Theorem 1 into an ordinal setting would therefore be to find a suitable topology on the set of preference relations. We leave the consideration of this for future research.[12]

The arguments in the proof of Theorem 1 can be used to show that, in our model, every equilibrium distribution of every continuum game is the limit of some sequence of finite-player games and corresponding strict pure strategy Nash equilibria, in the sense of the following definition.

**Definition.** Let $\nu \in \mathcal{G}$ be a continuum game and $\tau$ an equilibrium distribution for $\nu$. A sequence $\langle (I_n, G_n)\rangle_{n\in\mathbb{N}}$ of finite-player games such that $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$ is said to *asymptotically implement* $(\nu, \tau)$ if for all $n$ larger than some $N \in \mathbb{N}$, $(I_n, G_n)$ has a strict pure strategy Nash equilibrium $f_n$ such that the sequence of distributions of the maps $G_n \times f_n$ converges to $\tau$ narrowly. We say that $(\nu, \tau)$ is *asymptotically implementable* if it can be asymptotically implemented by some sequence $\langle (I_n, G_n)\rangle_{n\in\mathbb{N}}$ with $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$.

**Theorem 2.** *Every $(\nu, \tau)$, where $\nu \in \mathcal{G}$ is a continuum game and $\tau$ is an equilibrium distribution for $\nu$, is asymptotically implementable by a sequence $\langle (I_n, G_n)\rangle_{n\in\mathbb{N}}$ of finite-player games such that $\nu_{G_n} \in \mathcal{G}$ for each $n$.*

(See Section 4.4 for the proof.)

We emphasize that every continuum game can be taken in Theorem 2, not just one from a generic set. Thus Theorem 2 shows that, in our model, no equilibrium distribution of any continuum game is an artifact of having continuum many players.

The idea of the proof is to show first that any continuum game $\nu \in \mathcal{G}$, and any equilibrium distribution $\tau$ of $\nu$, can be approximated by $\nu'$s belonging to the generic set $\mathcal{G}^*$ from Theorem 1, together with equilibrium distributions witnessing this. We then extend the arguments of the proof of Theorem 1 and show that, given any $\nu' \in \mathcal{G}^*$, and any equilibrium distribution $\tau'$ witnessing this, along every sequence $\langle (I_n, G_n)\rangle$ of finite-player games such that $\nu_{G_n} \to \nu'$ there are strict pure strategy Nash equilibria $f_n$ if $n$ is large enough such that the distributions of the maps $G_n \times f_n$ converge to $\tau'$ narrowly. Coupling these arguments with an argument (based on the law of large numbers) showing that sequences of finite player games converging to a given continuum game do exist, Theorem 2 follows by an obvious diagonal argument.

The notion in the following definition strengthens the notion of asymptotic implementability.

[12]For a formulation of continuum games in ordinal terms without differentiability assumptions, see Khan and Sun (1990).

**Definition.** Let $\nu \in \mathcal{G}$ be a continuum game, and $\tau$ an equilibrium distribution for $\nu$. We say that $(\nu, \tau)$ is *asymptotically robust* if *each* sequence $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ of finite-player games such that $\#(I_n) \to \infty$, $\nu_{G_n} \in \mathcal{M}$ for each $n$, and $\nu_{G_n} \to \nu$ asymptotically implements $(\nu, \tau)$.

**Theorem 3.** *There is an open dense subset $\mathcal{G}^*$ of $\mathcal{G}$ such that whenever $\nu \in \mathcal{G}^*$, then there is an equilibrium distribution $\tau$ for $\nu$ such that $(\nu, \tau)$ is asymptotically robust.*

(See Section 4.5 for the proof.)

In Theorem 3 we cannot have $\mathcal{G}^* = \mathcal{G}$. This follows from the example after the statement of Theorem 1, because any continuum game has an equilibrium distribution. Indeed, in that example, the distribution of players' characteristics is independent of the number of players, therefore convergent if this number grows towards infinity. Given this observation, Theorem, 3 follows, with the set $\mathcal{G}^*$ from Theorem 1, by showing, as in the proof of Theorem 2, that if $\nu \in \mathcal{G}^*$ and $\tau$ is an equilibrium distribution witnessing this, then along every sequence $\langle (I_n, G_n) \rangle$ of finite-player games such that $\nu_{G_n} \to \nu$ there are strict pure strategy Nash equilibria $f_n$ if $n$ is large enough such that the distributions of the maps $G_n \times f_n$ converge to $\tau$ narrowly.

In much of the literature on continuum games it is assumed, following Schmeidler (1973) and Mas-Colell (1984), that all players have the same action set. In our model, the set of continuum games in which all players have the same action set is not open in the set of all continuum games, so Theorems 1–3 do not apply to this special case. We will address this case with another theorem. To this end, it is convenient to settle some additional notation.

Let $\mathcal{C}$ be the set of all closed subsets $C$ of the actions universe $A$ such that $C$ has dense interior. For each $C \in \mathcal{C}$, let $S_C = \{u \in \mathcal{U} \colon A_u = C\}$. Note that $S_C$ is closed in $\mathcal{U}$ for each $C \in \mathcal{C}$. For each $C \in \mathcal{C}$, let $\mathcal{M}_C = \{\nu \in \mathcal{M} \colon \mathrm{supp}(\nu) \subseteq S_C\}$ and $\mathcal{G}_C = \mathcal{G} \cap \mathcal{M}_C$.

**Theorem 4.** *Given any $C \in \mathcal{C}$, every $(\nu, \tau)$, where $\nu \in \mathcal{G}_C$ is a continuum game and $\tau$ is an equilibrium distribution for $\nu$, is asymptotically implementable by a sequence $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ of finite-player games such that $\nu_{G_n} \in \mathcal{M}_C$ for each $n$.*

*Moreover, there is a relatively open dense subset $\mathcal{G}_C^*$ of $\mathcal{G}_C$ such that the following are true.*

*(i) Whenever $\nu \in \mathcal{G}_C^*$ and $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ is a sequence of finite-player games such that $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$, then there is an $N \in \mathbb{N}$ such that $(I_n, G_n)$ has a strict pure strategy Nash equilibrium if $\#(I_n) \geq N$.*

*(ii) Whenever $\nu \in \mathcal{G}_C^*$, then there is an equilibrium distribution $\tau$ for $\nu$ such that $(\nu, \tau)$ is asymptotically robust.*

(See Section 4.6 for the proof.)

The ideas of the proof are the same as those of the proofs of Theorems 1–3, modulo that concerning asymptotic implementation of equilibrium distributions of continuum games one has to take care to choose finite-player games in which all players have the correct action set.

### 3.8 An application

In some applications, one is interested in a class of games that forms a proper subset of $\mathcal{G}$. An example, which we consider in this section, is a model of Cournot oligopoly where an inverse demand function is given and the payoff function of different players (i.e., firms) can differ only if their cost functions differ. That is, only payoff functions that can be written as the difference between some cost function and the revenue function defined from the given inverse demand function are relevant. In this case, it is desirable to obtain a generic subset of distributions of cost functions having certain properties, not a generic subset of distributions of all possible payoff functions.[13]

We assume that the set of possible outputs that can be produced by an individual firm is the interval $[0, m]$, where $m > 0$; the number $m > 0$ can be interpreted as a capacity constraint. Let $X$ be the set of all twice continuously differentiable functions on $[0, m]$, equipped with the topology of $C^2$-uniform convergence. The inverse demand function is given in terms of output per firm (independently of the actual number of firms in an oligopoly) and specified by an element $p$ of $X$ with $p(e) > 0$ for all $e \in [0, m]$ and $p(0) > p(m) > 0$. Let $Z$ be the subset of $X$ consisting of the elements $v$ of $X$ with $v(0) = 0$ and $Dv(a) > 0$ for all $a \in [0, m]$. Give $Z$ the subspace topology induced by the topology of $X$ and let $\mathcal{K}$ be a non-empty open subset of $Z$ such that for each $v \in \mathcal{K}$, $Dv(0) < p(e) < v(m)/m$ for all $e \in [0, m]$. The elements of $\mathcal{K}$ are the possible cost functions. A Cournot oligopoly is a pair $(I, G)$ where $I$ is a finite set of firms, with $\#(I) \geq 2$, and $G\colon I \to \mathcal{K}$ is a map assigning cost functions to firms. A strategy profile $f\colon I \to [0, m]$ is a Cournot equilibrium of $(I, G)$ if

$$\begin{aligned} p\Bigg(\tfrac{1}{\#(I)} f(i) + \tfrac{1}{\#(I)} \sum_{j \in I\setminus\{i\}} f(j)\Bigg) f(i) - G(i)(f(i)) \\ \geq p\Bigg(\tfrac{1}{\#(I)} a + \tfrac{1}{\#(I)} \sum_{j \in I\setminus\{i\}} f(j)\Bigg) a - G(i)(a) \end{aligned}$$

for each $i \in I$ and each $a \in [0, m]$. Let $\mathcal{M}_{\mathcal{K}}$ be the set of all Borel probability measures on $\mathcal{K}$, with compact support. Give $\mathcal{M}_{\mathcal{K}}$ the topology analogous to that of $\mathcal{M}$.

**Theorem 5.** *There is an open dense subset $\mathcal{M}^*_{\mathcal{K}}$ of $\mathcal{M}_{\mathcal{K}}$ such that if $\nu \in \mathcal{M}^*_{\mathcal{K}}$ and $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ is a sequence of Cournot oligopolies such that $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$, then there is an $N \in \mathbb{N}$ such that $(I_n, G_n)$ has a Cournot equilibrium if $\#(I_n) \geq N$.*

(See Section 4.7 for the proof.)

[13]Another setting where the approach of this section might be useful is that of monopolistic competition considered, amongst others, in Chipman (1970), Hart (1979), or Páscoa (1993).

# 4 Proofs

This section contains the proofs of our results. Here is a short roadmap.

In Section 3.4 we have made a convention on the topology of the space $\mathcal{U}$ of payoff functions. In Lemma 1 below we prove the existence of this topology, and in Lemma 2 that $\mathcal{U}$ (with this topology) is separable.

We continue with some more preparing lemmata. We mention Lemma 7, which proves the fact announced in Remark 3 that, in our model, every continuum game can be approximated by finite-player games, a fact that is a prerequisite for asymptotic implementability of equilibrium distributions of continuum games.

The set $\mathcal{G}^*$ in the statement of Theorem 1 is defined in (c) of the proof of this theorem. The proof that this set is dense in $\mathcal{G}$ is delegated to a separate lemma (Lemma 9 after the proof of Theorem 1). This lemma is stated in greater generality than is actually needed in the proof of Theorem 1, so that it can also be used in the proof of Theorem 2.

To avoid confusion, the theorems from Section 3 are restated when we come to prove them. It will be convenient in the proofs to have fixed some additional terminology, which is done next.

## 4.1 Continuous correspondences

Let $X$, $Y$ be topological spaces. A correspondence $\theta\colon X \to 2^Y$, i.e., set-valued map from $X$ to $Y$, is called continuous if for each $x \in X$ and each open set $O \subseteq Y$ the following are true: (a) whenever $\theta(x) \subseteq O$, then there is a neighborhood $V$ of $x$ such that $\theta(x') \subseteq O$ for all $x' \in V$, and (b) whenever $\theta(x) \cap O \neq \emptyset$, then there is a neighborhood $V$ of $x$ such that $\theta(x') \cap O \neq \emptyset$ for all $x' \in V$. Note that if $X$, $Y$ are metric spaces and $\theta(x)$ is non-empty and compact for each $x \in X$, then this definition is equivalent to saying that whenever $x \in X$ and $x_k \in X$, $k \in \mathbb{N}$, are such that $x_k \to x$, then $\rho_H(\theta(x), \theta(x_k)) \to 0$ (see Hildenbrand, 1974, B.III, Problem 4).

## 4.2 Preliminaries

**Lemma 1.** *There is unique metrizable topology on $\mathcal{U}$ (with which $\mathcal{U}$ is regarded as being endowed) such that a sequence $\langle u_k \rangle$ in $\mathcal{U}$ converges to some $u \in \mathcal{U}$ if and only if*

*(a)* $\rho_H(\operatorname{dom} u, \operatorname{dom} u_k) \to 0$ *and* $\rho_H(\partial A_u, \partial A_{u_k}) \to 0$*;*

*(b) if* $(a, e) \in \operatorname{dom} u$ *and* $(a_k, e_k) \in \operatorname{dom} u_k$, $k \in \mathbb{N}$, *are such that* $(a_k, e_k) \to (a, e)$, *then* $u_k(a_k, e_k) \to u(a, e)$, $Du_k(a_k, e_k) \to Du(e, a)$, *and* $D^2 u_k(a_k, e_k) \to D^2 u(e, a)$.

*Proof.* It suffices to find one metric $\rho$ on $\mathcal{U}$ for which convergence of any sequence in $\mathcal{U}$ is equivalent to the truth of (a) and (b). Write $\Gamma_f$ for the graph of a function $f$ and define $\rho$ by setting

$$\begin{aligned}\rho(u, u') = {} & \rho_H(\operatorname{dom} u, \operatorname{dom} u') + \rho_H(\partial A_u, \partial A_{u'}) \\ & + \rho_H(\Gamma_u, \Gamma_{u'}) + \rho_H(\Gamma_{Du}, \Gamma_{Du'}) + \rho_H(\Gamma_{D^2 u}, \Gamma_{D^2 u'})\end{aligned}$$

for $u$, $u' \in \mathcal{U}$. Clearly $\rho$ is a metric on $\mathcal{U}$. Let $u \in \mathcal{U}$ and $\langle u_k \rangle$ a sequence in $\mathcal{U}$.

Assume $\rho(u, u_k) \to 0$. Directly from the definition of $\rho$ we see that (a) must be true. As for (b), let $(a, e)$ and $(a_k, e_k)$, $k \in \mathbb{N}$, be as hypothesized. Note that $\rho_H(\Gamma_u, \Gamma_{u_k}) \to 0$ because $\rho(u, u_k) \to 0$. Thus, by the definition of $\rho_H$, the sequence $\langle (a_k, e_k, u_k(a_k, e_k)) \rangle$ must be bounded and any of its cluster points must belong to $\Gamma_u$ and therefore (by the definition of graph) must be of the form $(a, e, u(a, e))$ since $(a_k, e_k) \to (a, e)$. Thus the assertion of (b) follows (because, in a Euclidean space, a bounded sequence has a cluster point, $x$ say, and is convergent to $x$ if $x$ is the only cluster point). Similarly, the other assertions of (b) follow.

Assume that (a) and (b) are true. Combining the first of the facts in (b with the fact that $\rho_H(\operatorname{dom} u, \operatorname{dom} u_k) \to 0$, we see that $\Gamma_u \subseteq \operatorname{Li} \Gamma_{u_k}$.[14] Suppose $(a, e, r) \in \operatorname{Ls} \Gamma_{u_k}$. Then for some sequence $\langle n_i \rangle_{i \in \mathbb{N}}$ in $\mathbb{N}$ there are points $(a_{k_i}, e_{k_i}) \in \operatorname{dom} u_{k_i}$, $i \in \mathbb{N}$, such that $(a_{k_i}, e_{k_i}, u_{k_i}(a_{k_i}, e_{k_i})) \to (a, e, r)$. From the fact that $\rho_H(\operatorname{dom} u, \operatorname{dom} u_k) \to 0$ we see that $(a, e) \in \operatorname{dom} u$. Again by this fact, there is a sequence $\langle (a_k, e_k) \rangle_{k \in \mathbb{N}}$ such that $(a_k, e_k) \to (a, e)$ and $(a_k, e_k) \in \operatorname{dom} u_k$ for each $k$. Define a sequence $\langle (a'_k, e'_k) \rangle_{k \in \mathbb{N}}$ by setting $(a'_k, e'_k) = (a_{k_i}, e_{k_i})$ if $k = k_i$ for some $i$, and $(a'_k, e'_k) = (a_k, e_k)$ otherwise. Then $(a'_k, e'_k) \in \operatorname{dom} u_k$ for each $k$ and $(a'_k, e'_k) \to (a, e)$. Consequently (b) implies that $u_k(a'_k, e'_k) \to u(a, e)$. In particular, we have $u_{k_i}(u_{k_i}, e_{k_i}) \to u(a, e)$ and therefore $r = u(a, e)$. Thus $\operatorname{Ls} \Gamma_{u_k} \subseteq \Gamma_u$ and it follows that $\Gamma_u = \operatorname{Ls} \Gamma_{u_k} = \operatorname{Li} \Gamma_{u_k}$. Now because $\operatorname{dom} u$ and $\operatorname{dom} u_k$, $k \in \mathbb{N}$, are all included in the compact set $A \times E$, and because the maps $u$ and $u_k$ are continuous, (a) and (b) imply, in particular, that the sets $\Gamma_u$ and $\Gamma_{u_k}$, $k \in \mathbb{N}$, are commonly included in a compact subset of the ambient Euclidean space, so the fact that $\Gamma_u = \operatorname{Ls} \Gamma_{u_k} = \operatorname{Li} \Gamma_{u_k}$ implies that $\rho_H(\Gamma_u, \Gamma_{u_k}) \to 0$.

Similarly, we see that both $\rho_H\big(\Gamma_{Du}, \Gamma_{Du_k}\big)$ and $\rho_H\big(\Gamma_{D^2u}, \Gamma_{D^2u_k}\big)$ converge to 0 as $k \to \infty$. By the definition of $\rho$, we conclude that $\rho(u, u_k) \to 0$. □

**Lemma 2.** *$\mathcal{U}$ is separable.*

*Proof.* Let $\mathcal{F}_0$ be the set of all non-empty compact subsets of $\mathbb{R}^{n+m}$, $\mathcal{F}_1$ the set of all non-empty compact subsets of $\mathbb{R}^{n+m} \times \mathbb{R}$, $\mathcal{F}_2$ the set of all non-empty compact subsets of $\mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$, and $\mathcal{F}_3$ the set of all non-empty compact subsets of $\mathbb{R}^{n+m} \times \mathbb{R}^{(n+m)^2}$. For each $i = 0, 1, 2, 3$, give $\mathcal{F}_i$ the Hausdorff metric topology, so that each $\mathcal{F}_i$ becomes a separable metric space. Write $\mathcal{F} = \mathcal{F}_0 \times \mathcal{F}_0 \times \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{F}_3$ and give $\mathcal{F}$ the product topology. Then $\mathcal{F}$ is a separable metrizable topological space. Consider the map $\phi\colon \mathcal{U} \to \mathcal{F}$ defined by setting

$$\phi(u) = \big(\operatorname{dom} u, \partial A_u, \Gamma_u, \Gamma_{Du}, \Gamma_{D^2u}\big)$$

for each $u \in \mathcal{U}$. By definition of the topology of $\mathcal{U}$, $\phi$ is a homeomorphism from $\mathcal{U}$ onto $\phi(\mathcal{U})$. As $\mathcal{F}$ is separable and metrizable, any subset of $\mathcal{F}$ is separable (in the subspace topology). In particular $\phi(\mathcal{U})$ is separable, and it follows that $\mathcal{U}$ is separable. □

[14] Here an below, $\operatorname{Li} \Gamma_{u_k}$ is the set of limits of sequences $\langle (a_k, e_k, r_k) \rangle$ such that $(a_k, e_k, r_k) \in \Gamma_{u_k}$ for all $k$, and $\operatorname{Ls} \Gamma_{u_k}$ the set of cluster points of such sequences.

**Lemma 3.** *Let $C$ and $C_k$, $k \in \mathbb{N}$, be compact subsets of $\mathbb{R}^\ell$, all with non-empty interior, such that both $\rho_H(C, C_k) \to 0$ and $\rho_H(\partial C, \partial C_k) \to 0$ as $k \to \infty$. Let $x \in \operatorname{int} C$, and suppose $\langle x_k \rangle$ is a sequence in $\mathbb{R}^\ell$ such that $x_k \to x$. Then $x_k \in \operatorname{int} C_k$ for all sufficiently large $k$.*

*Proof.* Otherwise, passing to a subsequence, if necessary, we can assume $x_k \notin \operatorname{int} C_k$ for each $k$. As $\rho_H(C, C_k) \to 0$, we can find a $y_k \in C_k$ for each $k$ so that $y_k \to x$. Now using the fact that the $C_k$'s are closed, we can select a $\lambda_k \in [0, 1]$ for each $k$ so that $z_k = (1 - \lambda_k)y_k + \lambda_k x_k \in \partial C_k$. As $x_k \to x$ as well as $y_k \to x$, we also have $z_k \to x$. As $\rho_H(\partial C, \partial C_k) \to 0$, it follows that $x \in \partial C$, contradicting the hypothesis about $x$. □

**Lemma 4.** *Let $C$ and $C_k$, $k \in \mathbb{N}$, be compact subsets of $\mathbb{R}^\ell$, all with non-empty interior, such that both $\rho_H(C, C_k) \to 0$ and $\rho_H(\partial C, \partial C_k) \to 0$ as $k \to \infty$. Let $K$ be a compact subset of $\operatorname{int} C$. Then $K \subseteq \operatorname{int} C_k$ for all sufficiently large $k$.*

*Proof.* Otherwise, passing to a subsequence, if necessary, for each $k$ we can find an $x_k \in K$ such that $x_k \notin \operatorname{int} C_k$. As $K$ is compact, we can assume that $x_k \to x$, again passing to a subsequence, if necessary. Now $x \in K \subseteq \operatorname{int} C$, so by Lemma 3 we must have $x_k \in \operatorname{int} C_k$ for large $k$, thus getting a contradiction. □

**Lemma 5.** *(a) For every $\nu \in \mathcal{M}$, $E(\nu)$ is a compact convex subset of $E$ with non-empty interior in $\mathbb{R}^m$. (b) If $\nu_n \to \nu$ in $\mathcal{M}$, then $\rho_H(E(\nu_n), E(\nu)) \to 0$.*

*Proof.* Clearly $E(\nu)$ is convex for each $\nu \in \mathcal{M}$. As for compactness, fix $\nu \in \mathcal{M}$ and set $Y = \operatorname{supp}(\nu)$. By the definition of $\mathcal{M}$, $Y$ is compact, and by hypothesis, so is the actions universe $A$. Thus the set $Z$ of all Borel probability measures on $Y \times A$ is narrowly compact. Evidently the set of those Borel probability measures which matter in the definition of $E(\nu)$ can be regarded as a narrowly closed subset of $Z$, and thus $E(\nu)$ must be compact, because $g$ is continuous.

For the other claims, consider the correspondence $\theta\colon \mathcal{U} \to 2^{\mathbb{R}^m}$ defined by setting

$$\theta(u) = \operatorname{co} g(A_u)$$

for each $u \in \mathcal{U}$. Then $\theta$ has non-empty compact convex values, all with non-empty interior by Lemma 10(b). The fact that $\theta$ has convex values implies that $\int \theta \, d\nu$ is convex for all $\nu \in \mathcal{M}$, and the fact that $\theta$ has compact values, all included in the compact set $\operatorname{co} g(A)$, implies that $\int \theta \, d\nu$ is compact for all $\nu \in \mathcal{M}$ (see Hildenbrand, 1974, D.II.4, Proposition 7). Note that the correspondence $u \mapsto A_u\colon \mathcal{U} \to 2^A$ is continuous. Because the map $g$ is continuous, this implies that the correspondence $u \mapsto g(A_u)\colon \mathcal{U} \to 2^{\mathbb{R}^m}$ is continuous. By Hildenbrand (1974, B.III, Propositions 6 and 10), it follows that $\theta$ is continuous.

We claim that $E(\nu) = \int \theta(u) \, d\nu(u)$ for each $\nu \in \mathcal{M}$. To see this, fix any $\nu \in \mathcal{M}$ and any $p \in \mathbb{R}^m$. Note that the map $p \circ g$ from $A$ to $\mathbb{R}$ is continuous. Consequently, since the correspondence $u \mapsto A_u$ is continuous, with non-empty compact values, it has a measurable selection $h$ such that $(p \circ g)h(u) = \max(p \circ g)A_u$ for each $u \in \mathcal{U}$

(use the maximum theorem together with Hildenbrand, 1974, B.III, Proposition 1 and D.II.2, Lemma 1). We must therefore have $\max pE(\nu) = \int_{\mathcal{U}} \max pg(A_u)\, d\nu(u)$, by the definition of $E(\nu)$, and also

$$\int_{\mathcal{U}} \max pg(A_u)\, d\nu(u) = \int_{\mathcal{U}} \max p \operatorname{co} g(A_u)\, d\nu(u) = \max p \int_{\mathcal{U}} \theta(u)\, d\nu(u)\,.$$

As $p$ is an arbitrary element of $\mathbb{R}^m$, and both $E(\nu)$ and $\int_{\mathcal{U}} \theta(u)\, d\nu(u)$ are compact and convex, it follows that $E(\nu) = \int_{\mathcal{U}} \theta(u)\, d\nu(u)$, as claimed.

Now from this equality we can see that $\operatorname{int} E(\nu) \neq \emptyset$ for each $\nu \in \mathcal{M}$. Indeed, pick any $\nu \in \mathcal{M}$ and any $u' \in \operatorname{supp}(\nu)$. By what has been noted above, $\operatorname{int} \theta(u') \neq \emptyset$, so there is a compact set $K \subseteq \operatorname{int} \theta(u')$ such that $\operatorname{int} K \neq \emptyset$. By Lemma 4, there is an open neighborhood $V$ of $u'$ such that $K \subseteq \theta(u'')$ for each $u'' \in V$. As $u' \in \operatorname{supp}(\nu)$, $\nu(V) > 0$, so the set $\nu(V)K$ has non-empty interior. Now

$$\nu(V)K + \int_{\mathcal{U}\setminus V} \theta(u)\, d\nu(u) \subseteq \int_{V} \theta(u)\, d\nu(u) + \int_{\mathcal{U}\setminus V} \theta(u)\, d\nu(u) = \int_{\mathcal{U}} \theta(u)\, d\nu(u)\,,$$

showing that $\operatorname{int} \int_{\mathcal{U}} \theta(u)\, d\nu(u) \neq \emptyset$. Finally, to see that (b) of the lemma is true, note that since $\theta$ is continuous, with non-empty compact values, for each $p \in \mathbb{R}^m$ the map $u \mapsto \max p\theta(u) \colon \mathcal{U} \to \mathbb{R}$ is continuous, by the maximum theorem. Moreover, this map is bounded because the values of $\theta$ are included in the compact set $E \subseteq \mathbb{R}^m$. Hence, for each $p \in \mathbb{R}^m$, the map $\nu \mapsto \int_{\mathcal{U}} \max p\theta(u)\, d\nu(u) : \mathcal{M} \to \mathbb{R}$ is continuous. By the facts used above, we see that $\theta$ has a measurable selection $h$ such that $h(u) = \max p\theta(u)$ for each $u \in \mathcal{U}$, implying that $\int_{\mathcal{U}} \max p\theta(u)\, d\nu(u) = \max p \int_{\mathcal{U}} \theta(u)\, d\nu(u)$, and it follows that for each $p \in \mathbb{R}^m$ the map $\nu \mapsto \max p \int_{\mathcal{U}} \theta(u)\, d\nu(u) : \mathcal{M} \to \mathbb{R}$ is continuous. Because $\int_{\mathcal{U}} \theta(u)\, d\nu(u)$ is non-empty convex and compact for each $\nu \in \mathcal{M}$, it follows from this that the map $\nu \mapsto \int_{\mathcal{U}} \theta(u)\, d\nu(u)$ is continuous for the Hausdorff metric on the set of all non-empty compact subsets of $\mathbb{R}^m$ (see Castaing and Valadier, 1977, II-23). Thus we get (b), again by the equality $E(\nu) = \int_{\mathcal{U}} \theta(u)\, d\nu(u)$ established above. □

**Lemma 6.** *Let $\nu \in \mathcal{G}$ and let $\langle (I_n, G_n) \rangle_{n\in\mathbb{N}}$ be a sequence of finite-player games such that $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$ in $\mathcal{M}$. Let $W$ be a compact subset of* $\operatorname{int} E(\nu)$*. Then there is an $N \in \mathbb{N}$ such that $W \subseteq E_{G_n(i)}$ for each $i \in I_n$ whenever $n \geq N$.*

*Proof.* In this proof, we write $E_{n,i}$ for $E_{G_n(i)}$, and $l_n$ for $\#(I_n) - 1$.

**(a)** There are numbers $N_1 \in \mathbb{N}$ and $\epsilon > 0$ such that whenever $n \geq N_1$ and $i \in I_n$, then $W \subseteq \operatorname{co}E_{n,i}$ but $\operatorname{dist}(e, \partial \operatorname{co}E_{n,i}) \geq \epsilon$ for each $e \in W$. To see this, for each $n$ and each $i \in I_n$ define $\nu_{n,i} \in \mathcal{M}$ by setting $\nu_{n,i} = (1/l_n) \sum_{j\in I_n\setminus\{i\}} \delta_{G_n(j)}$. Note that $E(\nu_{n,i}) = \operatorname{co}E_{n,i}$ for each $n$ and each $i \in I_n$. Consider any sequence $\langle i_n \rangle$ with $i_n \in I_n$ for each $n$. The hypothesis that $\nu_{G_n} \to \nu$ in $\mathcal{M}$ implies that $\nu_{n,i_n} \to \nu$ in $\mathcal{M}$. By Lemma 5(b), it follows that $\operatorname{co}E_{n,i_n} \to E(\nu)$, and therefore, by Lemma 5(a) and Lemma 4, that $W \subseteq \operatorname{int} \operatorname{co}E_{n,i}$ if $n$ is sufficiently large. Thus, would the claim be wrong, there would be a subsequence $\langle I_{n_k} \rangle$ of the sequence $\langle I_n \rangle$ such that $\operatorname{dist}(e_k, \partial \operatorname{co}E_{n_k,i_k}) \to 0$ for some points $i_k \in I_{n_k}$ and $e_k \in W$, $k \in \mathbb{N}$. Now because all the sets $E(\nu)$ and $\mathrm{co}E_{n_k,i_k}$, $k \in \mathbb{N}$, are compact, convex, and have non-empty interior, it follows from the material in Wills (2007) that $\rho_H(\partial E(\nu), \partial \mathrm{co}E_{n_k,i_k})) \to 0$, and therefore that $\mathrm{dist}(e_k, \partial E(\nu)) \to 0$. Because $W$ is compact we can assume that $e_k \to e$ for some $e \in W$. But then, because $\partial E(\nu)$ is closed, we must have $e \in W \cap \partial E(\nu)$, contradicting the hypothesis that $W \subseteq \mathrm{int}\, E(\nu)$.

**(b)** There are numbers $\eta > 0$ and $N_2$, with $N_2 \geq m$ ($m$ being the dimension of the ambient Euclidean space of the externalities universe), such that if $l_n \geq N_2$, then for each $i \in I_n$ and each $J \subseteq I_n \backslash \{i\}$ with $\#(J) = m$ the set $\sum_{j \in J} g(A_{G_n(j)})$ includes a ball of radius $\eta$. To see this, consider the correspondence $\theta\colon \mathrm{supp}(\nu) \to 2^A$, given by setting $\theta(u) = A_u$ for each $u \in \mathrm{supp}(\nu)$. Then $\theta$ is continuous, with non-empty compact values, all with non-empty interior. Moreover, $\rho_H(\partial A_u, \partial A_{u_k}) \to 0$ whenever $u_k \to u$ in $\mathrm{supp}(\nu)$. Using Lemma 4 and the fact that $\mathrm{supp}(\nu)$ is compact, we see from these properties that there is a non-empty finite set $\mathcal{C}$ of compact balls in $A$ (all with radius $> 0$), such that for each $u \in \mathrm{supp}(\nu)$, $\mathrm{int}\, A_u$ includes some member of $\mathcal{C}$. As $\rho_H(\mathrm{supp}\, \nu, \mathrm{supp}(\nu_{\mathcal{G}_n})) \to 0$, another invocation of Lemma 4 and the fact that $\mathrm{supp}(\nu)$ is compact now shows that there is an $N_2' \in \mathbb{N}$ such that whenever $n \geq N_2'$, then for each $i \in I_n$, $A_{G_n(i)}$ includes some member of $\mathcal{C}$. Let $\mathcal{F}$ be the set of all families $F = (B_{F,1}, \ldots, B_{F,m})$ where $B_{F,i} = C$ for some $C \in \mathcal{C}$, $i = 1, \ldots, m$. Then $\mathcal{F}$ is finite, and by Lemma 10 in the appendix, for each $F \in \mathcal{F}$, $\sum_{i=1}^m g(B_{F,i})$ includes a ball of radius $\eta_F > 0$. Set $\eta = \min\{\eta_F \colon F \in \mathcal{F}\}$. As $\mathcal{F}$ is finite, $\eta > 0$. Thus, setting $N_2 = \max\{N_2', m\}$, the claim follows.

**(c)** Because the actions universe is compact and $g$ is continuous, there is number $\delta' > 0$ such that $\mathrm{diam}(g(A_{G_n(i)})) \leq \delta'$ for all $i \in I_n$ and $n$. Set $\delta_0 = m\delta'$, so that we have $\mathrm{diam}(g(A_{G_n(i)})) \leq \delta_0$ for all $i \in I_n$ and $n$ as well as $\mathrm{diam}(\sum_{j \in J} g(A_{G_n(j)})) \leq \delta_0$ if $J$ is as above. Choose a number $h \in \mathbb{N}$ with $h\eta \geq m\delta_0$. Note that for all $n$ and $i \in I_n$,

$$l_n E_{n,i} = \sum_{j \in I_n \backslash \{i\}} g(A_{G_n(j)}),$$

and that if $l_n \geq hm$, the latter sum can be written in the form

$$\sum_{j \in J_1} g(A_{G_n(j)}) + \cdots + \sum_{j \in J_h} g(A_{G_n(j)}) + \sum_{j \notin \bigcup_{k=1}^h J_k} g(A_{G_n(j)}),$$

where the $J_k$'s, $k = 1, \ldots, h$, are pairwise disjoint and $\#(J_k) = m$ for all $k = 1, \ldots, h$. By Howe (1979, Proposition 2), it follows that if $n \geq N_2$ is such that $l_n \geq hm$, and $z \in \mathrm{co}(l_n E_{n,i})$ is such that $\mathrm{dist}(z, \partial \mathrm{co}(l_n E_{n,i})) \geq h\delta_0$, then $z \in l_n E_{n,i}$.[15]

Note that (a) implies that whenever $n \geq N_1$ and $i \in I_n$, then $l_n W \subseteq \mathrm{co}(l_n E_{n,i})$ and for each $e \in l_n W$ we have $\mathrm{dist}(e, \partial \mathrm{co}(l_n E_{n,i})) \geq l_n \epsilon$. By the previous paragraph, it follows that if $n \geq \max\{N_1, N_2\}$ is such that both $l_n \geq hm$ and $l_n \epsilon \geq h\delta_0$, then $l_n W \subseteq l_n E_{n,i}$ for each $i \in I_n$, and thus $W \subseteq E_{n,i}$ for each $i \in I_n$. As $n \to \infty$ implies $l_n \to \infty$, this establishes the lemma. □

[15] Note that the $h$ here corresponds to the $m$ in Howe (1979, Proposition 2), while $\dim V$ and $\nu$ there are what is called $m$ and $\eta$ here respectively.

**Lemma 7.** *Given $\nu \in \mathcal{G}$, there is a sequence $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ of finite-player games such that $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$ in $\mathcal{M}$.*

*Proof.* **(a)** Let $\nu \in \mathcal{G}$ be given. Write $X = \operatorname{supp}(\nu)$. By the law of large numbers (Glivenko-Cantelli version) there is a sequence $\langle u_n \rangle$ in $X$ such that the sequence $\langle \nu_n \rangle$, defined by setting $\nu_n = 1/(n+1) \sum_{i=0}^{n} \delta_{u_i}$ for each $n \in \mathbb{N}$, converges to $\nu$ narrowly. Since $\operatorname{supp}(\nu_n) \subseteq X = \operatorname{supp}(\nu)$, narrow convergence of $\langle \nu_n \rangle$ to $\nu$ implies that we also have $\rho_H(\operatorname{supp}(\nu_n), \operatorname{supp}(\nu)) \to 0$. Thus we have $\nu_n \to \nu$ in the topology of $\mathcal{M}$. For each $n \in \mathbb{N} \backslash \{0\}$ and each $0 \le i \le n$ define $\nu_{n,i} \in \mathcal{M}$ by setting $\nu_{n,i} = 1/n \sum_{j \in J_{n,i}} \delta_{u_j}$ where $J_{n,i} = \{0, \dots, n\} \backslash \{i\}$; set $\nu_{0,0} = \nu_0$. Note that for each $n \in \mathbb{N} \backslash \{0\}$ and each $0 \le i \le n$ we have $\|\nu_n - \nu_{n,i}\|_V \le 2/n$, writing $\|\cdot\|_V$ for the variation norm on $\mathcal{M}$. Consequently, because $\nu_n \to \nu$, we have $\nu_{n,i_n} \to \nu$ as $n \to \infty$ whenever $\langle i_n \rangle$ is a sequence in $\mathbb{N}$ with $0 \le i_n \le n$ for each $n$. By Lemma 5, it follows that $\rho_H(E(\nu), E(\nu_{n,i_n})) \to 0$ whenever $\langle i_n \rangle$ is as in the previous sentence.

Fix $b \in \operatorname{int} E(\nu)$. By Lemma 3 it follows from the conclusion of the previous paragraph that $b \in \operatorname{int} E(\nu_{n,i})$ for each $0 \le i \le n$ if $n$ is large enough; we can assume that this is true for all $n$. Now, for each $n$, we can define $r_n > 0$ to be the largest real number $r \le 1$ such that $r\big(E(\nu_{n,i}) - \{b\}\big) + \{b\} \subseteq E(\nu)$ for each $0 \le i \le n$. We must have $r_n \to 1$. To see this, fix $0 < r < 1$. Since $E(\nu)$ is convex and $b \in \operatorname{int} E(\nu)$, we have $r\big(E(\nu) - \{b\}\big) + \{b\} \subseteq \operatorname{int} E(\nu)$. Using the fact that $\rho_H(E(\nu), E(\nu_{n,i_n})) \to 0$ whenever $\langle i_n \rangle$ is a sequence in $\mathbb{N}$ with $0 \le i_n \le n$ for each $n$, it follows that if $n$ is large, then $r\big(E(\nu_{n,i}) - \{b\}\big) + \{b\} \subseteq \operatorname{int} E(\nu)$ for all $0 \le i \le n$. Thus $r_n \ge r$ for such $n$. As $0 < r < 1$ is arbitrary, we conclude that $r_n \to 1$.

**(b)** For each $n$ and each $i = 0, \dots, n$, define a map $\tilde{u}_{n,i} \colon A_{u_i} \times E(\nu_{n,i_n}) \to \mathbb{R}$ by setting $\tilde{u}_{n,i}(a, e) = u_i(a, r_n(e - b) + b)$ for $(a, e) \in A_{u_i} \times E(\nu_{n,i})$ and note that $\tilde{u}_{n,i} \in \mathcal{U}$. We claim that for any $\epsilon > 0$ there is an $n_\epsilon$ such that whenever $n > n_\epsilon$, then $\rho(\tilde{u}_{n,i}, u_i) < \epsilon$ for all $i = 0, \dots, n$ (where $\rho$ is the metric on $\mathcal{U}$ chosen in Section 3.4). Indeed, otherwise there are points $\tilde{u}_{n_k,i_k}$ and $u_{i_k}$, $k \in \mathbb{N}$, such that $n_k \to \infty$ as $k \to \infty$ and $\rho(\tilde{u}_{n_k,i_k}, u_{i_k}) \ge \epsilon > 0$ for each $k$. Because $u_{i_k} \in \operatorname{supp}(\nu)$ and $\operatorname{supp}(\nu)$ is compact, we can assume that $u_{i_k} \to \bar{u}$ for some $\bar{u} \in \operatorname{supp}(\nu)$. But then, using Lemma 1, together with the facts that $\rho_H(E(\nu), E(\nu_{n_k,i_{n_k}})) \to 0$ and $r_{n_k} \to 1$, it follows that also $\tilde{u}_{n_k,i_k} \to \bar{u}$, and we get a contradiction.

**(c)** For each $n \in \mathbb{N} \backslash \{0\}$ and each $0 \le i \le n$, set $E_{n,i} = 1/n \sum_{j \in J_{n,i}} g(A_j)$. Note that $E(\nu_{n,i}) = 1/n \sum_{j \in J_{n,i}} \operatorname{co} g(A_j)$ (cf. the proof of Lemma 5); thus $E_{n,i} \subseteq E(\nu_{n,i})$. Let $u_{n,i} \colon A_{u_{n,i}} \times E_{n,i} \to \mathbb{R}$ be the restriction of $\tilde{u}_{n,i}$ to $A_{u_{n,i}} \times E_{n,i}$. By Lemma 10(a) and the hypothesis on $g$ made in Section 3.2, there is an $\bar{n} \in \mathbb{N}$ such that if $n \ge \bar{n}$ then for all $0 \le i \le n$, $\operatorname{int} E_{n,i}$ is dense in $E_{n,i}$ and thus $u_{n,i} \in \mathcal{U}$. Because all the sets $E_{n,i}$ are included in the compact convex externalities universe $E$, it follows from the Shapley-Folkman theorem that for each $\epsilon > 0$ there is a $n'_\epsilon \in \mathbb{N}$ such that $\rho_H(E_{n,i}, E(\nu_{n,i})) < \epsilon$ for all $0 \le i \le n$ if $n \ge n'_\epsilon$. Using Lemma 1, it follows from this and (b) that for each $\epsilon > 0$ there is an $n''_\epsilon \in \mathbb{N}$ such that $\rho(u_{n,i}, u_i) < \epsilon$ for all $0 \le i \le n$ whenever $n \ge \max\{\bar{n}, n''_\epsilon\}$, because $u_{n,i}$ is just the restriction of $\tilde{u}_{n,i}$ to $A_{u_{n,i}} \times E_{n,i}$.

Now, for each $n \in \mathbb{N}$ with $n \ge \bar{n}$, set $I_n = \{0, 1, \dots, n\}$ and define $G_n \colon I_n \to \mathcal{U}$ by setting $G(i) = u_{n,i}$ for each $i \in I_n$. For $n < \bar{n}$, let $(I_n, G_n)$ be an arbitrary finite-player game. By (a), $\nu_n = 1/(n+1)\sum_{i=0}^{n} \delta_{u_i} \to \nu$ narrowly and $\rho_H(\text{supp}(\nu_n), \text{supp}(\nu)) \to 0$, so from the end of the previous paragraph we see that $\rho_H(\text{supp}(\nu_{G_n}), \text{supp}(\nu)) \to 0$ and that $\int h\, d\nu_{G_n} \to \int h\, d\nu$ whenever $h\colon \mathcal{U} \to \mathbb{R}$ is a bounded uniformly continuous function. By Billingsley (1968, Theorem 2.1), the latter fact implies that $\nu_{G_n} \to \nu$ narrowly. We conclude that $\nu_{G_n} \to \nu$ in the topology of $\mathcal{M}$. □

**Remark 6.** Straightforward adaptations of the arguments of the proof of Lemma 7 show that the set of those elements of $\mathcal{G}$ which have finite support is dense in $\mathcal{G}$: just argue with $E(\nu_n)$ instead of the sets $E(\nu_{n,i})$, and observe that several details become unnecessary; in particular, the first paragraph in (c) can be dropped.

**Remark 7.** Inspecting the proof of Lemma 7 shows that, given any $\nu \in \mathcal{G}$, the sequence $\langle (I_n, G_n) \rangle_{n\in\mathbb{N}}$ as guaranteed by Lemma 7 can be chosen so that for each $n$, $u \in \text{supp}(\nu_{\mathcal{G}_n})$ implies $A_u = A_{u'}$ for some $u' \in \text{supp}(\nu)$. This observation is useful in regard to the proof of Theorem 4.

**Lemma 8.** *The set of $\nu \in \mathcal{G}$ with ordinally non-equivalent payoff functions is dense in $\mathcal{G}$.*

*Proof.* By what was noted in Remark 6, it suffices to show that if $\nu \in \mathcal{G}$ is of the form $\nu = \sum_{i=1}^{m} \alpha_i \delta_{u_i}$, then there is a sequence $\nu_n \in \mathcal{G}$ such that $\nu_n \to \nu$ and each $\nu_n$ has ordinally non-equivalent payoff functions. Let such a $\nu$ be given.

Fix any $i \in \{1, \ldots, m\}$. By (U3) in the assumptions on payoff functions there is a compact set $K_i \subseteq \text{int}\, A_{u_i}$ such that $\varphi(u_i, e) \subseteq \text{int}\, K_i$ for each $e \in E(\nu)$. Let $\tilde{u}_i$ be an extension of $u_i$ to a twice continuously differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}$. Let $f$ be the vector $f = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Choose a sequence $\langle c_{i,n} \rangle_{n\in\mathbb{N}}$ of real numbers such that $c_{i,n} \to 0$ and for each $n$, $c_{i,n} > 0$ but small enough so that, for each $t \in [0, 1]$, the function $u_{i,n,t}\colon A_{u_i} \times E(\nu) \to \mathbb{R}$, defined by setting

$$u_{i,n,t}(a, e) = \tilde{u}_i(a + (c_{i,n}t)f, e),$$

satisfies (U3) in the assumptions on payoff functions, and therefore belongs to $\mathcal{U}$.

Using Lemma 1 we see that the map $t \mapsto u_{i,n,t}$ from $[0, 1]$ to $\mathcal{U}$ is continuous for each $n$. Thus we can speak of the distribution of Lebesgue measure on $[0, 1]$ under this map; moreover, writing $\nu_{i,n}$ for this distribution, $\text{supp}(\nu_{i,n})$ is compact. Again using Lemma 1, we see that $u_{i,n,t} \to u_i$ in $\mathcal{U}$ for each $t \in [0, 1]$ as $n \to \infty$, so $\nu_{i,n} \to \delta_{u_i}$ narrowly for each $t \in [0, 1]$. In fact, $u_{i,n,t_n} \to u_i$ whenever $\langle t_n \rangle$ is a sequence in $[0, 1]$, from which we see that $\rho_H(\text{supp}(\nu_i), \text{supp}(\nu_{i,n})) \to 0$. Thus $\nu_{i,n} \to \delta_{u_i}$ in the topology of $\mathcal{M}$. Because $\varphi(u_i, e) \subseteq \text{int}\, K_i$ for each $e \in E(\nu)$, for large $n$ the following inclusions must be true for each $e \in E(\nu)$ and each $t \in [0, 1]$: $\varphi(u_i, e) \pm \{(c_{i,n}t)f\} \subseteq \text{int}\, K_i$, $\varphi(u_{i,n,t}, e) \subseteq \text{int}\, K_i$, and $\varphi(u_{i,n,t}, e) \pm \{(c_{i,n}t)f\} \subseteq \text{int}\, K_i$. Consequently, for large $n$, we must have $\varphi(u_{i,n,t}, e) = \varphi(u_i, e) - \{(c_{i,n}t)f\}$ for each $e \in E(\nu)$ and each $t \in [0, 1]$. We may assume that this is true actually for every $n$.

Do this construction for each $i = 1, \ldots, m$. Set $\nu_n = \sum_{i=1}^{m} \alpha_i \nu_{n,i}$ for each $n$. Then $E(\nu_n) = E(\nu)$ for each $n$ and thus $\nu_n \in \mathcal{G}$ for each $n$. Moreover, $\nu_n \to \nu$. By what was noted above, we have $\varphi(u_{i,n,t}, e) = \varphi(u_i, e) - \{(c_{i,n}t)f\}$ for each $e \in E(\nu)$, each $n \in \mathbb{N}$, each $i = 1, \ldots, m$, and each $t \in [0, 1]$. Because the map $t \mapsto c_{i,n}t$ is an injection for each $i$ and each $n$, and because $\varphi(u, e) = \varphi(u', e)$ for two ordinally equivalent payoff functions $u$ and $u'$ in any of the continuum games $\nu_n$, it follows that the sequence $\langle \nu_n \rangle$ is as desired. □

## 4.3 Proof of Theorem 1

**Theorem 1.** *There is an open and dense subset $\mathcal{G}^*$ of $\mathcal{G}$ such that whenever $\nu \in \mathcal{G}^*$ and $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ is a sequence of finite-player games such that $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$, then there is an $N \in \mathbb{N}$ such that $(I_n, G_n)$ has a strict pure strategy Nash equilibrium if $\#(I_n) \geq N$.*

*Proof.* **(a)** As noted in Section 3.6, every continuum game $\nu \in \mathcal{G}$ has an equilibrium distribution, i.e., there is a Borel probability measure $\tau$ on $\mathcal{U} \times A$ such that $\tau_{\mathcal{U}} = \nu$ and $\operatorname{supp}(\tau) \subseteq \{(u, a) \in \mathcal{U} \times A \colon a \in \varphi(u, e(\tau_A))\}$. By (U3) in the assumptions on payoff functions, we have $\varphi(u, e) \subseteq \operatorname{int} A_u$ for each $u \in \mathcal{U}$ and each $e \in E(\nu)$. Thus if $\tau$ is an equilibrium distribution of $\nu \in \mathcal{G}$, then $a \in \operatorname{int} A_u$ for each $(u, a) \in \operatorname{supp}(\tau)$.

**(b)** Write $\mathcal{G}_1$ for the subset of $\mathcal{G}$ consisting of those $\nu$ such that for some equilibrium distribution $\tau$ of $\nu$,

(i) $\#(\varphi(u, e(\tau_A))) = 1$ for each $u \in \operatorname{supp}(\nu)$;

(ii) for each $u \in \operatorname{supp}(\nu)$, $D_a^2 u(a_u, e(\tau_A))$ is negative definite, where $a_u$ is the unique element of $\varphi(u, e(\tau_A))$ (note that $a_u \in \operatorname{int} A_u$, so $D_a^2 u(a_u, e(\tau_A))$ is defined);

(iii) $e(\tau_A) \in \operatorname{int} E(\nu)$.

Write $\mathcal{U}_c = \{u \in \mathcal{U} \colon E_u \text{ is convex}\}$. We claim that given any $\nu \in \mathcal{G}_1$ and any equilibrium distribution $\tau$ of $\nu$ such that (i)–(iii) are true, there are open neighborhoods $\tilde{V}$ of $\operatorname{supp}(\nu)$ in $\mathcal{U}$ and $W$ of $e(\tau_A)$ in $\mathbb{R}^m$, with $\operatorname{cl} W \subseteq \operatorname{int} E(\nu)$, such that, setting $\hat{V} = \{u \in \tilde{V} \colon W \subseteq E_u\}$ and $V = \tilde{V} \cap \mathcal{U}_c$, the following hold: $V \subseteq \hat{V}$, and on $\hat{V} \times W$, the best replies of $u$ against $e$ are given by a continuous map $h \colon \hat{V} \times W \to A$ such that (1) $h(u, \cdot)$ is differentiable for each $u \in \hat{V}$, (2) the derivative of $h(u, \cdot)$ depends continuously on $(u, e)$, and (3) $D_a^2 u(h(u, e), e)$ is negative definite for each $(u, e) \in \hat{V} \times W$.

To see that this claim is true, choose a compact neighborhood $W_1$ of $e(\tau_A)$ such that $W_1 \subseteq \operatorname{int} E(\nu)$, which is possible by (iii). Then by compactness of $\operatorname{supp}(\nu)$, Lemma 1, and Lemma 4, there is a (relatively) open neighborhood $V_1$ of $\operatorname{supp}(\nu)$ in $\mathcal{U}_c$ such that $W_1 \subseteq \operatorname{int} E_u$ for all $u \in V_1$. Now pick any $u \in \operatorname{supp}(\nu)$. As above, let $a_u \in \operatorname{int} A_u$ be the unique element of $\varphi(u, e(\tau_A))$. Then there is a compact and convex neighborhood $U_{a_u}$ of $a_u$ in $\operatorname{int} A_u$ such that $D_a^2 u(a, e(\tau_A))$ is negative definite for every $a \in U_{a_u}$. Now we can find numbers $r_1$, $r_2$ such that $u(a_u, e(\tau_A)) > r_1 > r_2 > u(a, e(\tau_A))$ for each

$a \in A_u \backslash U_{a_u}$. In particular, we must have $r_1 > u(a, e(\tau_A))$ for all $a \in \mathrm{cl}\,(A_u \backslash U_{a_u})$. Using Lemma 1 and Lemma 4 we see that there is a neighborhood $\tilde{V}'_u$ of $u$ in $\mathcal{U}$ such that $U_{a_u} \subseteq \mathrm{int}\, A_{u'}$ for each $u' \in \tilde{V}'_u$. Because the actions universe $A$ is compact, Lemma 1 now shows that there are open neighborhoods $\tilde{V}_u$ of $u$ in $\mathcal{U}$ and $W_u$ of $e(\tau_A)$ in $\mathbb{R}^m$, with $\tilde{V}_u \cap \mathcal{U}_c \subseteq \tilde{V}'_u \cap V_1$ and $W_u \subseteq W_1$, such that, setting $\hat{V}_u = \{u' \in \tilde{V}_u \colon W_u \subseteq E_{u'}\}$, $u'(a_u, e) > r_1 > u'(a, e)$ for each $u' \in \hat{V}_u$, $e \in W_u$, and $a \in A_{u'} \backslash U_{a_u}$, and such that $D_a^2 u'(a, e)$ is negative definite for each $u' \in \hat{V}_u$, $e \in W_u$, and $a \in U_{a_u}$. In particular, for each $u' \in \hat{V}_u$ and $e \in W_u$, $u'(\cdot, e)$ is strictly concave on $U_{a_u}$. Consequently for each $u' \in \hat{V}_u$ and $e \in W_u$, the best reply of $u'$ against $e$ is unique. Apply this argument to each $u \in \mathrm{supp}(\nu)$. Then by compactness of $\mathrm{supp}(\nu)$ there are $u_1, \ldots, u_k \in \mathrm{supp}(\nu)$ such that $\mathrm{supp}(\nu) \subseteq \tilde{V} = \bigcup_{i=1}^k \tilde{V}_{u_i}$. Set $W = \bigcap_{i=1}^k W_{u_i}$ and $K = \bigcup_{i=1}^k U_{a_i}$. Then for $V$ and $\hat{V}$ (defined relative to the sets $\tilde{V}$ and $W$ according to the previous paragraph), we have $V \subseteq \hat{V}$, and for each $(u, e) \in \hat{V} \times W$, the best reply of $u$ against $e$ is unique and belongs to $K$. Thus, on $\hat{V} \times W$, the best reply correspondence $\varphi$ can be identified with a function $h$ taking values in $K$. Using the fact that $K$ is compact we see that $h$ is continuous. Note that by construction, $D_a^2 u(h(u, e), e)$ is negative definite for each $(u, e) \in \hat{V} \times W$, i.e., we have (3). In view of this, the implicit function theorem applied to the maps $(a, e) \mapsto D_a u(a, e)$, $u \in \hat{V}$, shows that (1) is true. Using Lemma 1 we see that the evaluation maps $(u, a, e) \mapsto D_a^2 u(a, e)$ and $(u, a, e) \mapsto D_e D_a u(a, e)$, which are defined on the set $\{(u, a, e) \in \mathcal{U} \times A \times E \colon a \in \mathrm{int}\, A_u,\ e \in E_u\}$, are continuous. From this we see that (2) is true.

Let $\nu \in \mathcal{G}_1$ and $\tau$ an equilibrium distribution for $\nu$ such that (i)–(iii) are satisfied. Let $W$ correspond to $\tau$ as above. We can then define a map $\xi_\tau \colon W \to \mathbb{R}^m$ by setting

$$\xi_\tau(e) = \int g(h(u, e))\, d\nu(u) - e$$

for each $e \in W$; then by the generalized version of Leibniz' rule in Schwartz (1967, Chap IV.11, Theorem 115), $\xi_\tau$ is continuously differentiable on $W$, and we have $D\xi_\tau(e) = \int D_e(g \circ h)(u, e)\, d\nu(u) - I$ where $I$ is the $(m \times m)$-identity matrix.

**(c)** Let $\mathcal{G}^*$ be the subset of $\mathcal{G}$ consisting of those $\nu \in \mathcal{G}$ such that for some equilibrium distribution $\tau$ of $\nu$, (i)–(iii) of (b) are satisfied and $D\xi_\tau(e(\tau_A))$ has full rank, where $\xi_\tau$ is associated with $\tau$ as above. (Note that while the choice of the neighborhood $W$ of $e(\tau_A)$, i.e., the domain of $\xi_\tau$, involves some arbitrariness, $D\xi_\tau(e(\tau_A))$ is uniquely determined.) By Lemma 9 below, $\mathcal{G}^*$ is dense in $\mathcal{G}$, and we are now going to show that $\mathcal{G}^*$ is open in $\mathcal{G}$.

**(d)** Fix $\nu \in \mathcal{G}^*$. We need to show that $\nu$ has a neighborhood $U$ in $\mathcal{G}$ such that $U$ is included in $\mathcal{G}^*$. Let $\tau$ be an equilibrium distribution for $\nu$, witnessing that $\nu \in \mathcal{G}^*$. Let $W$, $V$, $h$, and $\xi_\tau$ be associated with $\tau$ as in (b).

**(i)** Pick a compact neighborhood $W_1$ of $e(\tau_A)$ with $W_1 \subseteq W$. Then there is a $k \in \mathbb{N}$ and a neighborhood $V_1$ of $\mathrm{supp}(\nu)$ in $\mathcal{U}_c$, with $V_1 \subseteq V$, such that $\|D_e(g \circ h)(u, e)\| \leq k$ for each $(u, e) \in V_1 \times W_1$. Indeed, otherwise, for each $k \in \mathbb{N} \backslash \{0\}$, we can find points $e_k \in W_1$ and $u_k \in V$ such that $\|D_e(g \circ h)(u_k, e_k)\| > k$ but $\mathrm{dist}(u_k, \mathrm{supp}(\nu)) < 1/k$.

Since $W_1$ and $\text{supp}(\nu)$ are compact we may assume that $(u_k, e_k) \to (u, e)$ for some $(u, e) \in \text{supp}(\nu) \times W_1$. Now $D_e(g \circ h)(u_k, e_k) = Dg(h(u_k, e_k))D_e h(u_k, e_k)$, and because $Dg$, $h$, and $D_e h$ are continuous, it follows that $D_e(g \circ h)(u_k, e_k) \to D_e(g \circ h)(u, e)$, and we get a contradiction.

**(ii)** Write $W_2$ for the interior of $W_1$ in $\mathbb{R}^m$. Choose an open neighborhood $U_1$ of $\nu$ in $\mathcal{G}$ such that $\text{supp}(\nu') \subseteq V_1$ for each $\nu' \in U_1$. Note that $W_2 \subseteq E(\nu')$ for each $\nu' \in U_1$. We can therefore define a map $\xi_U \colon U_1 \times W_2 \to \mathbb{R}^m$ by setting

$$\xi_U(\nu', e) = \int g(h(u, e))\, d\nu'(u) - e$$

for each $\nu' \in U_1$ and $e \in W_2$. As above we see that for each fixed $\nu' \in U_1$, $\xi_U(\nu', \cdot)$ is continuously differentiable on $W_2$, with $D_e \xi_U(\nu', e) = \int D_e(g \circ h)(u, e)\, d\nu'(u) - I$ where $I$ is the $(m \times m)$-unit matrix. Now $\xi_U$ is continuous and $D_e \xi_U(\nu', e)$ depends continuously on $(\nu', e)$. Indeed, suppose that $e_k \to e$ in $W_2$ and $u_k \to u$ in $V_1$. Then $(g \circ h)(u_k, e_k) \to (g \circ h)(u, e)$, because $h$ and $g$ are continuous, and as in (i) we see that $D_e(g \circ h)(u_k, e_k) \to D_e(g \circ h)(u, e)$. Thus, uniformly on compact subsets of $V_1$, we have both $(g \circ h)(\cdot, e_k) \to (g \circ h)(\cdot, e)$ and $D_e(g \circ h)(\cdot, e_k) \to D_e(g \circ h)(\cdot, e)$. Using Billingsley (1968, Theorem 5.5) it follows that if $\nu_k \to \nu'$ in $U_1$, then the corresponding sequences of distributions of the maps $(g \circ h)(\cdot, e_k)$ and $D_e(g \circ h)(\cdot, e_k)$ converge narrowly to the distributions of $(g \circ h)(\cdot, e)$ and $D_e(g \circ h)(\cdot, e)$ respectively. As $g \circ h$ takes values in the compact set $E$, we can now use change of variables to see that $\xi_U(\nu_k, e_k) \to \xi_U(\nu', e)$. Similarly, by (1), we see that $D_e \xi_U(\nu_k, e_k) \to D_e \xi_U(\nu', e)$. Thus, on $U_1 \times W_2$, $\xi_U$ is continuous and $D_e \xi_U(\nu', e)$ depends continuously on $(\nu', e)$, as claimed.

Now as $\tau$ is an equilibrium distribution for $\nu$, we have $\xi_U(\nu, e(\tau_A)) = 0$, and since $\nu \in \mathcal{G}^*$, $D_e \xi_U(\nu, e(\tau_A)) \equiv D\xi_\tau(e(\tau_A))$ has full rank. Hence, by a version of the implicit function theorem (see Schwartz, 1967, Chap. III.8, Theorem 25, or Mas-Colell, 1985, Chap. 1, C.3.3), there is an open neighborhood $U$ of $\nu$ in $\mathcal{G}$, with $U \subseteq U_1$, and a continuous map $\nu' \mapsto e(\nu') \colon U \to W_2$ such that for each $\nu' \in U$, $\xi_U(\nu', e(\nu')) = 0$. Also, since $D_e \xi_U(\nu', e)$ depends continuously on $(\nu', e)$, $D_e \xi_U(\nu', e(\nu'))$ has full rank for each $\nu' \in U$, shrinking $U$ if need be.

Fix any $\nu' \in U$ and set $\tau' = \nu' \circ \big(id \times h(\cdot, e(\nu'))\big)^{-1}$. Then

$$\text{supp}(\tau') \subseteq \{(u, a) \in \mathcal{U} \times A \colon a \in \varphi(u, e(\nu'))\},$$

by the choice of $h$, and

$$e(\tau'_A) = \int g(h(u, e(\nu')))\, d\nu'(u) = \xi_U(\nu', e(\nu')) + e(\nu') = e(\nu').$$

Thus $\tau'$ is an equilibrium distribution for $\nu'$. By the choices of $V$ and $W$, and since $e(\nu'') \in W_2 \subseteq W$ for each $\nu'' \in U$, (i)–(iii) of (b) are true for $\tau'$. Let $V'$, $W'$, $h'$, and $\xi_{\tau'}$ be associated with $\tau'$ as in (b). Then $V' \cap V_1$ is a neighborhood of $\text{supp}(\nu')$ in $\mathcal{U}_c$, and $W' \cap W_2$ a neighborhood of $e(\tau'_A) = e(\nu')$. Moreover, $h$ and $h'$ agree on

$(V'\cap V_1)\times(W'\cap W_2)$, and hence so do $\xi_{\tau'}$ and $\xi_U(\nu',\cdot)$. Thus $D\xi_{\tau'}(e(\tau'_A))$ has maximal rank. It follows that every $\nu'\in U$ belongs to $\mathcal{G}^*$. As $\nu\in\mathcal{G}^*$ is arbitrary, $\mathcal{G}^*$ is open.

**(e)** Let $\nu\in\mathcal{G}^*$ and let $\langle(I_n,G_n)\rangle_{n\in\mathbb{N}}$ be a sequence of finite-player games such that $\#(I_n)\to\infty$ and $\nu_{G_n}\to\nu$ in $\mathcal{M}$. For each $n$ we can write $I_n=\{1,\dots,k_n\}$, where $k_n=\#(I_n)$. Let $\hat{A}$ be the convex hull of the actions universe $A$, and identify $g$ with a continuous extension to $\hat{A}$. For any map $f\colon I_n\to\hat{A}$, and any $i\in I_n$, we write $\tau_{\hat{A},f}$ for the probability measure on $\hat{A}$ given by setting $\tau_{\hat{A},f}(B)=\#\{j\in I_n\colon f(j)\in B\}/\#(I_n)$ for each Borel set $B\subseteq\hat{A}$, and $\tau_{\hat{A},f,i}$ for the probability measure on $\hat{A}$ which is given by setting $\tau_{\hat{A},f,i}(B)=\#\{j\in I_n\backslash\{i\}\colon f(j)\in B\}/(\#(I_n)-1)$ for each Borel set $B\subseteq\hat{A}$.

Write $\|\cdot\|_V$ for the variation norm on the space $M(\hat{A})$ of all signed Borel measures on $\hat{A}$. Note that for any $n\in\mathbb{N}$ and any $f\colon I_n\to\hat{A}$, $\|\tau_{\hat{A},f,i}-\tau_{\hat{A},f}\|_V\le 2/\#(I_n)$ for each $i\in I_n$. Because $g$ is bounded on $\hat{A}$, it follows that for any $\delta>0$ there is an $N_\delta\in\mathbb{N}$ such that if $n\ge N_\delta$ then $\left\|\int g(a)\,d\tau_{\hat{A},f,i}(a)-\int g(a)\,d\tau_{\hat{A},f}(a)\right\|<\delta$ for each $f\colon I_n\to\hat{A}$ and each $i\in I_n$.

Let $\tau$ be an equilibrium distribution for $\nu$, witnessing that $\nu\in\mathcal{G}^*$. Let $\tilde{V}$, $\hat{V}$, $W$, and $h$ be as in the paragraph after the statement of (i)–(iii) in (b).

As $\nu\in\mathcal{G}^*$, the derivative of $\xi_\tau$ at $e(\tau_A)$ has full rank, which implies that on some convex compact neighborhood $W_1$ of $e(\tau_A)$ in $\mathbb{R}^m$, with $W_1\subseteq W$, $\xi_\tau(e)=0$ if and only if $e=e(\tau_A)$. Let $W_2$ be a convex compact neighborhood of $e(\tau_A)$ in $\mathbb{R}^m$ such that $W_2\subseteq\operatorname{int}W_1$.

Now because $\nu_{G_n}\to\nu$, and therefore $\rho_H(\operatorname{supp}(\nu_{G_n}),\operatorname{supp}(\nu))\to 0$, Lemma 6 implies that there is an $N\in\mathbb{N}$ such that whenever $n\ge N$, then $\operatorname{supp}(\nu_{G_n})\subseteq\hat{V}$ (i.e., $\operatorname{supp}(\nu_{G_n})\subseteq\tilde{V}$ and $W\subseteq E_{G_n(i)}$ for each $i\in I_n$). By what was noted in the second paragraph of this part of the proof, we can assume that $N$ is so large that if $n\ge N$, then for each $f\in\hat{A}^{I_n}$ and $i=1\dots,k_n$,

$$\int g(a)\,d\tau_{\hat{A},f,i}(a)-\int g(a)\,d\tau_{\hat{A},f}(a)+e\in W_1$$

whenever $e\in W_2$. For $n\ge N$, consider the function $\Lambda\colon\hat{A}^{I_n}\times W_2\to\hat{A}^{I_n}\times\mathbb{R}^m$ defined by setting

$$\begin{aligned}&\Lambda(f,e)\\&=\Bigg(h_1\bigg(\int g(a)\,d\tau_{\hat{A},f,1}(a)-\int g(a)\,d\tau_{\hat{A},f}(a)+e\bigg),\dots,\\&\qquad h_{k_n}\bigg(\int g(a)\,d\tau_{\hat{A},f,k_n}(a)-\int g(a)\,d\tau_{\hat{A},f}(a)+e\bigg),\int g(a)\,d\tau_{\hat{A},f}(a)\Bigg),\end{aligned}$$

writing $h_i(\cdot)$ in place of $h(G(i),\cdot)$ for each $i\in\{1,\dots,k_n\}$. Then a fixed point of $\Lambda$ gives a strict pure strategy Nash equilibrium of $(I_n,G_n)$.

We claim that there is an $N_1\ge N$ such that for $n\ge N_1$ the fixed point theorem stated in the appendix as Theorem 6 applies to $\Lambda$. Clearly $\Lambda$ is continuous, and for $X=\hat{A}^{I_n}$ and $Y=W_2$ the requirements of Theorem 6 on $X$ and $Y$ are satisfied. With the map $\xi_\tau$ it is also clear that we have (a) of Theorem 6. Let $\gamma > 0$ be such that $\|\xi_\tau(e)\| \geq \gamma$ for each $e \in \partial W_2$. We need to show that for some $N_1 \geq N$ also (b) of that theorem is satisfied for $\Lambda$ and $\xi_\tau$ if $n \geq N_1$.

To this end, fix $n \geq N$ and suppose that $f \in \hat{A}^{I_n}$ and $e \in \partial W_2$ are such that

$$f = \Big(h_1\Big(\int g(a)\,d\tau_{\hat{A},f,1}(a) - \int g(a)\,d\tau_{\hat{A},f}(a) + e\Big),$$
$$\ldots, h_{k_n}\Big(\int g(a)\,d\tau_{\hat{A},f,k_n}(a) - \int g(a)\,d\tau_{\hat{A},f}(a) + e\Big)\Big).$$

Note that

$$\Big\|\frac{1}{k_n}\sum_{i=1}^{k_n} g(f(i)) - e - \xi_\tau(e)\Big\| = \Big\|\frac{1}{k_n}\sum_{i=1}^{k_n} g(f(i)) - e - \Big(\int g(h(u,e))\,d\nu(u) - e\Big)\Big\|$$
$$= \Big\|\frac{1}{n_k}\sum_{i=1}^{n_k} g(f(i)) - \int g(h(u,e))\,d\nu(u)\Big\|$$
$$\leq \Big\|\frac{1}{k_n}\sum_{i=1}^{k_n} g(f(i)) - \frac{1}{k_n}\sum_{i=1}^{k_n} g(h_i(e))\Big\| + \Big\|\frac{1}{k_n}\sum_{i=1}^{k_n} g(h_i(e)) - \int g(h(u,e))\,d\nu(u)\Big\|$$
$$= \Big\|\frac{1}{k_n}\sum_{i=1}^{k_n} g(f(i)) - \frac{1}{k_n}\sum_{i=1}^{k_n} g(h_i(e))\Big\| + \Big\|\int g(h(u,e))\,d\nu_{G_n}(u) - \int g(h(u,e))\,d\nu(u)\Big\|.$$

Since $\operatorname{supp}(\nu)$ is compact, there are a neighborhood $\tilde{V}_1$ of $\operatorname{supp}(\nu)$ in $\mathcal{U}$, with $\tilde{V}_1 \subseteq \tilde{V}$, and a $\delta > 0$ such that for $z \in \mathbb{R}^m$ with $e + z \in W$, $\|g(h(u,e+z)) - g(h(u,e))\| < \gamma/3$ whenever $u \in \tilde{V}_1 \cap \hat{V}$, $e \in W_1$, and $\|z\| < \delta$, and by what was noted earlier, if $n$ is large then $\big\|\int g(a)\,d\tau_{\hat{A},f,i}(a) - \int g(a)\,d\tau_{\hat{A},f}(a)\big\| < \delta$ for each $f \in \hat{A}^{I_n}$ and each $i \in I_n$. Hence, as $\rho_H(\operatorname{supp}(\nu_{G_n}), \operatorname{supp}(\nu)) \to 0$, we have $\big\|\frac{1}{k_n}\sum_{i=1}^{k_n} g(f(i)) - \frac{1}{k_n}\sum_{i=1}^{k_n} g(h_i(e))\big\| < \gamma/3$ for large $n$ whenever $f \in \hat{A}^{I_n}$ and $e \in \partial W_2$ are as above. On the other hand, combining the first part of the penultimate sentence with the fact that $\nu_{G_n} \to \nu$ narrowly, we see that each $e \in \partial W_2$ has a neighborhood $U$ in $\partial W_2$ such that for large $n$ we have $\|\int g(h(u,e'))\,d\nu_{G_n}(u) - \int g(h(u,e'))\,d\nu(u)\| < (2\gamma)/3$ for each $e' \in U$. Thus, since $\partial W_2$ is compact, if $n$ is large then $\big\|\int g(h(u,e))\,d\nu_{G_n}(u) - \int g(h(u,e))\,d\nu(u)\big\| < (2\gamma)/3$ for every $e \in \partial W_2$. It follows that for some $N_1 \geq N$, $\big\|\frac{1}{k_n}\sum_{i=1}^{k_n} g(f(i)) - e - \xi_\tau(e)\big\| < \gamma$ for $n \geq N_1$ whenever $f \in \hat{A}^{I_n}$ and $e \in \partial W_2$ are as above. Consequently, because $\frac{1}{k_n}\sum_{i=1}^{k_n} g(f(i)) = \int g(a)\,d\tau_{\hat{A},f}(a)$, (b) of Theorem 6 is satisfied if $n \geq N_1$.

We can conclude that for $n \geq N_1$, $\Lambda$ has a fixed point and thus $(I_n, G_n)$ has a strict pure strategy Nash equilibrium. □

**Lemma 9.** *Let $\mathcal{G}^*$ be defined as in the proof of Theorem 1. Let $\nu \in \mathcal{G}$ and $\tau$ an equilibrium distribution for $\nu$. Then there is a sequence $\langle \nu_k \rangle$ of elements of $\mathcal{G}^*$ and a sequence $\langle \tau_k \rangle$ of corresponding equilibrium distributions such that $\nu_k \to \nu$ in the topology of $\mathcal{G}$, $\tau_k \to \tau$ narrowly, and for each $k$, $\tau_k$ witnesses that $\nu_k \in \mathcal{G}^*$.*

*Proof.* In the sequel, for elements $\nu$ and $\nu_k$, $k \in \mathbb{N}$, in $\mathcal{G}$, we write $\nu_k \to \nu$ to mean convergence of the sequence $\langle \nu_k \rangle$ to $\nu$ in the topology of $\mathcal{G}$; for equilibrium distributions $\tau$ and $\tau_k$, $k \in \mathbb{N}$, we write $\tau_k \to \tau$ to mean convergence of the sequence $\langle \tau_k \rangle$ to $\tau$ in the narrow topology. Given $\nu \in \mathcal{G}$, we often write $Y$ for $\mathrm{supp}(\nu)$. The set $\mathcal{G}_1 \subseteq \mathcal{G}$ is defined as in the proof of Theorem 1. We write $\mathcal{G}_2$ for the set of elements of $\mathcal{G}$ which have an equilibrium distribution satisfying (i) and (ii) in the definition of $\mathcal{G}_1$, and $\mathcal{G}_3$ for the subset of $\mathcal{G}_2$ consisting of the elements $\nu$ of $\mathcal{G}_2$ such that for some $\bar{u} \in \mathrm{supp}(\nu)$ there is a decreasing sequence $\langle W_l \rangle$ of compact subsets of $Y$, with $A_u = A_{\bar{u}}$ for each $u \in W_0$, such that $\bigcap_{l=0}^{\infty} W_l = \{\bar{u}\}$, and $\nu(W_l) > 0$ for each $l$. The proof of the lemma is organized in the following steps.

(a) If $\nu \in \mathcal{G}$ and $\tau$ is an equilibrium distribution for $\nu$, there is a sequence $\langle \nu_k \rangle$ in $\mathcal{G}_2$ and a sequence $\langle \tau_k \rangle$ of equilibrium distributions for the $\nu_k$'s such that $\nu_k \to \nu$, $\tau_k \to \tau$, and for each $k$, $\tau_k$ witnesses that $\nu_k \in \mathcal{G}_2$.

(b) If $\nu \in \mathcal{G}_2$ and $\tau$ is an equilibrium distribution for $\nu$, witnessing that $\nu \in \mathcal{G}_2$, then there is a sequence $\langle \nu_k \rangle$ in $\mathcal{G}_3$ and a sequence $\langle \tau_k \rangle$ of equilibrium distributions for the $\nu_k$'s such that $\nu_k \to \nu$, $\tau_k \to \tau$, and for each $k$, $\tau_k$ witnesses that $\nu_k \in \mathcal{G}_2$.

(c) If $\nu \in \mathcal{G}_3$ and $\tau$ is an equilibrium distribution for $\nu$, witnessing that $\nu \in \mathcal{G}_2$, then there is a sequence $\langle \nu_k \rangle$ in $\mathcal{G}_1$ and a sequence $\langle \tau_k \rangle$ of equilibrium distributions for the $\nu_k$'s such that $\nu_k \to \nu$, $\tau_k \to \tau$, and for each $k$, $\tau_k$ witnesses that $\nu_k \in \mathcal{G}_1$.

(d) If $\nu \in \mathcal{G}_1$ and $\tau$ is an equilibrium distribution for $\nu$, witnessing that $\nu \in \mathcal{G}_1$, then there is a sequence $\langle \nu_k \rangle$ in $\mathcal{G}^*$ and a sequence $\langle \tau_k \rangle$ of equilibrium distributions for the $\nu_k$'s such that $\nu_k \to \nu$, $\tau_k \to \tau$, and for each $k$, $\tau_k$ witnesses that $\nu_k \in \mathcal{G}^*$.

Putting (a)–(d) together, proves the lemma.

**(a)** Let $\nu \in \mathcal{G}$ and $\tau$ any equilibrium distribution for $\nu$. Choose a twice continuously differentiable map $\rho\colon \mathbb{R}^n \to \mathbb{R}$ such that $\rho(0) = 0 > \rho(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $D^2\rho(0)$ is negative definite. For each $k \in \mathbb{N}$, $a \in A$, and $u \in Y$ define a function $u_{k,a}\colon A_u \times E(\nu) \to \mathbb{R}$ by setting

$$u_{k,a}(a', e) = u(a', e) + \tfrac{1}{k+1}\big(\rho(a - a')\big)$$

for each $(a', e) \in A_u \times E(\nu)$. Evidently (U1) and (U2) in the definition of payoff functions are satisfied for each $u_{k,a}$. Using Lemmata 1 and 3, we see from compactness of $Y$, $A$, and $E(\nu)$ that there is a $k_0 \in \mathbb{N}$ such that also (U3) in the definition of payoff functions is satisfied whenever $k \geq k_0$. We can assume that $k_0 = 0$.

Now, for each $k$, define a map $\lambda_k\colon Y \times A \to \mathcal{U}$ by setting $\lambda_k(u, a) = u_{k,a}$ for each $u \in Y$ and $a \in A$. Write $\mathrm{proj}_Y$ for the projection of $Y \times A$ onto $Y$. Using Lemma 1, we see that $\lambda_k$ is continuous for each $k$, and that the sequence $\langle \lambda_k \rangle$ converges uniformly to $\mathrm{proj}_Y$ as $k \to \infty$. Because $A$ and $Y$ are compact and $\lambda_k$ is continuous for each $k$, we can define an element $\nu_k$ in $\mathcal{M}$ for each $k$ by setting $\nu_k = \tau \circ \lambda_k^{-1}$. Evidently $E(\nu_k) = E(\nu)$ for each $k$, so each $\nu_k$ actually belongs to $\mathcal{G}$. The fact that $\langle \lambda_k \rangle$ converges uniformly to $\mathrm{proj}_Y$ implies that $\nu_k \to \nu$, because $\tau_{\mathcal{U}} = \nu$ by the fact that $\tau$ is an equilibrium distribution for $\nu$.

Write $\text{proj}_A$ for the projection of $\mathcal{U} \times A$ onto $A$. For each $k$, define a continuous map $\kappa_k : Y \times A \to \mathcal{U} \times A$ by setting $\kappa_k = \lambda_k \times \text{proj}_A$ and set $\tau_k = \tau \circ \kappa_k^{-1}$. Then $\tau_{k,\mathcal{U}} = \nu_k$ and $\tau_{k,A} = \tau_A$ for each $k$. As $\langle \lambda_k \rangle$ converges uniformly to $\text{proj}_Y$, $\langle \kappa_k \rangle$ converges uniformly to the identity on $Y \times A$, and thus we have $\tau_k \to \tau$. Fix any $k$. Note that because $Y \times A$ is compact and $\kappa_k$ is continuous, $\text{supp}(\tau_k) = \{(\kappa_k(u,a) \colon (u,a) \in Y \times A\}$; that is, $\text{supp}(\tau_k) = \{(u_{k,a}, a) \colon (u,a) \in Y \times A\}$. Thus if $(u', a') \in \text{supp}(\tau_k)$, then for some $(u,a) \in \text{supp}(\tau)$, $u' = u_{k,a}$ and $a' = a$. But if $(u,a) \in \text{supp}(\tau)$, then $a \in \varphi(u, e(\tau_A))$ since $\tau$ is an equilibrium distribution; in particular, $D_a^2 u(a, e(\tau_A))$ is negative semi-definite. Consequently, by the choice of the functions $u_{k,a}$, if $(u', a') \in \text{supp}(\tau_k)$, then $\varphi(u', e(\tau_A)) = \{a'\}$ and $D_a^2 u'(a', e(\tau_A))$ is negative definite. As $\tau_{k,\mathcal{U}} = \nu_k$ and $\tau_A = \tau_{k,A}$, it follows that $\tau_k$ is an equilibrium distribution for $\nu_k$ and that $\nu_k \in \mathcal{G}_2$.

**(b)** Let $\nu \in \mathcal{G}_2$ and $\tau$ an equilibrium distribution for $\nu$, witnessing that $\nu \in \mathcal{G}_2$. If there is a $u \in Y$ with $\nu(\{u\}) > 0$, we can simply set $\nu_k = \nu$ and $\tau_k = \tau$ for each $\kappa$. Suppose that $\nu(\{u\}) = 0$ for each $u \in Y$. Pick an arbitrary point $\bar{u}$ in $Y$. By Lemma 12 in the appendix, there is an increasing sequence $\langle A_{\bar{u}_k} \rangle$ of non-empty compact subsets of $A_{\bar{u}}$, all with dense interior, such that $A_{\bar{u}_k} \subseteq \text{int}\, A_{\bar{u}}$ for each $k$ and both $\rho_H(A_{\bar{u}}, A_{\bar{u}_k}) \to 0$ and $\rho_H(\partial A_{\bar{u}}, \partial A_{\bar{u}_k}) \to 0$. Now by Lemma 4, for each $k$ there is a closed neighborhood $V_k'$ of $\bar{u}$ in $\mathcal{U}$ such that $A_{\bar{u}_k} \subseteq \text{int}\, A_u$ for each $u \in V_k'$. For each $k$, set $V_k = V_k' \cap Y$. Then $\nu(V_k) > 0$ for each $k$, because $\bar{u} \in Y$ ($\equiv \text{supp}(\nu)$). We can assume that the sequence $\langle V_k \rangle$ is decreasing and that $\bigcap_{k=0}^{\infty} V_k = \{\bar{u}\}$.

For each $k$ and each $u \in Y$ define a map $u_k'$ by setting $u_k' = u \restriction (A_{\bar{u}_k} \times E(\nu))$ if $u \in V_k$ and $u_k' = u$ otherwise. Then all these maps are twice continuously differentiable on their domains. As $\rho_H(A_{\bar{u}}, A_{\bar{u}_k}) \to 0$ and $\rho_H(\partial A_{\bar{u}}, \partial A_{\bar{u}_k}) \to 0$, we can see, as in (a), that if $k$ is sufficiently large, then (U3) in the assumptions on payoff functions is satisfied by all the maps $u_k'$. We can assume that this is true for each $k$, so that we can define a map $\lambda_k' : Y \to \mathcal{U}$ for each $k$ by setting $\lambda_k'(u) = u_k'$ for each $u \in Y$. Note that the restrictions of $\lambda_k'$ to $V_k$ and $Y \setminus V_k$ are both continuous. Thus for each $k$, $\lambda_k'$ is measurable, so $\nu_k' = \nu \circ \lambda_k'$ is defined. Evidently, for each $k$, $\text{supp}(\nu_k')$ is compact, so $\nu_k' \in \mathcal{M}$. Using Lemma 1 and the choice of the sequence $\langle A_{\bar{u}_k} \rangle$ it follows that the sequence $\langle \lambda_k' \rangle$ converges uniformly to the identity on $Y$, which implies that $\nu_k' \to \nu$ in the topology of $\mathcal{M}$.

Now by Lemma 5, $\rho_H(E(\nu), E(\nu_k')) \to 0$. Moreover, $E(\nu_k') \subseteq E(\nu)$ for each $k$. Indeed, fix any $k$. Then

$$\int_{V_k} \text{co}\, g(A_{\bar{u}_k})\, d\nu = \nu(V_k) \text{co}\, g(A_{\bar{u}_k}) = \nu_k'(\{\bar{u}_k'\}) \text{co}\, g(A_{\bar{u}_k}),$$

so, by the proof of Lemma 5 and the choice of $\nu_k'$,

$$\begin{aligned} E(\nu) = \int_Y \text{co}\, g(A_u)\, d\nu(u) &\supseteq \int_{Y \setminus V_k} \text{co}\, g(A_u)\, d\nu(u) + \int_{V_k} \text{co}\, g(A_{\bar{u}_k})\, d\nu \\ &= \int_{Y \setminus V_k} \text{co}\, g(A_u)\, d\nu_k'(u) + \nu_k'(\{\bar{u}_k'\}) \text{co}\, g(A_{\bar{u}_k}) = E(\nu_k'). \end{aligned}$$

For each $u \in Y$, we can therefore define a map $u_k\colon A_u \times E(\nu'_k) \to \mathbb{R}$ by setting $u_k(a,e) = u(a,e)$ for $(a,e) \in A_u \times E(\nu'_k)$, and for each $u \in V_k$, we can define a map $\tilde{u}_k\colon A_{\bar{u}_k} \times E(\nu'_k) \to \mathbb{R}$ by setting $\tilde{u}_k(a,e) = u(a,e)$ for $(a,e) \in A_{\bar{u}_k} \times E(\nu'_k)$. For each $k$, define $\lambda_k\colon Y \to \mathcal{U}$ by setting $\lambda_k(u) = u_k$ for $u \in Y \backslash V_k$, and $\lambda_k(u) = \tilde{u}_k$ for $u \in V_k$. As with $\lambda'_k$, we see that $\lambda_k$ is measurable for each $k$, so $\nu_k = \nu \circ \lambda_k^{-1}$ is defined. Clearly $\mathrm{supp}(\nu_k)$ is compact and $E(\nu_k) = E(\nu'_k)$ for each $k$, so $\nu_k$ belongs to $\mathcal{G}$. Since $\rho_H(E(\nu), E(\nu'_k) \to 0$, we see that $\langle \lambda_k \rangle$ converges uniformly to the identity on $Y$, using Lemma 1 and the choice of the sequence $\langle A_{\bar{u}_k} \rangle$. Thus $\nu_k \to \nu$.

Since $\nu \in \mathcal{G}_2$ and $\tau$ witnesses this, we see as in (i) in the proof of Theorem 1 that there is continuous map $f\colon Y \to A$ such that $\varphi(u, e(\tau_A)) = \{f(u)\}$ for all $u \in Y$; in particular, $\tau = \nu \circ (id_Y \times f)^{-1}$. By (U3) in the assumptions on payoff functions, we have $f(\bar{u}) \in \operatorname{int} A_{\bar{u}}$. Consequently, by Lemma 3 and the choice of the sequence $\langle A_{\bar{u}_k} \rangle$, we must have $f(\bar{u}) \in \operatorname{int} A_{\bar{u}_k}$ for large $k$, and therefore, by the choice of the sequence $\langle V_k \rangle$, continuity of $f$ implies that $f(u) \in A_{\bar{u}_k}$ for each $u \in V_k$ if $k$ is large enough. We can assume that this is true for all $k$. Now for each $k$, set $\tau_k = \nu \circ (\lambda_k \times f)^{-1}$. Then $\tau_{k,\mathcal{U}} = \nu_k$ for each $k$. Because $Y$ and $V_k \subseteq Y$ are compact, and the maps $f$, $u \mapsto u_k\colon Y \to \mathcal{U}$, and $u \mapsto \tilde{u}_k\colon V_k \to \mathcal{U}$ are continuous, the set $\{(u_k, f(u))\colon u \in Y\} \cup \{(\tilde{u}_k, f(u))\colon u \in V_k\}$ is closed and therefore includes $\mathrm{supp}(\tau_k)$. Note that $e(\tau_{k,A}) = \int g(f(u))\, d\nu(u) = e(\tau_A)$. By the hypothesis that $\tau$ is an equilibrium distribution for $\nu$ satisfying (i) and (ii) in the definition of $\mathcal{G}_1$, together with the choice of the $u_k$'s and $\tilde{u}_k$'s, it follows that $\tau_k$ is an equilibrium distribution for $\nu_k$, also satisfying (i) and (ii) in the definition of $\mathcal{G}_1$. Clearly the sequence $\langle \lambda_k, f \rangle$ converges uniformly to $id_Y \times f$, so $\tau_k \to \tau$, because $\tau = \nu \circ (id_Y \times f)^{-1}$.

Finally to see that each $\nu_k$ actually belongs to $\mathcal{G}_3$, fix any $k$. Note first that $\lambda_k(\bar{u}) \in \mathrm{supp}(\nu_k)$. Indeed, note that $\lambda_k$ is continuous on $V_k$ and that $\bar{u}$ belongs to the relative interior of $V_k$ in $Y$. Hence whenever $O$ is an open neighborhood of $\lambda_k(\bar{u})$, there is a relatively open neighborhood $U$ of $\bar{u}$ in $Y$, with $U \subseteq V_k$, such that $\lambda_k(U) \subseteq O$. Consequently, for any such $O$ and $U$,

$$\nu_k(O) = \nu(\{u \in Y\colon \lambda_k(u) \in O\} \geq \nu(U) > 0,$$

since $\bar{u} \in Y$ ($\equiv \mathrm{supp}(\nu)$).

Now set $W_l = \lambda_k(V_{k+l}) \cap \mathrm{supp}(\nu_k)$ for each $l \in \mathbb{N}$, so that each $W_l$ is a closed subset of $\mathrm{supp}(\nu_k)$, because $V_{k+l}$ is compact and $\lambda_k$ is continuous on $V_{k+l}$. By the choice of the sequence $\langle V_k \rangle$, we have $\bigcap_{l=0}^{\infty} V_{k+l} = \{\bar{u}\}$, therefore $\bigcap_{l=0}^{\infty} \lambda_k(V_{k+l}) = \{\lambda_k(\bar{u})\}$ since $\lambda_k$ is continuous on $V_k$, and hence $\bigcap_{l=0}^{\infty} W_l = \{\lambda_k(\bar{u})\}$. Also, for each $l$, the relative interior of $V_{k+l}$ in $Y$ is non-empty, therefore $\nu(V_{k+l}) > 0$ since $V_{k+l} \subseteq \mathrm{supp}(\nu)$, and hence $\nu_k(\lambda_k(V_{k+l})) > 0$. Thus $\nu_k(W_l) > 0$ for each $l$. By construction, $A_u = A_{\lambda_k(\bar{u})}$ for each $u \in W_0$, and it follows that $\nu_k$ satisfies the requirements to be a member of $\mathcal{G}_3$.

**(c)** Let $\nu \in \mathcal{G}_3$ and $\tau$ an equilibrium distribution for $\nu$, witnessing that $\nu \in \mathcal{G}_2$. As in (b), there is continuous map $f\colon Y \to A$ such that $\varphi(u, e(\tau_A)) = \{f(u)\}$ for all $u \in Y$, and we have $e(\tau_A) = \int g(f(u))\, d\nu(u)$ and $\tau = \nu \circ (id_Y \times f)^{-1}$.

**(i)** Suppose first that there is a $\bar{u} \in Y$ with $\nu(\{\bar{u}\}) > 0$. Write $\alpha$ for $\nu(\{\bar{u}\})$.

Note that $f(\bar u) \in \operatorname{int} A_{\bar u}$. Let $\langle W_k \rangle$ be a non-increasing sequence of compact convex neighborhoods of $f(\bar u)$ in $A_{\bar u}$ such that $\bigcap_{k=1} W_k = \{f(\bar u)\}$. By Lemma 10(b), $\operatorname{int} \operatorname{co} g(W_k)$ is non-empty for each $k$. For each $k$ fix a point $e_{\bar u,k} \in \operatorname{int} \operatorname{co} g(W_k)$. Using Caratheodory's theorem, for each $k$ we can find points $a_{k,h}$, $h = 1, \dots, m+1$, in $W_k$ such that $e_{\bar u,k} = \sum_{h=1}^{m+1} \beta_{k,h} g(a_{k,h})$ for some numbers $\beta_{k,h}$ with $\beta_{k,h} \geq 0$ and $\sum_{h=1}^{m+1} \beta_{k,h} = 1$. Note that $E(\nu) = \alpha \operatorname{co}(g(A_{\bar u})) + \int_{Y \setminus \{\bar u\}} \operatorname{co} g(A_u)\, d\nu(u)$. Thus, setting $e_k = \alpha e_{\bar u,k} + \int_{Y \setminus \{\bar u\}} g(f(u))\, d\nu(u)$ we have $e_k \in \operatorname{int} E(\nu)$ for each $k$. Also, $e_k \to e(\tau_A)$, by continuity of $g$, since $a_{k,h} \to f(\bar u)$ for each $h$ if $k \to \infty$ by choice of the points $a_{k,h}$.

As $f(\bar u) \in \operatorname{int} A_{\bar u}$, we can find numbers $0 < r_1 < r_2$ such that $\bar B(f(\bar u), r_1) \subseteq \bar B(f(\bar u), r_2) \subseteq \operatorname{int} A_{u_i}$. Let $\rho \colon \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable map such that $\rho(a) = 1$ if $a \in B(f(\bar u), r_1)$, $0 \leq \rho(a) \leq 1$ for all $a \in \mathbb{R}^n$, and $\rho(a) = 0$ if $a \notin B(f(\bar u), r_2)$. As $a_{k,h} \to f(\bar u)$ for each $h = 1, \dots, m+1$ if $k \to \infty$, we can assume for each $k$ and each $h$ that $a_{k,h} \in B(f(\bar u), r_1)$ and that $a + \rho(a)(f(\bar u) - a_{k,h}) \in \operatorname{int} A_{\bar u}$ whenever $a \in B(f(\bar u), r_2)$. Define a map $\bar u_{k,h} \colon A_{\bar u} \times E(\nu) \to \mathbb{R}$ for each $h = 1, \dots, m+1$ and each $k$ by setting

$$\begin{aligned} \bar u_{k,h}(a, e) =& \bar u\big(a + \rho(a)(f(\bar u) - a_{k,h}), e\big) \\ &- \bar u\big(a + \rho(a)(f(\bar u) - a_{k,h}), e_k\big) + \bar u\big(a + \rho(a)(f(\bar u) - a_{k,h}), e(\tau_A)\big) \end{aligned}$$

for each $(a, e) \in A_{\bar u} \times E(\nu)$; for each $u \in Y$ and each $k$, define a map $u_k \colon A_u \times E(\nu) \to \mathbb{R}$ by setting

$$u_k(a, e) = u\big(a, e\big) - u(a, e_k) + u(a, e(\tau_A))$$

for each $(a, e) \in A_u \times E(\nu)$. Then, for each $k$, all the maps $u_k$ and $\bar u_{k,h}$, $h = 1, \dots, m+1$, are twice continuously differentiable on their domains. Using the facts that $e_k \to e(\tau_A)$ and that $a_{k,h} \to f(\bar u)$ for each $h = 1, \dots, m+1$, we can assume, as in (a), that they all satisfy (U3) in the assumptions on payoff functions.

Now for each $k$ define $\lambda_k \colon Y \to \mathcal{U}$ by setting $\lambda_k(u) = u_k$ for each $u \in Y$ and note that $\lambda_k$ is continuous. For each $k$, set $\nu' = \nu - \alpha \delta_{\bar u}$ and $\nu_k = \alpha \sum_{h=1}^{m+1} \beta_{k,h} \delta_{\bar u_{k,h}} + \nu' \circ \lambda_k^{-1}$. Because $Y$ is compact and $\lambda_k$ is continuous, $\operatorname{supp}(\nu' \circ \lambda_k^{-1})$ is compact for each $k$, and hence so is $\operatorname{supp}(\nu_k)$ for each $k$. Thus $\nu_k \in \mathcal{M}$ for each $k$. Because the distribution of action sets induced by $\nu$ is the same as that induced by $\nu_k$, we have $E(\nu_k) = E(\nu)$, and thus $\nu_k$ actually belongs to $\mathcal{G}$ for each $k$. Using Lemma 1 we see that $\lambda_k$ converges uniformly to the identity on $Y$, and from this that $\nu' \circ \lambda_k^{-1} \to \nu'$ narrowly and that $\rho_H(\operatorname{supp}(\nu'), \operatorname{supp}(\nu' \circ \lambda_k^{-1})) \to 0$. Also $\alpha \sum_{h=1}^{m+1} \beta_{k,h} \delta_{\bar u_{k,h}} \to \alpha \delta_{\bar u}$ narrowly and $\rho_H(\operatorname{supp}(\alpha \delta_{\bar u}), \operatorname{supp}(\alpha \sum_{h=1}^{m+1} \beta_{k,h} \delta_{\bar u_{k,h}})) \to 0$, by the facts that $a_{k,h} \to f(\bar u)$ for each $h = 1, \dots, m+1$ and that $e_k \to e(\tau_A)$. Consequently we have $\nu_k \to \nu$.

Set $\tau_k = \alpha \sum_{h=1}^{m+1} \beta_{k,h} \delta_{\bar u_{k,h}, a_{k,h}} + \nu' \circ (\lambda_k \times f)^{-1}$ for each $k$. We claim that if $k$ is large enough, then $\tau_k$ is an equilibrium distribution for $\nu_k$ such that such that (i)–(iii) in the definition of $\mathcal{G}_1$ are true. Indeed, it is clear that $\tau_{k,\mathcal{U}} = \nu_k$. Note next that by the facts that $f$ and $\lambda_k$ are continuous and $Y$ is compact, the set

$$\{(\bar u_{k,h}, a_{k,h}) \colon h = 1, \dots, m+1\} \cup \{(\lambda_k(u), f(u)) \colon u \in Y\}$$

is closed and must therefore include $\text{supp}(\tau_k)$. Note also that

$$e(\tau_{k,A}) = \alpha \sum_{h=1}^{m+1} \beta_{k,h} g(a_{k,h}) + \int g(f(u))\, d\nu'(u) = e_k.$$

Now for any $u \in Y$, we have $\lambda_k(u)(a, e_k) = u(a, e(\tau_A))$ for each $a \in A_u$, by the choice of $\lambda_k(u)$; hence $\varphi(\lambda_k(u), e_k) = \{f(u)\}$ and $D^2\lambda_k(u)(f(u), e_k)$ is negative definite, because $\tau$ is an equilibrium distribution for $\nu$ satisfying (i) and (ii) in the definition of $\mathcal{G}_1$. From the second property of $\tau$ we also see that there is a compact neighborhood $\bar{U}$ of $f(\bar{u})$ in $\text{int}\, A_{\bar{u}}$ such that $D_a^2 \bar{u}_{k,h}(a, e_k)$ is negative definite for each $a \in \bar{U}$ and each $h = 1, \ldots, m+1$ if $k$ is large enough, because $\bar{u}_{k,h} \to \bar{u}$ for each $h = 1, \ldots, m+1$. Since $a_{k,h} \to f(\bar{u})$ and $D_a \bar{u}_{k,h}(a_{k,h}, e_k) = D_a \bar{u}(f(\bar{u})), e(\tau_A)) = 0$ for each $h = 1, \ldots, m+1$, it now follows from first of the properties noted for the equilibrium distribution $\tau$ that $\varphi(\bar{u}_{k,h}, e_k) = \{a_{kh}\}$ for each $h = 1, \ldots, m+1$ if $k$ is large enough. By construction, $e(\tau_{k,A}) = e_k \in \text{int}\, E(\nu) = \text{int}\, E(\nu_k)$ for each $k$, and thus the above claim is true.

Clearly, the sequence $\langle \alpha \sum_{h=1}^{m+1} \beta_{k,h} \delta_{\bar{u}_{k,h}, a_{k,h}} \rangle$ of measures on $\mathcal{U}$ converges narrowly to $\alpha\delta_{\bar{u}, f(\bar{u})}$, and since $\langle \lambda_k \times f \rangle$ converges uniformly to $id_Y \times f$, the sequence $\langle \nu' \circ (\lambda_k \times f)^{-1} \rangle$ of measures on $\mathcal{U}$ converges narrowly to $\nu' \circ (id_Y \times f)^{-1}$. Since $\alpha\delta_{\bar{u}, f(\bar{u})} = \alpha\delta_{\bar{u}} \circ (id_Y \times f)^{-1}$ and $\nu' \circ (id_Y \times f)^{-1} = (\nu - \alpha\delta_{\bar{u}})(id_Y \times f)^{-1}$, it follows that $\tau_k \to \nu \circ (id_Y \times f)^{-1}$ narrowly; that is $\tau_k \to \tau$, because $\tau = \nu \circ (id_Y \times f)^{-1}$. Thus the assertion of (c) is true in case there is a $\bar{u} \in Y$ with $\nu(\bar{u}) > 0$.

**(ii)** Now suppose $\nu(\{u\}) = 0$ for each $u \in Y$. As $\nu \in \mathcal{G}_3$, we can choose a $\bar{u} \in Y$ and a decreasing sequence $\langle W_l \rangle$ of compact subsets of $Y$ such that $\bigcap_{l=0}^{\infty} W_l = \{\bar{u}\}$, $\nu(W_l) > 0$ for each $l$, and $A_u = A_{\bar{u}}$ for each $u \in W_0$.

For each $l$, define a map $f_l \colon Y \to A$ by setting $f_l = 1_{Y \setminus W_l} f + 1_{W_l} f(\bar{u})$ and set $e_l = \int_{Y \setminus W_l} g(f(u))\, d\nu(u) + \nu(W_l) g(f(\bar{u}))$. Note that $e_l \to e(\tau_A)$ as $l \to \infty$ and that $e_l \in E(\nu)$ for each $l$. Now for each $l$ and each $u \in Y$, define a map $u_l \colon A_u \times E(\nu \to \mathbb{R}$ by setting

$$u_l(a, e) = u(a, e) - u(a, e_l) + u(a, e(\tau_A))$$

for $(a, e) \in A_u \times E(\nu)$, and for each $u \in W_l$ and each $l$, define $\bar{u}_l \colon A_{\bar{u}_l} \times E(\nu) \to \mathbb{R}$ by setting

$$\bar{u}_l(a, e) = \bar{u}(a, e) - \bar{u}(a, e_l) + \bar{u}(a, e(\tau_A))$$

for $(a, e) \in A_{\bar{u}_l} \times E(\nu)$. As in (i), all these maps are twice continuously differentiable on their domains and we may assume that they all satisfy (U3) in the assumptions on payoff functions. For each $l$, define $\lambda_l \colon Y \to \mathcal{U}$ by setting $\lambda_l(u) = u_l$ for $u \in Y \setminus W_l$, and $\lambda_l(u) = \bar{u}_l$ for $u \in W_l$. As in (b), $\lambda_l$ is measurable for each $i$, so $\nu_l = \nu \circ \lambda_l^{-1}$ is defined. As in (i) it follows that $\nu_l \in \mathcal{G}$ for each $l$. Using Lemma 1 and the fact that $e_l \to e(\tau_A)$, we see that $\langle \lambda_l \rangle$ converges uniformly to the identity on $Y$. This implies that $\nu_l \to \nu$. Moreover, we have $\nu_l(\{\bar{u}\}) \geq \nu(W_l) > 0$ for each $l$.

For each $l$, set $\tau_l = \nu \circ (\lambda_l \times f_l)^{-1}$. Then $\tau_{l,\mathcal{U}} = \nu_l$ for each $l$. For reasons as in (b), we have $\text{supp}(\tau_l) \subseteq \{(u_l, f(u)) \colon u \in Y\} \cup \{(\bar{u}_l, f(\bar{u})\}$. Noting that $e(\tau_{l,A}) = \int g \circ f_l\, d\nu = e_l$, it follows from the hypothesis that $\tau$ is an equilibrium distribution for $\nu$ satisfying

(i) and (ii) in the definition of $\mathcal{G}_1$, together with the choice of the $u_l$'s and $\bar{u}_l$'s, that $\tau_l$ is an equilibrium distribution for $\nu_l$, also satisfying (i) and (ii) in the definition of $\mathcal{G}_1$. Finally, note that $\tau = \nu \circ (id_Y \times f)^{-1}$ and that the sequence $\langle \lambda_l \times f_l \rangle$ converges uniformly to $id_Y \times f$. Consequently $\tau_l \to \tau$.

**(iii)** Combining (i) and (ii), proves the assertion of (c).

**(d)** Let $\nu \in \mathcal{G}_1$ and $\tau$ an equilibrium distribution for $\nu$, witnessing that $\nu \in \mathcal{G}_1$. Let $V$, $W$, $h$, and $\xi_\tau$ be associated with $\tau$ as in (b) of the proof of Theorem 1. Write $\bar{e} = e(\tau_A)$. If $\det D\xi_\tau(\bar{e}) \neq 0$, then $\nu \in \mathcal{G}^*$. Otherwise, pick any $0 < \lambda < 1$. For each $u \in Y$ define $u_\lambda \in \mathcal{U}$ by setting $u_\lambda(a, e) = u(a, (1-\lambda)\bar{e}+\lambda e)$ for each $(a, e) \in A_u \times E(\nu)$, so that, in particular, $\operatorname{dom} u_\lambda = \operatorname{dom} u$. Note that for each $u \in Y$ and $a \in A_u$, we have $u_\lambda(a, \bar{e}) = u(a, \bar{e})$. Define a map $\kappa_\lambda \colon Y \to \mathcal{U}$ by setting $\kappa_\lambda(u) = u_\lambda$. By Lemma 1, $\kappa_\lambda$ is continuous. Thus $\nu_\lambda = \nu \circ \kappa_\lambda^{-1}$ has compact support (since $\nu$ has) and therefore $\nu_\lambda$ belongs to $\mathcal{M}$. Because the distribution of action sets induced by $\nu_\lambda$ is the same as that induced by $\nu$, we have $E(\nu_\lambda) = E(\nu)$, and thus $\nu_\lambda$ actually belongs to $\mathcal{G}$. Set $\tau_\lambda = \nu \circ (\kappa_\lambda \times h(\cdot, \bar{e}))^{-1}$. Clearly $\tau_{\lambda,\mathcal{U}} = \nu_\lambda$. Moreover,

$$e(\tau_{\lambda,A}) = \int_{\mathcal{U}} g(h(u, \bar{e}))\, d\nu(u) = \bar{e}.$$

Further, if $(u', a') \in \operatorname{supp}(\tau_\lambda)$, then for some $u \in Y$, $u' = \kappa_\lambda(u) = u_\lambda$ and $a' = h(u, \bar{e})$. Since $u_\lambda(a, \bar{e}) = u(a, \bar{e})$ for each $u \in Y$ and $a \in A_u$, it follows that $\tau_\lambda$ is an equilibrium distribution for $\nu_\lambda$ satisfying (i) and (ii) in the definition of $\mathcal{G}_1$. Because $\bar{e} \in \operatorname{int} E(\nu)$ (by hypothesis), and since $E(\nu_\lambda) = E(\nu)$ and $e(\tau_{\lambda,A}) = \bar{e}$, also (iii) in the definition of $\mathcal{G}_1$ is true for $\tau_\lambda$. Thus $\tau_\lambda$ is an equilibrium distribution for $\nu_\lambda$, witnessing that $\nu_\lambda \in \mathcal{G}_1$.

Using Lemma 1 and the fact that $Y$ is compact we see that whenever $\langle \lambda_k \rangle$ is a sequence in $(0, 1)$ such that $\lambda_k \to 1$, then the sequence $\langle \kappa_{\lambda_k} \rangle$ converges uniformly to $id_Y$ and thus $\rho_H(\operatorname{supp}(\nu), \operatorname{supp}(\nu_{\lambda_k})) \to 0$. Hence, for large $\lambda \in (0, 1)$, $\operatorname{supp}(\nu_\lambda) \subseteq V$ and therefore $h(u_\lambda, \cdot) \colon W \to A$ is defined for each $u \in Y$; in particular, $D_e u_\lambda(h(u_\lambda, \bar{e}), \bar{e})$ must be defined. Clearly, for such $\lambda$, we have $h(u_\lambda, \bar{e}) = h(u, \bar{e})$ for each $u \in Y$, and in addition we must have $D_a^2 u_\lambda(h(u_\lambda, \bar{e}), \bar{e}) = D_a^2 u(h(u, \bar{e}), \bar{e})$ and $D_e u_\lambda(h(u_\lambda, \bar{e}), \bar{e}) = \lambda D_e u(h(u, \bar{e}), \bar{e}))$, therefore

$$D_e(g \circ h)(u_\lambda, \bar{e}) = \lambda D_e(g \circ h)(u, \bar{e}).$$

Consequently

$$D\xi_{\tau_\lambda}(\bar{e}) = \int_{\mathcal{U}} D_e(g \circ h)(u_\lambda, \bar{e})\, d\nu(u) - I = \lambda \int_{\mathcal{U}} D_e(g \circ h)(u, \bar{e})\, d\nu(u) - I.$$

Because the characteristic polynomial of the matrix $\int_{\mathcal{U}} D_e(g \circ h)(u, \bar{e})\, d\nu(u)$ can have only finitely may zeros, we have $\det(\int_{\mathcal{U}} D_e(g \circ h)(u, \bar{e})\, d\nu(u) - \frac{1}{\lambda} I) \neq 0$ for all sufficiently large $0 < \lambda < 1$, and hence $\det(\lambda \int_{\mathcal{U}} D_e(g \circ h)(u, \bar{e})\, d\nu(u) - I) \neq 0$ for such numbers $\lambda$.

As earlier, write $id_Y$ for the identity on $Y$. Let $\langle \lambda_k \rangle$ be a sequence in $(0, 1)$ such that $\lambda_k \to 1$. Then, as noted above, the sequence $\langle \kappa_{\lambda_k} \rangle$ converges uniformly to $id_Y$, so $\nu_{\lambda_k} \to \nu$. The fact that $\langle \kappa_{\lambda_k} \rangle$ converges uniformly to $id_Y$ implies also that the sequence

$\langle \kappa_{\lambda_k} \times h(\cdot, \bar{e})\rangle$ converges uniformly to $id_Y \times h(\cdot, \bar{e})$, and thus we have $\tau_{\lambda_k} \to \tau$. By what was also noted above, each $\tau_{\lambda_k}$ belongs to $\mathcal{G}_1$. Combining these facts with the conclusion of the previous paragraph, we see that there is a sequence $\langle \nu_k \rangle$ in $\mathcal{G}^*$ and a sequence $\langle \tau_k \rangle$ of equilibrium distributions for the $\nu_k$'s such that $\nu_k \to \nu$, $\tau_k \to \tau$, and for each $k$, $\tau_k$ witnesses that $\nu_k \in \mathcal{G}^*$. □

## 4.4 Proof of Theorem 2

**Theorem 2.** *Every $(\nu, \tau)$, where $\nu \in \mathcal{G}$ is a continuum game and $\tau$ is an equilibrium distribution for $\nu$, is asymptotically implementable by a sequence $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ of finite-player games such that $\nu_{G_n} \in \mathcal{G}$ for each $n$.*

*Proof.* **(a)** Let $\mathcal{G}^*$ be defined as in the proof of Theorem 1. Fix any $\nu \in \mathcal{G}^*$ and let $\tau$ be an equilibrium distribution for $\nu$ such that the requirements in (c) of the proof of Theorem 1 are satisfied. Suppose $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ is a sequence of finite-player games such that $\#(I_n) \to \infty$, $\nu_{G_n} \in \mathcal{M}$ for each $n$, and $\nu_{G_n} \to \nu$. Let $\hat{V}$, $W$, $W_1$, and $h\colon \hat{V} \times W \to A$ be as in (e) of the proof of Theorem 1. Observe that $\tau = \nu \circ (id_{\hat{V}} \times h(\cdot, e(\tau_A)))^{-1}$. Let $\langle W_{2,k} \rangle$ be a non-increasing sequence of compact convex neighborhoods of $e(\tau_A)$, with $W_{2,k} \subseteq \operatorname{int} W_1$ for each $k$, such that $\bigcap_{k=0}^{\infty} W_{2,k} = \{e(\tau_A)\}$. Instead with a fixed $W_2$, the argument in (f) of the proof of Theorem 1 can be applied with each member of the sequence $\langle W_{2,k} \rangle$ to yield an increasing sequence $\langle n_k \rangle$ in $\mathbb{N}$, and for each $k \in \mathbb{N}$ an equilibrium $f_k$ of the game $(I_{n_k}, G_{n_k})$ such that $f_k$ can be written in the form $f_k(i) = h(G_{n_k}(i), e_k + z_k(i))$, $i \in I_{n_k}$, where $e_k \in W_{2,k}$ and $\|z_k(i)\| \leq \epsilon_k$ for each $i \in I_{n_k}$, and $\epsilon_k \to 0$ as $k \to \infty$.

For each $k$, let $f'_k\colon I_{n_k} \to A$ be the map defined by setting $f'_k(i) = h(G_{n_k}(i), e_k)$ for each $i \in I_{n_k}$. Let $\tau_k$ be the distribution of $G_{n_k} \times f_k$, and $\tau'_k$ that of $G_{n_k} \times f'_k$. As $e_k \in W_{2,k}$ for all $k$ and $\bigcap_{k=0}^{\infty} W_{2,k} = \{e(\tau_A)\}$, we have $e_k \to e(\tau_A)$. From this we see that $id_V \times h(\cdot, e_k) \to id_V \times h(\cdot, e(\tau_A))$, uniformly on compact subsets of $V$, because $h$ is continuous. Consequently

$$\tau'_k = \nu_{G_{n_k}} \circ (id_V \times h(\cdot, e_k))^{-1} \to \nu \circ (id_V \times h(\cdot, e(\tau_A)))^{-1} = \tau,$$

i.e., the sequence $\langle \tau'_k \rangle$ of distributions of the maps $G_{n_k} \times f'_k$ converges to $\tau$ narrowly. Now note that if $\langle u'_k \rangle$ is any sequence in $\operatorname{supp}(\nu)$, and $\langle z_k \rangle$ a sequence in $\mathbb{R}^m$ such that $h(u'_k, e_k + z_k)$ is defined and $\|z_k\| \to 0$, then

$$\|h(u'_k, e_k + z_k) - h(u'_k, e_k)\| \to 0,$$

because $\operatorname{supp}(\nu)$ is compact, $h$ continuous, and $e_k \to e(\tau_A)$. Since $\nu_{G_n} \to \nu$ and thus $\rho_H(\operatorname{supp}(\nu_{G_{n_k}}), \operatorname{supp}(\nu)) \to 0$, it follows that for every $\epsilon' > 0$ there is a $k_{\epsilon'} \in \mathbb{N}$ such that whenever $k \geq k_{\epsilon'}$, then

$$\|h(G_{n_k}(i), e_k + z_k(i)) - h(G_{n_k}(i), e_k)\| \leq \epsilon'$$

for all $i \in I_{n_k}$, i.e., $\|f_k(i) - f'_k(i)\| \leq \epsilon'$ for all $i \in I_{n_k}$, and thus for some product metric $\tilde{\rho}$ on $\mathcal{U} \times A$ (recall that $\mathcal{U}$ can be regarded as a metric space) we have

$$\tilde{\rho}\big((G_{n_k}(i), f_k(i)), (G_{n_k}(i), f'_k(i))\big) \leq \epsilon'$$

for all $i \in I_{n_k}$ whenever $k \geq k_{\epsilon'}$. In view of this, we can conclude, using Billingsley (1968, Theorem 4.1), that the fact that the sequence $\langle \tau'_k \rangle$ of distributions of the maps $G_{n_k} \times f'_k$ converges narrowly to $\tau$ implies that the sequence $\langle \tau_k \rangle$ of distributions of the maps $G_{n_k} \times f_k$ converges narrowly to $\tau$, too.

**(b)** By Lemma 7, given $\nu \in \mathcal{G}$, a sequence $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ of finite player games such that $\nu_{G_n} \to \nu$, $\nu_{G_n} \in \mathcal{M}$ for each $n$, and $\#(I_n) \to \infty$ does exist. Putting this fact together with (a) and Lemma 9 proves the theorem. □

## 4.5 Proof of Theorem 3

**Theorem 3.** *There is an open dense subset $\mathcal{G}^*$ of $\mathcal{G}$ such that whenever $\nu \in \mathcal{G}^*$, then there is an equilibrium distribution $\tau$ for $\nu$ such that $(\nu, \tau)$ is asymptotically robust.*

*Proof.* Let $\mathcal{G}^*$ be defined as in the proof of Theorem 1 and apply (a) in the proof of Theorem 2. □

## 4.6 Proof of Theorem 4

**Theorem 4.** *Given any $C \in \mathcal{C}$, every $(\nu, \tau)$, where $\nu \in \mathcal{G}_C$ is a continuum game and $\tau$ is an equilibrium distribution for $\nu$, is asymptotically implementable by a sequence $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ of finite-player games such that $\nu_{G_n} \in \mathcal{M}_C$ for each $n$.*

*Moreover, there is a relatively open dense subset $\mathcal{G}^*_C$ of $\mathcal{G}_C$ such that the following are true.*

*(i) Whenever $\nu \in \mathcal{G}^*_C$ and $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ is a sequence of finite-player games such that $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$, then there is an $N \in \mathbb{N}$ such that $(I_n, G_n)$ has a strict pure strategy Nash equilibrium if $\#(I_n) \geq N$.*

*(ii) Whenever $\nu \in \mathcal{G}^*_C$, then there is an equilibrium distribution $\tau$ for $\nu$ such that $(\nu, \tau)$ is asymptotically robust.*

*Proof.* Fix $C \in \mathcal{C}$. Let $\mathcal{G}^*$ be defined as in the proof of Theorem 1. Set $\mathcal{G}^*_C = \mathcal{G}^* \cap \mathcal{G}_C$. Then $\mathcal{G}^*_C$ is relatively open in $\mathcal{G}_C$. As for density, note that the only step in the proof of Lemma 9 which requires perturbation of action set is step (b) there, and this step is not needed if $\nu \in \mathcal{G}_C$. Thus the assertion of Lemma 9 is true with $\mathcal{G}_C$ substituted for $\mathcal{G}$, and $\mathcal{G}^*_C$ for $\mathcal{G}^*$; in particular $\mathcal{G}^*_C$ is dense in $\mathcal{G}_C$. Now, with the appropriate substitutions, (e) of the proof of Theorem 1 yields (i) of the present theorem, and the proofs of Theorems 2 and 3, together with the fact noted in Remark 7, apply to establish the rest. □

## 4.7 Proof of Theorem 5

**Theorem 5.** *There is an open dense subset $\mathcal{M}_{\mathcal{K}}^*$ of $\mathcal{M}_{\mathcal{K}}$ such that if $\nu \in \mathcal{M}_{\mathcal{K}}^*$ and $\langle (I_n, G_n) \rangle_{n\in\mathbb{N}}$ is a sequence of Cournot oligopolies such that $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$, then there is an $N \in \mathbb{N}$ such that $(I_n, G_n)$ has a Cournot equilibrium if $\#(I_n) \geq N$.*

*Proof.* With $A = [0, m]$ and $g$ being the restriction to $A$ of the identity on $\mathbb{R}$, let $\mathcal{U}$, $\mathcal{M}$, and $\mathcal{G}$ be defined as in the general model. Note that $E = g(A) = [0, m]$. In particular, we have $E(\nu) = E$ for all $\nu \in \mathcal{M}$ because $E = g(A)$.

To each $v \in \mathcal{K}$ associate $u_v \in \mathcal{U}$ by setting $u_\nu(a, e) = p(e)a - v(a)$ for $a \in A$ and $e \in E$. Define a map $\kappa \colon \mathcal{K} \to \mathcal{U}$ by setting $\kappa(v) = u_v$ for each $v \in X$. Note that $\kappa$ is a homeomorphism from $\mathcal{K}$ onto $\kappa(\mathcal{K})$. Let $\tilde{\kappa} \colon \mathcal{M}_{\mathcal{K}} \to \mathcal{G}$ be defined by setting $\tilde{\kappa}(\nu) = \nu \circ \kappa^{-1}$ for each $\nu \in \mathcal{M}_{\mathcal{K}}$. As $\kappa$ is a homeomorphism from $\mathcal{K}$ onto $\kappa(\mathcal{K})$, $\tilde{\kappa}$ is a homeomorphism from $\mathcal{M}_{\mathcal{K}}$ onto $\tilde{\kappa}(\mathcal{M}_{\mathcal{K}})$.

Let $\mathcal{G}^* \subseteq \mathcal{G}$ be as guaranteed by Theorem 1. Then $\mathcal{M}_{\mathcal{K}}^* = \tilde{\kappa}^{-1}(\mathcal{G}^*)$ is open in $\mathcal{M}_{\mathcal{K}}$, because $\tilde{\kappa}$ is continuous. Let $\nu \in \mathcal{M}_{\mathcal{K}}^*$ and let $\langle (I_n, G_n) \rangle_{n\in\mathbb{N}}$ be a sequence of Cournot oligopolies such that $\#(I_n) \to \infty$ and $\nu_{G_n} \to \nu$. By continuity of $\tilde{\kappa}$, we have $\tilde{\kappa}(\nu_{G_n}) \to \tilde{\kappa}(\nu)$. For each $n$ and each $i \in I_n$, define $u_{n,i} \in \mathcal{U}$ by setting $u_{i,n}(a, e) = p\big(\frac{1}{\#(I_n)}a + \frac{\#(I_n - 1)}{\#(I_n}e\big)a - G_n(i)(a)$. For each $n$, let $\tilde{\nu}_n = \sum_{i \in I_n} \frac{1}{\#(I_n)} \delta_{u_{i,n}}$, so that $\tilde{\nu}_n \in \mathcal{G}$. Note that since $\tilde{\kappa}(\nu_n) \to \tilde{\kappa}(\nu)$ and $\#(I_n) \to \infty$, we also have $\tilde{\nu}_n \to \tilde{\kappa}(\nu)$ (use Billingsley (1968, Theorem 4.1)). Because $\tilde{\kappa}(\nu) \in \mathcal{G}^*$, it follows from Theorem 1 (and the definition of Nash equilibrium for finite-player games as stated before the formulation of Theorem 1) that $(I_n, G_n)$ has a Cournot equilibrium if $n$ is large.

It remain to show that $\mathcal{M}_{\mathcal{K}}^*$ is dense in $\mathcal{M}_{\mathcal{K}}$. As $\tilde{\kappa}$ is a homeomorphism, this means to show that $\mathcal{G}^* \cap \tilde{\kappa}(\mathcal{M}_{\mathcal{K}})$ is dense in $\tilde{\kappa}(\mathcal{M}_{\mathcal{K}})$. For this, the proof of Lemma 9 can be used, with the following modifications.

As the action sets of the players now coincide, step (b) of that proof is not needed. The same is true for step (c). Indeed, because $g$ is now the identity on the common action set $A$, (iii) of the definition of $\mathcal{G}_1$ holds automatically for any $\nu$ and any equilibrium distribution $\tau$ of $\nu$. (To see this, note that $g$ being the identity on $A$ implies that $E(\nu) = \int_{\mathcal{U}} \mathrm{co} A \, d\nu(u)$ and that, by condition (U3) in the definition of payoff functions, if $\tau$ is an equilibrium distribution for $\nu$, then $(u, a) \in \mathrm{supp}(\tau)$ implies that $a \in \mathrm{int}\, A$. Now use the facts that, for any $p \in \mathbb{R}^n$, $\max pE(\nu) = \int_{\mathcal{U}} \max pA \, d\nu(u)$ and that for any non-empty compact set $B \subseteq \mathbb{R}^n$, $\max pB$ is attained at a boundary point of $B$. For the former fact, see the proof of Lemma 5.)

Now adjustments are needed only because of the requirement that costs be zero at $a = 0$. The following changes in (a) and (d) of the proof of Lemma 9 are sufficient (the space $X$ which matters was defined in the setup of the example under consideration).

As for step (a), let $\rho \colon A \to X$ be a continuous map such that $\rho(a)(0) = 0$ for each $a \in A$ and if $a > 0$, then at $a' = a$, $\rho(a)$ has a unique maximum and $D^2\rho(a)(a')$ is negative definite. Then for each $k \in \mathbb{N}$, $a \in A$, and $u \in Y$ define $u_{k,a} \colon A \times E(\nu) \to \mathbb{R}$ by setting

$$u_{k,a}(a', e) = u(a', e) + \tfrac{1}{k+1}\rho(a)(a')$$

for $(a', e) \in A \times E(\nu)$.

As for step (d), make the following modification of the perturbation of the $u$'s into $u_\lambda$'s: Because $h(\cdot, \bar{e})\colon Y \to A$ is continuous and $Y$ is compact, we can find numbers $0 < r_1 < r_2$ such that $\bar{B}(h(u, \bar{e}), r_1) \subseteq B(h(u, \bar{e}), r_2) \subseteq \operatorname{int} A$ for each $u \in Y$, using Lemma 4. Let $\rho\colon Y \to X$ be a continuous map such that for each $u \in Y$, $\rho(u)(a) = 1$ if $a \in B(h(u, \bar{e}), r_1)$, $0 \leq \rho(u)(a) \leq 1$ for all $a \in A$, and $\rho(u)(a) = 0$ if $a \notin B(h(u, \bar{e}), r_2)$. Now for each $u \in Y$ and each $0 < \lambda < 1$ define $u_\lambda\colon A \times E(\nu)$ by setting $u_\lambda(a, e) = u(a, e) + \lambda\rho(u)(a)u(a, \bar{e})$ for $(a, e) \in A \times E(\nu)$. Arguing as in (d) of the proof of Lemma 9, we see that $D\xi_{\tau_\lambda}(\bar{e}) = \frac{1}{1+\lambda}\int_{\mathcal{U}} D_e h(u, \bar{e})\, d\nu(u) - I$, and we can finish as in (d) of that proof. □

# 5 Appendix

**Lemma 10.** *Let $A$ be a non-empty subset of $\mathbb{R}^n$, with dense interior, and $g\colon A \to \mathbb{R}^m$ a continuously differentiable function. Suppose that $g(O)$ affinely spans $\mathbb{R}^m$ whenever $O$ is a non-empty open set in $\mathbb{R}^n$ such that $O \subseteq A$. Then:*

*(a) If $A_1, \ldots, A_l$ are non-empty closed subsets of $A$ such that $\operatorname{int} A_i$ is dense in $A_i$ for each $i = 1, \ldots, l$, then $\operatorname{int} \frac{1}{l}\sum_{i=1}^{l} g(A_i)$ is dense in $\frac{1}{l}\sum_{i=1}^{l} g(A_i)$ if $l \geq m$.*

*(b) If $A'$ is a non-empty closed subset of $A$ such that $\operatorname{int} A'$ is dense in $A'$, then $\operatorname{int} \operatorname{co} g(A') \neq \emptyset$.*

*Proof.* (a) Fix an integer $l \geq m$. Let $\tilde{g}\colon A^l \to \mathbb{R}^m$ be the function defined by setting $\tilde{g}(a_1, \ldots, \alpha_l) = \sum_{i=1}^{l} g(a_i)$ for each $(a_1, \ldots, a_l) \in A^l$. Then $\tilde{g}$ is continuously differentiable, with derivative $D\tilde{g}(a_1, \ldots, a_l) = (Dg(a_1), \ldots, Dg(a_l))$ at each $(a_1, \ldots, a_l) \in A^l$. Let $\tilde{A}$ be the set of elements of $\operatorname{int} A^l$ at which the derivative of $\tilde{g}$ has rank $m$. Then $\tilde{A}$ is open. Moreover the restriction of $\tilde{g}$ to $\tilde{A}$ is an open map (see Guillemin and Pollack, 1974, p.25, Exercise 1).

We next show that $\tilde{A}$ is dense in $\operatorname{int} A^l$. We first claim that given any non-empty open and convex set $O \subseteq \mathbb{R}^n$ such that $O \subseteq A$, the set $C$ of all column vectors of the matrices $Dg(a)$ as $a$ runs over $O$ linearly spans $\mathbb{R}^m$. Suppose, if possible, otherwise. Then $C$ is included in a linear subspace $L$ of $\mathbb{R}^m$ with $\dim L < m$. Fix any $a_0 \in O$. Note that as $O$ is convex, we have $ta + (1-t)a_0 \in O$ for all $a \in O$. Consequently, for each $a \in O$ we have

$$g(a) - g(a_0) = \int_{[0,1]} \frac{d}{dt} g(ta + (1-t)a_0)\, dt = \int_{[0,1]} Dg(ta + (1-t)a_0)(a - a_0)\, dt \in L.$$

But this implies that $g(O) \subseteq L + \{g(a_0)\}$, contradicting the hypotheses about $g$.

Let $(a_1, \ldots, a_l) \in \operatorname{int} A^l$, and $U \subseteq \operatorname{int} A^l$ a neighborhood of this point. We can assume that $U$ is of the form $O_1 \times \cdots \times O_l$ where $O_i$ is convex neighborhood of $a_i$, $i = 1, \ldots, l$. By the previous paragraph we can choose an $a_1' \in O_1$ such that $Dg(a_1')$ has a non-zero column vector $v_1$. Again by the previous paragraph, we can choose an $a'_2 \in O_2$ such that $Dg(a'_2)$ has a column vector $v_2$ such that $v_1$ and $v_2$ are linearly independent. Continuing in the fashion we find points $a'_1, \ldots, a'_m$ such that each matrix $Dg(a'_i)$ has a column vector $v_i$ such that the matrix $(v_1, \ldots, v_m)$ has rank $m$. If $l > m$, let $a'_i$ be an arbitrary point of $O_i$ for $m < i \leq l$. Then the matrix $(Dg(a'_1), \ldots, Dg(a'_l))$ has rank $m$. Thus $\tilde{A}$ is dense in int $A^l$.

Now let $A_1, \ldots, A_l$ be non-empty closed subsets of $A$, all with dense interior. Then $\operatorname{int}(A_1 \times \cdots \times A_l) = \operatorname{int} A_1 \times \cdots \times \operatorname{int} A_l \neq \emptyset$, so $\operatorname{int}(A_1 \times \cdots \times A_l)$ is dense in $A_1 \times \cdots \times A_l$, and thus $\tilde{A} \cap \operatorname{int}(A_1 \times \cdots \times A_l)$ is open and dense in $A_1 \times \cdots \times A_l$. Because $g$ is continuous, it follows that the set $\{\frac{1}{l}\sum_{k=1}^{l} g(a_i) \colon (a_1, \ldots, a_l) \in \tilde{A} \cap \operatorname{int}(A_1 \times \cdots \times A_l)\}$ is dense in $\frac{1}{l}\sum_{i=1}^{l} g(A_i)$, and since $\tilde{g}$ is an open map, it follows that the former set is open. We conclude that int $\frac{1}{l}\sum_{i=1}^{l} g(A_i)$ is dense in $\frac{1}{l}\sum_{i=1}^{l} g(A_i)$.

(b) By (a), int $\frac{1}{m}\sum_{i=1}^{m} g(A') \neq \emptyset$, and of course $\frac{1}{m}\sum_{i=1}^{m} g(A') \subseteq \operatorname{co} g(A')$. □

**Lemma 11.** *Let $A$ be a non-empty subset of $\mathbb{R}^n$, with dense interior, let $m = kn$, where $k \in \mathbb{N}$, and let $g \colon A \to \mathbb{R}^m$ be given by setting*

$$g(a) = (a_{(1)}, a_{(1)}^2 \ldots a_{(1)}^k, a_{(2)}, a_{(2)}^2 \ldots a_{(2)}^k, \ldots, a_{(n)}, \ldots, a_{(n)}^k)$$

*for each $a \in A$, where the subscript $(h)$ means the $h$th coordinate of $a$, $h = 1, \ldots, n$. Then $g(O)$ affinely spans $\mathbb{R}^m$ whenever $O$ is a non-empty open set in $\mathbb{R}^n$ with $O \subseteq A$.*

*Proof.* Fix a non-empty open set $O \subseteq \mathbb{R}^n$ with $O \subseteq A$ and choose elements $a_1, \ldots, a_k$ in $O$ such that for each coordinate $h = 1, \ldots, n$ the points $a_{1,(h)}, a_{2,(h)}, \ldots, a_{k,(h)}$ are distinct. Reorder the columns of the matrix $\big(Dg(a_1), \ldots, Dg(a_k)\big)$ so as to get a block diagonal matrix

$$\begin{pmatrix} B_1 & 0 & \ldots & 0 \\ 0 & B_2, & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & B_n \end{pmatrix}$$

where $B_h$, $h = 1, \ldots, n$, is a $(k \times k)$-matrix of the form

$$B_h = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 2a_{1,(h)} & 2a_{2,(h)} & 2a_{3,(h)} & \ldots & 2a_{k,(h)} \\ 3a_{1,(h)}^2 & 3a_{2,(h)}^2 & 3a_{3,(h)}^2 & \ldots & 3a_{k,(h)}^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ka_{1,(h)}^{k-1} & ka_{2,(h)}^{k-1} & ka_{3,(h)}^{k-1} & \ldots & ka_{k,(h)}^{k-1} \end{pmatrix}.$$

Now for each $h$, the determinant of the matrix $B_h$ is just a positive multiple of the Vandermonde determinant, and thus non-zero because the points $a_{1,(h)}, a_{2,(h)}, \ldots, a_{k,(h)}$ are distinct. This shows that the matrix $\big(Dg(a_1), \ldots, Dg(a_k)\big)$ has rank $kn$, and therefore, since $kn = m$, that $g(O)$ cannot be included in an affine subspace of dimension smaller than $m$. □

**Lemma 12.** *Let $K$ be a non-empty compact subset of $\mathbb{R}^n$, with dense interior. Then there is a non-deceasing sequence $\langle K_k \rangle$ of non-empty compact subsets of $K$, all with dense interior, such that $K_k \subseteq \operatorname{int} K$ for each $k$ and both $\rho_H(K, K_k) \to 0$ and $\rho_H(\partial K, \partial K_k) \to 0$.*

*Proof.* For each $k \in \mathbb{N}$, let $K'_k = \{x \in K \colon \operatorname{dist}(x, \partial K)\} > 1/(k+1)$, and $K_k = \operatorname{cl} K'_k$. Then $K'_k$ is open for each $k$. Also, for sufficiently large $k$, $K'_k$ is non-empty; we may assume that this is true for each $k$. Further, $\partial K_k \subseteq \{x \in K \colon \operatorname{dist}(x, \partial K) = 1/(k+1)\}$ for each $k$, so, as $\partial K$ is compact, we need only show that given any $y \in \partial K$ and any $\epsilon > 0$, we have $\operatorname{dist}(y, \partial K_k) < \epsilon$ if $k$ is large enough. To see this, fix $y \in \partial K$ and $\epsilon > 0$. As $\operatorname{int} K$ is dense in $K$, there is an $x \in B(y, \epsilon) \cap \operatorname{int} K$. In particular, there is a $\bar{k} \in \mathbb{N}$ such that $x \in K'_k$ if $k > \bar{k}$. Pick any $k > \bar{k}$. Consider the line segment $Z = \{\lambda x + (1-\lambda) y \colon 0 \leq \lambda \leq 1\}$. Clearly $Z \subseteq B(y, \epsilon)$. But also, $Z \cap K_k$ is closed, and since $x \in K'_k$ and $y \notin K_k$, $Z$ must contain a boundary point of $K_k$. □

The following theorem is a special version of a fixed point result due to Mas-Colell (1983).

**Theorem 6.** *Let $X \subseteq \mathbb{R}^\ell$ and $Y \subseteq \mathbb{R}^m$ be compact convex sets with non-empty interior. Let $\Lambda \colon X \times Y \to X \times \mathbb{R}^m$ be a continuous function; write $\Lambda_X$ for $\operatorname{proj}_X \circ \Lambda$ and $\Lambda_Y$ for $\operatorname{proj}_{\mathbb{R}^m} \circ \Lambda$. Suppose there is an open set $U \subseteq \mathbb{R}^m$, with $Y \subseteq U$, and a continuously differentiable function $\zeta \colon U \to \mathbb{R}^m$ such that, setting $\gamma = \min\{\|\zeta(y)\| \colon y \in \partial Y\}$,*

*(a) for some $y^* \in \operatorname{int} Y$, $D\zeta(y^*)$ has full rank and $\zeta(y) = 0$ if and only if $y = y^*$ (so that, in particular, $\gamma > 0$);*

*(b) if $y \in \partial Y$ and $x = \Lambda_X(x, y)$, then $\|\Lambda_Y(x, y) - y - \zeta(y)\| < \gamma$.*

*Then $\Lambda$ has a fixed point, i.e., there is an $(x, y) \in X \times Y$ such that $\Lambda(x, y) = (x, y)$.*

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