An exact Fatou's lemma for Gelfand integrals by means of Young measure theory*

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Abstract

We show that an exact version of Fatou's lemma for Gelfand integrable functions can be obtained by combining Young measure techniques and results due to Balder (2000, 2002) with a purification result in Greinecker and Podczeck (2015).

Keywords: Gelfand integral, Fatou's lemma, Purification

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1 Introduction

In a seminal paper, Schmeidler (1970) established a version of Fatou's lemma for functions with values in a finite dimensional vector space. Subsequently, various authors established versions of Fatou's lemma for functions with values in a Banach space, both for the Bochner integral and the Gelfand integral. See Khan and Majumdar (1986), Balder (1988), Yannelis (1988, 1989, 1991), Balder and Hess (1995) for the case of Bochner integrable functions, and Cornet and Medecin (2002), Balder (2002), Cornet and Martins-da Rocha (2004), Balder and Sambucini (2005) for the case of Gelfand integrable functions. However, the versions of Fatou's lemma provided in these papers are approximate versions, "approximate" meaning that with a non-atomic measure space as domain the conclusion holds only in terms of a convex closure operation applied to the pointwise limiting sets of the given sequence of functions (see Remark 5 below). The reason why the convex closure operation is needed is the fact that Liapounoff's theorem fails in infinite-dimensional spaces (see the discussion in Section 7).

An exact version of Fatou's lemma for functions with an infinite-dimensional codomain needs special assumptions about the domain. In fact, under the hypothesis that the domain space be super-atomless (see Definition 3 below), an

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exact version of Fatou's lemma has been established by Khan and Sagara (2014) for Bochner integrable functions. Using nonstandard analysis, Loeb and Sun (2007) have established an exact version of Fatou's lemma for Gelfand integrable functions defined on an atomless Loeb measure space. Under additional boundedness assumptions on the sequence of functions, Khan, Sagara, and Suzuki (2015) recently established an exact version of Fatou's lemma for Gelfand integrable functions on general super-atomless probability spaces. We briefly discuss this latter result at the end of this introduction.

In this paper, we look at Fatou's lemma for Gelfand integrable functions from the perspective of Young measure theory. The fundamental idea of Young measure theory is that in many contexts the natural limit of a sequence of measurable functions is not an ordinary function, but rather a measure-valued function, a so-called Young measure. Indeed, as follows from the analysis in Balder (2002) or Balder and Sambucini (2005), an exact version of Fatou's lemma holds if one is content with the limiting object being a Young measure (see Theorem 1 below). Viewed that way, an exact Fatou lemma in terms of limit functions holds precisely when it is possible to replace the Young measure shown to exist in the limit by an equivalent function. A recent "purification"-result of the authors shows that this can be done when the underlying measure space allows for sufficient independent variation with respect to the limiting Young measure. More precisely, when the domain (Ω, Σ, μ) is atomless, there has to exist a sub-σ-algebra of Σ that is, probabilistically speaking, stochastically independent of the σ-algebra generated by the Young measure and on which the measure μ is still atomless. This relative notion of non-atomicity has resurfaced in various guises in measure theory, ergodic theory, and probability theory.[1] Our main results based on this notion are Theorems 2 and 4 in Section 4. Exact versions of Fatou's lemma when the domain is a hyperfinite Loeb probability space, as in Loeb and Sun (2007), or a general super-atomless probability space, can be obtained as easy corollaries of these results (see Sections 4–6).

As a typical application, we prove a result on weak*-compactness of the Gelfand integral of a multifunction on a super-atomless probability space (see the end of Section 6). The proof proceeds along classical lines, without the tensor product machinery employed in Podczeck (2008). Another difference is that no measurability assumption on the multifunction is needed.

In closing the introduction, we note that the method of proof employed in our paper to get an exact version of Fatou's lemma for the Gelfand integral differs significantly from that in Khan et al. (2015). The argument in Khan et al. (2015) combines a result from Cornet and Medecin (2002) on sequential compactness in spaces of Gelfand integrable functions with a result from Podczeck (2008) on the convexity and compactness of the Gelfand integral of a multifunction, this latter result being applied to the multifunction given by the pointwise limit sets of the given sequence of functions. This approach requires the sequence of functions to be integrably bounded. The approach in our paper uses

[1] See Maharam (1942), Rohlin (1952), Hanen and Neveu (1966), and Hoover and Keisler (1984).

the notion of tightness in spaces of Young measures and applies results from Balder (2000) to get a Young measure as a limit of a sequence of Gelfand integrable functions; this allows for a considerable relaxation of the boundedness assumptions in Khan et al. (2015). Purification then gives a function so that the Fatou type conclusion holds. Besides weakening the boundedness assumptions in Khan et al. (2015), our approach of combining Young measure convergence with purification is of independent interest and can be used to obtain other topological results for spaces of measurable functions on super-atomless probability spaces.

2 Notation and terminology

(1) Let Z be a Hausdorff topological space.

(a) If $\langle x_n \rangle$ is a sequence of points of Z, then $\operatorname{Ls} x_n$ is the set of $x \in Z$ such that $x = \lim x_k$ for some subsequence $\langle x_k \rangle$ of $\langle x_n \rangle$; if $\langle A_n \rangle$ is a sequence of subsets of Z, then $\operatorname{Ls} A_n$ is the set of $x \in Z$ such that $x = \lim x_k$ where $x_k \in A_k$ for all k and $\langle A_k \rangle$ is a subsequence of $\langle A_n \rangle$.

(b) $\mathcal{B}(Z)$ denotes the Borel σ-algebra of Z. $M_+(Z)$ denotes the set of all (non-negative) tight Borel measures on Z, and $M_+^1(Z)$ the subset consisting of the elements that are probability measures; "tight" means inner regular with respect to the compact subsets of Z.

(c) Below we will work with the narrow topology on $M_+(Z)$, or on subsets of $M_+(Z)$, like $M_+^1(Z)$, but only in a context in which Z is completely regular. In this case, the narrow topology on $M_+(Z)$ is the topology of pointwise convergence on the bounded continuous functions on Z, evaluation being given by integration.

(d) For $z \in Z$, δ_z denotes Dirac measure at z.

(e) A sequence $\langle \nu_n \rangle$ in $M_+(Z)$ is called (uniformly) tight if for every $\varepsilon > 0$ there is a compact $K \subseteq Z$ such that $\nu_n(Z \setminus K) < \varepsilon$ for all n.

(f) Let (Ω, Σ) be a measurable space. A Young measure from Ω to Z is a map $\gamma \colon \Omega \to M_+^1(Z)$ such that $\omega \mapsto \gamma(B)$ is measurable for each $B \in \mathcal{B}(Z)$. The set of all Young measures from Ω to Z is denoted by $\mathcal{R}(\Omega, Z)$.

(2) Let (Ω, Σ, μ) be a probability space, and X the dual of a Banach space Y.

(a) We regard Y as a set of functionals on X, and in this sense use the notation $y(x)$ if $y \in Y$ and $x \in X$.

(b) (X, weak^*) means X endowed with its weak*-topology; for a set $A \subseteq X$, (A, weak^*) means A with the subspace topology defined from (X, weak^*), and $\text{w}^*\text{-c}\ell\, A$ means closure of A in (X, weak^*). We note the following fact (see Schwartz, 1973, Corollary 2, p. 102).

Fact 1. *If Y is separable, then $(X, weak^*)$ is a Souslin space.*[2]

(c) $\text{w}^*\text{-}\operatorname{Ls} x_n$ and $\text{w}^*\text{-}\operatorname{Ls} A_n$ for sequences of points and subsets of X, respectively, mean $\operatorname{Ls} x_n$ and $\operatorname{Ls} A_n$ with respect to (X, weak^*).

[2]Recall that a topological space Z is a Souslin space if it is Hausdorff and if there is a continuous surjection from a Polish space onto Z.

(d) A map $f\colon \Omega \to X$ is Gelfand integrable if for each $y \in Y$ the map $y \circ f$ is integrable. In this case, for each $E \in \Sigma$ there is an element $x_E \in X$ such that $y(x_E) = \int_E y \circ f \, \mathrm{d}\mu$ for each $y \in Y$; the element x_Ω is called the Gelfand integral of f, and is denoted by $\int_\Omega f(\omega)\, \mathrm{d}\mu(\omega)$, or shortly by $\int f \, \mathrm{d}\mu$. We will use the following fact (see Thomas, 1975, Theorem 1).

Fact 2. *If Y is separable and (Ω, Σ, μ) is complete, then a Gelfand integrable function $f\colon \Omega \to X$ is $(\Sigma, \mathcal{B}(X, \textit{weak}^*))$-measurable.*[3]

(e) If $F\colon \Omega \to 2^X$ is a multifunction, then $\int_\Omega F(\omega)\, \mathrm{d}\mu(\omega)$, or $\int F \, \mathrm{d}\mu$, denotes the set $\{\int f \, \mathrm{d}\mu \colon f$ is a Gelfand integrable a.e. selection of $F\}$. We do not require a multifunction to have non-empty values or a measurable graph. In particular, $\int F \, \mathrm{d}\mu$ may be empty.

(f) If $f\colon \Omega \to \mathbb{R}$ is any function, then the *positive part* f^+ and the *negative part* f^- are the functions on Ω defined pointwise by $f^+(\omega) = \max\{0, f(\omega)\}$ and $f^-(\omega) = \max\{0, -f(\omega)\}$ respectively.

3 Preliminaries

Lemma 1. *Let X be the dual of a separable Banach space, and $\langle x_k \rangle_{k\in\mathbb{N}}$ a sequence in X. For $p > 0$, write B_p for the closed ball in X of center 0 and radius p. Then*

$$\text{w}^*\text{-Ls}\, x_k = \bigcup_{p\in\mathbb{N}\setminus\{0\}} \bigcap_{j\in\mathbb{N}} \text{w}^*\text{-c}\ell \Big\{ x_k \colon x_k \in B_p,\ k \geq j \Big\};$$

in particular, $\text{w}^*\text{-Ls}\, x_k$ *is a Borel set in* $(X, \textit{weak}^*)$.

Proof. For the fact that $\text{w}^*\text{-Ls}\, x_k$ is included in the other set in the claimed equality, just note that a weak*-convergent sequence in a dual Banach space is norm-bounded. For the reverse inclusion, note that in the dual of a separable Banach space, norm-bounded subsets are weak*-metrizable, which implies that a weak*-cluster point of a bounded sequence in such a space is the limit of a weak*-convergent subsequence. □

Remark 1. In the context of Lemma 1, $\text{w}^*\text{-Ls}\, x_k$ need not be weak*-closed. In fact, it need not even be weak*-sequentially closed. For instance, let X be the space of all bounded signed Borel measures on $[0,1] \times [0,1]$, so that X is the dual of the separable Banach space $C([0,1] \times [0,1])$ of all continuous functions on $[0,1] \times [0,1]$, endowed with the sup-norm. For each $m, n \in \mathbb{N}\setminus\{0\}$, let $x_{m,n} \in X$ be given by $x_{m,n} = (n+1)\delta_{(1/n,0)} - n\delta_{(1/n,1/m)}$. Re-index the family $\langle x_{m,n} \rangle$ as an ordinary sequence $\langle x_k \rangle$. Observe that $\delta_{(1/n,0)} \in \text{w}^*\text{-Ls}\, x_k$ for each $n \in \mathbb{N}\setminus\{0\}$. Now $\delta_{(1/n,0)} \to \delta_{(0,0)}$ in the weak*-topology of X, but since a weak*-convergent sequence in X must be bounded, $\delta_{(0,0)}$ cannot belong to $\text{w}^*\text{-Ls}\, x_k$.

[3] Completeness of (Ω, Σ, μ) is needed because integrability of a function from Ω to $\mathbb{R}$ requires measurability only for the completion of (Ω, Σ, μ).

Lemma 2. *Let X be the dual of a separable Banach space, and $\langle x_k \rangle_{k\in\mathbb{N}}$ a sequence in X. For each $m \in \mathbb{N}$, let $\nu_m \in M_+^1(X, \text{weak}^*)$ be given by $\nu_m = \frac{1}{m+1}\sum_{k=0}^{m} \delta_{x_k}$. Suppose the sequence $\langle \nu_m \rangle_{m\in\mathbb{N}}$ is tight and that for some $\nu \in M_+^1(X, \text{weak}^*)$, $\nu_m \to \nu$ in the narrow topology. Then $\nu(\mathrm{w}^*\text{-}\mathrm{Ls}\, x_k) = 1$.*

Proof. As above, for $p > 0$ let B_p be the closed ball in X of center 0 and radius p. Note that B_p is both compact and metrizable in the weak*-topology. Now because weak*-compact subsets of X are norm-bounded, and because the sequence $\langle B_p \rangle_{p\in\mathbb{N}\setminus\{0\}}$ is increasing, tightness of the sequence $\langle \nu_m \rangle$ implies that there is a sequence $\langle \varepsilon_p \rangle_{p\in\mathbb{N}}$ of non-negative real numbers such that $\varepsilon_p \to 0$ and $\nu_m(B_p) \geq 1 - \varepsilon_p$ for all $m \in \mathbb{N}$ and all $p \in \mathbb{N}\setminus\{0\}$. Down to the last paragraph of this proof, let p be a fixed but arbitrary element of $\mathbb{N}\setminus\{0\}$.

For every $m \in \mathbb{N}$, let $\nu_m^p \in M_+(B_p, \text{weak}^*)$ be the restriction of ν_m to $\mathcal{B}(B_p, \text{weak}^*)$.[4] The fact that (B_p, weak^*) is compact and metrizable implies that the set $\{\nu' \in M_+(B_p, \text{weak}^*) \colon \nu'(B_p) \leq 1\}$ is compact and metrizable for the narrow topology, so there are a $\nu^p \in M_+(B_p, \text{weak}^*)$ and a subsequence $\langle \nu_i^p \rangle_{i\in\mathbb{N}}$ of the sequence $\langle \nu_m^p \rangle_{m\in\mathbb{N}}$ such that $\nu_i^p \to \nu^p$ in the narrow topology of $M_+(B_p, \text{weak}^*)$. Set $H = \bigcap_{j\in\mathbb{N}} \mathrm{w}^*\text{-}\mathrm{c}\ell\big\{x_k \colon x_k \in B_p,\ k \geq j\big\}$. Let $x \in B_p \setminus H$. Then there are a $j \in \mathbb{N}$ and an open subset U_x of (B_p, weak^*), with $x \in U_x$, such that $x_k \notin U_x$ for $k > j$. By the definition of the measures ν_m^p and ν_m, it follows that $\nu_i^p(U_x) \to 0$, so $\nu^p(U_x) = 0$ because $\nu_i^p \to \nu^p$ in the narrow topology of $M_+(B_p, \text{weak}^*)$ and U_x is open in (B_p, weak^*).

Thus we have $B_p \setminus H = \bigcup_{x\in B_p\setminus H} U_x$, where each U_x is open in (B_p, weak^*) with $\nu^p(U_x) = 0$. As each weak*-compact $K \subseteq B_p \setminus H$ is included in the union of finitely many of these sets, tightness of ν^p implies $\nu^p(B_p \setminus H) = 0$. Now because $\nu_i^p \to \nu^p$ narrowly in $M_+(B_p, \text{weak}^*)$, and because $\nu_i^p(B_p) \geq 1 - \varepsilon_p$ for each i, we must have $\nu^p(B_p) \geq 1 - \varepsilon_p$, therefore $\nu^p(H) \geq 1 - \varepsilon_p$ because $\nu^p(B_p \setminus H) = 0$.

Let $\bar{\nu}^p \in M_+(B_p, \text{weak}^*)$ be the restriction of ν to $\mathcal{B}(B_p, \text{weak}^*)$. We claim that $\bar{\nu}^p \geq \nu^p$ as elements of $M_+(B_p, \text{weak}^*)$. To see this, it suffices to show that $\int q \,\mathrm{d}\bar{\nu}^p \geq \int q \,\mathrm{d}\nu^p$ whenever $q \colon B_p \to \mathbb{R}_+$ is continuous. Given such a q, the map $\tilde{q} \colon X \to \mathbb{R}_+$, defined by setting $\tilde{q}(x) = q(x)$ if $x \in B_p$ and $\tilde{q}(x) = 0$ otherwise, is upper semi-continuous on (X, weak^*), so we have $\int \tilde{q} \,\mathrm{d}\nu \geq \overline{\lim}_m \int \tilde{q} \,\mathrm{d}\nu_m$ since $\nu_m \to \nu$ narrowly in $M_+^1(X, \text{weak}^*)$ by hypothesis. By the choice of $\tilde{q}$, it follows that $\int q \,\mathrm{d}\bar{\nu}^p \geq \overline{\lim}_m \int q \,\mathrm{d}\nu_m^p$. On the other hand, for the subsequence $\langle \nu_i^p \rangle$ of $\langle \nu_m^p \rangle$, we have $\nu_i^p \to \nu^p$ in the narrow topology of $M_+(B_p, \text{weak}^*)$, and therefore $\int q \,\mathrm{d}\nu^p = \lim_i \int q \,\mathrm{d}\nu_i^p \leq \overline{\lim}_m \int q \,\mathrm{d}\nu_m^p$. Thus $\int q \,\mathrm{d}\bar{\nu}^p \geq \int q \,\mathrm{d}\nu^p$.

In particular, we must have $\nu(H) = \bar{\nu}^p(H) \geq \nu^p(H)$. Consequently, from the penultimate paragraph, we see that $\nu(H) \geq 1 - \varepsilon_p$.

Thus, as $p \in \mathbb{N}\setminus\{0\}$ was arbitrary in the arguments so far, we must have $\nu\big(\bigcap_{j\in\mathbb{N}} \mathrm{w}^*\text{-}\mathrm{c}\ell\{x_k \colon x_k \in B_p,\ k \geq j\}\big) \geq 1 - \varepsilon_p$ for all $p \in \mathbb{N}\setminus\{0\}$. As $\varepsilon_p \to 0$, we conclude by Lemma 1 that $\nu(\mathrm{w}^*\text{-}\mathrm{Ls}\, x_k) = 1$. □

[4] Recall that if Z is a topological space and $A \in \mathcal{B}(Z)$, then $\mathcal{B}(A) = \{B \in \mathcal{B}(Z) \colon B \subseteq A\}$.

Lemma 3. *Let (Ω, Σ, μ) be a complete probability space, X the dual of a separable Banach space, and $\langle f_n \rangle$ a sequence of $(\Sigma, \mathcal{B}(X, weak^*))$-measurable maps from Ω to X. Then the graph of the multifunction $\omega \mapsto \text{w}^*\text{-Ls}\, f_n(\omega)$, i.e., the set $\{(\omega, x) \in \Omega \times X \colon x \in \text{w}^*\text{-Ls}\, f_n(\omega)\}$, belongs to $\Sigma \otimes \mathcal{B}(X, weak^*)$.*

Proof. See Balder and Sambucini (2005, Lemma 4.5). □

Definition 1. Let (Ω, Σ, μ) be a probability space, and $\langle f_n \rangle$ a sequence of maps from Ω to a topological space Z. The sequence $\langle f_n \rangle$ is said to be *tight* if for each $\varepsilon > 0$ there is a compact $K \subseteq Z$ such that $\mu^*(f_n^{-1}(Z \setminus K)) < \varepsilon$ for all n, where μ^* is the outer measure defined from μ.

Convention 1. In the rest of this paper, for X a dual Banach space and $\langle f_n \rangle$ a sequence of maps from a probability space to X, we simply write tight to mean tight with respect to (X, weak^*).

Recall that if (Ω, Σ, μ) is a probability space, then a sequence $\langle \rho_n \rangle$ of functions from Ω to $\mathbb{R}$ is uniformly integrable if (1) ρ_n is integrable for each n, and (2) $\lim_{a \to \infty} \sup_n \int_{\{\omega \in \Omega \colon |\rho_n(\omega)| \geq a\}} |\rho_n| \, \mathrm{d}\mu = 0$; recall also that, in this case, the sequence $\langle \int \rho_n \, \mathrm{d}\mu \rangle$ is bounded (see Fremlin, 2001, 246B, C, I).

The following theorem summarizes facts established by Balder (2002), but slightly extended to include the case of tight sequences of Gelfand integrable functions.

Theorem 1. *Let (Ω, Σ, μ) be a complete probability space, X the dual of a separable Banach space Y, and $\langle f_n \rangle$ a sequence of Gelfand integrable functions from Ω to X. Write C for the set of all $y \in Y$ for which the sequence $\langle (y \circ f_n)^- \rangle$ is uniformly integrable. Suppose that the sequence $\langle f_n \rangle$ is tight. Then there are a subsequence $\langle f_k \rangle$ and a $\gamma \in \mathcal{R}(\Omega, (X, weak^*))$ such that:*

(a) $\underline{\lim} \int_\Omega (\rho \circ f_k)(\omega) \, \mathrm{d}\mu(\omega) \geq \int_\Omega \int_X \rho(x) \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\mu(\omega)$ *(in $\mathbb{R}_+ \cup \{+\infty\}$) for each weak*-lower semi-continuous function $\rho \colon X \to \mathbb{R}_+$;*

(b) for every $y \in C$, $\int_\Omega \int_X y(x) \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\mu(\omega)$ exists in $\mathbb{R} \cup \{+\infty\}$ and we have $\underline{\lim} \int_\Omega (y \circ f_k)(\omega) \, \mathrm{d}\mu(\omega) \geq \int_\Omega \int_X y(x) \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\mu(\omega)$;

(c) $\gamma(\omega)(\text{w}^*\text{-Ls}\, f_n(\omega)) = 1$ *a.e. in Ω.*

Proof. Each f_n is $(\Sigma, \mathcal{B}(X, \text{weak}^*))$-measurable (see)2(2)(d)). Therefore we get an element $\gamma_n \in \mathcal{R}(\Omega, (X, \text{weak}^*))$ for each n by setting $\gamma_n(\omega) = \delta_{f_n(\omega)}$, $\omega \in \Omega$.

The sequence $\langle \gamma_n \rangle$ is weak*-tight in the sense of Balder (2000, Definition 3.3 and Remark 3.4), i.e., for each $\varepsilon > 0$ there is a multifunction $G^\varepsilon \colon \Omega \to 2^X$, with weak*-sequentially compact values, such that $\int_\Omega^* \gamma_n(\omega)(X \setminus G^\varepsilon(\omega)) \, \mathrm{d}\mu(\omega) \leq \varepsilon$ for all n, where $\int^*$ means outer integral. Indeed, tightness of the sequence $\langle f_n \rangle$ says that for any $\varepsilon > 0$ there is a weak*-compact $K^\varepsilon \subseteq X$ with $\mu^*(f_n^{-1}(X \setminus K^\varepsilon)) \leq \varepsilon$ for all n; define $G^\varepsilon \colon \Omega \to 2^X$ by setting $G^\varepsilon(\omega) = K^\varepsilon$ for each $\omega \in \Omega$, and note that as $X = Y^*$ with Y separable, weak*-compact sets in X are weak*-sequentially compact.

Now by Balder (2000, proof of Theorem 3.8, Remark 4.2, and Theorem 4.7, (d)⇒(e)), weak*-tightness of the sequence $\langle \gamma_n \rangle$ and Fact 1 in 2(2)(b) imply that there are a $\gamma \in \mathcal{R}(\Omega, (X, \text{weak}^*))$ and a subsequence $\langle \gamma_k \rangle$ such that (1) for any $g\colon \Omega \times X \to \mathbb{R}$ such that $g(\omega, \cdot)$ is weak*-lower semi-continuous for each $\omega \in \Omega$ and such that the sequence of functions $\omega \mapsto g^-(\omega, f_k(\omega))\colon \Omega \to \mathbb{R}_+$, $k \in \mathbb{N}$, is uniformly integrable, we have, writing $g(\omega, f_k(\omega))$ for $\int_X g(\omega, x)\, \mathrm{d}\gamma_k(\omega)(x)$,

$$\underline{\lim} \int_\Omega^* g(\omega, f_k(\omega))\, \mathrm{d}\mu(\omega) \geq \int_\Omega^* \int_X g(\omega, x)\, \mathrm{d}\gamma(\omega)(x)\, \mathrm{d}\mu(\omega) > -\infty,$$

and (2), a.e. in Ω, the sequence $\langle \frac{1}{m+1} \sum_{k=0}^m \gamma_k(\omega) \rangle_{m \in \mathbb{N}}$ in $M_+^1(X, \text{weak}^*)$ is tight, with $\frac{1}{m+1} \sum_{k=0}^m \gamma_k(\omega) \to \gamma(\omega)$ narrowly in $M_+^1(X, \text{weak}^*)$. (We must have the "$> -\infty$" in (1), because if $g\colon \Omega \times X \to \mathbb{R}$ satisfies the hypotheses in (1), then so does g^-, and the uniform integrability hypothesis implies that the sequence $\langle \int_\Omega g^-(\omega, f_k(\omega))\, \mathrm{d}\mu(\omega) \rangle$ is bounded in $\mathbb{R}_+$; thus, for such a g, applying (1) to g^- gives $0 \leq \int_\Omega^* \int_X g^-(\omega, x)\, \mathrm{d}\gamma(\omega)(x)\, \mathrm{d}\mu(\omega) < \infty$.)

Note that if a map $g\colon \Omega \times X \to \mathbb{R}$, in addition to satisfying the hypotheses in (1), is $\Sigma \otimes \mathcal{B}(X, \text{weak}^*)$-measurable, then the outer integrals in the conclusion are integrals. Consequently, (1) applies to the maps $(\omega, x) \mapsto y(x)$ for $y \in C$ to give (b) of the theorem. Similarly, if $\rho\colon X \to \mathbb{R}_+$ is weak*-lower semi-continuous, (1) applies to $(\omega, x) \mapsto \rho(x)$ and we get (a); evidently, for a $g\colon \Omega \times X \to \mathbb{R}$ with non-negative values, the uniform integrability hypothesis in (1) is satisfied. Finally, note that by Lemma 2 and the fact that w*-Ls $f_k(\omega) \subseteq$ w*-Ls $f_n(\omega)$, (2) implies (c). □

For the purification argument below, the following fact is important.

Remark 2. In the context of Theorem 1, let Σ_1 be a sub-σ-algebra of Σ which is complete for $\mu \restriction \Sigma_1$ and assume that f_n is actually $(\Sigma_1, \mathcal{B}(X, \text{weak}^*))$-measurable for each n. Then, with the same C as originally given, Theorem 1 applies with $(\Omega, \Sigma_1, \mu \restriction \Sigma_1)$, so that, for the resulting Young measure γ, the map $\omega \mapsto \gamma(\omega)(B)$ is Σ_1-measurable for each $B \in \mathcal{B}(X, \text{weak}^*)$, but, of course, conclusions (a)–(c) continue to hold with (Ω, Σ, μ). (Just observe that $\mu \restriction \Sigma_1$-null sets are μ-null sets as well; in particular, concerning (a) and (b) of Theorem 1, a function defined $\mu \restriction \Sigma_1$-a.e. on Ω is also defined μ-a.e. on Ω. As for the tightness hypothesis, note that if f_n is $(\Sigma_1, \mathcal{B}(X, \text{weak}^*))$-measurable then, for any $B \in \mathcal{B}(X, \text{weak}^*)$, $(\mu \restriction \Sigma_1)^*(f_n^{-1}(B)) = (\mu \restriction \Sigma_1)(f_n^{-1}(B)) = \mu(f_n^{-1}(B)) = \mu^*(f_n^{-1}(B))$.)

Remark 3. As remarked in the introduction, Young measures can be interpreted as limits of sequences of measurable functions under suitable notions of convergence. In this regard, there are two intimately related notions of convergence that are relevant for the proof of Theorem 1, K-convergence and convergence coming from the narrow topology for Young measures; see Balder (2000) for details. The narrow topology for Young measures can be seen as generalization of the usual narrow topology for probability measures, and versions of Prohorov's theorem and Alexandroff's portmanteau theorem continue to hold. These are the principal tools in the proof of Theorem 1 for guaranteeing that the limiting Young measure exists and has the desired properties.

4 Main results

Our first main result below addresses the context in which versions of Fatou's lemma for Gelfand integrable functions were established in Balder (2002) and Cornet and Martins-da Rocha (2004). We follow Cornet and Martins-da Rocha (2004) in using the terminology introduced in the next definition.

Definition 2. Let (Ω, Σ, μ) be a probability space, X a dual Banach space, with norm $\|\cdot\|$, and $\langle f_n \rangle$ a sequence of Gelfand integrable functions from Ω to X.

(a) The sequence $\langle f_n \rangle$ is called *mean-norm-bounded* if $\sup_n \int \|\cdot\| \circ f_n \, \mathrm{d}\mu < \infty$ and the integrals involved are defined.

(b) The sequence $\langle f_n \rangle$ is called *uniformly integrable* if the sequence $\langle \|\cdot\| \circ f_n \rangle$ of functions from Ω to $\mathbb{R}_+$ is uniformly integrable.

(c) The sequence $\langle f_n \rangle$ is called *C-uniformly integrable* for a cone $C \subseteq Y$ if for each $y \in C$ the sequence $\langle (y \circ f_n)^- \rangle$ of functions from Ω to $\mathbb{R}$ is uniformly integrable.

In the sequel, for a subset C of a Banach space Y, C^0 denotes the negative polar cone in the dual space X, i.e., $C^0 = \{x \in X \colon y(x) \leq 0 \text{ for all } y \in C\}$.

Theorem 2. *Let (Ω, Σ, μ) be a complete probability space, X the dual of a separable Banach space Y, and $\langle f_n \rangle$ a sequence of Gelfand integrable functions from Ω to X. Suppose $\langle f_n \rangle$ is mean-norm-bounded and C-uniformly integrable for a cone $C \subseteq Y$. Let Σ_1 and Σ_2 be stochastically independent sub-σ-algebras of Σ. Suppose f_n is $(\Sigma_1, \mathcal{B}(X, \text{weak}^*))$-measurable for each n, and that $\mu \restriction \Sigma_2$ is atomless. Then*

$$\text{w}^*\text{-Ls} \int_\Omega f_n \, \mathrm{d}\mu \subseteq \int_\Omega \text{w}^*\text{-Ls}\, f_n(\omega) \, \mathrm{d}\mu(\omega) - C^0.$$

Proof. We will use the following notation. For $\gamma \in \mathcal{R}(\Omega, (X, \text{weak}^*))$, we write τ_γ for the uniquely determined probability measure with domain $\Sigma \otimes \mathcal{B}(X, \text{weak}^*)$ such that $\tau_\gamma(A \times B) = \int_A \gamma(\omega)(B) \, \mathrm{d}\mu(\omega)$ for each $A \times B \in \Sigma \otimes \mathcal{B}(X, \text{weak}^*)$, and for a $(\Sigma, \mathcal{B}(X, \text{weak}^*))$-measurable function $f \colon T \to X$, we write τ_f for the uniquely determined probability measure with domain $\Sigma \otimes \mathcal{B}(X, \text{weak}^*)$ such that $\tau_f(A \times B) = \mu(A \cap f^{-1}(B))$ for each $A \times B \in \Sigma \otimes \mathcal{B}(X, \text{weak}^*)$. Further, if τ is any probability measure with domain $\Sigma \otimes \mathcal{B}(X, \text{weak}^*)$, we write τ^X for the marginal measure on X.

Consider the σ-algebra $\Sigma_1 \subseteq \Sigma$. Let $\widetilde{\Sigma}_1$ be its completion for $\mu \restriction \Sigma_1$. Note that $\widetilde{\Sigma}_1 \subseteq \Sigma$, and that since Σ_1 and Σ_2 are stochastically independent, so are $\widetilde{\Sigma}_1$ and Σ_2. Replacing Σ_1 by $\widetilde{\Sigma}_1$, if necessary, we may therefore assume that Σ_1 is complete for $\mu \restriction \Sigma_1$.

Now suppose $\bar{x} \in \text{w}^*\text{-Ls} \int f_n \, \mathrm{d}\mu$ and let $\langle f_i \rangle$ be a subsequence of $\langle f_n \rangle$ such that $\int f_i \, \mathrm{d}\mu \to \bar{x}$ for the weak*-topology of X. Mean-norm-boundedness of the sequence $\langle f_n \rangle$ implies that the sequence $\langle f_i \rangle$ is tight. Hence, by Theorem 1 and what is stated in Remark 2, there are a subsequence $\langle f_k \rangle$ of $\langle f_i \rangle$ and a $\gamma \in \mathcal{R}(\Omega, (X, \text{weak}^*))$ such that (a)–(c) of that theorem hold for γ and $\langle f_k \rangle$ and such that for each $B \in \mathcal{B}(X, \text{weak}^*)$ the map $\omega \mapsto \gamma(\omega)(B)$ is Σ_1-measurable. Note also that we have $\int y \circ f_k \, \mathrm{d}\mu \to y(\bar{x})$ for each $y \in Y$ (so that, in particular, $\underline{\lim} \int_\Omega (y \circ f_k)(\omega) \, \mathrm{d}\mu(\omega)$ is the same as $\lim \int_\Omega (y \circ f_k)(\omega) \, \mathrm{d}\mu(\omega)$).

Because $\langle f_n \rangle$ is mean-norm-bounded, (a) of Theorem 1 implies that we have $\int_\Omega \int_X \|x\| \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\mu(\omega) < \infty$, so for each $y \in Y$, $\int_\Omega \int_X |y(x)| \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\mu(\omega)$ exists in $\mathbb{R}$. Therefore, by Tonelli's theorem for Young measures (see Neveu, 1965, Proposition III.2.1), the map $(\omega, x) \mapsto y(x) \colon \Omega \times X \to \mathbb{R}$ is τ_γ-integrable for each $y \in Y$, with $\int y(x) \, \mathrm{d}\tau_\gamma(\omega, x) = \int_\Omega \int_X y(x) \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\nu(\omega)$. Consequently the map $x \mapsto y(x) \colon X \to \mathbb{R}$ is τ_γ^X-integrable for each $y \in Y$, and $\int y(x) \, \mathrm{d}\tau_\gamma^X(x) = \int_\Omega \int_X y(x) \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\mu(\omega)$. Using (b) of Theorem 1, it follows that $\int y(x) \, \mathrm{d}\tau_\gamma^X(x) \le \underline{\lim} \int y \circ f_k \, \mathrm{d}\mu$ for each $y \in C$, and thus, from the previous paragraph, that $\int y(x) \, \mathrm{d}\tau_\gamma^X(x) \le y(\bar{x})$ if $y \in C$.

Since the map $\omega \mapsto \gamma(\omega)(B)$ is Σ_1-measurable for each $B \in \mathcal{B}(X, \text{weak}^*)$, the hypotheses on Σ_1 and Σ_2 imply that there is a $(\Sigma, \mathcal{B}(X, \text{weak}^*))$-measurable map $f \colon \Omega \to X$ such that τ_f and τ_γ agree on $\Sigma_1 \otimes \mathcal{B}(X, \text{weak}^*)$ (see Greinecker and Podczeck, 2015, Theorem 1), and thus τ_f^X and τ_γ^X agree on $\mathcal{B}(X, \text{weak}^*)$. By the penultimate paragraph, it follows that the map $\omega \mapsto (y \circ f)(\omega)$ is integrable for each $y \in Y$, and in particular that $\int (y \circ f)(\omega) \, \mathrm{d}\mu(\omega) = \int y(x) \, \mathrm{d}\tau_\gamma^X(x) \le y(\bar{x})$ for each $y \in C$. Thus f is Gelfand integrable, and $\int f \, \mathrm{d}\mu - \bar{x} \in C^0$.

Write $H = \{(\omega, x) \in \Omega \times X \colon x \in \text{w}^*\text{-Ls}\, f_k(\omega)\}$. Then $H \in \Sigma \otimes \mathcal{B}(X, \text{weak}^*)$ by Lemma 3, and by Fubini's theorem for Young measures, (c) of Theorem 1 implies $\tau_\gamma(H) = 1$. By hypothesis, each f_k is $(\Sigma_1, \mathcal{B}(X, \text{weak}^*))$-measurable, so by another application of Lemma 3, with $(\Omega, \Sigma_1, \mu \restriction \Sigma_1)$ in place of (Ω, Σ, μ), we see that, in fact, $H \in \Sigma_1 \otimes \mathcal{B}(X, \text{weak}^*)$. As τ_f and τ_γ agree on $\Sigma_1 \otimes \mathcal{B}(X, \text{weak}^*)$, it follows that $\tau_f(H) = 1$. Now $f(\omega) \in \text{w}^*\text{-Ls}\, f_k(\omega)$ if and only if $(\omega, f(\omega)) \in H$. As the map $\omega \mapsto (\omega, f(\omega))$ is measurable for Σ and $\Sigma \otimes \mathcal{B}(X, \text{weak}^*)$, it follows that $f(\omega) \in \text{w}^*\text{-Ls}\, f_k(\omega)$ a.e. in Ω. As $\text{w}^*\text{-Ls}\, f_k(\omega) \subseteq \text{w}^*\text{-Ls}\, f_n(\omega)$, we conclude that $\bar{x} \in \{\int f \, \mathrm{d}\mu\} - C^0 \subseteq \int_\Omega \text{w}^*\text{-Ls}\, f_n(\omega) \, \mathrm{d}\mu(\omega) - C^0$. □

To guarantee that the relative non-atomicity assumption in Theorem 2 is satisfied independently of the specific sequence $\langle f_n \rangle$, the next result assumes the underlying probability space to be super-atomless in the sense of the next definition. As follows from Section 7, this is the weakest assumption for this purpose.

Definition 3. A probability space (Ω, Σ, μ) is said to be *super-atomless* if for every $E \in \Sigma$ with $\mu(E) > 0$, the subspace of $L^1(\mu)$ consisting of the elements of $L^1(\mu)$ vanishing off E is non-separable.[5]

[5] See Podczeck (2008, Section 3) for equivalent definitions. Super-atomless probability spaces are called "saturated" in Khan and Sagara (2014).

Theorem 3. *Let (Ω, Σ, μ) be a complete probability space, and X the dual of a separable Banach space Y. Suppose (Ω, Σ, μ) is super-atomless. Then for any sequence $\langle f_n \rangle$ of Gelfand integrable functions from Ω to X which is mean-norm-bounded and C-uniformly integrable for a cone $C \subseteq Y$,*

$$\text{w}^*\text{-Ls} \int_\Omega f_n \, \mathrm{d}\mu \subseteq \int_\Omega \text{w}^*\text{-Ls}\, f_n(\omega) \, \mathrm{d}\mu(\omega) - C^0.$$

Proof. Given such a sequence $\langle f_n \rangle$, each f_n is $(\Sigma, \mathcal{B}(X, \text{weak}^*))$-measurable (see 2(2)(d)). Moreover, as Y is separable, (X, weak^*) is the union of a sequence of compact metrizable subspaces, so $\mathcal{B}(X, \text{weak}^*)$ is countably generated. There is therefore a countably generated σ-algebra $\Sigma_1 \subseteq \Sigma$ such that each f_n is actually $(\Sigma_1, \mathcal{B}(X, \text{weak}^*))$-measurable. Now if (Ω, Σ, μ) is super-atomless, then there is a σ-algebra $\Sigma_2 \subseteq \Sigma$ which is stochastically independent of Σ_1 and such that $\mu \restriction \Sigma_2$ is atomless (see Greinecker and Podczeck, 2015, Lemma 6). Thus the theorem follows from Theorem 2. □

Remark 4. If, in Theorems 2 and 3, it is assumed in addition that $\lim \int f_n \, \mathrm{d}\mu$ exists in (X, weak^*), then the conclusion of these theorems can be written in the form

$$\text{w}^*\text{-}\lim \int_\Omega f_n \, \mathrm{d}\mu \in \int_\Omega \text{w}^*\text{-Ls}\, f_n(\omega) \, \mathrm{d}\mu(\omega) - C^0,$$

where $\text{w}^*\text{-}\lim$ means limit in (X, weak^*).

Remark 5. If the probability space (Ω, Σ, μ) is atomless, but not super-atomless, then Theorem 3 holds approximate in the sense that the conclusion is

$$\text{w}^*\text{-Ls} \int_\Omega f_n \, \mathrm{d}\mu \subseteq \int_\Omega \text{w}^*\text{-}\overline{\text{co}}\; \text{w}^*\text{-Ls}\, f_n(\omega) \, \mathrm{d}\mu(\omega) - C^0$$

where $\text{w}^*\text{-}\overline{\text{co}}$ means weak*-closed convex hull. For this result, see Balder (2002), Balder and Sambucini (2005), and Cornet and Martins-da Rocha (2004).

Remark 6. The hypothesis in Theorems 2 and 3 that the sequence $\langle f_n \rangle$ be mean-norm-bounded is not minor. In fact, if the codomain X contains a copy of ℓ_∞, then, with any atomless probability space (Ω, Σ, μ), there is a Gelfand integrable function $f \colon \Omega \to X$ such that $\int \|\cdot\| \circ f \, \mathrm{d}\mu = \infty$.[6] Now given such an f, the constant sequence $\langle f_n \rangle$ where $f_n = f$ for each n provides an instance in which an exact version of Fatou's lemma holds trivially, but which is not covered by Theorems 2 or 3. On the other hand, such a constant sequence is tight in the sense of Definition 1 if $X = Y^*$ with Y separable. In fact, in this case, (X, weak^*) is a Souslin space by what was noted in 2(2)(b), so any Borel measure on (X, weak^*) is tight; in particular the (by 2(2)(d) well-defined) distribution of a Gelfand integrable function with values in X is tight.

[6] Let $\langle E_i \rangle_{i \in \mathbb{N}}$ be a partition of Ω into elements of Σ with $\mu(E_i) = 2^{-(i+1)}$, and define $f \colon \Omega \to \ell_\infty$ by $f(\omega) = 2^{i+1} e_i$ if $\omega \in E_i$, where e_i is the i-th unit base element of c_0; cf. Musiał (1991, Example 4.3).

Our second main result shows that, in the context of Theorem 2, for some specification of the range of the sequence $\langle f_n \rangle$ the hypothesis that this sequence be mean-norm-bounded may be relaxed into the hypothesis that it be tight. In this regard, recall that an ordered Banach space is a Banach space endowed with a vector order, and that the positive cone Y_+ in such a space Y is called generating if $Y_+ - Y_+ = Y$.

Theorem 4. *Let (Ω, Σ, μ) be a complete probability space, and X the dual of an ordered separable Banach space Y whose positive cone Y_+ is generating. Write X_+ for the dual positive cone in X. Let $\langle f_n \rangle$ and $\langle g_n \rangle$ be sequences of Gelfand integrable functions from Ω to X. Suppose that $\langle f_n \rangle$ is tight, that $\langle g_n \rangle$ is uniformly integrable, and that for every n, $f_n(\omega) - g_n(\omega) \in X_+$ a.e. in Ω. Let Σ_1 and Σ_2 be stochastically independent sub-σ-algebras of Σ. Suppose that f_n is $(\Sigma_1, \mathcal{B}(X, weak^*))$-measurable for each n, and that $\mu \upharpoonright \Sigma_2$ is atomless. Then*

$$\text{w}^*\text{-Ls} \int_\Omega f_n \, \mathrm{d}\mu \subseteq \int_\Omega \text{w}^*\text{-Ls} \, f_n(\omega) \, \mathrm{d}\mu(\omega) + X_+.$$

Proof. Suppose $\bar{x} \in \text{w}^*\text{-Ls} \int f_n \, \mathrm{d}\mu$ and let $\langle f_i \rangle$ be a subsequence of $\langle f_n \rangle$ such that $\int f_i \, \mathrm{d}\mu \to \bar{x}$ in the weak*-topology of X. Let $\langle g_i \rangle$ be the corresponding subsequence of $\langle g_n \rangle$. Of course, $\langle f_i \rangle$ is tight and $\langle g_i \rangle$ is uniformly integrable.

Note that for each $y \in Y_+$ the sequence $\langle (y \circ f_i)^- \rangle$ is uniformly integrable. This is so because the hypothesis that $f_n(\omega) - g_n(\omega) \in X_+$ a.e. in Ω for each n implies that for any $y \in Y_+$,

$$(y \circ f_i)^-(\omega) \le \left| y(g_i(\omega)) \right| \le \|y\| \, \|g_i(\omega)\|$$

a.e. in Ω for each $i \in \mathbb{N}$. Hence, since the sequence $\langle g_i \rangle$ is uniformly integrable, so is the sequence $\langle (y \circ f_i)^- \rangle$ for $y \in Y_+$.

Observe also that for each $y \in Y$ the sequence $\langle \int |y(f_i(\omega))| \, \mathrm{d}\mu(\omega) \rangle$ in $\mathbb{R}$ is bounded. Indeed, consider first a $y \in Y_+$. Then, again since $f_n(\omega) - g_n(\omega) \in X_+$ a.e. in Ω for each n, we have

$$\begin{aligned} \int |y(f_i(\omega))| \, \mathrm{d}\mu(\omega &\le \int |y(f_i(\omega)) - y(g_i(\omega))| \, \mathrm{d}\mu(\omega) + \int |y(g_i(\omega))| \, \mathrm{d}\mu(\omega) \\ &= \int y(f_i(\omega)) - y(g_i(\omega)) \, \mathrm{d}\mu(\omega) + \int |y(g_i(\omega))| \, \mathrm{d}\mu(\omega) \\ &\le \int y(f_i(\omega)) \, \mathrm{d}\mu(\omega) + 2\|y\| \int \|g_i(\omega)\| \, \mathrm{d}\mu(\omega) \end{aligned}$$

for all i. Now $\langle \int y(f_i(\omega) \, \mathrm{d}\mu(\omega) \rangle$ is bounded as $\langle \int f_i \, \mathrm{d}\mu \rangle$ is weak*-convergent, and as $\langle g_i \rangle$ is uniformly integrable, $\langle \int \|g_i(\omega)\| \, \mathrm{d}\mu(\omega) \rangle$ is also bounded. Thus $\langle \int |y(f_i(\omega))| \, \mathrm{d}\mu(\omega) \rangle$ is bounded for each $y \in Y_+$. As Y_+ is generating, each $y \in Y$ is a difference $y_1 - y_2$ for some y_1, $y_2 \in Y_+$, so for each $y \in Y$ we have $\int |y(f_i(\omega))| \, \mathrm{d}\mu(\omega) \le \int |y_1(f_i(\omega))| \, \mathrm{d}\mu(\omega) + \int |y_2(f_i(\omega))| \, \mathrm{d}\mu(\omega)$ for such y_1 and y_2, and it follows that $\langle \int |y(f_i(\omega))| \, \mathrm{d}\mu(\omega) \rangle$ is bounded for each $y \in Y$.

As in the proof of Theorem 2 we may assume that Σ_1 is complete for $\mu \upharpoonright \Sigma_1$. Now by Theorem 1 and Remark 2, it follows that there are a subsequence $\langle f_k \rangle$

of $\langle f_i \rangle$ and a $\gamma \in \mathcal{R}(\Omega, (X, \text{weak}^*))$ such that $\omega \mapsto \gamma(\omega)(B)$ is Σ_1-measurable for each $B \in \mathcal{B}(X, \text{weak}^*)$ and such that (a)–(c) of Theorem 1 hold for γ and $\langle f_k \rangle$, with $C \supseteq Y_+$ in (b).

By the penultimate paragraph, for each $y \in Y$, $\langle \int |y(f_i(\omega))| \, \mathrm{d}\mu(\omega) \rangle$ is a bounded sequence in $\mathbb{R}$, so by (a) of Theorem 1, $\int_\Omega \int_X |y(x)| \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\mu(\omega)$ exists in $\mathbb{R}$. Consequently, with τ_γ^X being defined as in the proof of Theorem 2, we may see as in that proof that the map $x \mapsto y(x) \colon X \to \mathbb{R}$ is τ_γ^X-integrable for each $y \in Y$, with $\int y(x) \, \mathrm{d}\tau_\gamma^X(x) \le y(\bar{x})$ for $y \in Y_+$. Arguing in a similar manner as in the proof of Theorem 2, we can conclude that there is a Gelfand integrable $f \colon \Omega \to X$ such that $\bar{x} \in \{\int f \, \mathrm{d}\mu\} + X_+ \subseteq \int_\Omega \text{w}^*\text{-Ls}\, f_n(\omega) \, \mathrm{d}\mu(\omega) + X_+$. □

Remark 7. A version of Theorem 4 with a super-atomless probability space as domain, similar to Theorem 3, can be obtained in the same way that Theorem 3 was obtained from Theorem 2.

5 Corollaries

In this section, we place the analysis from before into some specific contexts which are described in Definition 2(b) above and in Definitions 4 and 5 below. The results we will present are easy consequences of Theorem 3, and no further invocation of a purification argument is needed. We start with the case of a sequence of Gelfand integrable functions that is uniformly integrable (see Definition 2(b)). In this case, we actually get a result with an “upper closure”-type conclusion in addition to a Fatou-type one.

Theorem 5. *Let (Ω, Σ, μ) be a complete probability space, X the dual of a separable Banach space Y, and $\langle f_n \rangle$ a uniformly integrable sequence of Gelfand integrable functions from Ω to X. Let Σ_1 and Σ_2 be stochastically independent sub-σ-algebras of Σ. Suppose that f_n is $(\Sigma_1, \mathcal{B}(X, \text{weak}^*))$-measurable for each n, and that $\mu \restriction \Sigma_2$ is atomless. Then*

$$\emptyset \neq \text{w}^*\text{-Ls} \int_\Omega f_n \, \mathrm{d}\mu \subseteq \int_\Omega \text{w}^*\text{-Ls}\, f_n(\omega) \, \mathrm{d}\mu(\omega) \, .$$

Proof. To see that $\text{w}^*\text{-Ls} \int_\Omega f_n \, \mathrm{d}\mu$ is non-empty, note that uniform integrability of the sequence $\langle f_n \rangle$ implies mean-norm-boundedness, i.e., there is a $K > 0$ such that $\int \|f_n(\omega)\| \, \mathrm{d}\mu(\omega) \le K$ for all n. Hence, for any $y \in Y$, we have $|y(\int_\Omega f_n \, \mathrm{d}\mu)| \le \int_\Omega |y \circ f_n| \, \mathrm{d}\mu \le \|y\| K$ for all n. By the principle of uniform boundedness, it follows that the sequence $\langle \int f_n \, \mathrm{d}\nu \rangle$ is norm-bounded, so has a weak*-convergent subsequence because Y is separable.

For the inclusion, note that for all $y \in Y$, $\omega \in \Omega$, and n,

$$(y \circ f_n)^-(\omega) \le |(y \circ f_n)(\omega)| \le \|y\| \, \|f_n(\omega)\| .$$

From this, we see that uniform integrability of $\langle f_n \rangle$ implies that this sequence is C-uniformly integrable for $C = Y$. Thus, since $C^0 = \{0\}$ if $C = Y$, and since $\langle f_n \rangle$ is mean-norm-bounded, the inclusion follows from Theorem 2. □

Theorem 6. *Let (Ω, Σ, μ) be a super-atomless complete probability space, and X the dual of a separable Banach space Y. Then for any uniformly integrable sequence $\langle f_n \rangle$ of Gelfand integrable functions from Ω to X,*

$$\emptyset \neq \text{w}^*\text{-Ls} \int_\Omega f_n \,\mathrm{d}\mu \subseteq \int_\Omega \text{w}^*\text{-Ls}\, f_n(\omega)\,\mathrm{d}\mu(\omega)\,.$$

Proof. This follows from Theorem 5, in the same way as Theorem 3 was deduced from Theorem 2. □

Definition 4. Let X be the dual of a Banach space Y, and Λ a closed convex cone in (X, weak^*). Then Λ is said to have a *weak***-compact sole* if there is a $y \in Y$ with $y(x) > 0$ for all $x \in \Lambda \setminus \{0\}$ such that $\{x \in \Lambda\colon y(x) = 1\}$ is weak*-compact.

Definition 5. Let X be the dual of a Banach space Y, and Λ a closed convex cone in (X, weak^*). We say that Λ has the *norm approximation property* if there is a Λ-non-decreasing sequence $\langle y_i \rangle$ in Y such that $\|x\| = \sup_i y_i(x)$ for each $x \in \Lambda$ (where "Λ-non-decreasing" means that $y_i(x) \geq y_j(x)$ for every $x \in \Lambda$ if $i > j$).

Theorem 7. *Let (Ω, Σ, μ) be a complete probability space, X the dual of a separable Banach space Y, and $\langle f_n \rangle$ a sequence of Gelfand integrable functions from Ω to X. Let Σ_1 and Σ_2 be stochastically independent sub-σ-algebras of Σ, suppose that f_n is $(\Sigma_1, \mathcal{B}(X, \text{weak}^*))$-measurable for each n, and that $\mu \restriction \Sigma_2$ is atomless. Let Λ be a closed convex cone in (X, weak^*). Then*

$$\text{w}^*\text{-Ls} \int_\Omega f_n \,\mathrm{d}\mu \subseteq \int_\Omega \text{w}^*\text{-Ls}\, f_n(\omega)\,\mathrm{d}\mu(\omega) + \Lambda$$

if any one of the following two conditions is satisfied:

(1) (a) Λ has a weak*-compact sole and (b) for some uniformly integrable sequence $\langle \rho_n \rangle$ of functions $\rho_n\colon \Omega \to \mathbb{R}_+$, $f_n(\omega) \in \Lambda + \rho_n(\omega)B$ a.e. in Ω for all n, where B is the closed unit ball in X.

(2) (a) Λ has the norm approximation property and (b) for some uniformly integrable sequence $\langle g_n \rangle$ of Gelfand integrable functions $g_n\colon \Omega \to X$ we have $f_n(\omega) - g_n(\omega) \in \Lambda$ a.e. in Ω for all n.

Proof. Let $C = \{y \in Y\colon y(x) \geq 0 \text{ for all } x \in \Lambda\}$, so that $C^0 = -\Lambda$. Then (1)(b) as well as (2)(b) imply that the sequence $\langle f_n \rangle$ is C-uniformly integrable. Indeed, if (1)(b) holds, then for any $y \in C$, $(y \circ f_n)^-(\omega) \leq \|y\| \rho_n(\omega)$ a.e. in Ω for all n. Now (2)(b) implies (1)(b). To see this, just write $f_n = (f_n - g_n) + g_n$ and set $\rho_n = \|\cdot\| \circ g_n$, so that $g_n(\omega) \in \rho_n(\omega)B$ for all $\omega \in \Omega$.

Now suppose $\bar{x} \in \text{w}^*\text{-Ls} \int_\Omega f_n \,\mathrm{d}\mu$, and let $\langle f_k \rangle$ be a subsequence of $\langle f_n \rangle$ such that $\int f_k \,\mathrm{d}\mu \to \bar{x}$ in (X, weak^*). In particular, the sequence $\langle \int f_k \,\mathrm{d}\mu \rangle$ is bounded.

Suppose (1) holds. Then by Cornet and Martins-da Rocha (2004, Proposition 2.1(b)), weak*-convergence of $\langle \int f_k \,\mathrm{d}\mu \rangle$ and (1)(a) imply that the sequence $\langle f_k \rangle$ is mean-norm-bounded.

Suppose (2) holds and let $\langle y_i \rangle$ be chosen according to Definition 5. Note that uniform integrability of the sequence $\langle g_n \rangle$ implies that this sequence is mean-norm-bounded. Also, with D being a countable dense subset of the unit sphere of Y, we see that for each k,

$$\left\| \int g_k(\omega)\, \mathrm{d}\mu(\omega) \right\| = \sup_{y \in D} \left| y \left(\int g_k(\omega)\, \mathrm{d}\mu(\omega) \right) \right| = \sup_{y \in D} \left| \int y(g_k(\omega))\, \mathrm{d}\mu(\omega) \right|$$
$$\leq \int \sup_{y \in D} |y(g_k(\omega))|\, \mathrm{d}\mu(\omega) = \int \|g_k(\omega)\|\, \mathrm{d}\mu(\omega)\,.$$

Now if $h \colon \Omega \to X$ is any Gelfand integrable function with $h(\omega) \in \Lambda$ a.e. in Ω, then also $\int h\, \mathrm{d}\mu \in \Lambda$. Thus, using the monotone convergence theorem, we have

$$\left\| \int h\, \mathrm{d}\mu \right\| = \sup_i y_i \left(\int h\, \mathrm{d}\mu \right) = \sup_i \int y_i(h(\omega))\, \mathrm{d}\mu(\omega) = \int \sup_i y_i(h(\omega))\, \mathrm{d}\mu(\omega)$$

and therefore $\|\int h\, \mathrm{d}\mu\| = \int \|\cdot\| \circ h\, \mathrm{d}\mu$, by the choice of $\langle y_i \rangle$. In particular, we must have $\|\int f_k - g_k\, \mathrm{d}\mu\| = \int \|\cdot\| \circ (f_k - g_k)\, \mathrm{d}\mu$ for each k. Now, for each k,

$$\begin{aligned}
\int \|f_k(\omega)\|\, \mathrm{d}\mu(\omega) &\leq \int \|f_k(\omega) - g_k(\omega)\|\, \mathrm{d}\mu(\omega) + \int \|g_k(\omega)\|\, \mathrm{d}\mu(\omega) \\
&= \left\| \int (f_k(\omega) - g_k(\omega))\, \mathrm{d}\mu(\omega) \right\| + \int \|g_k(\omega)\|\, \mathrm{d}\mu(\omega) \\
&\leq \left\| \int f_k(\omega)\, \mathrm{d}\mu(\omega) \right\| + \left\| \int g_k(\omega)\, \mathrm{d}\mu(\omega) \right\| + \int \|g_k(\omega)\|\, \mathrm{d}\mu(\omega) \\
&\leq \left\| \int f_k(\omega)\, \mathrm{d}\mu(\omega) \right\| + 2 \int \|g_k(\omega)\|\, \mathrm{d}\mu(\omega)\,.
\end{aligned}$$

As both $\langle \|\int f_k\, \mathrm{d}\mu\| \rangle$ and $\langle \int \|\cdot\| \circ g_k\, \mathrm{d}\mu \rangle$ are bounded sequences in $\mathbb{R}$, it follows that the sequence $\langle f_k \rangle$ is mean-norm-bounded.

Thus, with any of (1) or (2), $\langle f_k \rangle$ is mean-norm-bounded. Also, since $\langle f_n \rangle$ is C-uniformly integrable, so is the subsequence $\langle f_k \rangle$. Thus, by Theorem 2, we have $\bar{x} \in \int_\Omega \text{w*-Ls}\, f_k(\omega)\, \mathrm{d}\mu(\omega) - C^0$, and therefore $\bar{x} \in \int_\Omega \text{w*-Ls}\, f_n(\omega)\, \mathrm{d}\mu(\omega) + \Lambda$, because $-C^0 = \Lambda$ and $\langle f_k \rangle$ is a subsequence of $\langle f_n \rangle$. □

Remark 8. (a) Condition (1)(a) in Theorem 7 holds whenever Y is an ordered Banach space whose positive cone has non-empty interior, and Λ is taken to be the positive cone in X for the dual order.

(b) A particular example in which both (1)(a) and (2)(a) of Theorem 7 hold is provided by taking for Y the space of continuous functions on a compact Hausdorff space K, endowed with the sup-norm, and for Λ the cone of all non-negative elements in the space of all bounded tight signed Borel measures on K.

(c) 2(b) in Theorem 7 implies (1)(b) there; see the proof of this theorem.

(d) A version of Theorem 7 for a super-atomless domain, analogous to Theorems 3 and 6, also holds.

6 Applications to multifunctions

In this section, we apply the previous results to multifunctions defined on a super-atomless probability space.[7] The theorems below on sequences of multifunctions all follow more or less directly from their counterparts in terms of functions; the point is just to look at the definitions of $\operatorname{Ls} x_i$ and $\operatorname{Ls} A_i$ for sequences of points and subsets of a topological space Z respectively.

Definition 6. Let (Ω, Σ, μ) be a probability space, and X a Banach space, with norm $\|\cdot\|$. A sequence $\langle F_n \rangle$ of multifunctions $F_n \colon \Omega \to 2^X$ is said to be *uniformly integrable* if there is a uniformly integrable sequence $\langle \rho_n \rangle$ of functions from Ω to $\mathbb{R}_+$ such that for each n, $\sup\{\|x\| : x \in F_n(\omega)\} \leq \rho_n(\omega)$ a.e. in Ω.

Theorem 8. *Let (Ω, Σ, μ) be a super-atomless complete probability space, X the dual of a separable Banach space, and $\langle F_n \rangle$ a uniformly integrable sequence of multifunctions $F_n \colon \Omega \to 2^X$. Then*

$$\text{w}^*\text{-Ls} \int_\Omega F_n(\omega)\, \mathrm{d}\mu(\omega) \subseteq \int_\Omega \text{w}^*\text{-Ls}\, F_n(\omega)\, \mathrm{d}\mu(\omega)\,,$$

and if $\int_\Omega F_n(\omega)\, \mathrm{d}\mu(\omega) \neq \varnothing$ for infinitely many n, then also

$$\text{w}^*\text{-Ls} \int_\Omega F_n(\omega)\, \mathrm{d}\mu(\omega) \neq \varnothing.$$

Proof. The theorem follows immediately from Theorem 6 together with (a) the facts that if $x \in \text{w}^*\text{-Ls} \int F_n(\omega)\, \mathrm{d}\mu(\omega)$ then $x \in \text{w}^*\text{-Ls} \int f_k(\omega)\, \mathrm{d}\mu$ for some sequence $\langle f_k \rangle$ of Gelfand integrable a.e. selections along a subsequence $\langle F_k \rangle$ of $\langle F_n \rangle$,[8] and that for such a sequence $\langle f_k \rangle$, $\text{w}^*\text{-Ls}\, f_k(\omega) \subseteq \text{w}^*\text{-Ls}\, F_n(\omega)$ for each $\omega \in \Omega$, and (b) the observation that uniform integrability of $\langle F_n \rangle$ implies that such a sequence $\langle f_k \rangle$ is uniformly integrable. □

The next definition is a translation of Definition 1 into terms of multifunctions; see also Remark 9 below.

Definition 7. Let (Ω, Σ, μ) be a probability space, and $\langle F_n \rangle$ a sequence of multifunction from Ω to a topological space Z. The sequence $\langle F_n \rangle$ is said to be *tight* if for each $\varepsilon > 0$ there is a compact $K \subseteq Z$ such that $\mu^*(F_n^-(Z \setminus K)) < \varepsilon$ for all n, where $F_n^-(Z \setminus K) = \{\omega \in \Omega \colon F_n(\omega) \cap (Z \setminus K) \neq \varnothing\}$ and μ^* is the outer measure defined from μ.

[7] We shall formulate results for multifunctions only in terms of a super-atomless domain, rather than also in the Σ_1-Σ_2 terms which matter in, say, Theorem 5, for the simple reason that the family of all Gelfand integrable selections of a multifunction F from a probability space (Ω, Σ, μ) to a dual Banach space X typically generates the entire σ-algebra Σ. For instance, suppose that $F(\omega) = B$ for all $\omega \in \Omega$, where B is the unit ball in X, and note that for any $E \in \Sigma$ and $x \in B$ the function $\omega \mapsto 1_E(\omega)x \colon \Omega \to X$ is a Gelfand integrable selection of F.

[8] We need to speak here in terms of a subsequence of $\langle F_n \rangle$ because $\int F_n(\omega)\, \mathrm{d}\mu(\omega)$ might be empty for some n.

Convention 2. In the sequel, for X a dual Banach space and $\langle F_n \rangle$ a sequence of multifunctions from a probability space to X, we simply write tight to mean tight with respect to (X, weak^*).

Theorem 9. *Let (Ω, Σ, μ) be a super-atomless complete probability space, and X the dual of an ordered separable Banach space Y whose positive cone Y_+ is generating. Write X_+ for the dual positive cone in X. Let $\langle F_n \rangle$ be a sequence of multifunction from Ω to X, and $\langle g_n \rangle$ a sequences of Gelfand integrable functions from Ω to X. Suppose that $\langle F_n \rangle$ is tight, that $\langle g_n \rangle$ is uniformly integrable, and that for every n, $F_n(\omega) - \{g_n(\omega)\} \subseteq X_+$ a.e. in Ω. Then*

$$\text{w}^*\text{-Ls} \int_\Omega F_n \, \mathrm{d}\mu \subseteq \int_\Omega \text{w}^*\text{-Ls}\, F_n(\omega) \, \mathrm{d}\mu(\omega) + X_+.$$

Proof. The theorem follows immediately from Theorem 4 and Remark 7, together with (a) of the previous proof and the observation that tightness of $\langle F_n \rangle$ implies that if $\langle F_k \rangle$ is a subsequence of $\langle F_n \rangle$ and f_k a Gelfand integrable a.e. selection of F_k for each k, then the sequence $\langle f_k \rangle$ is tight. □

Theorem 10. *Let (Ω, Σ, μ) be a complete probability space, X the dual of a separable Banach space Y, and $\langle F_n \rangle$ a sequence of multifunctions from Ω to X. Suppose any of (1) or (2) in Theorem 7 holds, but in (1)(b) with "$F_n(\omega) \subseteq \Lambda + \rho_n(\omega) B$" substituted for "$f_n(\omega) \in \Lambda + \rho_n(\omega) B$," and in (2)(b) with "$F_n(\omega) - \{g_n(\omega)\} \subseteq \Lambda$" substituted for "$f_n(\omega) - g_n(\omega) \in \Lambda$." If (Ω, Σ, μ) is a super-atomless, then*

$$\text{w}^*\text{-Ls} \int_\Omega F_n(\omega) \, \mathrm{d}\mu(\omega) \subseteq \int_\Omega \text{w}^*\text{-Ls}\, F_n(\omega) \, \mathrm{d}\mu(\omega) + \Lambda.$$

Proof. The theorem is a direct consequence of Theorem 7 and what is stated in Remark 8(d)), similar to the way Theorem 8 follows from Theorem 6. □

As a particular application of Theorem 6, we will now present a result on weak*-compactness of the Gelfand integral of a multifunction. For this, we need the following definition.

Definition 8. Let (Ω, Σ, μ) be a probability space, and X a Banach space, with norm $\|\cdot\|$. A multifunction $F \colon \Omega \to 2^X$ is said to be *integrably bounded* if there is an integrable function $\rho \colon \Omega \to \mathbb{R}_+$ such that $\sup\{\|x\| : x \in F(\omega)\} \leq \rho(\omega)$ for almost all $\omega \in \Omega$.

A result on convexity and weak*-compactness of the Gelfand integral of an integrably bounded multifunction was established in Podczeck (2008, Theorem 4). The argument there for the compactness part was rather involved. However, as the following proof shows, this part can be deduced easily by using Theorem 6. In particular, just as in the finite-dimensional context, no measurability assumptions on the multifunction are needed.

Theorem 11. *Let (Ω, Σ, μ) be a super-atomless complete probability space, X the dual of a separable Banach space, and $F \colon \Omega \to 2^X$ an integrably bounded multifunction such that $F(\omega)$ is weak*-closed for almost all $\omega \in \Omega$. Then $\int F \, \mathrm{d}\nu$ is convex and weak*-compact.*

Proof. By Podczeck (2008, Theorem 3), the hypothesis that (Ω, Σ, μ) is super-atomless implies that $\int F \, \mathrm{d}\nu$ is convex.[9] The hypothesis that F is integrably bounded implies that a sequence of Gelfand integrable a.e. selections of F is uniformly integrable. Consequently, by Theorem 6, $\int F \, \mathrm{d}\nu$ is weak*-sequentially closed, because $F(\omega)$ is weak*-closed a.e. in Ω. By the Krein-Šmulian theorem, a convex and weak*-sequentially closed subset of the dual of a separable Banach space is weak*-closed. Thus $\int F \, \mathrm{d}\nu$ is weak*-closed. Finally, by the principle of uniform boundedness, the hypothesis that F is integrably bounded implies that $\int F \, \mathrm{d}\mu$ is norm-bounded. We conclude that $\int F \, \mathrm{d}\mu$ is weak*-compact. □

We close this section with a remark relating Definition 7 to an alternative tightness definition for sequences of multifunctions, which is part of Definition 3.1 in Castaing and Saadoune (2009).

Remark 9. Let (Ω, Σ, μ) be a complete probability space, and X the dual of a separable Banach space. In this context, a sequence $\langle F_n \rangle$ of multifunctions from Ω to X is tight in the sense of Definition 7 if and only if is tight according to the following definition: For each $\varepsilon > 0$, there is a multifunction $G^\varepsilon \colon \Omega \to 2^X$, with weak*-compact values and graph belonging to $\Sigma \otimes \mathcal{B}(X)$, such that for each n there is an $A_n \in \Sigma$ with $\mu(A_n) < \varepsilon$ and such that $F_n(\omega) \subseteq G^\varepsilon(\omega)$ for all $\omega \in \Omega \setminus A_n$.[10] Indeed, to see that tightness according to Definition 7 implies tightness according to the alternative definition, just consider constant valued multifunctions. For the reverse implication, fix $\varepsilon > 0$ and let $G^\varepsilon \colon \Omega \to 2^X$ be chosen according to the alternative tightness definition. For each $p \in \mathbb{N} \setminus \{0\}$, let B_p be the closed ball in X of center 0 and radius p. Then for each $p \in \mathbb{N} \setminus \{0\}$, the set $E_p = \{\omega \in \Omega \colon G^\varepsilon(\omega) \cap X \setminus B_p \neq \emptyset\}$ belongs to Σ (use Castaing and Valadier, 1977, III.23, together with Fact 1 in 2(2)(b)). Observe that the sequence $\langle E_p \rangle_{p \in \mathbb{N} \setminus \{0\}}$ is non-increasing with $\bigcap_{p \in \mathbb{N} \setminus \{0\}} E_p = \emptyset$. There is therefore a $\bar{p}$ such that $\mu(E_{\bar{p}}) < \varepsilon$. Now for each n, $\{\omega \in \Omega \colon F_n(\omega) \cap X \setminus B_{\bar{p}} \neq \emptyset\} \subseteq A_n \cup E_{\bar{p}}$, so $\{\omega \in \Omega \colon F_n(\omega) \cap X \setminus B_{\bar{p}} \neq \emptyset\}$ must have outer measure smaller than 2ε for each n. Thus, as $\varepsilon > 0$ was arbitrary, and closed balls in X are weak*-compact, tightness according to Definition 7 follows.

7 Necessity

An exact version of Fatou's lemma in the forms of Theorems 3 or 6 fails if the underlying probability space (Ω, Σ, μ) is atomless but not super-atomless. This may be seen as a consequence of Liapounoff's theorem for finite-dimensional spaces together with the fact that Liapounoff's theorem fails in infinite-dimensional spaces. We show this here by an argument which amounts to a particular interpretation of Lemma 4 in Podczeck (2008), this latter result being based on a

[9] See also Greinecker and Podczeck (2013, Proposition 3(c)).

[10] If the graph of each F_n belongs to $\Sigma \otimes \mathcal{B}(X)$, this may be equivalently expressed by saying that $\mu(\{\omega \in \Omega \colon F_n(\omega) \subseteq G^\varepsilon(\omega)\}) \geq 1 - \varepsilon$ for each n.

construction that was used in Diestel and Uhl (1977, proof of Corollary IX.1.6) to show that Liapounoff's theorem fails in any infinite-dimensional Banach space.[11]

Let X be an infinite-dimensional Banach space, with norm $\|\cdot\|$, and (Ω, Σ, μ) an atomless probability space which is not super-atomless. Then there is a Bochner integrable function $f\colon \Omega \to X$ such that $1/2 \int_\Omega f \, d\mu \neq \int_E f \, d\mu$ for each $E \in \Sigma$ (see Podczeck, 2008, proof of Lemma 4). On the other hand, it is a consequence of Liapounoff's theorem for finite-dimensional spaces and the definition of the Bochner integral that the norm closure of the range of an indefinite Bochner integral is convex (see the proof of Theorem 6.10 in Diestel and Uhl, 1977), so there is a sequence $\langle A_n \rangle$ in Σ such that for the functions $f_n = f \times 1_{A_n}$ we have $\int_\Omega f_n \, d\mu \to 1/2 \int_\Omega f \, d\mu$ in the norm of X.

Now suppose X is actually the dual of a separable Banach space. Then, of course, each f_n is Gelfand integrable. By construction, a.e. in Ω we have $f_n(\omega) \in \{0, f(\omega)\}$ and therefore $\|\cdot\|\text{-Ls}\, f_n(\omega) = \text{w}^*\text{-Ls}\, f_n(\omega) \subseteq \{0, f(\omega)\}$. In addition, by the fact that norm convergence implies weak*-convergence, we have $\text{w}^*\text{-Ls} \int f_n \, d\mu = \|\cdot\|\text{-Ls} \int f_n \, d\mu = \{1/2 \int_\Omega f \, d\mu\}$. Now by 2(2)(d), any Gelfand integrable selection of the multifunction $\omega \mapsto \{0, f(\omega)\}$ must be measurable for $\mathcal{B}(X)$ and the completion of Σ, so must be equal a.e. in Ω to a function of the form $f \times 1_E$ for some $E \in \Sigma$. But for any $E \in \Sigma$, $f \times 1_E$ is Bochner integrable, so the Gelfand and the Bochner integral of $f \times 1_E$ are the same, and by the choice of f it follows that the multifunction $\omega \mapsto \text{w}^*\text{-Ls}\, f_n(\omega)$ has no Gelfand integrable selection h with $\int h \, d\mu \in \{1/2 \int_\Omega f \, d\mu\} = \text{w}^*\text{-Ls} \int f_n \, d\mu$. In particular,

$$\text{w}^*\text{-Ls} \int f_n \, d\mu \not\subseteq \int \text{w}^*\text{-Ls}\, f_n(\omega) \, d\mu(\omega) \, .$$

On the other hand, as f is Bochner integrable, the function $\omega \mapsto \|f(\omega)\|$ is integrable, and it follows that the sequence $\langle f_n \rangle$ is uniformly integrable in the sense of Definition 2. In particular, the sequence $\langle f_n \rangle$ is mean-norm-bounded and C-uniformly integrable with $C = Y$ (see the proof of Theorem 5). Thus Theorems 3 and 6 fail as a consequence of the assumption that (Ω, Σ, μ) is atomless but not super-atomless.

Remark 10. This argument shows also that an exact Fatou lemma for Bochner integrals fails if the underlying probability space is not super-atomless. This was already observed in Khan and Sagara (2014), where the argument is based in the same way as above on Podczeck (2008).

8 Extensions

The results in Sections 4 and 5 can be easily extended into results where, for a common function f, the conclusions hold against a sequence of probability measures on the domain of the functions f_n (see Theorems 13 and 14 below).

[11] This argument was also given in Khan et al. (2015). Since it is short and instructive, we repeat it for the reader's convenience.

In a Loeb space setting, results of this kind were demonstrated by Loeb and Sun (2007) using nonstandard analysis, and were shown to be useful in game theory.

For such an extension, the next theorem settles the basics in terms of Young measure theory. The theorem is a just a version of Theorem 1.

Theorem 12. *Let (Ω, Σ, μ) be a complete probability space, X the dual of a separable Banach space Y, and $\langle f_n \rangle$ a sequence of $(\Sigma, \mathcal{B}(X, weak^*))$-measurable functions from Ω to X. Write D for the set of all pairs (y, h), where $y \in Y$ and $h \colon \Omega \to \mathbb{R}_+$ is measurable, such that the sequence $\langle ((y \circ f_n) \times h)^- \rangle$ is uniformly integrable. Suppose that the sequence $\langle f_n \rangle$ is tight. Then there are a subsequence $\langle f_k \rangle$ and a $\gamma \in \mathcal{R}(\Omega, (X, weak^*))$ such that:*

(a) for each weak-lower semi-continuous function $\rho \colon X \to \mathbb{R}_+$, and each measurable function $h \colon \Omega \to \mathbb{R}_+$,*

$$\underline{\lim} \int_\Omega ((\rho \circ f_k) \times h)(\omega) \, \mathrm{d}\mu(\omega) \geq \int_\Omega \int_X \rho(x) h(\omega) \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\mu(\omega)$$

(in $\mathbb{R}_+ \cup \{+\infty\}$);

(b) for each $(y, h) \in D$, $\int_\Omega \int_X y(x) h(\omega) \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\mu(\omega)$ exists in $\mathbb{R} \cup \{+\infty\}$ and

$$\underline{\lim} \int_\Omega ((y \circ f_k) \times h)(\omega) \, \mathrm{d}\mu(\omega) \geq \int_\Omega \int_X y(x) h(\omega) \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\mu(\omega);$$

(c) $\gamma(\omega)(\mathrm{w}^*\text{-Ls}\, f_n(\omega)) = 1$ a.e. in Ω.

Proof. The proof of Theorem 1 applies, with the slight modification that (1) in that proof needs to be applied to maps of the form $(\omega, x) \mapsto \rho(x) h(\omega)$, and to the maps $(\omega, x) \mapsto y(x) h(\omega)$ for $(y, h) \in D$. □

Remark 11. In the context of Theorem 12, let Σ_1 be a sub-σ-algebra of Σ, and let $D_1 = \{(y, h) \in D \colon h \text{ is } \Sigma_1\text{-measurable}\}$. Suppose Σ_1 is complete for $\mu \restriction \Sigma_1$, and that f_n is $(\Sigma_1, \mathcal{B}(X, \mathrm{weak}^*))$-measurable for each n. Then, with D_1 in place of D, and Σ_1-measurable h's in (a), Theorem 12 is true with a Young measure γ such that for each $B \in \mathcal{B}(X, \mathrm{weak}^*)$, $\omega \mapsto \gamma(\omega)(B)$ is Σ_1-measurable; cf Remark 2.

Lemma 4. *Let (Ω, Σ, μ) be a complete probability space, $\langle \mu_j \rangle_{j \in \mathbb{N}}$ a sequence of probability measures with domain Σ, Z a Hausdorff space, and $\langle f_n \rangle$ a sequence of $(\Sigma, \mathcal{B}(Z))$-measurable functions from Ω to Z. Suppose that for each $j \in \mathbb{N}$ the sequence $\langle f_n \rangle$ is tight for μ_j, and that $\mu_j(E) > 0$ for some $j \in \mathbb{N}$ whenever $E \in \Sigma$ satisfies $\mu(E) > 0$. Then the sequence $\langle f_n \rangle$ is tight for μ.*

Proof. By the Vitali-Hahn-Saks theorem, setting $\bar{\mu} = \sum_{j=0}^{\infty} 2^{-(j+1)} \mu_j$ defines another probability measure with domain Σ. Now $\langle f_n \rangle$ is tight for $\bar{\mu}$. To see this, fix any $\varepsilon > 0$. There is an $i \in \mathbb{N}$ such that $\sum_{j=i+1}^{\infty} 2^{-(j+1)} \leq (1/2)\varepsilon$. As $\langle f_n \rangle$ is tight for each μ_j, there are compact subsets $K_0, \ldots, K_i$ of Z such that for each $j = 0, \ldots, i$, $\mu_j(f_n^{-1}(Z \setminus K_j)) < (1/2)\varepsilon$ for each n. Now $K = K_0 \cup \cdots \cup K_i$ is compact, and $\bar{\mu}(f_n^{-1}(Z \setminus K)) < \varepsilon$ for each n. As $\varepsilon > 0$ was arbitrary, $\langle f_n \rangle$ is tight for $\bar{\mu}$.

Now to see that $\langle f_n \rangle$ is tight for μ, again fix any $\varepsilon > 0$. Note that the hypotheses on μ and $\langle \mu_j \rangle_{j\in\mathbb{N}}$ imply that μ is absolutely continuous with respect to $\bar{\mu}$. Thus there is an $\varepsilon' > 0$ such that $\mu(E) < \varepsilon$ whenever $E \in \Sigma$ satisfies $\bar{\mu}(E) < \varepsilon'$. As $\langle f_n \rangle$ is tight for $\bar{\mu}$, there is a compact $K \subseteq Z$ such that $\bar{\mu}(f_n^{-1}(Z \setminus K)) < \varepsilon'$ for each n. Now $\mu(f_n^{-1}(Z \setminus K)) < \varepsilon$ for each n. As $\varepsilon > 0$ was arbitrary, $\langle f_n \rangle$ is tight for μ. $\square$

We need also the following, more or less well-known fact.

Fact 3. *Let (Ω, Σ, μ) be a probability space, and $\mu' \colon \Sigma \to \mathbb{R}$ a probability measure with a density h with respect to μ. If a set A' of functions from Ω to $\mathbb{R}$ is uniformly integrable for μ', then the set $A = \{\rho \times h \colon \rho \in A'\}$ is uniformly integrable for μ.*

(One way to see this is to note that the map $f \mapsto f \times h \colon L_1(\mu') \to L_1(\mu)$ is continuous for the weak topologies of $L_1(\mu')$ and $L_1(\mu)$, and that subsets of $L_1(\mu')$ or of $L_1(\mu)$ are uniformly integrable if and only if they are relatively weakly compact.)

Now as an example of an application of Theorem 12, we present the following variant of Theorem 5.

Theorem 13. *Let (Ω, Σ, μ) be a complete probability space, $\langle \mu_j \rangle$ a sequence of probability measures with domain Σ, X the dual of a separable Banach space Y, and $\langle f_n \rangle$ a sequence of $(\Sigma, \mathcal{B}(X, weak^*))$-measurable functions from Ω to X. Suppose:*

(i) For each $j \in \mathbb{N}$, each f_n is Gelfand integrable for μ_j and $\int f_n \,\mathrm{d}\mu_j \to \bar{x}_j$ in $(X, weak^*)$ for some $\bar{x}_j \in X$.

(ii) For each $j \in \mathbb{N}$, the sequence $\langle f_n \rangle$ is uniformly integrable for μ_j.

(iii) Each μ_j is absolutely continuous with respect to μ, and whenever $E \in \Sigma$ satisfies $\mu(E) > 0$, then $\mu_j(E) > 0$ for some j.

(iv) There are stochastically independent sub-σ-algebras Σ_1 and Σ_2 of Σ such that f_n is $(\Sigma_1, \mathcal{B}(X, weak^*))$-measurable for each n, μ_j has a Σ_1-measurable density h_j with respect to μ for each j, and $\mu \restriction \Sigma_2$ is atomless.

Then there is a function $f \colon \Omega \to X$ such that $f(\omega) \in \mathrm{w}^*\text{-}\mathrm{Ls}\, f_n(\omega)$ μ-a.e. in Ω and such that for each j, f is Gelfand integrable for μ_j, with $\int f \,\mathrm{d}\mu_j = \bar{x}_j$.

Proof. Note first that for each $j \in \mathbb{N}$, (ii) implies that $\sup_{n\in\mathbb{N}} \int \|\cdot\| \circ f_n \,\mathrm{d}\mu_j < k_j$ for some number $k_j < \infty$, so for each $\varepsilon > 0$, $\mu_j(\{\omega \in \Omega \colon \|f_n(\omega)\| > k_j/\varepsilon\}) < \varepsilon$ for every n, i.e., the sequence $\langle f_n \rangle$ is tight for μ_j, because closed balls in X are weak*-compact. Hence, by (iii) and Lemma 4, $\langle f_n \rangle$ is tight for μ.

Next, concerning the functions h_j from (iv), it follows from (i) that for each j and each n, the function $f_n \times h_j \colon \Omega \to X$ is Gelfand integrable for μ, and that $\int (y \circ f_n) \times h_j \,\mathrm{d}\mu = \int y \circ f_n \,\mathrm{d}\mu_j \to y(\bar{x}_j)$ for each $y \in Y$. Moreover, by Fact 3, (ii) implies that for each j the sequence $\langle (\|\cdot\| \circ f_n) \times h_j \rangle$ is uniformly integrable for μ. In particular, for each j and each $y \in Y$, the sequence $\langle (y \circ f_n) \times h_j \rangle$ is uniformly integrable for μ, and hence so is the sequence $\langle((y \circ f_n) \times h_j)^-\rangle$. Note that the fact that the sequence $\langle(\|\cdot\| \circ f_n) \times h_j\rangle$ is uniformly integrable for μ implies that the sequence $\langle\int(\|\cdot\| \circ f_n) \times h_j \, \mathrm{d}\mu\rangle$ in $\mathbb{R}$ is bounded.

As in the proof of Theorem 2 we may assume that Σ_1 is complete for $\mu \restriction \Sigma_1$. Now because of the measurability assumptions in (iv) on the maps f_n and h_j, it follows from Theorem 12 and Remark 11 that there is a $\gamma \in \mathcal{R}(\Omega, (X, \text{weak}^*))$ such that (1) for each $B \in \mathcal{B}(X, \text{weak}^*)$ the map $\omega \mapsto \gamma(\omega)(B)$ is Σ_1-measurable, (2) $\gamma(\omega)(\text{w}^*\text{-Ls}\, f_n(\omega)) = 1$ μ-a.e. in Ω, (3) $\int_\Omega \int_X \|x\| h_j(\omega) \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\mu(\omega) < \infty$ for each j (since $\langle\int(\|\cdot\| \circ f_n) \times h_j \, \mathrm{d}\mu\rangle$ is a bounded sequence in $\mathbb{R}$), and (4), $\int_\Omega \int_X y(x) h_j(\omega) \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\mu(\omega) = y(\bar{x}_j)$ for each j and each $y \in Y$ (the equalities here follow because the inequalities that correspond to Theorem 12(b) hold for both y and $-y$).

With τ_γ and τ_f defined as in the proof of Theorem 2, it now follows as in that proof that there is a $(\Sigma, \mathcal{B}(X, \text{weak}^*))$-measurable function $f \colon \Omega \to X$ such that τ_f and τ_γ agree on $\Sigma_1 \otimes \mathcal{B}(X, \text{weak}^*)$. As in the proof of Theorem 2, we see from this that $f(\omega) \in \text{w}^*\text{-Ls}\, f_n(\omega)$ μ-a.e. in Ω. Now, for each j and each $y \in Y$, the map $(\omega, x) \mapsto y(x) h_j(\omega)$ is $\Sigma_1 \otimes \mathcal{B}(X, \text{weak}^*)$-measurable, and by (3) we see that $\int_\Omega \int_X |y(x) h_j(\omega)| \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\mu(\omega)$ exists in $\mathbb{R}$. Hence, for each j and each $y \in Y$, the map $(\omega, x) \mapsto y(x) h_j(\omega)$ is τ_γ-integrable,[12] and we have

$$\begin{aligned}\int_\Omega \int_X y(x) h_j(\omega) \, \mathrm{d}\gamma(\omega)(x) \, \mathrm{d}\mu(\omega) &= \int_{\Omega \times X} y(x) h_j(\omega) \, \mathrm{d}\tau_\gamma(\omega, x) \\ &= \int_{\Omega \times X} y(x) h_j(\omega) \, \mathrm{d}\tau_f(\omega, x) = \int_\Omega (y \circ f) \times h_j \, \mathrm{d}\mu\,,\end{aligned}$$

because τ_f and τ_γ agree on $\Sigma_1 \otimes \mathcal{B}(X, \text{weak}^*)$. By the previous paragraph, it follows that $\int y \circ f \, \mathrm{d}\mu_j = \int (y \circ f) \times h_j \, \mathrm{d}\mu = y(\bar{x}_j)$ for each j and each $y \in Y$. We conclude that f is Gelfand integrable for each μ_j, with $\int f \, \mathrm{d}\mu_j = \bar{x}_j$. □

For a super-atomless probability space as domain, we have the following version of Theorem 13.

Theorem 14. *Let (Ω, Σ, μ) be a super-atomless complete probability space, and X the dual of a separable Banach space. Then, given any sequence $\langle f_n\rangle$ of $(\Sigma, \mathcal{B}(X, \text{weak}^*))$-measurable functions from Ω to X, and any sequence $\langle\mu_j\rangle$ of probability measures with domain Σ, if (i)–(iii) of Theorem 13 hold for $\langle\mu_j\rangle$ and $\langle f_n\rangle$, then there is a function $f \colon \Omega \to X$ such that $f(\omega) \in \text{w}^*\text{-Ls}\, f_n(\omega)$ μ-a.e. in Ω and such that for each j, f is Gelfand integrable for μ_j, with $\int f \, \mathrm{d}\mu_j = \bar{x}_j$.*

Proof. Let $\langle f_n\rangle$ and $\langle\mu_j\rangle$ be given. By hypothesis, each μ_j has a density h_j with respect to μ; let Σ_j be a countably generated sub-σ-algebra of Σ such that h_j is Σ_j-measurable. As in the proof of Theorem 3, there is a countably generated sub-σ-algebra of Σ, say Σ', such that f_n is Σ'-measurable for each n. Let Σ_1 be a countably generated sub-σ-algebra of Σ which includes Σ' and each Σ_j. Again as in the proof of Theorem 3, there is a sub-σ-algebra Σ_2 of Σ which is

[12] As in the proof of Theorem 2, this follows by Tonelli's theorem for Young measures.

stochastically independent of Σ_1 and such that $\mu \upharpoonright \Sigma_2$ is atomless. The theorem now follows from Theorem 13. □

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