Large Incomplete-Information Games*

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Abstract

We consider Bayes-Nash equilibria of large semi-anonymous games (i.e., each player's payoff is determined by his type, his action, and the distribution of the realized types and choices of the others). In a model with finite type and action spaces, we provide a characterization of limits of sequences of Bayes-Nash equilibria as the number of players goes to infinity. This gives a suitable notion of Bayes-Nash equilibria for non-atomic games with incomplete information. Bases on this, we show that strict pure-strategy Bayes-Nash equilibria exist in all sufficiently large finite-player games for generic distributions of players' payoff functions and type distributions.

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1 Introduction

Many economic problems feature a large number of participants, each of them having incomplete information about the characteristics of the others, and these characteristics matter, in general, for his/her well-being. Examples include electoral rules that determine the composition of parliament, income taxes that determine how much people work and save, and school allocation rules that determine which school will each student attend.

The analysis of such problems is simplified by assuming a continuum of participants. In this paper we provide a formalization of incomplete information games with a continuum of players, and in particular of an equilibrium notion for such games, by characterizing limits of mixed strategy Bayes-Nash equilibria of finite-player games with incomplete information as the number of players goes to infinity. This is done in a setting where, as in Kalai (2004), there are finite type and action spaces.

A central point that we use is that players' type distributions are independent, so that aggregate uncertainty disappears asymptotically if the number of agents in finite-player games becomes large. Exploiting this fact, we obtain an equilibrium notion for games with a continuum of players and incomplete information (but without aggregate uncertainty) that is suitable to capture limits of finite-player games with a growing number of participants. This equilibrium notion actually generalizes Mas-Colell's (1984) notion of an equilibrium distribution to the case of incomplete information, taking into account that the characteristics of a player consist now of a distribution over types and a payoff function depending on the joint distribution of realized types and actions chosen by the other players.[1]

In contrast, there is a literature that uses an individualistic approach to incom-

[1]Our equilibrium notion for incomplete-information games with a continuum of players differs from that introduced in Remark 4 of Mas-Colell (1984). Roughly, the difference is that while we focus on the joint distribution over types and actions, his notion focus on the distribution over measurable functions from a state space (e.g. the (uncountable) product of players' type spaces) to actions. This feature of Mas-Colell's (1984) approach creates technical difficulties, which we have not explored in this paper.

plete information games with a continuum of players, i.e. an approach with an explicit set of players, e.g. Sun (2006), and Sun and Zhang (2009). However, with a continuum of players, the individualistic approach requires the introduction of "rich" probability spaces of players and of states of the world that satisfy some technical requirements (which, in particular, imply that they cannot be the standard unit interval with Lebesgue measure). The notion of equilibrium distributions, which we focus on, dispenses with these elements and side steps the technical difficulties surrounding them. This make equilibrium distributions rather easy to use. We illustrate this point in Section 4 by considering a simplified (2-period) version of Krusell and Smith's (1998) macroeconomic model.

On top of the above, a point of our paper is that, by using our distributional equilibrium concept for games with a continuum of players and incomplete information, new results for large finite-player games can be obtained. Indeed, we establish an existence result for our equilibrium notion of games with a continuum of players. From this we deduce that sufficiently large finite-player games with a distribution of characteristics (i.e., payoff functions and type distributions) that belongs to a *generic* set of such distributions have a strict pure strategy Bayes-Nash equilibrium. This should be contrasted with Kalai (2004) and subsequent papers by Carmona (2008), Gradwohl and Reingold (2010), Carmona and Podczeck (2012) and Deb and Kalai (2015) where it is shown that, with finitely many players, mixed strategy Bayes-Nash equilibria can be approximately purified if the number of players is sufficiently large.

Using the generic asymptotic existence of pure strategy Bayes-Nash equilibria in large finite-player games, we obtain asymptotic implementability of the equilibria of games with a continuum of players and incomplete information. That is, all equilibria of such games turn out to be exactly the limit points of pure strategy Bayes-Nash equilibria of large finite-player incomplete-information games, i.e., are not artifacts of having continuum many players. In particular, the limits of mixed strategy Bayes-Nash equilibria along sequence of increasing finite-player games with incomplete information are the same as the limits of pure strategy Bayes-Nash equilibria. Thus, for characterizing Bayes-Nash equilibria of games with a continuum of players in terms of limits of Bayes-Nash equilibria of finite-player games, one can dispense with the notion of mixed strategy.

In summary, the main contributions of the paper are: first, to obtain a formalization and an equilibrium notion for games with a continuum of players and incomplete information which, at the same time, are easy to use in economic applications and connect well with mixed strategy Bayes-Nash equilibria of large finite-player games; and second, to obtain the generic asymptotic existence of pure strategy Bayes-Nash equilibria in large finite-player games and to show that pure strategies are enough to characterize the limits of all Bayes-Nash equilibria of such games.

The latter two results generalize analogous ones for Nash equilibria of large finite-player games with complete information obtained in our papers Carmona and Podczeck (2019) and Carmona and Podczeck (2020). A particular aspect of this generalization is, as pointed out above, to get a suitable notion of equilibrium distribution to capture limits of outcomes along sequences of finite-player games with an increasing number of players. This is done based on Carmona and Podczeck (2012), which uses material from the theory of large deviations (see Theorem 1 below).

Pure strategy Bayes-Nash equilibria have been shown to exist in finite-player incomplete-information games by Milgrom and Weber (1985) and He and Sun (2019), provided that each player has a continuum of possible types. In contrast, in our framework, players have finite type spaces and thus those results do not apply. In this respect, the importance of our results comes from the fact that in many applications just two types of players are considered, e.g., agents are of "high ability" or "low ability."

The paper is organized as follows. Our model and the results are presented in Section 2. Proofs can be found in Section 3. In Section 4 we consider Krusell and Smith's (1998) model.

2 The model and the results

In this section we describe our model and formulate our results. The framework we consider is the class of *semi-anonymous games* where each player has a finite set of types and actions and his payoff function is determined by his type, his choice and on the distribution of realized types and actions of the other players. The emphasis will be on whether there is a finite set of players or a continuum of them; in particular, we will omit the semi-anonymous qualifier. Moreover, for simplicity, we will speak of "equilibrium," rather than "Bayes-Nash equilibrium."

2.1 Finite-player games

A *finite-player game* is defined as follows. There is a finite set I of players, with $\#(I) \geq 2$, a nonempty finite type space T, common to all players, and a nonempty finite action space A, also common to all players. For each $i \in I$, player i's payoff function depends on the distribution of realized types and actions of the other players. The collection of all these distributions is the set $M(T \times A)$ of all probability measures on $T \times A$. The set $M(T \times A)$ is identified with the standard unit simplex of $\mathbb{R}^{\#(T)\#(A)}$ and endowed with its usual norm. Player i's payoff function is a continuous function $u_i : T \times A \times M(T \times A) \to \mathbb{R}$, with $u_i(t, a, \pi)$ being player i's payoff when he is of type t, plays action a and faces the distribution π on $T \times A$ induced by the types and actions of all other players. The distribution on $T \times A$ induced by a type-action profile $(\tilde{t}_{-i}, \tilde{a}_{-i})$ of all players other than i is the probability measure $\frac{1}{n-1} \sum_{j \in I \setminus \{i\}} 1_{(\tilde{t}_j, \tilde{a}_j)}$; here and elsewhere in this paper, $1_{(t,a)}$ denotes the Dirac measure at $\{(t, a)\}$.[2] For each $i \in I$, τ_i denotes player i's type distribution. A central assumption in our analysis is that type realizations are independent among players, so that the probability of a type realization $\tilde{t} \in T^I$ is the number $\prod_{i \in I} \tau_i(\tilde{t}_i)$. A mixed strategy (in distributional form) of player i is an element σ_i of $M(T \times A)$ such that the marginal on T is τ_i.[3]

[2]The set of distributions of realized types and actions of the players other that i that can actually occur is $E_i = \{\pi \in M(T \times A) : \pi = \sum_{j \in I \setminus \{i\}} \frac{1_{(t_j, a_j)}}{\#(I)-1}$ where $(t_j, a_j) \in T \times A$ for all $j \in I \setminus \{i\}\}$. Thus, as in Carmona and Podczeck (2020), we could have let the domain of player i's payoff function be $T \times A \times E_i$. All our results extend to this case; letting the domain of player i's payoff function be $T \times A \times M(T \times A)$ is made only for simplicity.

[3]This way of representing mixed or behavioral strategies was introduced into the literature by Milgrom and Weber (1985); see that paper for a discussion.

Given a profile $\sigma = \langle \sigma_i \rangle_{i \in I}$ of mixed strategies, the (expected) payoff of player i is

$$U_i(\sigma) = \int_{(T\times A)^I} u_i\left(\tilde{t}_i, \tilde{a}_i, \tfrac{1}{\#(I)-1} \sum_{j\in I\setminus\{i\}} 1_{(\tilde{t}_j,\tilde{a}_j)}\right) d\tilde{\sigma}(\tilde{t},\tilde{a}),$$

where $\tilde{\sigma}$ is the product measure defined from the σ_i's. This concludes the definition of a finite-player game. In short, a finite-player game can be described by a finite set I of players, with $\#(I) \geq 2$, and a function G assigning to each player his payoff function and type distribution, i.e., $G(i) = (u_i, \tau_i)$. We thus write (I, G) for such a game.

Let a finite-player game (I, G) be given. If σ is a mixed strategy profile, and σ'_i any mixed strategy of player $i \in I$, we write (σ'_i, σ_{-i}) for the profile in which player i chooses the mixed strategy σ'_i and the choices of the other players are the same as in σ. Now a mixed strategy profile σ is a *mixed strategy equilibrium* if $U_i(\sigma) \geq U_i(\sigma'_i, \sigma_{-i})$ for each $i \in I$ and σ'_i. It is a *strict equilibrium* if $U_i(\sigma) > U_i(\sigma'_i, \sigma_{-i})$ for each $i \in I$ and σ'_i such that $\sigma'_i \neq \sigma_i$. Note that a strict equilibrium is actually *pure*, i.e., for each player $i \in I$, the support of σ_i is the graph of a function from T to A.

The setting we consider in this paper is quite general and can therefore accommodate a wealth of applications. In particular, it can accommodate all the applications in Kalai (2004) as the two setting are, effectively, the same.[4]

2.2 Games with different numbers of players

To relate games with different numbers of players, we need to introduce a space of distributions of players' characteristics. For this, let $\mathcal{U}$ be the set of payoff functions according to Section 2.1, endowed with the sup-norm, and $M(T)$ the set of all type distributions, identified with the standard unit simplex of $\mathbb{R}^{\#(T)}$, the latter being endowed with its usual norm. Thus the space of players' characteristics is $\mathcal{U} \times M(T)$. We require that if the number of agents in finite-player games grows towards infinity, then these characteristics do not become too disperse. This can be accommodated by restricting the space of distributions of players' characteristics to the space of the Borel probability measures on $\mathcal{U} \times M(T)$ with compact support and by giving this

[4]Strictly speaking, the setting of Kalai (2004) is the one described in Footnote 2.

space the topology such that a sequence $\nu_n \to \nu$ of elements converges to an element ν if and only if both $\nu_n \to \nu$ in the narrow topology and $\operatorname{supp}(\nu_n) \to \operatorname{supp}(\nu)$ in the Hausdorff metric topology. We write $\mathcal{M}$ for this space.

The following notation will be used in the sequel. Let (I, G) be a finite-player game. We write $\nu_{(I,G)}$ for the element of $\mathcal{M}$ given as $\nu_{(I,G)} = \mu \circ G^{-1}$ where μ is the normalized counting measure of I, i.e. $\mu(J) = \#(J)/\#(I)$ for each $J \subseteq I$; thus $\nu_{(I,G)}$ is the distribution of players' characteristics in the game (I, G). Given a profile σ of mixed strategies, $\rho_{(I,G,\sigma)}$ is the probability measure on $\mathcal{U} \times M(T) \times T \times A$ determined by the condition that $\rho_{(I,G,\sigma)}(E \times B) = \int_{G^{-1}(E)} \sigma_i(B)\, d\mu(i)$ for all measurable sets $E \subseteq \mathcal{U} \times M(T)$ and $B \subseteq T \times A$. Note that the marginals of $\rho_{(I,G,\sigma)}$ on $\mathcal{U} \times M(T)$ and $T \times A$ are just $\nu_{(I,G)}$ and $1/\#(I) \sum_{i \in I} \sigma_i$ respectively. When σ is a mixed strategy equilibrium of (I, G), we refer to $\rho_{(I,G,\sigma)}$ as an *equilibrium distribution of* (I, G).

As pointed out in the introduction, the central idea of our analysis is that aggregate uncertainty disappears asymptotically if the number of players in finite-player games becomes large. The following result, which may be read off from Kalai (2004, Lemma 4) or Carmona and Podczeck (2012, Lemma 2), gives this a precise meaning. It is in this result where the assumption that type realizations are independent among players becomes crucial.

Lemma 1. *Let $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ be a sequence of finite-player games with $\#(I_n) \to \infty$, and $\langle \sigma_n \rangle$ a corresponding sequence of mixed strategy profiles. Then for each $0 < \varepsilon < 1$ there is an $N \in \mathbb{N}$ such that if $n \geq N$, then*

$$\Pi_{i \in I_n} \sigma_{n,i} \left(\left\{ (\tilde{t}, \tilde{a}) \in (T \times A)^{I_n} : \left\| \tfrac{1}{\#(I_n)} \sum_{i \in I_n} 1_{(\tilde{t}_i, \tilde{a}_i)} - \rho_{(I_n, G_n, \sigma_n), T \times A} \right\| > \varepsilon \right\} \right) < \varepsilon.$$

The proof uses results from combinatorial measure theory, which are in the spirit of the law of large numbers; see Carmona and Podczeck (2012) for the details.

Based on this result, the following theorem states necessary conditions for limit points of equilibrium distributions of finite-player games if the number of players grows towards infinity.

Theorem 1. *Let $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ be a sequence of finite-player games with $\#(I_n) \to \infty$ and $\nu_{(I_n, G_n)} \to \nu$ in $\mathcal{M}$, let $\langle \sigma_n \rangle$ be a corresponding sequence of mixed strategy equi-*

libria, and ρ a probability measure on $\mathcal{U} \times M(T) \times T \times A$. If $\rho_{(I_n,G_n,\sigma_n)} \to \rho$ narrowly, then ρ satisfies all of the following: (i) $\rho_{\mathcal{U}\times M(T)} = \nu$; (ii) for every measurable rectangle $E \times F$ in $(\mathcal{U} \times M(T)) \times T$, $\rho(E \times F \times A) = \int_E \tau(F)\, d\nu(u,\tau)$; and (iii) for every $(u,\tau,t,a) \in \operatorname{supp}(\rho)$, $u(t,a,\rho_{T\times A}) = \max_{a'\in A} u(t,a',\rho_{T\times A})$.

See Section 3.2 for a proof.

Theorem 1 suggests the following two definitions.

Definition 1. A continuum game (short for game with a continuum of players) is an element ν of $\mathcal{M}$, i.e., a probability measure on $\mathcal{U} \times M(T)$ with compact support.

Definition 2. An equilibrium distribution for a continuum game $\nu \in \mathcal{M}$ is a probability measure ρ on $\mathcal{U} \times M(T) \times T \times A$ such that (i) $\rho_{\mathcal{U}\times M(T)} = \nu$; (ii) for every measurable rectangle $E \times F$ in $(\mathcal{U} \times M(T)) \times T$, $\rho(E \times F \times A) = \int_E \tau(F)\, d\nu(u,\tau)$; and (iii) for ρ-almost every (u,τ,t,a), $u(t,a,\rho_{T\times A}) = \max_{a'\in A} u(t,a',\rho_{T\times A})$.

Conditions (i) and (iii) are as usual in the notion of equilibrium distributions. Condition (ii) is a consistency condition: In the context of Theorem 1, the equality $\rho_{(I_n,G_n,f_n)}(E\times F\times A) = \int_E \tau(F)\, d\nu_{(I_n,G_n)}(u,\tau)$ is necessarily satisfied for every measurable rectangle $E\times F$ in $(\mathcal{U}\times M(T))\times T$ (see the proof of Theorem 1). It requires that, relative to every measurable set E of players' characteristics, the marginal measure $\rho(E,\cdot,A)$ over types defined from ρ equals the mean distribution of players' types.[5]

Theorem 2. *Every continuum game $\nu \in \mathcal{M}$ has an equilibrium distribution.*

See Section 3.3 for the proof. It follows that of Mas-Colell (1984, Theorem 1), just taking care in addition of condition (ii) in Definition 2.

The next theorem gives an individualistic interpretation to equilibrium distributions of continuum games. Specifically, in such an individualistic interpretation, a

[5]Evidently the condition can be equivalently expressed by requiring that for each measurable set $E \subseteq \mathcal{U} \times M(T)$, $(\nu\lfloor E)_T = \int_E \tau\, d\nu_{M(T)}(\tau)$, where $\nu\lfloor E$ is the measure on $\mathcal{U} \times M(T)$ given by setting $(\nu\lfloor E)(B) = \nu(B \cap E)$ for each $B \subseteq \mathcal{U} \times M(T)$. If types are identically distributed across almost all players, say with distribution τ_0, the condition boils down to requiring that $\rho_T = \tau_0$. Since T is finite, the integral in the condition can be interpreted as the usual one for a finite-dimensional valued map. If T can be any compact metric space, the Gelfand integral has to be taken.

continuum game is represented by an explicit set of players, an explicit set of states of world which determine players' types, and a measurable function that assigns characteristics to players.

Theorem 3. *There is a probability space* $(I, \mathcal{I}, \mu)$ *of players and a probability space* $(\Omega, \Sigma, \lambda)$ *of states of nature such that for each* $\nu \in \mathcal{M}$ *and each equilibrium distribution* ρ *for* ν *there are a measurable map* $G\colon I \to \mathcal{U} \times M(T)$ *(depending only on* ν*) and measurable maps* $f\colon I \times \Omega \to A$ *and* $h\colon I \times \Omega \to T$ *such that*

(a) $\mu \circ G^{-1}$ *is defined and equals* ν*,*

(b) $\lambda \circ h(i, \cdot)^{-1} = \tau_i$ *for each* $i \in I$*,*

(c) for almost all $\omega \in \Omega$*,* $\mu \circ (h(\cdot, \omega) \times f(\cdot, \omega))^{-1}$ *is defined and equals* $\rho_{T \times A}$*,*

(d) For almost all $\omega \in \Omega$*,* $f(i, \omega) = \operatorname{argmax}_{a \in A} u_i(h(i, \omega), a, \rho_{T \times A})$ *for almost all* $i \in I$*.*

In particular, f *is an equilibrium pure strategy profiled in the game* (I, G)*, interpreting elements of* Ω *as abstract types and elements of* T *as payoff relevant types. Moreover, for almost every* $\omega \in \Omega$*, the strategy profile* $f(\cdot, \omega)$ *is a pure strategy equilibrium for the complete information game determined by the types realization* ω*.*

See Section 3.4 for the proof. It is based on the exact law of large numbers due to Sun (2006).

Remark 1. The compact support assumption on the elements of $\mathcal{M}$ is not needed for Theorems 2 and 3 to be valid.

The next theorem extends Carmona and Podczeck (2020, Theorem 1) to the case of incomplete information.

Theorem 4. *There is an open and dense subset* $\mathcal{M}^*$ *of* $\mathcal{M}$ *such that if* $\nu \in \mathcal{M}^*$ *and* $\langle (I_n, G_n) \rangle_{n \in \mathbb{N}}$ *is a sequence of finite-player games with* $\nu_{(I_n, G_n)} \to \nu$ *and* $\#(I_n) \to \infty$*, then there is* $N \in \mathbb{N}$ *such that* (I_n, G_n) *has a strict equilibrium whenever* $\#(I_n) \geq N$*.*

The idea of the proof of Theorem 4 is as follows (see Section 3.5 for the details). Let $\mathcal{M}^*$ be the subset of $\mathcal{M}$ consisting of those continuum games ν that have a strict equilibrium distribution, i.e., an equilibrium distribution ρ such that $\#(\varphi(u,t,\rho_{T\times A})) = 1$ for each $u \in \text{supp}(\nu_{\mathcal{U}})$ and $t \in T$. Denseness of $\mathcal{M}^*$ in $\mathcal{M}$ is established by perturbing the payoff functions in $\text{supp}(\nu)$ of any $\nu \in \mathcal{M}$. Openness of $\mathcal{M}^*$ is a consequence of the hypothesis that A and T are finite. The fact that $\mathcal{M}^*$ is open, together with an invocation of Lemma 1 (in the form stated as Lemma 3 below), is used to establish the theorem. The invocation of Lemma 1 is the principal difference to the proof of Carmona and Podczeck (2020, Theorem 1). The need arises due to the presence of incomplete information, which implies that there is some randomness on the profile of types and actions that realizes.

The final theorem gives a characterization of equilibrium distributions of continuum games in terms of equilibria of finite-player games. In particular it shows that no equilibrium distributions of any continuum game is a pure artifact of having continuum many players.

Theorem 5. *If ρ is an equilibrium distribution of a continuum game $\nu \in \mathcal{M}$, then there is a sequence $\langle (I_n, G_n)\rangle_{n\in\mathbb{N}}$ of finite-player games such that $\#(I_n) \to \infty$ and $\nu_{(I_n,G_n)} \to \nu$, and a strict equilibrium σ_n of (I_n, G_n) for each $n \in \mathbb{N}$ such that $\rho_{(I_n,G_n,\sigma_n)} \to \rho$ narrowly.*

See Section 3.6 for the proof. The basic idea is to show that any equilibrium distribution ρ of any continuum game ν can be approximated by equilibrium distributions ρ' for continuum games ν' in $\mathcal{M}^*$ such that ρ' witnesses that ν' belongs to $\mathcal{M}^*$. If this has been established, the assertion follows easily by a diagonal argument using Theorem 4 and the Glivanko-Cantelli-theorem.

3 Proofs

3.1 Lemmata

Lemma 2. *Let* $\langle \nu_n \rangle_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{M}$ *and* $\nu \in \mathcal{M}$. *If* $\nu_n \to \nu$ *in the narrow topology and for each* $\varepsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that* $\operatorname{supp}(\nu_n) \subseteq B_\varepsilon(\operatorname{supp}(\nu))$ *for all* $n \geq N$, *then* $\operatorname{supp}(\nu_n) \to \operatorname{supp}(\nu)$ *in the Hausdorff metric topology.*

Proof. See Carmona and Podczeck (2020). □

Lemma 3. *Let* $\langle (I_n, G_n) \rangle_{n\in\mathbb{N}}$ *be a sequence of finite-player games with* $\#(I_n) \to \infty$ *such that* $\nu_{(I_n,G_n)}$ *converges in* $\mathcal{M}$, *and let* $\langle \sigma_n \rangle$ *be a corresponding sequence of mixed strategy profiles. Then for each* $\varepsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that whenever* $n \geq N$ *then, for each* $i \in I_n$ *and each* $(t, a) \in T \times A$,

$$\Bigg| u_i\bigg((t, a, \tfrac{1}{\#(I_n)-1} \sum_{j \in I_n \setminus \{i\}} \sigma_j \bigg) \\ - \int_{(T\times A)^{I_n \setminus \{i\}}} u_i\bigg(t, a, \tfrac{1}{\#(I_n)-1} \sum_{j \in I_n \setminus \{i\}} 1_{(\tilde{t}_j, \tilde{a}_j)} \bigg) \, d\tilde{\sigma}_{n,-i}(\tilde{t}_{-i}, \tilde{a}_{-i}) \Bigg| < \varepsilon.$$

Proof. This can be proved as a consequence of Lemma 1; see Carmona and Podczeck (2012, Lemma 5) for details. □

Lemma 4. *Let* $\langle \rho_n \rangle$ *be a sequence of probability measures on* $\mathcal{U} \times M(T) \times T \times A$ *such that* $\rho_n \to \rho$ *narrowly for some probability measures* ρ *on* $\mathcal{U} \times M(T) \times T \times A$. *Write* ν *for the marginal measure of* ρ *on* $\mathcal{U} \times M(T)$, *and* ν_n *for that of* ρ_n *on* $\mathcal{U} \times M(T)$, $n \in \mathbb{N}$. *Suppose that for each* n, $\rho_n(E \times F \times A) = \int_E \tau(F)\, d\nu_n(u, \tau)$ *for every measurable rectangle* $E \times F$ *in* $(\mathcal{U} \times M(T)) \times T$. *Then also* $\rho(E \times F \times A) = \int_E \tau(F)\, d\nu(u, \tau)$ *for every such set* $E \times F \subseteq (\mathcal{U} \times M(T)) \times T$.

Proof. Write ρ_1 for the marginal measure of ρ on $\mathcal{U} \times M(T) \times T$, and $\rho_{1,n}$ for that of ρ_n on $\mathcal{U} \times M(T) \times T$, $n \in \mathbb{N}$. Note that for each bounded continuous function $p\colon \mathcal{U} \times M(T) \times T \to \mathbb{R}$, the map $(u, \tau) \mapsto \int_T p(u, \tau, t)\, d\tau(t)$ is continuous, and that $\int_{\mathcal{U}\times M(T)\times T} p\, d\rho_{1,n} = \int_{\mathcal{U}\times M(T)} \int_T p(u, \tau, t)\, d\tau(t)\, d\nu_n(u, \tau)$ for each n by hypothesis. Since narrow convergence of a sequence of measures on a product space implies narrow convergence of the corresponding sequences of marginal measures, it follows that $\int_{\mathcal{U}\times M(T)\times T} p\, d\rho_1 = \int_{\mathcal{U}\times M(T)} \int_T p(u,\tau,t)\, d\tau(t)\, d\nu(u,\tau)$ for each bounded continuous function $p\colon \mathcal{U}\times M(T)\times T \to \mathbb{R}$. This implies that $\rho_1(E\times F) = \int_E \tau(F)\, d\nu(u,\tau)$ for each measurable rectangle $E\times F$ in $(\mathcal{U}\times M(T))\times T$, and therefore we have $\rho(E\times F\times A) = \int_E \tau(F)\, d\nu(u,\tau)$ for such sets $E\times F$. □

3.2 Proof of Theorem 1

(i): Narrow convergence of $\langle \rho_{(I_n,G_n,\sigma_n}\rangle_{n\in\mathbb{N}}$ to ρ implies narrow convergence of the marginal measures of $\langle \rho_{(I_n,G_n,\sigma_n}\rangle_{n\in\mathbb{N}}$ to those of ρ. In particular, (i) must be true.

(ii): For every measurable rectangle $E\times(F\times A)$ in $\mathcal{U}\times M(T)\times T\times A$ and each n we have

$$\begin{aligned}\rho_{(I_n,G_n,\sigma_n)}(E\times F\times A) &= \int_{G_n^{-1}(E)} \sigma_{n,i}(F\times A)\, d\mu_n(i) = \int_{G_n^{-1}(E)} \tau_{n,i}(F)\, d\mu_n(i)\\ &= \int_{G_n^{-1}(E)} (\mathrm{proj}_{M(T)} \circ G_n)(F)\, d\mu_n = \int_E \tau(F)\, d\nu_n(u,\tau)\,.\end{aligned}$$

Thus, by Lemma 4, (ii) is true.

(iii): Note first that $\mathrm{supp}(\rho_{(I_n,G_n,\sigma_n)}) = \bigcup_{i\in I_n}(G_n(i)\times \mathrm{supp}(\sigma_{n,i}))$ for each n. That is, for each $(u,\tau,t,a)\in \mathrm{supp}(\rho_{(I_n,G_n,\sigma_n})$ there is an $i\in I_n$ such that $(u,\tau)=G_n(i)$ and $(t,a)\in\mathrm{supp}(\sigma_{n,i})$. Since σ_n is a mixed strategy equilibrium for every n, we have

$$\begin{aligned}&\int_{(T\times A)^{I_n\setminus\{i\}}} u(a_n, t_n, \tfrac{1}{\#(I_n)-1}\sum_{j\in I_n\setminus\{i\}} 1_{\tilde t_j,\tilde a_j})\, d\tilde\sigma_{n,-i}(\tilde t_{-i},\tilde a_{-i})\\ &\qquad = \max_{a'\in A}\int_{(T\times A)^{I_n\setminus\{i\}}} u(a', t_n, \tfrac{1}{\#(I_n)-1}\sum_{j\in I_n\setminus\{i\}} 1_{\tilde t_j,\tilde a_j})\, d\tilde\sigma_{n,-i}(\tilde t_{-i},\tilde a_{-i})\end{aligned}$$

for each $i\in I_n$ and $(t,a)\in\mathrm{supp}(\sigma_{n,i})$, where $\tilde\sigma_{n,-i}$ is the product measure defined from the σ_j's with $j\neq i$, $j\in I_n$. Combining these facts with Lemma 3, the fact that for any $(u,\tau,t,a)\in\mathrm{supp}(\rho)$ there is a sequence $\langle(u_n,\tau_n,t_n,a_n)\rangle_{n\in\mathbb{N}}$, converging to (u,τ,t,a), such that $(u_n,\tau_n,t_n,a_n)\in\mathrm{supp}(\rho_{(I_n,G_n,\sigma_n)})$ for each n, and with the fact that $\langle\rho_{(I_n,G_n,\sigma_n),A\times T}\rangle$ converges to $\rho_{T\times A}$ narrowly, we see that (iii) must be true.

3.3 Proof of Theorem 2

Fix any $\nu \in \mathcal{M}$. Let K be the set of those elements of $M(\mathcal{U} \times M(T) \times T \times A)$ for which (i) and (ii) in the definition of an equilibrium distribution are true for the given ν. Then K is a nonempty, closed, and convex subset of the set H of all probability measures on the compact set $\text{supp}(\nu) \times M(T) \times T \times A$, endowed with the narrow topology. (To see that K is closed in H, use Lemma 4; to see that K is non-empty, let ρ_1 be the measure on $\mathcal{U} \times M(T) \times T$ determined by requiring that $\rho_1(E \times F) = \int_E \tau(F)\, d\nu(u,\tau)$ for each measurable rectangle $E \times F$ in $(\mathcal{U} \times M(T)) \times T$ and consider the measure $\rho_1 \times 1_{a_o}$ where a_0 is an arbitrary element of A.)

For each $\rho \in K$, let

$$B_\rho = \{(u,\tau,t,a) \in \text{supp}(\nu) \times M(T) \times T \times A : a \in \varphi(u,t,\rho_{T\times A})\}.$$

Then B_ρ is closed and non-empty for each $\rho \in K$. We can therefore define a correspondence $\Phi : K \to 2^K$ by setting

$$\Phi(\rho) = \{\rho' \in K : \rho'(B_\rho) = 1\}$$

for each $\rho \in K$. Then Φ is closed with convex and non-empty-values (to see that Φ is closed, use Lemma 4 again; to see that Φ has non-empty values, fix any $\rho \in K$ and note that the correspondence $\varphi(\cdot,\cdot,\rho_{T\times A})$ has a measurable selection, which implies that there is measurable map $h : \text{supp}(\nu) \times M(T) \times T \to A$ whose graph is included in B_ρ, so that $\rho_{\mathcal{U}\times M(T)\times T} \circ (id_{\mathcal{U}\times M(T)\times T} \times h)^{-1} \in \Phi(\rho)$).[6] Hence, since H can be viewed as a compact convex subset of a locally convex vector space, Φ has a fixed point.

3.4 Proof of Theorem 3

Let $(I,\mathcal{I},\mu)$ be a super-atomless probability space. In particular, $(I,\mathcal{I},\mu)$ is atomless, so there is a measurable map $G\colon I \to \mathcal{U} \times M(T)$ such that $\nu = \mu \circ G^{-1}$. Write $\tau_i = \tau_{G(i)}$ for each $i \in I$ and let κ be the probability measure on $I \times T$ determined by the condition that $\kappa(E \times F) = \int_E \tau_i(F)\, d\mu(i)$ for every measurable rectangle

[6] Given any set X, id_X denotes the identity on X.

$E \times F \subseteq I \times T$. By the second condition in the definition of equilibrium distributions, we have $\rho_{\mathcal{U}\times M(T)\times T} = \kappa \circ (G \times id_T)^{-1}$. As μ is super-atomless, so is κ,[7] and thus there is a measurable function $g\colon I \times T \to A$ such that $\rho = \kappa \circ (G \times id_T \times g)^{-1}$. Again because μ is super-atomless, there is a probability space $(\Omega, \Sigma, \lambda)$ such that the product measure $\mu \times \lambda$ on $I \times \Omega$ has a rich Fubini extension, η say. Consequently there is a η-measurable map $h\colon I \times \Omega \to T$ such that the family $\langle h(i, \cdot)\rangle_{i\in I}$ is essentially pairwise independent and for each $i \in I$, $\lambda \circ h(i, \cdot)^{-1} = \tau_i$. In particular, the map $(i, \omega) \mapsto (i, h(i, \omega))$ is η-measurable with distribution κ. Thus, writing $\tilde{h}$ for this map, the composition $(G \times id_T \times g) \circ \tilde{h}$ is η-measurable with distribution ρ. Define $f\colon I \times \Omega \to A$ by setting $f(i, \omega) = g(i, h(i, \omega))$ for each $(i, \omega) \in I \times \Omega$. Then, since the family $\langle h(i, \cdot)\rangle_{i\in I}$ is essentially pairwise independent, so is the family $\langle h(i, \cdot) \times f(i, \cdot)\rangle_{i\in I}$.[8] By the exact law of large numbers, the distribution of $h(\cdot, \omega) \times f(\cdot, \omega)$ against μ is $\rho_{T\times A}$ for λ-almost all $\omega \in \Omega$. Thus h and f are as required.

3.5 Proof of Theorem 4

(a) Let $\mathcal{M}^*$ be the subset of $\mathcal{M}$ consisting of those ν which have a strict equilibrium distribution, i.e., an equilibrium distribution ρ such that $\#(\varphi(u, t, \rho_{T\times A})) = 1$ for each $u \in \operatorname{supp}(\nu_{\mathcal{U}})$ and $t \in T$.

(b) $\mathcal{M}^*$ is dense in $\mathcal{M}$. To see this, fix $\nu \in \mathcal{M}$ and let ρ be an equilibrium distribution of ν. Note that $\operatorname{supp}(\rho)$ is compact. Because A is finite, for each $u \in \mathcal{U}$, $a \in A$, and $n \in \mathbb{N}$ we can define $u_{a,n} \in \mathcal{U}$ by setting

$$u_{a,n}(t', a', \pi) = \begin{cases} u(t', a', \pi) + \frac{1}{n+1} & \text{if } a' = a \\ u(t', a', \pi) & \text{otherwise} \end{cases}$$

for each $(t', a', \pi) \in T \times A \times M(T \times A)$. Define $\kappa_n\colon \mathcal{U} \times M(T) \times T \times A \to \mathcal{U} \times M(T) \times T \times A$ by setting $\kappa_n(u, \tau, t, a) = (u_{a,n}, \tau, t, a)$ for each $(u, \tau, t, a) \in \mathcal{U} \times M(T) \times T \times A$. Note that each κ_n is continuous and that $\kappa_n \to id_{\mathcal{U}\times M(T)\times T\times A}$ uniformly as $n \to \infty$ (because $d(u_{a,n}, u) \leq 1/(n + 1)$ for each $u \in \mathcal{U}$, $t \in T$, $a \in A$ and $n \in \mathbb{N}$).

[7] If $\langle E_j\rangle_{j\in J}$ is a stochastically independent family in $\mathcal{I}$ for μ, then $\langle E_j \cap T\rangle_{j\in J}$ is a stochastically independent family in $\mathcal{I} \times 2^T$ for κ. Now use Podczeck (2010, Remark 1).

[8] This may be checked elementary.

Let $\rho_n = \rho \circ \kappa_n^{-1}$ and ν_n the marginal measure of ρ_n on $\mathcal{U} \times M(T)$. Because $\text{proj}_{T\times A} \circ \kappa_n = \text{proj}_{T\times A}$ for each n, we have $\rho_{n,T\times A} = \rho_{T\times A}$ for each n. Using the fact that $\text{supp}(\rho)$ is compact, it is easily seen that $\text{supp}(\rho_n) \subseteq \kappa_n(\text{supp}(\rho))$. Thus if $(u', \tau', t', a') \in \text{supp}(\rho_n)$, then for some $(u, \tau, t, a) \in \text{supp}(\rho)$, $u' = u_{a,n}$, $\tau' = \tau$, $t' = t$ and $a' = a$; but whenever $(u, \tau, t, a) \in \text{supp}(\rho)$, then $a \in \varphi(u, t, \rho_{T\times A})$ and thus $\{a\} = \varphi(u_{a,n}, t, \rho_{T\times A})$ by the choice of $u_{a,n}$, i.e., we have $\{a'\} = \varphi(u', t', \rho_{n,T\times A})$ since $\rho_{T\times A} = \rho_{n,T\times A}$. It follows that, for each n, ρ_n is a strict equilibrium distribution of ν_n and that $\nu_n \in \mathcal{M}^*$. To see that (ii) in Definition 2 is true for ρ_n, we can argue as follows. For each $a \in A$ define $h_{n,a}\colon \mathcal{U} \times M(T) \to \mathcal{U} \times M(T)$ by setting $h_{n,a}(u, \tau) = (u_{n,a}, \tau)$ for $(u, \tau) \in \mathcal{U} \times M(T)$ and. Fix any measurable set $E \subseteq \mathcal{U} \times M(T)$ and any $F \subseteq T$. Then

$$\rho_n(E \times F \times A) = \rho(\kappa_n^{-1}(E \times T \times A)) = \sum_{a\in A} \rho(h_{n,a}^{-1}(E) \times T \times A).$$

On the other hand,

$$\begin{aligned}
\int_E \tau(F)\, d\nu_n(u,\tau) &= \int_{E\times T\times A} \tau(F)\, d\rho_n(u,\tau,t,a) \\
&= \sum_{a\in A} \int_{E\times T\times A} 1_{\{a\}} \tau(F)\, d\rho_n(u,\tau,t,a) = \sum_{a\in A} \int_E 1_{\{a\}} \tau(F)\, d\nu_n(u,\tau) \\
&= \sum_{a\in A} \int_{h_{n,a}^{-1}(E)} \tau(F)\, d\nu(u,\tau)\,.
\end{aligned}$$

By (ii) in Definition 2, we have $\int_{h_{n,a}^{-1}(E)} \tau(F)\, d\nu(u,\tau) = \rho(h_{n,a}^{-1}(E) \times F \times A)$ for each $a \in A$, and it follows that (ii) in Definition 2 is true for ρ_n,

Finally, note that the fact that $\kappa_n \to id_{\mathcal{U}\times M(T)\times T\times A}$ uniformly as $n \to \infty$ implies that $\rho_n \to \rho$ narrowly. In particular, $\nu_n \to \nu$ narrowly. As with $\text{supp}(\rho_n)$, we can see that $\text{supp}(\nu_n) \subseteq (\text{proj}_{\mathcal{U}\times M(T)} \circ \kappa_n)(\text{supp}(\nu))$ for each n. Also, we have $\text{proj}_{\mathcal{U}\times M(T)} \circ \kappa_n \to id_{\mathcal{U}\times M(T)}$ uniformly; hence for each $\varepsilon > 0$, $\text{supp}(\nu_n) \subseteq B_\varepsilon(\text{supp}(\nu))$ for all n sufficiently large. By Lemma 2, it follows that $\text{supp}(\nu_n) \to \text{supp}(\nu)$. Thus $\nu_n \to \nu$ in the topology of $\mathcal{M}$.

(c) $\mathcal{M}^*$ is an open subset of $\mathcal{M}$. To see this, fix $\nu \in \mathcal{M}^*$ and let ρ be an equilibrium distribution of ν witnessing that $\nu \in \mathcal{M}^*$. Because $\text{supp}(\nu)$ is compact and T and A are finite, there is are an $\epsilon > 0$, neighborhoods V of $\text{supp}(\nu_{\mathcal{U}})$ in $\mathcal{U}$ and W of $\rho_{T\times A}$ in $M(T \times A)$, and a map $h\colon V \times T \times W \to A$ such that whenever $(u, \kappa) \in V \times W$ then

$\varphi(u,t,\kappa) = \{h(u,t,\kappa)\}$ and $u(t, h(u,t,\kappa), \kappa) > u(t,a,\kappa) + \epsilon$ for each $t \in T$ and $a \in A$ with $a \neq h(u,t,\kappa)$. Because φ is upper hemi-continuous, h is continuous, and hence so is the map $\tilde{h}\colon V \times M(T) \times T \times W \to A$ given by setting $\tilde{h}(u,\tau,t,\kappa) = h(u,t,\kappa)$ for each $(u,\tau,t,\kappa) \in V \times M(T) \times T \times W$. Let $\rho_1 = \rho_{\mathcal{U}\times M(T)\times T}$. Note that we have $\rho = \rho_1 \circ (id_{\mathcal{U}\times M(T)\times T} \times \tilde{h}(\cdot, \rho_{T\times A}))^{-1}$.

Let U be the set of all $\nu \in \mathcal{M}$ with $\operatorname{supp}(\nu_{\mathcal{U}}) \subseteq V$. For each $\nu' \in U$ let $\rho_{1,\nu'}$ be the probability measure on $\mathcal{U} \times M(T) \times T$ determined by $\rho_{1,\nu'} = \int_E \tau(F)\, d\nu'(\tau)$ for every measurable rectangle $E \times F$ in $(\mathcal{U} \times M(T)) \times T$. By Lemma 4, the map $\nu' \mapsto \rho_{1,\nu'}$ is continuous. For each $\nu' \in U$ let $\rho_{\nu'} = \rho_{1,\nu'} \circ (id_{\mathcal{U}\times M(T)\times T} \times \tilde{h}(\cdot, \rho_{T\times A}))^{-1}$. (Thus the assignment of actions to elements of $\mathcal{U} \times M(T) \times T$ is the same as in ρ.) Moreover, for each $\nu' \in U$ and $\kappa \in W$, let $\rho_{\nu'}(\kappa) = \rho_{1,\nu'} \circ (id_{\mathcal{U}\times M(T)\times T} \times \tilde{h}(\cdot, \kappa))^{-1}$. Then the map $(\kappa, \nu') \mapsto \rho_{\nu'}(\kappa)$ is continuous. As $\rho = \rho_1 \circ (id_{\mathcal{U}\times M(T)\times T} \times \tilde{h}(\cdot, \rho_{T\times A}))^{-1}$, it follows that there is a neighborhood U_1 of ν, with $U_1 \subseteq U$, such that $\rho_{\nu', T\times A} \in W$ for each $\nu' \in U_1$ and $\kappa \in W$. Thus $\mathcal{M}^*$ is open in $\mathcal{M}$, by the choices of $\tilde{h}$ and W.

(d) Let $\nu \in \mathcal{M}^*$ and $\langle (I_n, G_n) \rangle_{n\in\mathbb{N}}$ a sequence of finite-player games such that $\nu_{(I_n,G_n)} \to \nu$ and $\#(I_n) \to \infty$. Let ρ be an equilibrium distribution of ν, witnessing that $\nu \in \mathcal{M}^*$. Let U_1, ϵ, and $\tilde{h}$ be as in (c). As $\nu_{(I_n,G_n)} \to \nu$ there is an $N \in \mathbb{N}$ such that $\nu_{(I_n,G_n)} \in U_1$ for $n \geq N$.

Fix such an n. For each $i \in I_n$, let $h_{n,i}\colon T \to T \times A$ be the map defined by setting $h_{n,i}(t) = \big(t, \tilde{h}(G_n(i), t, \rho_{\nu, T\times A})\big)$ for each $t \in T$ and set $\sigma_{n,i} = \tau_{n,i} \circ h_{n,i}^{-1}$. Then, for any measurable rectangle $F \times H$ in $T \times A$,

$$
\begin{aligned}
\tfrac{1}{\#(I_n)} \sum\nolimits_{i\in I_n} \sigma_{n,i}(F\times H) &= \int_{I_n} \sigma_{n,i}(F \times H)\, d\mu_n(i) \\
&= \int_{I_n} \tau_{n,i}(\{t \in F\colon \tilde{h}(G_n(i), t, \rho_{\nu,T\times A}) \in H\})\, d\mu_n(i) \\
&= \int_{\mathcal{U}\times M(T)} \tau(\{t \in F\colon \tilde{h}(u,\tau,t,\rho_{\nu,T\times A}) \in H\})\, d\nu_{(I_n,G_n)}(u,\tau) \\
&= \rho_{\nu_{(I_n,G_n)}}(\{(u,\tau,t,a) \in \mathcal{U} \times M(T) \times T \times A\colon t \in F,\ \tilde{h}(u,\tau,t,\rho_{\nu,T\times A}) \in H\}) \\
&= \rho_{\nu_{(I_n,G_n)}}(\{(u,\tau,t,a) \in \mathcal{U} \times M(T) \times T \times A\colon t \in F,\ a \in H\}) \\
&= \rho_{\nu_{(I_n,G_n)} T\times A}(F \times H),
\end{aligned}
$$

where the fourth equality follows from condition (ii) of Definition 2. Thus we have

$\frac{1}{\#(I_n)} \sum_{i \in I_n} \sigma_{n,i} = \rho_{\nu_{(I_n,G_n)} T \times A}$. Consequently, for any $i \in I_n$, if $(t, a) \in \text{supp}(\sigma_{n,i})$ then

$$u_i(t, a, \frac{1}{\#(I_n)} \sum_{i \in I_n} \sigma_{n,i}) > u_i(t, a', \frac{1}{\#(I_n)} \sum_{i \in I_n} \sigma_{n,i}) + \epsilon$$

for $a' \neq a$, because $(t, a) \in \text{supp}(\sigma_{n,i})$ implies that $a = \tilde{h}(u, \tau, t, \frac{1}{\#(I_n)} \sum_{i \in I_n} \sigma_{n,i})$. It follows that there is an $N_1 \geq N$ such that for each $i \in I_n$, if $a' \neq a$ then

$$u_i(t, a, \frac{1}{\#(I_n)-1} \sum_{i \in I_n} \sigma_{n,i}) > u_i(t, a', \frac{1}{\#(I_n)-1} \sum_{i \in I_n} \sigma_{n,i}) + \epsilon$$

whenever $n \geq N_1$. Using Lemma 3, it follows from this that there is an $N_2 \geq N_1$ such that if $n \geq N_2$, then σ_n is a strict equilibrium strategy profile in the game (I_n, G_n).

3.6 Proof of Theorem 5

Let $\nu \in \mathcal{M}$ and ρ an equilibrium distribution of ν. Inspecting (b) in the proof of Theorem 4 shows that there is a sequence $\langle \nu_n \rangle$ in $\mathcal{M}^*$ and a corresponding sequence $\langle \rho_n \rangle$ of strict equilibrium distributions such that $\rho_n \to \rho$ narrowly. By the law of large numbers (Glivanko-Cantelli version), for each n there is a sequence $\langle (I_k, G_k \rangle$ of finite-player games such that $\#(I_k) \to \infty$ and $\nu_{(I_k,G_k)} \to \nu_n$. Inspecting (c) and (d) in the proof of Theorem 4 shows that for each n there is a sequence σ_k of strategy profiles for the games (I_k, G_k) such that if k is large enough then σ_k is a strict equilibrium strategy profile and $\rho_{(I_k,G_k,\sigma_k)} \to \nu_n$ narrowly. Combining these facts proves the theorem by applying a diagonal argument.

4 An application

In this section, we consider a two-period version of the macroeconomic model of Krusell and Smith (1998) and show how it can be formulated as a game with incomplete information and a continuum of players. There is a continuum of individuals, each of whom have k units of capital and an endowment of one unit of time. Each individual is, in each period, subject to an idiosyncratic shock that determines the amount e_t of time he will work in period t, with $t \in \{1, 2\}$ and $e_t \in \{0, 1\}$; an individual with $e_t = 0$ is interpreted as being unemployed and an individual with $e_t = 1$

is interpreted as being employed in period t. In period 1, each individual rents his capital k at competitive gross rate of return R net of depreciation and receives wage W if he works; his wealth is then $Rk+We_1$. He then decides how much of this wealth to leave as capital for period 2, denoted by a; the remaining $Rk+We_1-a$ is consumed in period 1. The variable a is the action of an individual; we let $A \subseteq \mathbb{R}_+$ be the set of possible second-period capital levels and assume that it is finite. In period 2, the rate of return on capital and the wage rate both depend on the choice of capital made in period 1, via the distribution π of actions. Consumption in period 2 is $R(\pi)a + W(\pi)e_2$. The type of each individual is his first period idiosyncratic shock and thus we write $T = \{0, 1\}$ for the type space. There is a common type distribution q such that, for each $e_2 \in T$, $\sum_{e\in\{0,1\}} q(e)p(e, e_2) = q(e_2)$ where $p(e_1, \cdot) \in M(\{0, 1\})$,[9] so that q is a stationary distribution of "employment states", with $q(0)$ denoting the fraction of unemployed people; $p(e_1, e_2)$ is the probability of each individual having employment state e_2 in period 2 given that he had employment state e_1 in period 1.

Let $K \subseteq \mathbb{R}_+$ denote the set of initial capital stocks. We assume that $0 < a \leq Rk$ for each $a \in A$ and $k \in K$ to avoid negative consumption in the first period. The individual preferences are described as follows: There is a common utility function $U : \mathbb{R}_+ \to \mathbb{R}$ which gives the utility of consumption in each period. The function U is assumed to be continuous, and so are the functions $\pi \mapsto R(\pi)$ and $\pi \mapsto W(\pi)$. The utility of second period consumption is discounted by a common discount factor $\beta \in (0, 1)$. Thus, the utility of someone with initial capital $k \in K$ is, for each $(t, a, \pi) \in T \times A \times M(A)$,

$$u_k(t, a, \pi) = U(Rk + Wt - a) + \beta \sum_{e=0}^{1} p(t, e)U(R(\pi)a + W(\pi)e).$$

The final element of the model is the initial distribution of capital, which we denote by $\Gamma \in M(K)$.

The above framework is expressed as continuum game ν by setting $\nu = (\Gamma \circ V^{-1}) \times 1_q$, where $V(k) = u_k$ for each $k \in K$. This shows how to formulate a two-period version of macroeconomic model of Krusell and Smith (1998) as a game with

[9] For each nonempty subset X of $\mathbb{R}$, $M(X)$ is the set of all Borel probability measures on X.

incomplete information and a continuum of players.

Theorem 2 readily implies that an equilibrium distribution exists. The computation of an equilibrium distribution is facilitated by assuming that the set K of initial capital stocks is finite and by the following characterization. For each $(k, e, \pi) \in K \times \{0, 1\} \times M(A)$, let $F(k, t, \pi)$ be the set of solutions to

$$\max_{a \in A} U(Rk + Wt - a) + \beta \sum_{e=0}^{1} p(t, e) U(R(\pi)a + W(\pi)e);$$

then ρ is an equilibrium distribution of ν if and only if, for each $(k, t, a) \in K \times T \times A$, $\rho(u_k, q, t, a) = 0$ if $a \notin F(k, t, \rho_A)$, $0 \leq \rho(u_k, q, t, a) \leq \Gamma(k)\tau(t)$ if $a \in F(k, t, \rho_A)$ and $\sum_{a \in A} \rho(u_k, q, t, a) = \Gamma(k)\tau(t)$. Due to the finiteness of A, it is not difficult to write an algorithm to (approximately) compute equilibrium distributions of Γ.

Theorem 3 also applies to yield an individualistic interpretation of the equilibrium distributions of ν. Furthermore, with the proper formulation of finite-player games, Theorems 1 and 5 interprets them as limits of equilibria of finite-player games. For these latter theorems to be applicable, we need to slightly reformulate the model as follows. Each individual's utility depends on the levels of his consumption in the two periods. Thus, from the previous paragraph, each individual's utility depends on his initial level of capital k, his type t, action a, and the distribution κ of the actions of the other individuals. The initial level of capital, being exogenous, can be ignored, or, in other words, the dependence of utility on the initial level of capital can be made part of a utility function. Because a distribution of action is just a marginal of a distribution of pairs of types and actions, we can now describe a utility function as a function with domain $T \times A \times M(T \times A)$. Assuming utility functions, i.e., payoff functions, to be continuous, we get a space $\mathcal{U}$ of these functions as in our original model. A game is now an element ν of $\mathcal{M}$ such that for each $(u, \tau) \in \text{supp}(\nu)$, u factors through the map sending each $\kappa \in M(T \times A)$ to its marginal measure on A. Equilibrium distributions are given as in our original model. Of course Theorems 1, 2, and 3 remain valid, and with the proper formulation of finite-player games, so do Theorems 4 and 5. For the latter two theorems, just note that the perturbations of payoff functions made in the proof of Theorem 4 involves only the dependence of payoffs on the action variable a.

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