The Core in a Distributional Economy

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Abstract

An economy, large or small, has traditionally been defined in terms of an explicit set of agents and an assignment of characteristics to each agent. But when individual agents are negligible, many economically relevant properties of an economy can be defined in terms of the distribution of characteristics alone. Agents need not be specified.

It has been frequently asserted that the distributional description of an economy is too sparse for core analysis. Notions of coalitions and blocking require the individualistic description of agents. This paper shows that this is not so. The presence of blocking coalitions can be directly identified in terms of distributions alone. Indeed, we give a purely distributional proof of the classical core-equivalence theorem that delivers the core-equivalence theorem for individualistic economies as a corollary.

Our methods have applications outside of general equilibrium theory. They apply to large matching markets and to analogs of the Shapley-value for atomless economies.

1 Introduction

Large economies have traditionally be defined as an assignment of characteristics to every individual trader taken from a continuum. However, competitive equilibria can be defined in terms of the distribution of characteristics alone, as has been done by Hart, Hildenbrand, and Kohlberg (1974) and Hildenbrand (1975), and no specification of individual traders is needed. The distributional description of an economy is more parsimonious, allows us to study continuity properties of the equilibrium-correspondence, and allows us to relate large economies to limits of economies with finitely many traders.

So far, the core has eluded a distributional formulation, and a number of authors have written that such a distributional formulation is not possible.[1] As we show in this paper, it is.

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[1] For example, Hart, Hildenbrand, and Kohlberg (1974) write that "[T]he concepts of 'coalition' and 'to improve upon' require the individualistic description of an economy as a mapping which assigns to every individual agent his characteristics." and Mas-Colell (1975) writes that "for the equivalence theorem it is indispensable to consider the economy in representation form."

To understand the underlying economic issues, it helps to look at economies with finitely many traders and at the information that gets lost when looking at the distribution alone. Information is lost about the identity of individual traders and about how many traders there are with specific characteristics; how many traders are represented by a single type. Without knowing the identity of a trader, we can not track in two allocations whether the trader is better off in one rather than the other. Without knowing how many traders are represented by a single type, we can neither judge how competitive the economy is nor can we rely on the convexifying effect of large numbers.

These are not just problems one needs to address in order to understand the core, these problems need to be addressed already to understand the simpler concept of Pareto efficiency. The problem of how to keep track of traders' identity has a simple solution: One simply looks at the whole joint distribution of characteristics and the commodity bundles received in two allocations. This is enough to identify Pareto-rankings; see Proposition 1 below. The problem that we do not know how many traders a type represents becomes moot if the underlying space of traders is sufficiently competitive. This can be achieved using a strengthened nonatomicity assumption on the space of traders, the space must be superatomless; see Proposition 2 below.

We also show what the usual Pareto-ordering means in a purely distributional setting without looking at the whole joint distribution of characteristics and the commodity bundles received in two allocations. For short, let us refer to a measure on the product of a space of trader's characteristics and the commodity space as a distributional allocation, provided that its marginal on the former space coincides with a given distribution of trader's characteristics. Reif and Wiesmeth (1978) have introduced an ordering on distributional allocations in the spirit of first-order stochastic dominance that is meant to capture the distributional content of the Pareto ordering. They showed that if two allocations can be Pareto ordered, then the induced distributional allocations satisfy their order. Reif and Wiesmeth also conjectured that a converse holds for some individualistic representations of the distributional allocations, at least when preferences are strictly monotone. We show that this conjecture is indeed correct, even without any monotonicity assumption, in Theorem 2. A celebrated theorem of Strassen (1965) constructs appropriate joint distributions for stochastically ordered distributions. Lindvall (1999) has pointed out that the result holds also for stochastic pre-orderings, and this is the main tool to prove Theorem 2.

These results have analogs for the core with some modifications. In particular, we can meaningfully define the core of a distributional economy so that an allocation is in the classical core if and only if its induced distribution is in the distributional core. Importantly, even though we need a strengthened nonatomicity condition on an underlying space of traders to represent conceivable coalitions in the distribution exactly, which we do in Theorem 3, the presence of a conceivable blocking coalition in the distribution implies the existence of a blocking coalition even when the underlying space of traders is merely atomless; see Theorem 4. Our tool for proving this is Lyapunov's theorem for Young measures, a result due to Balder (2000).

In order to show that our notion of a distributional core is workable, we present a distributional core-equivalence theorem; Theorem 6. We only assume that preferences are continuous, irreflexive, and strictly monotone, and that the aggregate endowment is strictly positive. No transitivity assumption is needed. Our proof makes no use of an ancillary individualistic representation. One can obtain the general core-equivalence theorem of Hildenbrand (1982) for individualistic economies with an atomless space of traders as a corollary. Our distributional methods are as powerful as the individualistic approach.

The way we define blocking coalitions in a distributional economy is closely related to various notions of generalized coalitions that have been employed in the literature in equilibrium theory, value theory, and the theory of stable matchings. We show that our approach allows us to give a natural classical individualistic interpretation of the "ideal sets" of Aumann and Shapley (1974), the "fuzzy coalitions" of Aubin (1979), the "convex set of agents" of Gretsky and Ostroy (1985), and the "subpopulations" of Gretsky, Ostroy, and Zame (1997) and Che, Kim, and Kojima (2019). They can all be interpreted as distributions induced by classical coalitions in a sufficiently rich space of agents; the original set of agents can then be interpreted as a space of types.

2 Mathematical Preliminaries

For some of our results, we need a strengthened nonatomicity condition for the space of agents. In what follows, we assume all measures to be finite and nonzero. The notion of a superatomless measure space, as introduced by Podczeck (2008), is most naturally expressed in the language of measure algebras. Measure algebras are the result of identifying measurable sets whose symmetric difference has measure zero. By identifying measurable sets with their indicator functions, this is a special case of identifying measurable functions that agree almost everywhere. We use this identification to formulate superatomlessness in more familiar terms.[2] Let (T, Σ, ν) be a measure space and $L_1(\nu)$ be the corresponding space of equivalence classes of integrable functions endowed with the usual L_1-norm $\|\cdot\|_1$. For each $A \in \Sigma$, let $L_1(A, \nu)$ be the subspace of $L_1(\nu)$ consisting of elements vanishing outside of A. Now (T, Σ, ν) or ν is *superatomless* if $L_1(A, \nu)$ is non-separable for all $A \in \Sigma$ with $\nu(A) > 0$.[3] There is a very useful property, termed *saturation* by Hoover and Keisler (1984), that characterizes superatomless probability spaces:

A measure space (T, Σ, ν) is superatomless if and only if whenever X and Y are Polish spaces and κ is a Borel measure on $X \times Y$ such that its marginal on X equals

[2] The equivalence to the original definition is proven in Podczeck (2008).

[3] A wide variety of conditions equivalent to being superatomless can be found in the literature. Footnote 4 in Wang and Zhang (2012) lists most of the known equivalent conditions.

the distribution of some measurable function $f : T \to X$, there is a measurable function $g : T \to Y$ such that κ is the distribution of the function $t \mapsto (f(t), g(t))$.[4]

3 Economic Environment

The commodity space is $\mathbb{R}^l$ and each trader has the same consumption set $E = X = \mathbb{R}^l_+$. We also use notation like X' to distinguish different occurrences of X in a product space. We let $\mathcal{P}$ be the set of irreflexive relations on X with a relatively open graph, endowed with the coarsest topology such that the set

$$\{(\succ, x, y) \in \mathcal{P} \times X \times X : x \succ y\}$$

is open in the product topology. Under this topology, $\mathcal{P}$ is compact and metrizable. The subspace $\mathcal{P}^*$ of $\mathcal{P}$ consisting of asymmetric and negatively transitive relations is a G_δ-subset of $\mathcal{P}$, therefore a Polish subspace, by Alexandroff's lemma. Similarly, the subspace $\mathcal{P}_{\mathrm{mo}}$ of strictly monotone preferences, those preferences for which $x > y$ implies $x \succ y$, is a G_δ-subset of $\mathcal{P}$, therefore also a Polish subspace. Also, we let $\mathcal{P}^*_{\mathrm{mo}} = \mathcal{P}^* \cap \mathcal{P}_{\mathrm{mo}}$. These results follow from Hildenbrand (1974, 1.2).

An *individualistic economy* consists of a probability space (T, Σ, ν) of traders and a measurable function $\mathcal{E} : T \to \mathcal{P} \times E$ such that the second coordinate function is ν-integrable. We let c be the first coordinate function and η be the second coordinate function of $\mathcal{E}$. We also write $\succ_t = c(t)$ and $e_t = \eta(t)$. We call $\nu \circ \mathcal{E}^{-1}$ the *distribution* of the economy. An *individualistic allocation* f for the economy is a measurable function $f : T \to X$ such that

$$\int f \, \mathrm{d}\nu = \int \eta \, \mathrm{d}\nu.$$

We also introduce distributional versions of these concepts. A *distributional economy* is a Borel probability measure μ on $\mathcal{P} \times E$ such that the canonical projection $\pi_E : \mathcal{P} \times E \to E$ is μ integrable. A *distributional allocation* for the distributional economy μ is a Borel Probability measure α on $\mathcal{P} \times X$ such that α and μ have the same $\mathcal{P}$-marginal, and such that

$$\int \pi_X \, \mathrm{d}\alpha = \int \pi_E \, \mathrm{d}\mu,$$

with π_X being the projection onto X. We also define a *reallocation* for the distributional economy μ to be a Borel probability measure on $\mathcal{P} \times E \times X$ whose $\mathcal{P} \times E$-marginal is μ and whose $\mathcal{P} \times X$-marginal is a distributional allocation for the distributional allocation μ.

[4]The equivalence follows from Hoover and Keisler (1984, Corollary 4.5).

4 Pareto Efficiency

Fix an individualistic economy and let f and f' be individualistic allocations. We write $f \succsim_P f'$ if the set of t such that $f'(t) \succ_t f(t)$ has ν-measure zero. The asymmetric part $\succ_P$ of $\succsim_P$ represents the usual relation of *Pareto dominance* and a $\succsim_P$-maximal allocation is *Pareto efficient*. It is essential for a treatment of Pareto efficiency, that we allow for a measure zero exceptional set because feasibility is defined by integration. The following is straightforward.

Proposition 1. *Let f and f' be individualistic allocations. Then $f \succsim_P f'$ if and only if the set*

$$\{(\succ, x, x') : x' \succ x\}$$

has $\nu \circ (c, f, f')^{-1}$-measure zero.

The following is a direct consequence of the saturation property of superatomless probability spaces.

Proposition 2. *Let ν be superatomless. Let f be an individualistic allocation and μ be any probability measure on the set $\mathcal{P} \times X \times X'$, with $\mathcal{P} \times X$-marginal $\nu \circ (c, f)^{-1}$, such that*

$$\int \pi_{X'} \, \mathrm{d}\mu = \int \pi_X \, \mathrm{d}\nu,$$

where π_X and $\pi_{X'}$ are the projections onto X and X', respectively. Then there exists an individualistic allocation f' such that $\mu = \nu \circ (c, f, f')^{-1}$.

Together, the last two propositions show that for a superatomless space of agents, Pareto efficiency can be fully characterized in terms of the induced joint distributions; the joint distribution with a Pareto dominating distributional allocation can be exactly represented by an individualistic allocation. If (T, Σ, ν) is only assumed to be atomless, this is in general not possible. Nevertheless, Pareto efficiency can be defined in distributional terms as the following theorem shows.

Theorem 1. *Assume that (T, Σ, ν) is atomless. Let f be an individualistic allocation and μ a Borel probability measure on $\mathcal{P} \times E \times X \times X'$, with $\mathcal{P} \times E \times X$-marginal $\nu \circ (c, \eta, f)^{-1}$, such that the set*

$$\{(\succ, e, x, x') : x \succ x'\}$$

has μ-measure zero and such that the set

$$\{(\succ, e, x, x') : x' \succ x\}$$

has positive μ-measure. Then there exists an individualistic allocation f' such that $f' \succ_P f$.

It is crucial for the argument that one has information of the joint distribution of f and f'. Indeed, if μ is a measure on $\mathcal{P} \times X \times X'$ that puts zero probability on the set $\{(\succ, x, x') : x \succ x'\}$, and if f and f' are allocations such that $\nu \circ (c, f)^{-1}$ equals the $\mathcal{P} \times X$-marginal of μ and $\nu \circ (c, f')^{-1}$ equals the $\mathcal{P} \times X'$-marginal of μ, then we still do not know whether $f' \succeq_P f$. However, this is enough to show that there is another allocation f'' such that $\nu \circ (c, f')^{-1} = \nu \circ (c, f'')^{-1}$ and $f'' \succeq_P f'$. This is enough to determine whether f is $\succeq_P$-maximal, that is, Pareto efficient. Pareto efficiency can be characterized in distributional terms.

However, it would be desirable to make such comparisons without specifying a possible joint distribution. The following formulation in the spirit of stochastic dominance due to Reif and Wiesmeth (1978) will allow us to do just that. Let α and α' be distributional allocations for the distributional economy μ. We write $x \succeq y$ for $y \nsucc x$ and let $B(\succeq, x) = \{y \in Y : y \succeq x\}$. We write $\alpha \succeq_D \alpha'$ if for some (and hence all) regular conditional distributions $g_\alpha : \mathcal{P} \to \Delta(X)$ and $g_{\alpha'} : \mathcal{P} \to \Delta(X)$ of α and α', respectively, and each $x \in X$ one has

$$g_\alpha(\succ)\Big(B(\succeq, x)\Big) \geq g_{\alpha'}(\succ)\Big(B(\succeq, x)\Big)$$

for $\mu_{\mathcal{P}}$-almost all $\succ$, where $\mu_{\mathcal{P}}$ is the $\mathcal{P}$-marginal of μ. Reif and Wiesmeth have shown that the usual Pareto ordering of individualistic allocations in an individualistic economy induces the ordering $\succeq_D$ on the corresponding distributional allocations and conjectured that the converse holds for some individualistic representation of these distributional allocations when preferences are monotone. We show that this conjecture is indeed true, even without any monotonicity assumption.

The interpretation of the ordering is as follows: Look at all agents with preference exactly $\succ$. Then for every x, "at least as many" agents with this preference relation get something at least as good as x under μ than under μ'. By reordering agents, we could make sure that every agent is at least as well off under μ than under μ'.

Theorem 2. *Let μ be a distributional economy such that $\mu(\mathcal{P}^* \times E) = 1$ has μ-measure zero. Let α and α' be two distributional allocations for the economy μ such that $\alpha \succeq_D \alpha'$. Then there exists a measure λ on $\mathcal{P} \times X \times X'$ such that the $\mathcal{P} \times X$-marginal of λ is α, the $\mathcal{P} \times X'$-marginal of λ is α' and*

$$\lambda\Big(\{(\succeq, x, x') : x' \succ x\}\Big) = 0.$$

5 The Core

We next go to the slightly more involved problem of analyzing blocking coalitions from a distributional point of view. Consider an individualistic economy with space of traders (T, Σ, ν) and preferences and endowments given by $\mathcal{E}$:

$T \to \mathcal{P} \times E$. An *individualistic coalition* is simply a measurable set $C \in \Sigma$ such that $\nu(C) > 0$. We say that the individualistic coalition C *blocks* the individualistic allocation f if there is an individualistic allocation f' such that $f'(t) \succ_t f(t)$ for ν-almost all $t \in C$ and

$$\int_C f' \, \mathrm{d}\nu = \int_C \eta \, \mathrm{d}\nu.$$

In that case, we say that C *blocks* f *by* f'. We can still work with distributions, but have to look at the distribution of the measure when restricted to C. The distribution of an individualistic allocation f restricted to an individualistic coalition C assigns to each Borel subset B of $\mathcal{P} \times E \times X$ the measure $\nu(c, \eta, f, f')^{-1}(B \cap C)$. This restricted distribution is not a probability measure unless $\nu(C) = 1$. The following is, again, straightforward.

Proposition 3. *Let f and f' be individualistic allocations and C an individualistic coalition. Let*

$$\rho(A) = \nu\big((c, \eta, f, f')^{-1}(A) \cap C\big).$$

Then C blocks f by f' if and only if the set

$$\{(\succ, e, x, x') : x' \not\succ x\}$$

has ρ-measure zero and

$$\int \pi_{X'} \, \mathrm{d}\rho = \int \pi_E \, \mathrm{d}\rho.$$

Much like distributional allocations, we can exactly represent "distributional blocking coalitions" when (T, Σ, ν) is superatomless.

Theorem 3. *Assume that (T, Σ, ν) is superatomless. Let f be an individualistic allocation and μ a Borel probability measure on $\mathcal{P} \times E \times X \times X'$ such that the set*

$$\{(\succ, e, x, x') : x' \not\succ x\}$$

has μ-measure zero, such that the $\mathcal{P} \times E \times X$-marginal of μ is setwise smaller than $\nu \circ (c, \eta, f)^{-1}$, and such that

$$\int \pi_{X'} \, \mathrm{d}\mu = \int \pi_E \, \mathrm{d}\mu.$$

Then there exists an individualistic coalition C and an individualistic allocation f' such that C blocks f by f' and for every Borel subsets A of $\mathcal{P} \times E \times X \times X'$ one has $\mu(A) = \nu\Big((c, \eta, f, f')^{-1}(A) \cap C\Big)$.

This result also shows that for ν superatomless, every coalition that might exist in the distribution actually exists as a proper individualistic coalition. If ν is only assumed to be atomless, the existence of such a μ still guarantees that some coalition can block the allocation f.

Theorem 4. *Assume that* (T, Σ, ν) *is atomless. Let* f *be an individualistic allocation and* μ *a Borel probability measure on* $\mathcal{P} \times E \times X \times X'$ *such that the set*

$$\{(\succ, e, x, x') : x' \nsucc x\}$$

has μ*-measure zero and such that the* $\mathcal{P} \times E \times X$*-marginal of* μ *is setwise smaller than* $\nu \circ (c, \eta, f)^{-1}$ *and such that*

$$\int \pi_{X'} \, \mathrm{d}\mu = \int \pi_E \, \mathrm{d}\mu.$$

Then there exists an individualistic coalition C *and an individualistic allocation* f' *such that* C *blocks* f *by* f'*.*

Just as in the case of Pareto optimality, it is possible to compare coalitions in distributional terms without an *a priori* specification of a joint distribution. For coalitional blocking, we need to make somewhat stronger assumptions. We do not just want to have $f'(t) \succeq_t f(t)$ for almost all $t \in C$, we want $f'(t) \succ_t f(t)$ for almost all $t \in C$. To do so, we take preferences to be in $\mathcal{P}^*_{\text{mo}}$. Then, if we can show that $f'(t) \succeq_t f(t)$ for almost all $t \in C$ but not $f(t) \succeq_t f'(t)$ for almost all $t \in C$, then there must be a positive measure set $E \subseteq C$ such that $f'(t) \succ_t f(t)$ for almost all $t \in E$ and, by a standard argument using strict monotonicity, one can redistribute then between the members of C so that almost everyone is better off than under f. The other new complication comes from the fact that we are not just considering the grand coalition, we have to keep track of the total endowment of a coalition. To make the argument more transparent, we look, restricted to a distributional counterpart of a coalition, at the original allocation with values in X, an allocation that makes some agents better off and almost nobody worse off with values in X', and an allocation with values in X'' that makes almost everyone in the coalition strictly better off.

Theorem 5. *Suppose that* μ *and* μ' *are Borel probability measures (we just renormalize coalitions) on* $\mathcal{P}^*_{\text{mo}} \times E \times X$*, with the same* $\mathcal{P}^*_{\text{mo}} \times E$*-marginal, such that*

$$\int \pi_{X'} \, \mathrm{d}\mu' = \int \pi_E \, \mathrm{d}\mu',$$

and such that for regular conditional probabilities

$$d : \mathcal{P}^*_{\text{mo}} \times E \to \Delta(X)$$

and

$$d' : \mathcal{P}^*_{\text{mo}} \times E \to \Delta(X')$$

of μ *and* μ'*, respectively, we have*

$$d(\succ, e)\Big(B(\succeq, x)\Big) \geq d'(\succ, e)\Big(B(\succeq, x)\Big)$$

for almost all $(\succ, e)$ *with respect to the corresponding marginal of* μ *but not vice versa. Then there exists a measure* λ *on* $\mathcal{P}^*_{\text{mo}} \times E \times X \times X' \times X''$ *such that*

$$\lambda\Big(\{(\succ, e, x, x', x'') : x \succ x' \text{ or } x' \succeq x''\}\Big) = 0,$$

such that the $\mathcal{P}^*_{\text{mo}}\times E\times X$-marginal of λ coincides with μ, such that the $\mathcal{P}^*_{\text{mo}}\times E\times X'$-marginal of λ coincides with μ' and such that

$$\int \pi_{X''}\,\mathrm{d}\lambda = \int \pi_E\,\mathrm{d}\lambda.$$

6 Core-Equivalence

So far, we have shown that one can meaningfully define the relevant concepts of Pareto efficiency and the core in distributional terms. It is less clear that the distributional notions are operational, that one can actually use them without specifying an ancillary probability space of individual agents. We show that these notions are workable by using them to prove a fairly general core-equivalence theorem. For this we need to introduce some new economic terms.

A reallocation τ for the distributional economy μ is *Walrasian* if there exists $p \in \mathbb{R}^l$ such that

$$\tau\Big(\{(\succ, e, x) : p\cdot x \leq p\cdot e \text{ and } x' \succ x \text{ implies } p\cdot x' \text{ for all } x' \in X\}\Big) = 1.$$

The set involved in this definition is easily seen to be measurable, just note that the second condition needs only to be checked for x' in a countable dense subset of X. A reallocation τ is a *core reallocation* if there exists no measure μ on $\mathcal{P}\times E\times X\times X'$ such that the set

$$\{(\succ, e, x, x') : x' \not\succ x\}$$

has μ-measure zero and such that the $\mathcal{P}\times E\times X$-marginal of μ is setwise smaller than τ and such that

$$\int \pi_{X'}\,\mathrm{d}\mu = \int \pi_E\,\mathrm{d}\mu.$$

Theorem 6. *For a distributional economy μ such that $\mu(\mathcal{P}_{\text{mo}}\times E) = 1$ and $\int \pi_E\, d\mu \gg 0$, a reallocation is Walrasian if and only if it is a core reallocation.*

By combining Theorem 6 with Proposition 3, Theorem 4, and the well-known fact that an individualistic allocation is Walrasian if and only if the induced reallocation is Walrasian, one has an alternative proof of the very general core-equivalence theorem for an atomless economy of Hildenbrand (1982). The original core-equivalence theorem of Aumann (1964) does not assume preferences to be irreflexive, and though Aumann makes the same continuity assumption, his proof only requires all $\succ$-better sets to be open. The only role the stronger assumptions play in our proof and the proof of Hildenbrand (1982) is in guaranteeing that there is a well-defined measurable space of preferences. Aumann does not define an economy as a measurable function into a measurable space of characteristics and can, therefore, directly specify the minimal continuity and measurability assumptions needed for the proof to go through; see the discussion in Greinecker and Podczeck (2016).

Our proof of Theorem 5 does not use any ancilliary representation of the distributional economy as an individualistic economy; the distributional point of view is entirely appropriate for core analysis.

7 Generalized Coalitions

Many authors have found at some point a need to generalize the classical notion of a coalition as a measurable set of agents with positive measure. We show that our distributional point of view allows for a unified perspective on the existing notions of generalized coalitions.

Suppose you could order a continuum of players at random. For a given set S of players, the mass of players in S who end up in the top half under the random ordering should be one half of the mass of players in S almost surely. The set S would be "evenly spread" under the random ordering, to use the language of Aumann and Shapley (1974). An entirely satisfactory notion of such a random ordering is impossible, as shown by Aumann and Shapley, and even the concept of an "evenly spread measurable set" turns out to be empty in their framework. However, they provide a convincing intuition of the value of a nonatomic game with side-payments in terms of evenly spread sets. In order to translate this intuition into mathematics, Aumann and Shapley introduce *ideal sets*, which are simply measurable functions from the space of players to the unit interval. If g is such an ideal set and i a player, $g(i)$ tells us how much player i belongs to the ideal set g. Ordinary measurable sets are simply identified with their indicator functions.

With infinite dimensional commodity spaces, a non-atomic measure space of agents may not be sufficient for perfect competition, markets might fail to be "thick." An interesting test for thick markets was introduced in Gretsky and Ostroy (1985). Gretzky and Ostroy extend the underlying nonatomic probability space of agents to a *convex space of agents* consisting of simple functions with values in the unit interval. Markets are then considered thick when the space of allocations remains essentially unchanged when going from the usual space of agents to the convex space of agents. The latter is meant to express perfect divisibility on the level of coalitions and can be seen as a continuum replica version of the original economy.

A distributional approach is taken in Gretsky, Ostroy, and Zame (1997) and Che, Kim, and Kojima (2019). In both papers, there is a compact metric space of characteristics K and the population of agents is given by a nonnegative Borel measure μ on K. A *subpopulation* in these papers is then a nonnegative Borel measure ν on K such that $\nu(A) \leq \mu(A)$ for every Borel set $A \subseteq K$. The actual space of agents is not formalized. By the Radon-Nikodym theorem, subpopulations can be identified with (equivalence classes of) measurable functions with values in the unit interval.

We let $(X, \mathcal{X}, \lambda)$ be a probability measure space. A *generalized coalition* is a λ-equivalence class of measurable functions $c : X \to [0, 1]$. A *coalition* is a λ-equivalence class of measurable functions $c : X \to \{0, 1\}$. In Aumann and Shapley (1974), generalized coalitions are *ideal sets*. In Gretsky and Ostroy (1985), the subclass of generalized coalitions corresponding to simple functions are added to ordinary coalitions, to enrich the *indivisible space of agents* to the *con-*

vex space of agents. In Aubin (1979), generalized coalitions are *fuzzy coalitions*, a terminology inspired by the corresponding notion of a *fuzzy set* in Zadeh (1965).

We show that generalized coalitions on a probability space $(X, \mathcal{X}, \lambda)$ can be viewed as distributions induced by individualistic coalitions when one interprets elements of X as types of individuals instead of as individuals proper.[5] The setting we look at is that there is an underlying probability space of agents (T, Σ, ν) and a measurable function $\phi : T \to X$ with distribution λ. Under this distributional point of view, generalized coalitions are naturally treated as subpopulations and we do so in what follows. We show that under a suitable nonatomicity hypothesis, every subpopulation of $(X, \mathcal{X}, \lambda)$ corresponds to a proper coalition in Σ.

The relevant nonatomicity notion goes back to Maharam (1942) and a lemma from that paper (Lemma 3 in the next section) will be our main tool. Let (T, Σ, ν) be a finite measure space and $\Sigma' \subseteq \Sigma$ a sub-σ-algebra. We say that ν is *relatively atomless* over Σ' if for every $A \in \Sigma$ such that $\nu(A) > 0$, there exists $B \in \Sigma$ such that $B \subseteq A$ and $\nu(B \Delta C) > 0$ for all $C \in \Sigma'$.

Theorem 7. *Let (T, Σ, ν) be a probability space, $(X, \mathcal{X})$ a measurable space and $\phi : T \to X$ a measurable function. Suppose ν is relatively atomless over $\sigma(\phi)$. If μ is a measure on $(X, \mathcal{X})$ such that $\mu(A) \leq \nu \circ \phi^{-1}(A)$ for all $A \in \mathcal{X}$, then there exists a set $C \in \Sigma$ such that*

$$\mu(A) = \nu(\phi^{-1}(A) \cap C)$$

for all $A \in \mathcal{X}$.

It is relatively straightforward to show that a probability space (T, Σ, ν) is superatomless if and only if ν is relatively atomless over any countably generated sub-σ-algebra Σ' of Σ. So if $(X, \mathcal{X})$ is, for example, a Polish space as in most of the applications mentioned, one can take any superatomless probability space as the underlying space of agents. The existence of a measurable function ϕ with the appropriate distribution is then guaranteed too. However, relative atomlessness allows for a sharper result. The assumption that ν is relatively atomless over $\sigma(\phi)$ cannot be weakened; the conclusion of Theorem 7 is actually equivalent to ν being relatively atomless over $\sigma(\phi)$. The converse to Theorem 7 is the following proposition.

Proposition 4. *Let (T, Σ, ν) be a probability space, $(X, \mathcal{X})$ a measurable space and $\phi : T \to X$ a measurable function. If for every measure μ on $(X, \mathcal{X})$ such that $\mu(A) \leq \nu \circ \phi^{-1}(A)$ for all $A \in \mathcal{X}$ there exists a set $C_\mu \in \Sigma$ such that*

$$\mu(A) = \nu(\phi^{-1}(A) \cap C_\mu)$$

for all $A \in \mathcal{X}$, then ν is relatively atomless over $\sigma(\phi)$.

[5] A similar distributional interpretation of generalized coalitions has already been given in Hüsseinov (1994) in the special case of a finite space X.

8 Proofs

Proof of Theorem 1. $\kappa_\mu : \mathcal{P} \times E \times X \to \Delta(X')$ be a disintegration of μ. Define $\kappa : T \to \Delta(X')$ by $\kappa = \kappa_\mu \circ (c, \eta, f)$. Let $g_0 : T \times X' \to \mathbb{R}$ be given by $g_0(t, x) = 1$ if $f(t) \succ_t x$ and 0 otherwise. Let $g_1 : T \times X' \to \mathbb{R}$ be given by $g_1(t, x) = 1$ if $x \succ_t f(f)$. Finally, let $g_2 : T \times X' \to \mathbb{R}$ be given by $g_2(t, x) = x - e_t$. The integrand g_0 evaluated at κ is 0, the integrand g_1 evaluated at κ is positive, and the integrand g_2 evaluated at κ is 0 By Lyapunov's theorem for Young measures, Balder (2000, Theorem 5.10), there is a function $f' : T \to X'_\emptyset$ such that

$$\int g_0(t, f'(t))\ \mathrm{d}\nu = \int \int g_0(t, x)\ \mathrm{d}\kappa(t)\ \mathrm{d}\nu = 0,$$

$$\int g_1(t, f'(t))\ \mathrm{d}\nu = \int \int g_1(t, x)\ \mathrm{d}\kappa(t)\ \mathrm{d}\nu = \mu\Big(\{(\succ, e, x, x') : x' \succ x\}\Big) > 0$$

and

$$\int g_2(t, f'(t))\ \mathrm{d}\nu = \int \int g_2(t, x)\ \mathrm{d}\kappa(t)\ \mathrm{d}\nu = 0.$$

It follows from the last condition that f' is an individualistic allocation, from the first condition that $f' \succeq_P f$, and from the second condition that, indeed, $f' \succ_P f$. □

The following lemma is stated in Kamae, Krengel, and O'Brien (1977) without proof; we supply a proof for the readers' convenience.

Lemma 1. *Let $(M, \mathcal{M})$ be a measurable space, $\succcurlyeq$ a preorder on M such that all $\succcurlyeq$-lower sections are measurable, and $f : M \to [0,1]$ a $\succcurlyeq$-nondecreasing function. Then there exists a sequence $\langle f_n \rangle$ of simple functions $f_n : M \to [0,1]$ that converges pointwise to f such that each f_n is a positive linear combination of indicator functions of sets of the form $B(\succcurlyeq, x)$.*

Proof. Let f_0 be the constant function with value 0. For $n \geq 1$, let

$$K_n = \{k \in \mathbb{N} : 0 \leq k \leq n-1, f(x) \in [k/n, (k+1)/n] \text{ for some } x \in M\}.$$

For $n \geq 1$ and $k \in K_n$ choose some $x_n^k \in M$ such that $f(x_n^k) \in (k/n, (k+1)/n]$ and let

$$A_n^k = B(\succcurlyeq, x_n^k) \Big\backslash \bigcup_{k' > k} B(\succcurlyeq, x_n^{k'}).$$

Now let

$$f_n = \sum_{k \in K_n} k/n 1_{A_n^k}$$

Then $|f(x) - f_n(x)| \leq 1/n$ for all $x \in M$, and the result follows. □

Proof of Theorem 2. Define a relation $\succcurlyeq$ on $\mathcal{P} \times X$ such that $(\succ, x) \succcurlyeq (\succ', x')$ if and only if $\succ = \succ'$ and $x \succeq x'$. It is readily verified that $\succcurlyeq$ is a preorder with a closed graph. Let $f : \mathcal{P} \times X \to \mathbb{R}$ be a $\succcurlyeq$-nondecreasing measurable function. We ca assume in the following without loss of generality that the range of f is included in $[0,1]$. For each $\succ \in \mathcal{P}$, the section $f_\succ : X \to [0,1]$ is $\succeq$-nondecreasing and measurable. So using $\mu \succeq_D \mu'$, Lemma 1, the dominated convergence theorem, and Fubini's theorem for regular conditional probabilities, we get

$$\int f \, \mathrm{d}\nu = \int \int f_\succ \, \mathrm{d}g_\mu(\succ) \, \mathrm{d}\mu_{\mathcal{P}} \geq \int \int f_\succ \, \mathrm{d}g_{\mu'}(\succ) \, \mathrm{d}\mu_{\mathcal{P}} = \int f \, \mathrm{d}\mu'.$$

We can, therefore, apply the pre-order version of Strassen's theorem in Lindvall (1999) to obtain a probability measure τ on $(\mathcal{P} \times X) \times (\mathcal{P} \times X')$ that is supported on the graph of $\succcurlyeq$ and has marginals α and α', respectively. One obtains λ from τ by taking the appropriate marginal. □

Lemma 2. *Let (T, Σ, ν) be a superatomless probability space and let μ be a probability measure on (T, Σ) that is absolutely continuous with respect to ν. Then μ is superatomless too.*

Proof. Let $h : T \to \mathbb{R}$ be a Radon-Nikodym derivative of μ with respect to ν and let $E = \{t \in T \mid h(t) \neq 0\}$. Clearly, $\nu(E) > 0$. Define $\phi : L_1(\nu, E) \to L_1(\mu)$ by $\phi(f)(t) = fh^{-1}(t)$ for $t \in E$ and $\phi(f)(t) = 0$ for $t \notin E$. The operator ϕ is a linear isometry, so $L_1(\mu)$ is separable if and only if $L_1(E, \nu)$ is. The same construction can be applied to subsets, showing that μ is superatomless. □

Proof of Theorem 3. Let h be a Radon-Nikodym derivative of the $\mathcal{P} \times E \times X$-marginal of μ with respect to $\nu \circ (c, \eta, f)^{-1}$. Without loss of generality, we can take h to have values in the unit interval. Let λ be the measure on (T, Σ) with Radon-Nikodym derivative $h \circ (c, \eta, f)$ with respect to ν. Since λ is absolutely continuous with respect to a super-atomless measure, λ is superatomless too by Lemma 2. Then $\lambda \circ (c, \eta, f)^{-1}$ equals the $\mathcal{P} \times E \times X$-marginal of μ. By the saturation property of superatomless measure spaces, there exists $g : T \to X'$ such that $\mu = \lambda \circ (c, \eta, f, g)$. Let $\Sigma' = \sigma(c, \eta, f, g)$. Since ν is superatomless, there exists $C \in \Sigma$ such that $\lambda(A) = \nu(A \cap C)$ for all $A \in \Sigma'$. By the usual machinery, $\int j \, \mathrm{d}\lambda = \int_C j \, \mathrm{d}\nu$ for j a nonnegative Σ-measurable real function. Applying this coordinatewise, we get

$$\int_C g \, \mathrm{d}\nu = \int g \, \mathrm{d}\lambda = \int \pi_{X'} \, \mathrm{d}\mu = \int \pi_E \, \mathrm{d}\mu = \int \eta \, \mathrm{d}\lambda = \int_C \eta \, \mathrm{d}\mu.$$

Since c is Σ'-measurable, the set of t such that $g(t) \succ_t f(t)$ lies in Σ'. Since

$$0 = \mu\Big(\{(\succ, e, x, x') : x' \nsucc x\}\Big) = \nu\Big(C \cap \{(\succ, e, x, x') : x' \nsucc x\}\Big),$$

we have $g(t) \succ_t f(t)$ for ν-almost all $t \in C$. Finally, we define f' such that f' coincides with g on C and η outside C. It is straightforward that f' is an allocation with the desired properties. □

Proof of Theorem 4. Write λ for the marginal measure of μ on $\mathcal{P} \times E \times X$ and let $h : \mathcal{P} \times E \times X \to [0,1]$ be a Radon-Nikodym derivative of λ with respect to $\nu \circ (c, \eta, f)^{-1}$. Let $\kappa_\mu : \mathcal{P} \times E \times X \to \Delta(X')$ be a disintegration of μ. Let $X'_\varnothing = X' \cup \{\varnothing\}$

with $\o \notin X$ and endow it with the Polish topology that makes $\o$ an isolated point and X' have the original topology as a subspace. Now define $\kappa : T \to \Delta(X'_\o)$ by

$$\kappa(t) = h(\succ(t), e(t), f(t))\kappa_\mu(\succ(t), e(t), f(t)) + \Big(1 - h(\succ(t), e(t), f(t))\Big)\delta_\o.$$

Let $g_0 : T \times X \to \mathbb{R}$ be given by $g_0(\o) = 0$ and $g_0(x) = 1$ otherwise. Let $g_1 : T \times X'_\o \to \mathbb{R}^l$ be given by $g_1(t, \o) = 0$ and $g_1(t) = e(t) - x'$ for $x' \neq \o$ and $g_2 : T \times X_\o \to \mathbb{R}$ be given by $g_2(t, x') = 0$ if $x' = \o$ or $x' \succ (t) f(t)$ and 1 otherwise. The integrand g_0 evaluated at κ is $\mu(\mathcal{P} \times E \times X \times X')$, and the integrands g_1 and g_2 evaluated at κ are 0. By Lyapunov's theorem for Young measures, Balder (2000, Theorem 5.10), there is a function $f' : T \to X'_\o$ such that

$$\int g_0(t, f'(t))\, \mathrm{d}\nu = \int\int g_0(t, x)\, \mathrm{d}\kappa(t)\, \mathrm{d}\nu = \mu(\mathcal{P} \times E \times X \times X') > 0,$$

$$\int g_1(t, f'(t))\, \mathrm{d}\nu = \int\int g_1(t, x)\, \mathrm{d}\kappa(t)\, \mathrm{d}\nu = 0,$$

and

$$\int g_2(t, f'(t))\, \mathrm{d}\nu = \int\int g_2(t, x)\, \mathrm{d}\kappa(t)\, \mathrm{d}\nu = 0.$$

Let $C = f'^{-1}(X)$. The first integral equality implies that $\nu(C) > 0$, the second integral equality implies that

$$\int_C f'(t)\mathrm{d}\nu = \int_C e(t)\mathrm{d}\nu,$$

and the third integral equality implies that $f'(t) \succ (t) f(t)$ for almost all $t \in C$. It follows that C blocks f with the individualistic allocation that assigns $f'(t)$ to each $t \in C$ and $e(t)$ to each $t \notin C$. □

Proof of Theorem 5. Most of the proof is analogous to the proof of Theorem 2. What is new is how we construct the marginal on X''. Suppose we have already constructed the $\mathcal{P}^*_{\mathrm{mo}} \times E \times X \times X'$-marginal of λ. Call it τ. Then we must have

$$\tau\Big(\{(\succ, e, x, x') : x' \succ x\}\Big) > 0,$$

for otherwise we would get the converse ordering. For some i with $1 \le i \le l$, the i^{th} coordinate of $\int \pi_E\, \mathrm{d}\lambda$ must be positive. Moreover, there must be some positive n such that

$$\epsilon_n = \tau\Big(\{(\succ, e, x, x') : x' - 1/n e_i \succ x\}\Big) > 0,$$

where e_i is the i^{th} unit vector in $\mathbb{R}^l$ and $x' - 1/ne_i \succ x$ is taken to imply that $x' - 1/ne_i \ge 0$ so that the expression is well defined. Fix such an n. If

$$\beta = \tau\Big(\{(\succ, e, x, x') : x' - 1/ne_i \preceq x\}\Big) = 0,$$

we are done and we can just take $x'' = x'$. Otherwise, define a transition probability $\kappa : \mathcal{P}^*_{\mathrm{mo}} \times E \times X \times X \to \Delta(X'')$ by $\kappa(\succ, e, x, x') = \delta_{x' - 1/ne_i}$ if $x' - e_i \succ x$ and $\kappa(\succ, e, x, x') = x' + \epsilon_n/\delta e_i$. If we construct λ from τ using κ, we get the desired conclusion. □

Proof of Theorem 6. We prove the nontrivial direction. Let ν be a core reallocation and let

$$N = \Big\{z \in \mathbb{Q}^l : \nu\Big(\{(\succ, e, x) : z + e \geq 0, z + e \succ x\}\Big) > 0\Big\}.$$

By monotonicity, $N \neq \emptyset$. We show that $\mathrm{con}(N) \cap \mathrm{int}(-\mathbb{R}^l_+) = \emptyset$. Suppose for the sake of contradiction that there are $z_1, \ldots, z_K \in N$ and $\lambda_1, \ldots, \lambda_K \in \mathbb{R}_+$ such that $\sum_{i=1}^K \lambda_i = 1$ and $z = \sum_{i=1}^K \lambda_i z_i \ll 0$. For $i = 1, \ldots, K$, let

$$E_i = \{(\succ, e, x) : z_i + e \succ x\}.$$

Since preferences are monotone and $z \ll 0$,

$$\mathcal{P} \times E \times X = \bigcup_{n=1}^{\infty} \{(\succ, e, x) : e - nz \succ x\},$$

which implies

$$\lim_{n \to \infty} \nu\Big(\{(\succ, e, x) : e - nz \succ x\}\Big) = \nu(\mathcal{P} \times E \times X),$$

so for some N,

$$\nu\Big(\{(\succ, e, x) : e - Nz \succ x\}\Big) > 0.$$

Fix such an N and let

$$E_0 = \{(\succ, e, x) : e - Nz \succ x\}.$$

For $i = 0, \ldots, K$ let ν_i be the measure that has Radon-Nikodym derivative 1_{E_i} with respect to ν and let $\alpha_i = \nu_i(\mathcal{P} \times E \times X)$. For $\rho > 0$ small enough, the measure

$$\rho(\alpha_0 N)^{-1}\nu_0 + \rho \sum_{i=1}^K \alpha_i^{-1}\lambda_i \nu_i$$

is setwise smaller than ν.

Let $\psi_i : \mathcal{P} \times E \times X \to \mathcal{P} \times E \times X \times X'$ be given by

$$\psi_0(\succ, e, x) = (\succ, e, x, e - Nz)$$

and for $i = 1, \ldots, K$, define $\psi_i : \mathcal{P} \times E \times X \to \mathcal{P} \times E \times X \times X'$ by

$$\psi_i(\succ, e, x) = (\succ, e, x, e + z_i).$$

Then

$$\kappa = \rho(\alpha_0 N)^{-1}\nu_0 \circ \psi_0^{-1} + \rho \sum_{i=1}^K \alpha_i^{-1}\lambda_i \nu_i \circ \psi_i^{-1}$$

witnesses to ν being blocked. This contradiction shows that $\mathrm{con}(N) \cap \mathrm{int}(-\mathbb{R}^l_+) = \emptyset$.

By the separation theorem, there exists $p \neq 0$ such that $px \geq py$ for all $x \in \text{con}(N)$ and $y \in \text{int}(-\mathbb{R}^l_+)$. Clearly, $p \geq 0$. We have

$$\nu\Big(\big\{(\succ, e, x) : e + z \succ x \text{ with } z \in \mathbb{Q}^l \text{ implies } p(e+z) \geq pe\big\}\Big) = 1$$

and, since preferences are continuous, if $y \succ x$, there is a sequence $\langle z_n \rangle$ in $\mathbb{Q}_l$ such that $y = \lim_{n\to\infty} e + z_n$ and $e + z_n \succ x$ for all n. Therefore,

$$\nu\Big(\big\{(\succ, e, x) : y \succ x \text{ implies } py \geq pe\big\}\Big) = 1.$$

Since preferences are monotone and every point in X is the limit of a sequence of larger points, $\nu\big(\{(\succ, e, x) : px \geq pe\}\big) = 1$. If it would be the case that $\nu\big(\{(\succ, e, x) : px > pe\}\big) > 0$, then

$$p\int \pi_E \, \mathrm{d}\nu = \int p\pi_E \, \mathrm{d}\nu > \int p\pi_X \, \mathrm{d}\nu = p\int \pi_X \, \mathrm{d}\nu,$$

in contradiction to $\int \pi_E \, \mathrm{d}\nu = \int \pi_X \, \mathrm{d}\nu$. So $\nu\big(\{(\succ, e, x) : px = pe\}\big) = 1$. Since $p \geq 0$ and $p \neq 0$ and $\int \pi_E \, d\mu \gg 0$, we have $\nu\big(\{(\succ, e, x) : pe > 0\}\big) > 0$. The usual cheaper point argument implies then that $py > pe$ whenever $y \succ x$. By strict monotonicity, this is only possible if $p \gg 0$, which is therefore the case. But then pe is not positive only when $e = 0$, in which case we must have $x = 0$ and $y \succ x = 0$ implies then that $py > 0 = pe$. So $px \leq py$ and $y \succ x$ implies $py > pe$ holds for ν-almost all $(\succ, e, x)$ and ν is a Walrasian reallocation. □

The following lemma, translated from the abstract setting of measure algebras to the setting of measure spaces, lies at the heart of the proof of Maharam's celebrated characterization result for measure algebras in Maharam (1942). For an exceptionally clear proof, see Fremlin (1989, Lemma 3.2).

Lemma 3. *Let (T, Σ, ν) be a finite measure space, $\Sigma' \subseteq \Sigma$ a sub-σ-algebra such that ν is relatively atomless over Σ', and let μ be a measure on (T, Σ') such that $\mu(A) \leq \nu(A)$ for all $A \in \Sigma'$. Then there exists a set $C \in \Sigma$ such that $\mu(A) = \nu(A \cap C)$ for all $A \in \Sigma'$.*

Proof of Theorem 7. Clearly, μ is absolutely continuous with respect to $\nu \circ \phi^{-1}$ and has a Radon-Nikodym derivative $g : X \to [0, 1]$ with respect to $\nu \circ \phi^{-1}$. Let κ be the measure on Σ that has Radon-Nikodym derivative $g \circ \phi$ with respect to ν. Clearly, $\kappa(B) \leq \nu(B)$ for all $B \in \sigma(\phi)$. By the Lemma 3, there exists $C \in \Sigma$ such that $\kappa(B) = \nu(B \cap C)$ for all$B \in \sigma(\phi)$. Let $A \in \mathcal{X}$,

$$\mu(A) = \int_A g \, d\nu \circ \phi^{-1} = \int g 1_A \, d\nu \circ \phi^{-1} = \int g 1_A \circ \phi \, d\nu$$

$$= \int (1_A \circ \phi)(g \circ \phi) \, d\nu = \int 1_A \circ \phi \, d\kappa = \int 1_{\phi^{-1}(A)} \, d\kappa = \kappa \circ \phi^{-1}(A)$$

$$= \nu(C \cap \phi^{-1}(A)).$$

□

Proof of Proposition 4. Suppose for the sake of contradiction that there exists $A \in \Sigma$ with $\nu(A) > 0$ such that for every $B \in \Sigma$ with $B \subseteq A$, there exists $C \in \sigma(\phi)$ such that $\nu(B \Delta C) = 0$. In particular, there must exist $G \in \sigma(\phi)$ such that $\nu(A \Delta G) = 0$. Let $\mu = 1/2\ \nu \circ \phi^{-1}$. It follows that $\nu(C \cap C_\mu) = 1/2\nu(C)$ for all $C \in \sigma(\phi)$. Therefore, $\nu(A \cap C_\mu) = \nu(G \cap C_\mu) > 0$. Since $A \cap C_\mu \in \Sigma$ and $A \cap C_\mu \subseteq A$, there is an $F \in \sigma(\phi)$ such that $\nu\big((A \cap C_\mu)\Delta F\big) = 0$. But then

$$1/2\nu(F) = \nu(F \cap C_\mu) = \nu\big((A \cap C_\mu) \cap C_\mu\big) = \nu\big(A \cap C_\mu\big) = \nu(F) > 0,$$

which is absurd. □

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