LOCALIZATION OPERATORS AND TIME-FREQUENCY ANALYSIS

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Abstract. Localization operators have been object of study in quantum mechanics, in PDE and signal analysis recently. In engineering, a natural language is given by time-frequency analysis. Arguing from this point of view, we shall present the theory of these operators developed so far. Namely, regularity properties, composition formulae and their multilinear extension shall be highlighted. Time-frequency analysis will provide tools, techniques and function spaces. In particular, we shall use modulation spaces, which allow "optimal" results in terms of regularity properties for localization operators acting on $L^2(\mathbb{R}^d)$.

Contents

1. Introduction and Definitions

The name "localization operators" goes back to 1988, when I. Daubechies [17] first used these operators as a mathematical tool to localize a signal on the time-frequency plane. Localization operators with Gaussian windows were already known in physics: they were introduced as a quantization rule by Berezin [4]

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in 1971 and called anti-Wick operators. Since their first appearance, they have been extensively studied as an important mathematical tool in signal analysis and other applications (see [18, 37, 44] and references therein). Beyond signal analysis and the anti-Wick quantization procedure [4, 38], we recall their employment as approximation of pseudodifferential operators ("wave packets") [16, 27]. Besides, in other branches of mathematics, localization operators are also named Toeplitz operators (see, e.g., [19]) or short-time Fourier transform multipliers [25].

The objective of this chapter is to report on recent progress on localization operators and to present the state-of-the-art. We complement the "First survey of Gabor multipliers" [25] by Feichtinger and Nowak. Since the appearance of their survey our understanding of localization operators has expanded considerably, and many open questions have since been resolved satisfactorily.

The very definition of localization operators is carried out by time-frequency tools and representations, see for example [28]. Indeed, we consider the linear operators of translation and modulation (so-called time-frequency shifts) given by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega t} f(t)\,. \tag{1}$$

These occur in the following time-frequency representation. Let g be a non-zero window function in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, then the short-time Fourier transform (STFT) of a signal $f \in L^2(\mathbb{R}^d)$ with respect to the window g is given by

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle = \int_{\mathbb{R}^d} f(t)\, \overline{g(t-x)}\, e^{-2\pi i \omega t}\, dt\,. \tag{2}$$

We have $V_g f \in L^2(\mathbb{R}^{2d})$. This definition can be extended to every pair of dual topological vector spaces, whose duality, denoted by $\langle \cdot, \cdot \rangle$, extends the inner product on $L^2(\mathbb{R}^d)$. For instance, it may be suited to the framework of distributions and ultra-distributions.

Just few words to explain the meaning of the previous "time-frequency" representation. If $f(t)$ represents a signal varying in time, its Fourier transform $\hat{f}(\omega)$ shows the distribution of its frequency ω, without any additional information about "when" these frequencies appear. To overcome this problem, one may choose a non-negative window function g well localized around the origin. Then, the information of the signal f at the instant x can be obtained by shifting the window g till the instant x under consideration, and by computing the Fourier transform of the product $f(x)g(t-x)$, that localizes f around the instant time x.

Once the analysis of the signal f is terminated, we can reconstruct the original signal f by a suitable inversion procedure. Namely, the reproducing formula related to the STFT, for every pairs of windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ with $\langle \varphi_1, \varphi_2\rangle \neq 0$, reads as follows

$$\int_{\mathbb{R}^{2d}} V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2 \, dx d\omega = \langle \varphi_2, \varphi_1\rangle f\,. \tag{3}$$

The function φ_1 is called the *analysis* window, because the STFT $V_{\varphi_1} f$ gives the time-frequency distribution of the signal f, whereas the window φ_2 permits to come back to the original f and, consequently, is called the *synthesis* window.

The signal analysis often requires to highlight some features of the time-frequency distribution of f. This is achieved by first multiplying the STFT $V_{\varphi_1} f$ by a suitable function $a(x,\omega)$ and secondly by constructing $\tilde{f}$ from the product the product $aV_{\varphi_2} f$. In other words, we recover a filtered version of the original signal f which we denote by $A_a^{\varphi_1,\varphi_2}$. This intuition motivates the definition of time-frequency localization operators.

Definition 1.1. *The localization operator $A_a^{\varphi_1,\varphi_2}$ with symbol $a \in \mathcal{S}(\mathbb{R}^{2d})$ and windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ is defined to be*

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2(t)\, dx d\omega\,, \quad f \in L^2(\mathbb{R}^d). \tag{4}$$

The preceding definition makes sense also if we assume $a \in L^\infty(\mathbb{R}^{2d})$, see below. In particular, if $a = \chi_\Omega$ for some compact set $\Omega \subseteq \mathbb{R}^{2d}$ and $\varphi_1 = \varphi_2$, then $A_a^{\varphi_1,\varphi_2}$ is interpreted as the part of f that "lives on the set Ω" in the time-frequency plane. This is why $A_a^{\varphi_1,\varphi_2}$ is called a *localization* operator.

Often it is more convenient to interpret the definition of $A_a^{\varphi_1,\varphi_2}$ in a weak sense, then (4) can be recast as

$$\langle A_a^{\varphi_1,\varphi_2} f, g\rangle = \langle a V_{\varphi_1} f, V_{\varphi_2} g\rangle = \langle a, \overline{V_{\varphi_1} f}\, V_{\varphi_2} g\rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d)\,. \tag{5}$$

If we enlarge the class of symbols to the tempered distributions, i.e., we take $a \in \mathcal{S}'(\mathbb{R}^{2d})$ whereas $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then (4) is a well-defined continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. The previous assertion can be proven directly using the weak definition. For every window $\varphi_1 \in \mathcal{S}(\mathbb{R}^d)$ the STFT V_{φ_1} is a continuous mapping from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^{2d})$ (see, e.g., [28, Theorem 11.2.5]). Since also $V_{\varphi_2} g \in \mathcal{S}(\mathbb{R}^{2d})$, the brackets $\langle a, \overline{V_{\varphi_1} f}\, V_{\varphi_2} g\rangle$ are well-defined in the duality between $\mathcal{S}'(\mathbb{R}^{2d})$ and $\mathcal{S}(\mathbb{R}^{2d})$. Consequently, the left-hand side of (5) can be interpreted in the duality between $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ and shows that $A_a^{\varphi_1,\varphi_2}$ is a continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. The continuity of the mapping $A_a^{\varphi_1,\varphi_2}$ is achieved by using the continuity of both the STFT and the brackets $\langle\cdot,\cdot\rangle$. Similar arguments can be applied for tempered ultra-distributions, as we are going to see later on.

If $\varphi_1(t) = \varphi_2(t) = e^{-\pi t^2}$, then $A_a = A_a^{\varphi_1,\varphi_2}$ is the classical anti-Wick operator and the mapping $a \to A_a^{\varphi_1,\varphi_2}$ is interpreted as a quantization rule [4, 38, 44].

Note that the time-frequency shifts $(x,\omega,\tau) \mapsto \tau T_x M_\omega$, $(x,\omega) \in \mathbb{R}^{2d}, |\tau| = 1$, define the Schrödinger representation of the Heisenberg group; for a deeper understanding of localization operators it is therefore natural to use the mathematical tools associated to harmonic analysis and time-frequency shifts, see [27, 28] and the next Section 2.

Localization operators can be viewed as a multilinear mapping

$$(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}, \tag{6}$$

acting on products of symbol and window spaces. The dependence of the localization operator $A_a^{\varphi_1,\varphi_2}$ on all three parameters has been widely studied in different functional frameworks. The start was given by subspaces of the tempered distributions. The basic subspace is $L^2(\mathbb{R}^d)$, but many other Banach and Hilbert spaces, as well as topological vector spaces, have been considered. We mention L^p spaces [6, 44], potential and Sobolev spaces [7], modulation spaces [10, 25, 35, 42, 43] and Gelfand-Shilov spaces [15] (the last ones in the ultra-distribution environment) as samples of spaces either for choosing symbol and windows or for defining the action of the related localization operator. The outcomes are manifold. The continuity of the mapping in (6) can be expressed by an inequality of the form

$$\|A_a^{\varphi_1,\varphi_2}\|_{op} \leq C\|a\|_{B_1}\,\|\varphi_1\|_{B_2}\,\|\varphi_2\|_{B_3}\,, \tag{7}$$

where B_1, B_2, B_3 are suitable spaces of symbols and windows. For example, if $a \in L^\infty(\mathbb{R}^d)$ and $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$, then

$$\begin{aligned}
\|A_a^{\varphi_1,\varphi_2}\|_{B(L^2)} &= \sup_{\|f\|_{L^2}=1}\,\sup_{\|g\|_{L^2}=1} |\langle A_a^{\varphi_1,\varphi_2} f, g\rangle| \\
&= \sup_{\|f\|_{L^2}=1}\,\sup_{\|g\|_{L^2}=1} |\langle a, \overline{V_{\varphi_1} f}\, V_{\varphi_2} g\rangle| \\
&\leq \sup_{\|f\|_{L^2}=1}\,\sup_{\|g\|_{L^2}=1} \|a\|_{L^\infty} \|\overline{V_{\varphi_1} f}\, V_{\varphi_2} g\|_{L^1} \\
&\leq \sup_{\|f\|_{L^2}=1}\,\sup_{\|g\|_{L^2}=1} \|a\|_{L^\infty} \|V_{\varphi_1} f\|_{L^2} \|V_{\varphi_2} g\|_{L^2} \\
&= \|a\|_{L^\infty} \|\varphi_1\|_{L^2} \|\varphi_2\|_{L^2}\,,
\end{aligned}$$

where the last inequality is achieved by using the orthogonality relations for the STFT

$$\|V_\varphi f\|_{L^2(\mathbb{R}^{2d})} = \|\varphi\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)}, \quad \forall \varphi, f \in L^2(\mathbb{R}^d).$$

Thus for this particular choice of symbol classes and window spaces we obtain the L^2 boundedness. The previous easy proof gives just a flavour of the boundedness results for localization operators, we shall see that the symbol class L^∞ can be enlarged significantly. Even a tempered distribution like δ may give the boundedness of the corresponding localization operator. Apart from continuity, estimates of the type (7) also supply Hilbert-Schmidt, Trace class and Schatten class properties for $A_a^{\varphi_1,\varphi_2}$ [11, 15].

Among the many function/(ultra-)distribution spaces employed, modulation spaces reveal to be the *optimal choice* for handling localization operators, see Section 3 below. As special case we mention Feichtinger's algebra $M^1(\mathbb{R}^d)$ defined by the norm

$$\|f\|_{M^1} := \|V_g f\|_{L^1(\mathbb{R}^{2d})}$$

for some (hence all) non-zero $g \in \mathcal{S}(\mathbb{R}^d)$ [23, 28]. Its dual space $M^\infty(\mathbb{R}^{2d})$ is a very useful subspace of tempered distributions and possesses the norm

$$\|f\|_{M^\infty} := \sup_{(x,\omega)\in\mathbb{R}^{2d}} |V_g f(x,\omega)|\,.$$

With these spaces the estimate (7) reads as follows:

Theorem 1.2. *If $a \in M^\infty(\mathbb{R}^{2d})$, and $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$, then $A_a^{\varphi_1,\varphi_2}$ is bounded on $L^2(\mathbb{R}^d)$, with operator norm at most*

$$\|A_a^{\varphi_1,\varphi_2}\|_{B(L^2)} \leq C\|a\|_{M^\infty} \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}\,.$$

The striking fact is the converse of the preceding result [10].

Theorem 1.3. *If $A_a^{\varphi_1,\varphi_2}$ is bounded on $L^2(\mathbb{R}^d)$ uniformly with respect to* all *windows $\varphi_1, \varphi_2 \in M^1$, i.e., if there exists a constant $C > 0$ depending only on the symbol a such that, for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$,*

$$\|A_a^{\varphi_1,\varphi_2}\|_{B(L^2)} \leq C\|\varphi_1\|_{M^1}\,\|\varphi_2\|_{M^1}\,, \tag{8}$$

then $a \in M^\infty$.

Similar statements hold true for Schatten class properties [11] and for weighted ultra-distributional modulation spaces [15]. A recent result in the study of localization operators [26] reveals the optimality of modulation spaces even for the compactness property. These topics shall be detailed in Sections 4 and 5.

In Section 6 we shall treat the composition of localization operators. Whereas the product of two pseudodifferential operators is again a pseudodifferential operator, in general the composition of two localization operators is no longer a localization operator. This additional difficulty has captured the interest of several authors, generating some remarkable ideas. An exact product formula for localization operators, obtained in [20], shall be presented. Notice, however, that it works only under very restrictive conditions and is unstable. In another direction, many authors have made resort to asymptotic expansions that realize the composition of two localization operators as a sum of localization operators and a controllable remainder [1, 14, 33, 40]. These contributions are mainly motivated by applications to PDEs and energy estimates, and therefore use smooth symbols defined by differentiability properties, such as the traditional Hörmander or Shubin classes, and Gaussian windows. In the context of time-frequency analysis, where modulation spaces can be employed, much rougher symbols and more general window functions are allowed to be used for localization operators. Consequently, the product formula in [14] has been extended to rougher spaces of symbols in [12], as we are going to show.

In the end (Section 7), we shall present a new framework for localization operators. Namely, the study of multilinear pseudodifferential operators [2, 3] motivates the definition of multilinear localization operators. For them, we shall present the sufficient and necessary boundedness properties together with connection with Kohn-Nirenberg operators [13].

Notation. We define $t^2 = t \cdot t$, for $t \in \mathbb{R}^d$, and $xy = x \cdot y$ is the scalar product on $\mathbb{R}^d$.

The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. We use the brackets $\langle f, g\rangle$ to denote the extension to $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ of the inner product $\langle f, g\rangle = \int f(t)\overline{g(t)}dt$ on $L^2(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\omega) = \mathcal{F}f(\omega) = \int f(t)e^{-2\pi i t\omega}dt$, the involution g^* is $g^*(t) = \overline{g(-t)}$.

The singular values $\{s_k(L)\}_{k=1}^\infty$ of a compact operator $L \in B(L^2(\mathbb{R}^d))$ are the eigenvalues of the positive self-adjoint operator $\sqrt{L^*L}$. Equivalently, for every $k \in$

$\mathbb{N}$, the singular value $\{s_k(L)\}$ is given by

$$s_k(L) = \inf\{\|L - T\|_{L^2} \,:\, T \in B(L^2(\mathbb{R}^d)) \quad \text{and} \quad \dim\ \mathrm{Im}\,(T) \le k\}.$$

For $1 \le p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in l^p. For consistency, we define $S_\infty := B(L^2(\mathbb{R}^d))$ to be the space of bounded operators on $L^2(\mathbb{R}^d)$. In particular, S_2 is the space of Hilbert-Schmidt operators, and S_1 is the space of trace class operators.

Throughout the paper, we shall use the notation $A \lesssim B$ to indicate $A \le cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ if $A \le cB$ and $B \le kA$, for suitable $c, k > 0$.

2. Time-Frequency Methods

First we summarize some concepts and tools of time-frequency analysis, for an extended exposition we refer to the textbooks [27, 28].

The time-frequency representations required for localization operators and the Weyl calculus are the *short-time Fourier transform* and the *Wigner distribution*.

The short-time Fourier transform (STFT) is defined in (2). The *cross-Wigner distribution* $W(f,g)$ of $f,g \in L^2(\mathbb{R}^d)$ is given by

$$W(f,g)(x,\omega) = \int f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}e^{-2\pi i\omega t}\,dt. \tag{9}$$

The quadratic expression $Wf = W(f,f)$ is usually called the Wigner distribution of f.

Both the STFT V_gf and the Wigner distribution $W(f,g)$ are defined for f,g in many possible pairs of Banach spaces. For instance, they both map $L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$ and $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^{2d})$. Furthermore, they can be extended to a map from $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^{2d})$.

For a non-zero $g \in L^2(\mathbb{R}^d)$, we write V_g^* for the adjoint of V_g, given by

$$\langle V_g^* F, f\rangle = \langle F, V_g f\rangle, \quad f \in L^2(\mathbb{R}^d),\ F \in L^2(\mathbb{R}^{2d}).$$

In particular, for $F \in \mathcal{S}(\mathbb{R}^{2d})$, $g \in \mathcal{S}(\mathbb{R}^d)$, we have

$$V_g^* F(t) = \int_{\mathbb{R}^{2d}} F(x,\omega) M_\omega T_x g(t)\,dx\,d\omega \in \mathcal{S}(\mathbb{R}^d). \tag{10}$$

Take $f \in \mathcal{S}(\mathbb{R}^d)$ and set $F = V_g f$, then

$$f(t) = \frac{1}{\|g\|_{L^2}^2}\int_{\mathbb{R}^{2d}} V_g f(x,\omega) M_\omega T_x g(t)\,dx\,d\omega \in \mathcal{S}(\mathbb{R}^d) = \frac{1}{\|g\|_{L^2}^2} V_g^* V_g f(t). \tag{11}$$

We refer to [28, Proposition 11.3.2] for a detailed treatment of the adjoint operator.

Representation of localization operators as Weyl/Kohn-Nirenberg operators. Let $W(g,f)$ be the cross-Wigner distribution as defined in (9). Then the Weyl operator L_σ of symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ is defined by

$$\langle L_\sigma f, g\rangle = \langle \sigma, W(g,f)\rangle, \qquad f,g \in \mathcal{S}(\mathbb{R}^d). \tag{12}$$

Every linear continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ can be represented as a Weyl operator, and a calculation in [7, 27, 38] reveals that

$$A_a^{\varphi_1,\varphi_2} = L_{a*W(\varphi_2,\varphi_1)}, \tag{13}$$

so the (Weyl) symbol of $A_a^{\varphi_1,\varphi_2}$ is given by

$$\sigma = a * W(\varphi_2,\varphi_1)\,. \tag{14}$$

To get boundedness results for a localization operator, it is sometimes convenient to write it in a different pseudodifferential form. Consider the *Kohn-Nirenberg* form of a pseudodifferential operator, given by

$$T_\tau f(x) = \int_{\mathbb{R}^d} \tau(x,\omega)\hat{f}(\omega)e^{2\pi i x\omega}\,d\omega, \quad f\in\mathcal{S}(\mathbb{R}^d), \tag{15}$$

where τ is a measurable function, or even a tempered distribution on $\mathbb{R}^{2d}$.
If we define the *rotation* operator $\mathcal{U}$ acting on a function F on $\mathbb{R}^{2d}$ by

$$\mathcal{U}F(x,\omega) = F(\omega,-x), \quad \forall\,(x,\omega)\in\mathbb{R}^{2d}, \tag{16}$$

then, the identity of operators below holds [13]:

$$A_a^{\varphi_1,\varphi_2} = T_\tau, \tag{17}$$

with the Kohn-Nirenberg symbol τ given by

$$\tau = a * \mathcal{U}\mathcal{F}(V_{\varphi_1}\varphi_2) \tag{18}$$

The expression $\mathcal{U}\mathcal{F}(V_{\varphi_1}\varphi_2)$ is usually called the Rihaczek distribution.

3. Function Spaces

Gelfand-Shilov spaces. The Gelfand-Shilov spaces were introduced by Gelfand and Shilov in [30]. They have been applied by many authors in different contexts, see, e.g. [9, 32, 34, 41]. For the sake of completeness, we recall their definition and properties in more generality than required.

Definition 3.1. *Let $\alpha,\beta\in\mathbb{R}_+^d$, and assume $A_1,\dots,A_d$, $B_1,\dots,B_d>0$. Then the Gelfand-Shilov space $\mathcal{S}_{\beta,A}^{\alpha,B}=\mathcal{S}_{\beta,A}^{\alpha,B}(\mathbb{R}^d)$ is defined by*

$$\mathcal{S}_{\beta,A}^{\alpha,B} = \{f\in C^\infty(\mathbb{R}^d)\mid(\exists C>0)\ \|x^p\partial^q f\|_{L^\infty}\le CA^p(p!)^\beta B^q(q!)^\alpha,\ \ \forall p,q\in\mathbb{N}_0^d\}.$$

We then consider projective and inductive limits denoted by

$$\Sigma_\beta^\alpha = \operatorname{proj}\lim_{A>0,B>0}\mathcal{S}_{\beta,A}^{\alpha,B};\quad \mathcal{S}_\beta^\alpha := \operatorname{ind}\lim_{A>0,B>0}\mathcal{S}_{\beta,A}^{\alpha,B}.$$

For a comprehensive treatment of Gelfand-Shilov spaces we refer to [30]. We limit ourselves to those features that will be useful for our study.

Proposition 3.2. *The next statements are equivalent* [15, 30, 31]*:*
(i) $f\in\mathcal{S}_\beta^\alpha(\mathbb{R}^d)$.
(ii) $f\in\mathcal{C}^\infty(\mathbb{R}^d)$ and there exist real constants $h>0,k>0$ such that:

$$\|fe^{h|x|^{1/\beta}}\|_{L^\infty}<\infty \quad \textit{and} \quad \|\mathcal{F}fe^{k|\omega|^{1/\alpha}}\|_{L^\infty}<\infty, \tag{19}$$

where $|x|^{1/\beta} = |x_1|^{1/\beta_1} + \cdots + |x_d|^{1/\beta_d}$, $|\omega|^{1/\alpha} = |\omega_1|^{1/\alpha_1} + \cdots + |\omega_d|^{1/\alpha_d}$.

(iii) $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ *and there exists* $C > 0, h > 0$ *such that*

$$\|(\partial^q f)e^{h|x|^{1/\beta}}\|_{L^\infty} \leq C^{|q|+1}(q!)^\alpha, \quad \forall q \in \mathbb{N}_0^d. \tag{20}$$

Gelfand-Shilov spaces enjoy the following embeddings:
(i) For $\alpha, \beta \geq 0$ [30],

$$\Sigma_\beta^\alpha \hookrightarrow \mathcal{S}_\beta^\alpha \hookrightarrow \mathcal{S}. \tag{21}$$

(ii) For every $0 \leq \alpha_1 < \alpha_2$ and $0 \leq \beta_1 < \beta_2$ [15],

$$\mathcal{S}_{\beta_1}^{\alpha_1} \hookrightarrow \Sigma_{\beta_2}^{\alpha_2}. \tag{22}$$

Furthermore, S_β^α is not trivial if and only if $\alpha + \beta > 1$ or $\alpha + \beta = 1$ and $\alpha\beta > 0$. The spaces Σ_α^α with $\alpha \geq 1/2$ are studied by Pilipović [34]. In particular, the case $\alpha = 1/2$ yields $\Sigma_{1/2}^{1/2} = \emptyset$.

The Fourier transform $\mathcal{F}$ is a topological isomorphism between S_α^β and S_β^α ($\mathcal{F}(S_\alpha^\beta) = S_\beta^\alpha$) and extends to a continuous linear transform from $(S_\alpha^\beta)'$ onto $(S_\beta^\alpha)'$. If $\alpha \geq 1/2$, then $\mathcal{F}(S_\alpha^\alpha) = S_\alpha^\alpha$. The Gelfand-Shilov spaces are invariant under time-frequency shifts:

$$T_x(\mathcal{S}_\alpha^\beta) = \mathcal{S}_\alpha^\beta \quad \text{and} \quad M_\omega(\mathcal{S}_\alpha^\beta) = \mathcal{S}_\alpha^\beta\,, \tag{23}$$

and similarly to the Σ_β^α.

Therefore the spaces S_α^α are a family of *Fourier transform and time-frequency shift invariant spaces* which are contained in the Schwartz class $\mathcal{S}$. Among these S_α^α the *smallest* non-trivial Gelfand-Shilov space is given by $\mathcal{S}_{1/2}^{1/2}$. A basic example is given by $f(x) = e^{-\pi x^2} \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$.

Another useful characterization of the space $\mathcal{S}_\alpha^\alpha$ involves the STFT: $f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ if and only if $V_g f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d})$ (see [32, Proposition 3.12] and reference therein). We will use the case $\alpha = 1/2$: for a non-zero window $g \in \mathcal{S}_{1/2}^{1/2}$ we have

$$V_g f \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^{2d}) \Leftrightarrow f \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d). \tag{24}$$

The strong duals of Gelfand-Shilov classes $\mathcal{S}_\beta^\alpha$ and Σ_β^α are spaces of tempered ultra-distributions of Roumieu and Beurling type and will be denoted by $(\mathcal{S}_\beta^\alpha)'$ and $(\Sigma_\beta^\alpha)'$, respectively.

Modulation Spaces. The modulation space norms traditionally measure the joint time-frequency distribution of $f \in \mathcal{S}'$, we refer, for instance, to [21], [28, Ch. 11-13] and the original literature quoted there for various properties and applications. In that setting it is sufficient to observe modulation spaces with weights which admit at most polynomial growth at infinity. However the study of ultra-distributions requires a more general approach that includes the weights of exponential growth.

Weight Functions. In the sequel v will always be a continuous, positive, even, submultiplicative function (submultiplicative weight), i.e., $v(0) = 1$, $v(z) = v(-z)$, and $v(z_1+z_2)\leq v(z_1)v(z_2)$, for all $z,z_1,z_2\in\mathbb{R}^{2d}$. Moreover, v is assumed to be even in each group of coordinates, that is, $v(\pm x,\pm\omega)=v(x,\omega)$, for all $(x,\omega)\in\mathbb{R}^{2d}$ and all choices of signs. Submultiplicativity implies that $v(z)$ is *dominated* by an exponential function, i.e.

$$\exists\, C,k>0 \quad \text{such that} \quad v(z)\leq Ce^{k|z|}, \quad z\in\mathbb{R}^{2d}. \tag{25}$$

For example, every weight of the form $v(z)=e^{a|z|^b}(1+|z|)^s\log^r(e+|z|)$ for parameters $a,r,s\geq 0$, $0\leq b\leq 1$ satisfies the above conditions.

Associated to every submultiplicative weight we consider the class of so-called *v-moderate* weights $\mathcal{M}_v$. A positive, even weight function m on $\mathbb{R}^{2d}$ belongs to $\mathcal{M}_v$ if it satisfies the condition

$$m(z_1+z_2)\leq Cv(z_1)m(z_2) \quad \forall z_1,z_2\in\mathbb{R}^{2d}\,.$$

We note that this definition implies that $\frac{1}{v}\lesssim m\lesssim v$, $m\neq 0$ everywhere, and that $1/m\in\mathcal{M}_v$.

For the investigation of localization operators the weights mostly used are defined by

$$\begin{aligned}
v_s(z) &= v_s(x,\omega)=\langle z\rangle^s=(1+x^2+\omega^2)^{s/2}, \quad z=(x,\omega)\in\mathbb{R}^{2d} && (26)\\
w_s(z) &= w_s(x,\omega)=e^{s|(x,\omega)|}, \quad z=(x,\omega)\in\mathbb{R}^{2d}\,, && (27)\\
\tau_s(z) &= \tau_s(x,\omega)=\langle\omega\rangle^s && (28)\\
\mu_s(z) &= \mu_s(x,\omega)=e^{s|\omega|}\,. && (29)
\end{aligned}$$

Definition 3.3. *Let m be a weight in $\mathcal{M}_v$, and g a non-zero* window *function in $\mathcal{S}^{1/2}_{1/2}$. For $1\leq p,q\leq\infty$ and $f\in\mathcal{S}^{1/2}_{1/2}$ we define the modulation space norm (on $\mathcal{S}^{1/2}_{1/2}$) by*

$$\|f\|_{M^{p,q}_m}=\|V_gf\|_{L^{p,q}_m}=\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_gf(x,\omega)|^pm(x,\omega)^p\,dx\right)^{q/p}d\omega\right)^{1/q},$$

(with obvious changes if either $p=\infty$ or $q=\infty$). If $p,q<\infty$, the modulation space $M^{p,q}_m$ is the norm completion of $\mathcal{S}^{1/2}_{1/2}$ in the $M^{p,q}_m$-norm. If $p=\infty$ or $q=\infty$, then $M^{p,q}_m$ is the completion of $\mathcal{S}^{1/2}_{1/2}$ in the weak topology. If $p=q$, $M^p_m:=M^{p,p}_m$, and, if $m\equiv 1$, then $M^{p,q}$ and M^p stand for $M^{p,q}_m$ and $M^{p,p}_m$, respectively.*

Notice that:
(i) If $f,g\in\mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$, the above integral is convergent thanks to (19) and (24). Namely, the constant h in (19) guarantees $\|V_gfe^{h|\cdot|^2}\|_{L^\infty}<\infty$ and, for $m\in\mathcal{M}_v$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_gf(x,\omega)|^pm(x,\omega)^p\,dx\right)^{q/p}d\omega\\
&\qquad\leq C\,\|(V_gf)e^{h|\cdot|^2}\|_{L^\infty}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|m(x,\omega)|^pe^{-hp|(x,\omega)|^2}\,dx\right)^{q/p}d\omega<\infty\,.
\end{aligned}$$

(ii) By definition, $M^{p,q}_m$ is a Banach space. Besides, it is proven for the subexponential case in [21] and for the exponential one in [15] that their definition does not depend on the choice of the window g, that can be enlarged to the modulation algebra M^1_v.
(iii) For $m \in \mathcal{M}_v$ of at most polynomial growth, $M^{p,q}_m \subset \mathcal{S}'$ and the definition 3.3 reads as [10, 28]:

$$M^{p,q}_m(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \ V_g f \in L^{p,q}_m(\mathbb{R}^{2d})\}.$$

(iv) For every weight $m \in \mathcal{M}_v$, $M^{p,q}_m$ is the subspace of ultra-distribution $(\Sigma^1_1)'$ defined in [15, Definition 2.1].
(iv) If m belongs to $\mathcal{M}_v$ and fulfills the GRS-condition $\lim_{n\to\infty} v(nz)^{1/n} = 1$, for all $z \in \mathbb{R}^{2d}$, the definition of modulation spaces is the same as in [12] (because the "space of special windows" $\mathcal{S}_\mathcal{C}$ is a subset of $\mathcal{S}^{1/2}_{1/2}$).
(v) For related constructions of modulation spaces, involving the theory of coorbit spaces, we refer to [22, 24].

The class of modulation spaces contains the following well-known function spaces:
Weighted L^2-spaces: $M^2_{\langle x\rangle^s}(\mathbb{R}^d) = L^2_s(\mathbb{R}^d) = \{f : f(x)\langle x\rangle^s \in L^2(\mathbb{R}^d)\}, s \in \mathbb{R}$.
Sobolev spaces: $M^2_{\langle \omega\rangle^s}(\mathbb{R}^d) = H^s(\mathbb{R}^d) = \{f : \hat{f}(\omega)\langle \omega\rangle^s \in L^2(\mathbb{R}^d)\}, s \in \mathbb{R}$.
Shubin-Sobolev spaces [38, 7]: $M^2_{\langle (x,\omega)\rangle^s}(\mathbb{R}^d) = L^2_s(\mathbb{R}^d) \cap H^s(\mathbb{R}^d) = Q_s(\mathbb{R}^d)$.
Feichtinger's algebra: $M^1(\mathbb{R}^d) = S_0(\mathbb{R}^d)$.

The characterization of the Schwartz class of tempered distributions is given in [31]: we have $\mathcal{S}(\mathbb{R}^d) = \bigcap_{s\geq 0} M^1_{\langle\cdot\rangle^s}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d) = \bigcup_{s\geq 0} M^\infty_{1/\langle\cdot\rangle^s}(\mathbb{R}^d)$. A similar characterization for Gelfand-Shilov spaces and tempered ultra-distributions was obtained in [15, Proposition 2.3]: Let $1 \leq p, q \leq \infty$, and let w_s be given by (27), then,

$$\mathcal{S}^1_1 = \bigcap_{s\geq 0} M^{p,q}_{w_s}, \qquad (\mathcal{S}^1_1)' = \bigcup_{s\geq 0} M^{p,q}_{1/w_s}. \tag{30}$$

$$\Sigma^1_1 = \bigcup_{s>0} M^{p,q}_{w_s}, \qquad (\Sigma^1_1)' = \bigcap_{s>0} M^{p,q}_{1/w_s}. \tag{31}$$

Potential spaces. For $s \in \mathbb{R}$ the Bessel kernel is

$$G_s = \mathcal{F}^{-1}\{(1 + |\cdot|^2)^{-s/2}\}, \tag{32}$$

and the *potential space* [5] is defined by

$$W^p_s = G_s * L^p(\mathbb{R}^d) = \{f \in \mathcal{S}', \, f = G_s * g, \, g \in L^p\}$$

with norm $\|f\|_{W^p_s} = \|g\|_{L^p}$.

For comparison we list the following embeddings between potential and modulation spaces [10].

Lemma 3.1. *We have*
(i) If $p_1 \leq p_2$ and $q_1 \leq q_2$, then $M^{p_1,q_1}_m \hookrightarrow M^{p_2,q_2}_m$.
(ii) For $1 \leq p \leq \infty$ and $s \in \mathbb{R}$

$$W^p_s(\mathbb{R}^d) \hookrightarrow M^{p,\infty}_{\tau_s}(\mathbb{R}^d).$$

Consequently, $L^p \subseteq M^{p,\infty}$, and in particular, $L^\infty \subseteq M^\infty$. But M^∞ contains all bounded measures on $\mathbb{R}^d$ and other tempered distributions. For instance, the point measure δ belongs to M^∞, because for $g \in \mathcal{S}$ we have

$$|V_g\delta(x,\omega)| = |\langle \delta, M_\omega T_x g\rangle| = |\bar{g}(-x)| \leq \|g\|_{L^\infty}, \quad \forall (x,\omega) \in \mathbb{R}^{2d}.$$

Convolution Relations and Wigner Estimates. In view of the relation between the multiplier a and the Weyl symbol (14), we need to understand the convolution relations between modulation spaces and some properties of the Wigner distribution.

We first state a *convolution relation for modulation spaces* proven in [10], in the style of Young's theorem. Let v be an arbitrary submultiplicative weight on $\mathbb{R}^{2d}$ and m a v-moderate weight. We write $m_1(x) = m(x,0)$ and $m_2(\omega) = m(0,\omega)$ for the restrictions to $\mathbb{R}^d \times \{0\}$ and $\{0\} \times \mathbb{R}^d$, and likewise for v.

Proposition 3.4. *Let $\nu(\omega) > 0$ be an arbitrary weight function on $\mathbb{R}^d$ and $1 \leq p,q,r,s,t \leq \infty$. If*

$$\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}, \qquad \text{and} \qquad \frac{1}{t} + \frac{1}{t'} = 1\,,$$

then

$$M^{p,st}_{m_1\otimes\nu}(\mathbb{R}^d) * M^{q,st'}_{v_1\otimes v_2\nu^{-1}}(\mathbb{R}^d) \hookrightarrow M^{r,s}_m(\mathbb{R}^d) \tag{33}$$

with norm inequality $\|f*h\|_{M^{r,s}_m} \lesssim \|f\|_{M^{p,st}_{m_1\otimes\nu}} \|h\|_{M^{q,st'}_{v_1\otimes v_2\nu^{-1}}}$.

REMARKS: 1. Despite the large number of indices, the statement of this proposition has some intuitive meaning: a function $f \in M^{p,q}$ behaves like $f \in L^p$ and $\hat{f} \in L^q$; so the parameters related to the x-variable behave like those in Young's theorem for convolution, whereas the parameters related to ω behave like Hölder's inequality for pointwise multiplication.

2. A special case of Proposition 3.4 with a different proof is contained in [42].

The modulation space norm of a cross-Wigner distribution may be controlled by the window norms, as taken from [10, 15].

Proposition 3.5. *Let $1 \leq p \leq \infty$ and $s \geq 0$.*
i) If $\varphi_1 \in M^1_{v_s}(\mathbb{R}^d)$ and $\varphi_2 \in M^p_{v_s}(\mathbb{R}^d)$, then $W(\varphi_2,\varphi_1) \in M^{1,p}_{\tau_s}(\mathbb{R}^{2d})$, with

$$\|W(\varphi_2,\varphi_1)\|_{M^{1,p}_{\tau_s}} \lesssim \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^p_{v_s}}. \tag{34}$$

ii) If $\varphi_1 \in M^1_{w_s}(\mathbb{R}^d)$ and $\varphi_2 \in M^p_{w_s}(\mathbb{R}^d)$, then $W(\varphi_2,\varphi_1) \in M^{1,p}_{\mu_s}(\mathbb{R}^{2d})$ with

$$\|W(\varphi_2,\varphi_1)\|_{M^{1,p}_{\mu_s}} \lesssim \|\varphi_1\|_{M^1_{w_s}} \|\varphi_2\|_{M^p_{w_s}}. \tag{35}$$

4. Regularity Results

In this section, we first give general sufficient conditions for boundedness and Schatten classes of localization operators. Then we treat ultra-distributions with compact support as symbols, and finally we shall state a compactness result.

4.1. **Sufficient Conditions for Boundedness and Schatten Class.** Using the tools of time-frequency analysis in Section 3, we can now obtain the properties of localization operators with symbols in modulation spaces, by reducing the problem to the corresponding one for the Weyl calculus.

First, we recall a boundedness and trace class result for the Weyl operators in terms of modulation spaces.

Theorem 4.1. *(i) If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$, then L_σ is bounded on $M^{p,q}(\mathbb{R}^d)$, $1 \le p,q \le \infty$, with a uniform estimate $\|L_\sigma\|_{S_\infty} \lesssim \|\sigma\|_{M^{\infty,1}}$ for the operator norm. In particular, L_σ is bounded on $L^2(\mathbb{R}^d)$.*

(ii) If $\sigma \in M^1(\mathbb{R}^{2d})$, then $L_\sigma \in S_1$ and $\|L_\sigma\|_{S^1} \lesssim \|\sigma\|_{M^1}$.

(iii) If $1 \le p \le 2$ and $\sigma \in M^p(\mathbb{R}^{2d})$, then $L_\sigma \in S_p$ and $\|L_\sigma\|_{S_p} \lesssim \|\sigma\|_{M^p}$.

(iv) If $2 \le p \le \infty$ and $\sigma \in M^{p,p'}(\mathbb{R}^{2d})$, then $L_\sigma \in S_p$ and $\|L_\sigma\|_{S_p} \lesssim \|\sigma\|_{M^{p,p'}}$.

One of many proofs of (i) can be found in [28, Thm. 14.5.2], the L^2-boundedness was first discovered by Sjöstrand [39]. The trace class property (ii) is proved in [29], whereas (iii) and (iv) follow by interpolation from the first two statements, since $[M^1, M^2]_\theta = M^p$ for $1 \le p \le 2$, and $[M^{\infty,1}, M^{2,2}]_\theta = M^{p,p'}$ for $2 \le p \le \infty$.

Based on the Thm. 4.1 and Prop. 3.4, we present the most general boundedness results for localization operators obtained so far. We detail the polynomial weight case, the exponential one is stated and proved by replacing the weight v_s by w_s and τ_s by μ_s (see [15, Theorem 3.2])

Theorem 4.2. *Let $s \ge 0$, $a \in M^\infty_{1/\tau_s}(\mathbb{R}^{2d})$, $\varphi_1, \varphi_2 \in M^1_{v_s}(\mathbb{R}^d)$. Then $A_a^{\varphi_1,\varphi_2}$ is bounded on $M^{p,q}(\mathbb{R}^d)$ for all $1 \le p,q \le \infty$, and the operator norm satisfies the uniform estimate*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_\infty} \lesssim \|a\|_{M^\infty_{1/\tau_s}} \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^1_{v_s}}.$$

Proof. See [10, Theorem 3.2]. To highlight the role of time-frequency analysis, we sketch the proof. An appropriate convolution relation is employed to show that the Weyl symbol $a * W(\varphi_2, \varphi_1)$ of $A_a^{\varphi_1,\varphi_2}$ is in $M^{\infty,1}$. Namely, if $\varphi_1, \varphi_2 \in M^1_{v_s}(\mathbb{R}^d)$, then by (34) we have $W(\varphi_2, \varphi_1) \in M^1_{\tau_s}(\mathbb{R}^{2d})$. Applying Proposition 3.4 in the form $M^\infty_{1/\tau_s} * M^1_{\tau_s} \subseteq M^{\infty,1}$, we obtain that the Weyl symbol $\sigma = a * W(\varphi_2, \varphi_1) \in M^{\infty,1}$. The result now follows from Theorem 4.1 (i). ∎

REMARK: To compare Theorem 4.2 to existing results, we recall that the standard condition for $A_a^{\varphi_1,\varphi_2}$ to be bounded is $a \in L^\infty(\mathbb{R}^{2d})$, see [44]. A more subtle result of Feichtinger and Nowak [25] shows that the condition a in the Wiener amalgam space $W(M, L^\infty)$ is sufficient for boundedness. Since we have the proper embeddings $L^\infty \subset W(M, L^\infty) \subset M^\infty \subset M^\infty_{1/\tau_s}$ for $s \ge 0$, Theorem 4.2 appears as a significant improvement. A special case of Theorem 4.2 follows also from Toft's work [43].

Since $\tau_s(z,\zeta) = \langle \zeta \rangle^s$ depends only on the frequency variable, the condition $a \in M^\infty_{1/\tau_s}$ describes the admissible roughness of a, while in some sense a remains bounded in z. On the other hand, if we allow the symbol a to grow in both time and frequency by choosing the "full" weight $v_s = \langle (z,\zeta)\rangle^s$, then we obtain a negative result [10, Proposition 3.3]:

Proposition 4.3. *For any $s > 0$ there exist symbols $a \in M^\infty_{1/v_s}(\mathbb{R}^{2d})$ and windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ such that $A_a^{\varphi_1,\varphi_2}$ is unbounded on $L^2(\mathbb{R}^d)$.*

These results demonstrate that bounded symbols with negative smoothness may still yield bounded localization operators, provided that the roughness of a is compensated by a suitable time-frequency localization of the windows. On the other hand, a smooth unbounded symbol cannot, in general, yield a bounded operator.

The Schatten class properties of localization operators with symbols in modulation spaces are achieved accordingly. Combining Proposition 3.4 with Theorem 4.1, almost optimal conditions for $A_a^{\varphi_1,\varphi_2} \in S_p$ are derived in [10, 15]. Again, we state the earliest result for weights of polynomial growth [10, Theorem 3.4].

Theorem 4.4. *(i) If $1 \le p \le 2$, then the mapping $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$ is bounded from $M^{p,\infty}_{1/\tau_s}(\mathbb{R}^{2d}) \times M^1_{v_s}(\mathbb{R}^d) \times M^p_{v_s}(\mathbb{R}^d)$ into S_p, in other words,*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{p,\infty}_{1/\tau_s}} \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^p_{v_s}} .$$

(ii) If $2 \le p \le \infty$, then the mapping $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$ is bounded from $M^{p,\infty}_{1/\tau_s} \times M^1_{v_s} \times M^{p'}_{v_s}$ into S_p, and

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{p,\infty}_{1/\tau_s}} \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^{p'}_{v_s}} .$$

Using the embeddings $W^p_{-s} \hookrightarrow M^{p,\infty}_{1/\tau_s}$ (Lemma 3.1) and $M^1_{v_s} \hookrightarrow M^p_{v_s}$, one obtains a slightly weaker statement for symbols in potential spaces. This result was already derived in [7, Thm. 4.7].

Corollary 4.5. *Let $a \in W^p_{-s}(\mathbb{R}^{2d})$ for some $s \ge 0$, $1 \le p \le \infty$, and $\varphi_1, \varphi_2 \in M^1_{v_s}(\mathbb{R}^d)$. Then*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{W^p_{-s}} \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^1_{v_s}} .$$

REMARK: By using other convolution relations provided by Proposition 3.4, interpolation and embedding properties of modulation spaces, one may derive many variations of Theorem 4.4. We only mention two small modifications that might be of interest.

(a) If $a \in M^{1,p}_{1/\tau_s}$ and $\varphi_1 \in M^1_{v_s}$, $\varphi_2 \in M^{p'}_{v_s}$, then $A_a^{\varphi_1,\varphi_2}$ is of trace class, because $M^{1,p}_{1/\tau_s} * M^{1,p'}_{\tau_s} \subseteq M^1$. Comparing to Theorem 4.4(i), we see that this result allows us to use a window φ_2 with less time-frequency concentration, however, at the price of a slightly smaller symbol class.

(b) If $(a, \varphi_1, \varphi_2) \in M^{p,\infty}_{1/\tau_s} \times M^q_{v_s} \times M^r_{v_s}$, where $1/q + 1/r - 1 = 1/p$ and $1 \le p \le 2$, then $A_a^{\varphi_1,\varphi_2} \in S_p$. To see this, we observe that Theorem 4.4(i) also holds with the role of the windows reversed, i.e., for $(\varphi_1, \varphi_2) \in M^p_{v_s} \times M^1_{v_s}$. The result then follows from the interpolation property $[M^1_{v_s} \times M^p_{v_s}, M^p_{v_s} \times M^1_{v_s}]_\theta = M^q_{v_s} \times M^r_{v_s}$ with $1/q + 1/r - 1 = 1/p$.

4.2. **Ultra-Distributions with Compact Support as Symbols.** As an application we present the result shown in [15, Section 4], in terms of ultra-distributions with compact support, denoted by $\mathcal{E}'_t$, $t>1$. Recall the embeddings:

$$\mathcal{E}'_t \subset (\mathcal{S}^t_t)' \subset (\Sigma^t_t)', \quad t>1.$$

We skip the precise definition of $\mathcal{E}'_t$, which can be found in many places, see e.g. [36, Definition 1.5.5] and subsequent anisotropic generalization. The following structure theorem, obtained by a slight generalization of [36, Theorem 1.5.6] to the anisotropic case, will be sufficient for our purposes.

Theorem 4.6. *Let $t\in\mathbb{R}^d$, $t>1$, i.e. $t=(t_1,\dots,t_d)$, with $t_1>1,\dots,t_d>1$. Every $u\in\mathcal{E}'_t$ can be represented as*

$$u=\sum_{\alpha\in\mathbb{N}_0^d}\partial^\alpha\mu_\alpha, \tag{36}$$

where μ_α is a measure satisfying

$$\int_K |d\mu_\alpha|\leq C_\epsilon \epsilon^{|\alpha|}(\alpha!)^{-t}, \tag{37}$$

for every $\epsilon>0$ and a suitable compact set $K\subset\mathbb{R}^d$, independent of α.

Using the preceding characterization, the STFT of an ultra-distribution with compact support is estimated as follows. [15, Proposition 4.2].

Proposition 4.7. *Let $t\in\mathbb{R}^d$, $t>1$, and $a\in\mathcal{E}'_t(\mathbb{R}^d)$. Then its STFT with respect to any window $g\in\Sigma^1_1$ satisfies the estimate*

$$|V_g a(x,\omega)|\lesssim e^{-h|x|}e^{t|2\pi\omega|^{1/t}},$$

for every $h>0$, cf. (19) *and below for the vectorial notation.*

The STFT estimate given in Proposition 4.7, is the key of the following trace class result for localization operators:

Corollary 4.8. *Let $t\in\mathbb{R}^d$, $t>1$. If $a\in\mathcal{E}'_t(\mathbb{R}^{2d})$ and $\varphi_1,\varphi_2\in\mathcal{S}^1_1(\mathbb{R}^d)$, then $A_a^{\varphi_1,\varphi_2}$ is a trace class operator.*

Proof. See [15, Corollary 4.3]; for sake of clarity we sketch the proof. If $\varphi_1,\varphi_2\in\mathcal{S}^1_1(\mathbb{R}^d)$, the characterization in (30) with $p=q=1$ implies that $\varphi_1,\varphi_2\in M^1_{w_\epsilon}(\mathbb{R}^d)$ for some (all) $\epsilon>0$. Since, for $|\omega|>C_\epsilon$ (where C_ϵ is a suitable positive constant depending on ϵ) we can write

$$t\cdot|2\pi\omega|^{1/t}=\sum_{i=1}^d t_i|2\pi\omega_i|^{1/t_i}\leq\epsilon|\omega|\,,$$

then the estimate of Proposition 4.7 gives $a\in M^{1,\infty}_{1/\mu_\epsilon}(\mathbb{R}^{2d})$. Finally, since $\varphi_1,\varphi_2\in M^1_{w_\epsilon}(\mathbb{R}^d)$ and $a\in M^{1,\infty}_{1/\mu_\epsilon}(\mathbb{R}^{2d})$, Theorem 4.4 (i), written for the case $p=1$ with τ_s replaced by μ_s, and v_s by w_s, implies that the operator $A_a^{\varphi_1,\varphi_2}$ is trace class. ∎

REMARK: Similar results show that tempered distributions with compact support give trace class operators, see [10, Corollary 3.7].

4.3. **Compactness of Localization Operators.** Localization operators with symbols and windows in the Schwartz class are compact [7]. If we define by M^0 the closed subspace of M^∞, consisting of all $f \in \mathcal{S}'$ such that its STFT $V_g f$ (with respect to a non-zero Schwartz window g) vanishes at infinity, it is easy to show that localization operators with symbols in M^0 and Schwartz windows are compact. Namely, let $a \in M^0(\mathbb{R}^{2d})$ and $g \in \mathcal{S}(\mathbb{R}^{2d})$, for simplicity normalized to be $\|g\|_{L^2} = 1$; consider then $V_g a$. The Schwartz class is dense in M^0, hence there exists a sequence F_n of Schwartz functions on $\mathbb{R}^{4d}$ that converge to $V_g a$ in the L^∞-norm. Define the sequence $a_n := V_g^* F_n$, $n \in \mathbb{N}$, where V_g^* is the adjoint operator defined in (10). Then $a_n \in \mathcal{S}(\mathbb{R}^{2d})$ and $a_n \to a$ in the M^∞-norm, since by (11)

$$\|a - a_n\|_{M^\infty} = \|V_g a - V_g V_g^* F_n\|_{L^\infty} = \|V_g a - F_n\|_{L^\infty} \to 0,$$

for $n \to \infty$. From Theorem 4.2 we have

$$\|A_{a_n}^{\varphi_1,\varphi_2} - A_a^{\varphi_1,\varphi_2}\|_{B(L^2)} = \|A_{(a_n-a)}^{\varphi_1,\varphi_2}\|_{B(L^2)} \leq C\|a - a_n\|_{M^\infty}\|\varphi_1\|_{M^1}\|\varphi_2\|_{M^1} \to 0.$$

Since compact operators are a closed subspace of the space of all bounded operators $B(L^2)$, then the localization operator $A_a^{\varphi_1,\varphi_2}$ is compact.

The symbol class $M^0(\mathbb{R}^{2d})$ is not optimal as the next simple example shows. Consider $a = \delta \notin M^0(\mathbb{R}^{2d})$. Since $V_g\delta(z,\zeta) = \bar{g}(z)$, it does not tend to zero when $z \in \mathbb{R}^{2d}$ is fixed and $|\zeta|$ goes to infinity. Hence $\delta \notin M^0(\mathbb{R}^{2d})$. However $A_\delta^{\varphi_1,\varphi_2}$ is a trace class operator for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, in fact, a rank-one operator, and therefore it is compact.

The example just mentioned has been the inspiration for the following compactness result [26, Proposition 3.6.]:

Proposition 4.9. *Let $g \in \mathcal{S}(\mathbb{R}^{2d})$ be given and $a \in M^\infty(\mathbb{R}^{2d})$. If $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ and*

$$\lim_{|z|\to\infty} \sup_{|\zeta|\leq R} |V_g a(z,\zeta)| = 0, \qquad \forall\, R > 0, \tag{38}$$

then $A_a^{\varphi_1,\varphi_2}$ is a compact operator.

5. Necessary Conditions

In this section we show that the sufficient conditions obtained so far are essentially optimal. This investigation requires different techniques and we limit ourselves to state the main results. A first attempt is done in Theorems 4.3, 4.4 of [10], where a converse for bounded and Hilbert-Schmidt operators is obtained for modulation spaces with polynomial weights:

Theorem 5.1. *i) Let $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and fix $s \geq 0$. If there exists a constant $C = C(a) > 0$ depending only on a such that*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_\infty} \leq C\,\|\varphi_1\|_{M^1_{v_s}}\|\varphi_2\|_{M^1_{v_s}}$$

for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then $a \in M^\infty_{1/\tau_s}$.

ii) Let $a \in \mathcal{S}'(\mathbb{R}^{2d})$. If there exists a constant $C = C(a) > 0$ depending only on a such that

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_2} \leq C\,\|\varphi_1\|_{M^1}\|\varphi_2\|_{M^1}$$

for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then $a \in M^{2,\infty}$.

Next, an extension for the boundedness necessary condition is given in [15, Theorem 3.3]:

Theorem 5.2. *Let $a \in (\Sigma_1^1)'(\mathbb{R}^{2d})$ and fix $s \geq 0$. If there exists a constant $C = C(a) > 0$ depending only on a such that*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_\infty} \leq C\,\|\varphi_1\|_{M^1_{w_s}}\|\varphi_2\|_{M^1_{w_s}}$$

for all $\varphi_1, \varphi_2 \in \Sigma_1^1(\mathbb{R}^d)$, then $a \in M^\infty_{1/\mu_s}$.

Necessary conditions for localization operators belonging to the Schatten class S_p have been obtained for unweighted modulation spaces in [11]:

Theorem 5.3. *Let $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and $1 \leq p \leq \infty$. Assume that $A_a^{\varphi_1,\varphi_2} \in S_p$ for* all *windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ and that there exists a constant $B > 0$ depending only on the symbol a such that*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \leq B\,\|\varphi_1\|_{M^1}\,\|\varphi_2\|_{M^1}, \quad \forall \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)\,,$$

then $a \in M^{p,\infty}$.

The techniques employed for the converse results are thoroughly different from the techniques for the sufficient conditions. Gabor frames and equivalent norms for modulation spaces are some of the crucial ingredients in the proofs. For the sake of completeness, we shall sketch the main features. First, by using the Gabor frame of the form

$$\{M_{\beta n}T_{\alpha k}\Phi\}_{k,n\in\mathbb{Z}^{2d}}, \quad 0 < \alpha, \beta < 1,$$

with the Gaussian window $\Phi(x,\omega) = 2^{-d}\,e^{-\pi(x^2+\omega^2)}$, the $M^{p,\infty}(\mathbb{R}^{2d})$-norm of a can be expressed by the equivalent norms

$$\|a\|_{M^{p,\infty}(\mathbb{R}^{2d})} \asymp \|\langle a, M_{\beta n}T_{\alpha k}\Phi\rangle_{n,k\in\mathbb{Z}^{2d}}\|_{\ell^{p,\infty}(\mathbb{Z}^{4d})}. \tag{39}$$

Then one relates the action of the localization operator on certain time-frequency shift of the Gaussian φ to the Gabor coefficients, and for a diligent choice of (x,ξ) and (u,η) one obtains that $\langle A_a^{\varphi_1,\varphi_2} M_\xi T_x\varphi, M_\eta T_u\varphi\rangle = \langle a, M_{\beta n}T_{\alpha k}\Phi\rangle$. The result is then obtained by using (39).

In view of the sufficient Schatten class results known so far, it is left as an exercise to show that the necessary conditions for the Schatten class can also be formulated for weighted modulation spaces.

We end up with the compactness necessary result of [26, Theorem 3.15].

Theorem 5.4. *Let $a \in M^\infty(\mathbb{R}^{2d})$ and $g \in \mathcal{S}(\mathbb{R}^{2d})$ be given. Then, the following conditions are equivalent:*

(a) The localization operator $A_a^{\varphi_1,\varphi_2} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *is compact for every pair* $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$.

(b) The symbol a satisfies condition (38).

6. Composition Formula

Given two localization operators, we want to compute their product and develop a symbolic calculus. It would be useful to express it in terms of localization operators. We shall present two different product formulae. The first one is an exact formula that expresses the composition of two localization operators again as a localization operator. However, the formula holds only for Gaussian windows and very special symbols. The second formula is much more general, but in this case the product of two localization operators is a sum of localization operators plus a remainder term, which can be expressed in either the Weyl or integral operator form.

6.1. **Exact Product.** We reformulate the result of [20] in the notation of [27, 28].

We consider the window functions $\varphi_1(t) = \varphi_2(t) = \varphi(t) = 2^{d/4}e^{-\pi t^2}$, $t \in \mathbb{R}^d$. In this case, the Wigner distribution of the Gaussian φ is a Gaussian as well: $W(\varphi,\varphi)(z) = 2^{d/2}e^{-2\pi z^2}$, $z \in \mathbb{R}^{2d}$. According to (14) the Weyl symbol σ of the operator $A_a^{\varphi,\varphi}$ is $\sigma(\zeta) = 2^{d/2}(a * e^{-2\pi z^2})(\zeta)$, $z, \zeta \in \mathbb{R}^{2d}$. We first recall the well-known composition of Weyl transforms from [27, Chp. 3.2] and then make the transition to localization operators. Let $[\cdot,\cdot]$ be the standard *symplectic form* on $\mathbb{R}^{2d}$ defined by

$$[(z_1, z_2), (\zeta_1, \zeta_2)] = z_1\zeta_2 - z_2\zeta_1, \quad \text{with} \quad z = (z_1, z_2),\ \zeta = (\zeta_1, \zeta_2)\,,$$

and let the twisted convolution $\natural$ on $\mathbb{R}^{2d}$ be given by

$$F \natural G(\zeta) = \iint_{\mathbb{R}^{2d}} F(z)\, G(\zeta - z) e^{\pi i [z,\zeta]}\, dz\,. \tag{40}$$

Then the composition of two Weyl transforms with symbols σ and τ can be written formally as

$$L_\sigma L_\tau = L_{\mathcal{F}^{-1}(\hat{\sigma} \natural \hat{\tau})}\,. \tag{41}$$

For any $f, g \in \mathcal{S}(\mathbb{R}^{2d})$, we define the $\natural^\flat$ product by

$$f \natural^\flat g(\zeta) = \iint_{\mathbb{R}^{2d}} f(z) g(z - \zeta) e^{\pi(z\zeta + i[z,\zeta])}\, e^{-\pi z^2}\, dz\,. \tag{42}$$

Then the product of localization operators is given by the following formula.

Theorem 6.1. *Let* $a, b \in \mathcal{S}(\mathbb{R}^{2d})$. *If there exists a symbol* $c \in \mathcal{S}'(\mathbb{R}^{2d})$ *such that*

$$\hat{c} = 2^{-d/2} \hat{a} \natural^\flat \hat{b}, \tag{43}$$

then we have

$$A_a^{\varphi,\varphi} A_b^{\varphi,\varphi} = A_c^{\varphi,\varphi}.$$

The proof is a straightforward consequence of relations (40) and (42). Indeed, one rewrites $A_a^{\varphi,\varphi}A_b^{\varphi,\varphi}$ in the Weyl form and uses relation (40) for the Weyl product. The result is the Weyl operator L_μ, where the Fourier transform of μ is given by

$$\begin{aligned}\hat{\mu}(\zeta) &= [\mathcal{F}(a*W(\varphi,\varphi))\natural\mathcal{F}(b*W(\varphi,\varphi))](\zeta)\\ &= 2^{-d}\iint_{\mathbb{R}^{2d}}\hat{a}(z)\hat{b}(z-\zeta)e^{-(\pi/2)z^2}e^{-(\pi/2)(\zeta-z)^2}e^{\pi i[z,\zeta]}\,dz\\ &= 2^{-d}e^{-(\pi/2)\zeta^2}\iint_{\mathbb{R}^{2d}}\hat{a}(z)\hat{b}(z-\zeta)e^{\pi(z\zeta+i[z,\zeta])}\,e^{-\pi z^2}\,dz.\end{aligned}$$

Hence, we have

$$\hat{\mu}(\zeta)=\hat{c}(\zeta)(2^{-d/2}e^{-(\pi/2)\zeta^2})=\mathcal{F}(c*W(\varphi,\varphi))(\zeta),$$

where $\hat{c}$ is given by relation (43).

6.2. **Asymptotic Product.** A second approach to the composition of two localization operators derives asymptotic expansions [1, 14, 33, 40, 12]. These realize the product as a sum of localization operators plus a controllable remainder. Most of these expansions were motivated by PDEs and energy estimates, and therefore use smooth symbols that are defined by differentiability properties, such as the Hörmander or Shubin classes. For applications in quantum mechanics and signal analysis, alternative notions of smoothness — "smoothness in phase-space" or quantitative measures of "time-frequency concentration" — have turned out to be useful. This point of view is pursued in [12], and we shall present the corresponding results.

The starting point is the following composition formula for two localization operators derived in [14]:

$$A_a^{\varphi_1,\varphi_2}A_b^{\varphi_3,\varphi_4}=\sum_{|\alpha|=0}^{N-1}\frac{(-1)^{|\alpha|}}{\alpha!}A_{a\partial^\alpha b}^{\Phi_\alpha,\varphi_2}+E_N. \tag{44}$$

The essence of this formula is that the product of two localization operators can be written as a sum of localization operators with suitably defined, new windows Φ_α and a remainder term E_N, which is "small".

In the spirit of the classical symbolic calculus, this formula was derived in [14, Thm. 1.1] for *smooth* symbols belonging to some Shubin class $S^m(\mathbb{R}^{2d})$ and for windows in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$.

In [12] the validity of (44) is established on the modulation spaces. The innovative features of this extension are highlighted below. Since the results are very technical, we do not give the detailed statements and proofs, but refer the reader to [12].

(i) *Rough symbols.* While in (44) the symbol b must be N-times differentiable, the symbol a only needs to be locally bounded. The classical results in symbolic calculus require both symbols to be smooth.

(ii) *Growth conditions on symbols.* The symbolic calculus in (44) can handle symbols with ultra-rapid growth (as long as it is compensated by a decay of b or vice versa). For instance, a may grow subexponentially as $a(z)\sim e^{\alpha|z|^\beta}$ for $\alpha>0$

and $0 < \beta < 1$. This goes far beyond the usual polynomial growth and decay conditions.

(iii) *General window classes.* A precise description of the admissible windows φ_j in (44) is provided. Usually only the Gaussian $e^{-\pi x^2}$ or Schwartz functions are considered as windows.

(iv) *Size of the remainder term.* Norm estimates for the size of the remainder term E_N are derived. They depend explicitly on the symbols a, b and the windows φ_j.

(v) *The Fredholm Property of Localization Operators.* By choosing $N = 1$, $\varphi_1 = \varphi_2 = \varphi$ with $\|\varphi\|_2 = 1$, $a(z) \neq 0$ for all $z \in \mathbb{R}^{2d}$, and $b = 1/a$, the composition formula (44) yields the following important special case:

$$A_a^{\varphi,\varphi} A_{1/a}^{\varphi,\varphi} = A_1^{\varphi,\varphi} + R = \mathrm{I} + R\,. \tag{45}$$

Under the following conditions on a:

(i) $|a| \asymp 1/m$ (in particular, $a \in L^\infty_m(\mathbb{R}^{2d})$,) where $m \in \mathcal{M}_v$,
(ii) $(\partial_j a)m \in L^\infty$ and vanishes at infinity for $j = 1, \dots, 2d$;

the remainder R is shown to be compact, and as a consequence, $A_a^{\varphi,\varphi}$ is a Fredholm operator between the two modulation spaces $M^{p,q}$ and $M^{p,q}_m$ (with different weights). This result works even for ultra-rapidly growing symbols such as $a(z) = e^{\alpha|z|^\beta}$ for $\alpha > 0$ and $0 < \beta < 1$. For comparison, the reduction of localization operators to standard pseudodifferential calculus requires elliptic or hypo-elliptic symbols, and the proof of the Fredholm property works only under severe restrictions, see [8].

7. Multilinear Localization Operators

Multilinear localization operators are introduced in [13]; they not only generalize the linear case but also yield a subclass of multilinear pseudodifferential operators. To understand their meaning, one can think of localizing m-fold products of functions. For the sake of clarity, we shall first introduce the bilinear case and show how the construction arises naturally from the framework of reproducing formulae and linear localization operators. The general case can be treated similarly.

Bilinear localization operators. Let $f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$, then the tensor product $(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1) f_2(x_2)$ is a function in $\mathcal{S}(\mathbb{R}^{2d})$. Given four window functions $\varphi_i \in \mathcal{S}(\mathbb{R}^d)$, $i = 1, \dots, 4$, with $\langle \varphi_1, \varphi_3 \rangle = \langle \varphi_2, \varphi_4 \rangle = 1$, the usual reproducing formula for the functions f_1, f_2 stated in (3) reads as follows:

$$f_1 = \int_{\mathbb{R}^{2d}} V_{\varphi_1} f_1(z_1, \zeta_1)\, M_{\zeta_1} T_{z_1} \varphi_3 \, d\zeta_1 \, dz_1. \tag{46}$$

$$f_2 = \int_{\mathbb{R}^{2d}} V_{\varphi_2} f_2(z_2, \zeta_2)\, M_{\zeta_2} T_{z_2} \varphi_4 \, d\zeta_2 \, dz_2. \tag{47}$$

The product of both sides of equalities (46) and (47) yields

$$
\begin{aligned}
(f_1\otimes f_2)(x_1,x_2) &= \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} V_{\varphi_1}f_1(z_1,\zeta_1)V_{\varphi_2}f_2(z_2,\zeta_2)M_{\zeta_1}T_{z_1}\varphi_3(x_1)M_{\zeta_2}T_{z_2}\varphi_4(x_2)d\zeta dz\\
&= \int_{\mathbb{R}^{4d}} V_{\varphi_1\otimes\varphi_2}(f_1\otimes f_2)(z,\zeta)\, M_\zeta T_z(\varphi_3\otimes\varphi_4)(x_1,x_2)\, d\zeta\, dz
\end{aligned}
$$

with $z=(z_1,z_2)$, $\zeta=(\zeta_1,\zeta_2)\in\mathbb{R}^{2d}$.

The previous reproducing formula for the function $f_1\otimes f_2$ can be localized in the time-frequency plane yielding a localization operator $A_a^{\varphi_1\otimes\varphi_2,\varphi_3\otimes\varphi_4}$ with symbol a (defined on $\mathbb{R}^{4d}$) and windows $\varphi_1\otimes\varphi_2$, $\varphi_3\otimes\varphi_4$. Formally, the action of the operator on the function $f_1\otimes f_2$ is given by

$$
\begin{aligned}
&A_a^{\varphi_1\otimes\varphi_2,\varphi_3\otimes\varphi_4}(f_1\otimes f_2)(x_1,x_2)\\
&\qquad= \int_{\mathbb{R}^{4d}} a(z,\zeta)V_{\varphi_1}f_1(z_1,\zeta_1)V_{\varphi_2}f_2(z_2,\zeta_2)M_{\zeta_1}T_{z_1}\varphi_3(x_1)M_{\zeta_2}T_{z_2}\varphi_4(x_2)dzd\zeta.
\end{aligned}
$$

For any symbol $a\in\mathcal{S}'(\mathbb{R}^{4d})$, and window functions φ_j on $\mathcal{S}(\mathbb{R}^d)$, the operator $A_a^{\varphi_1\otimes\varphi_2,\varphi_3\otimes\varphi_4}$ can be seen as a bilinear mapping from the 2-fold product of Schwartz spaces $\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)$ into the space $\mathcal{S}'(\mathbb{R}^{2d})$ of tempered distributions. Moreover, if we restrict now our attention to a smoother symbol $a\in\mathcal{S}(\mathbb{R}^{4d})$, we obtain a multilinear mapping from $\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^{2d})$.

In [13] the boundedness properties of the trace of $A_a^{\varphi_1\otimes\varphi_2,\varphi_3\otimes\varphi_4}$ on the diagonal $x_1=x_2$ are studied. This restriction leads to a new kind of localization operator.

Definition 7.1. *Let $f_1,f_2\in\mathcal{S}(\mathbb{R}^d)$. Given a symbol $a\in\mathcal{S}'(\mathbb{R}^{4d})$ and window functions $\varphi_i\in\mathcal{S}(\mathbb{R}^d)$, with $i=1,\dots,4$, the* bilinear localization operator A_a *is given by*

$$
A_a(f_1,f_2)(x)=\int_{\mathbb{R}^{4d}} a(z,\zeta)\,V_{\varphi_1}f_1(z_1,\zeta_1)\,V_{\varphi_2}f_2(z_2,\zeta_2)\,M_{\zeta_1}T_{z_1}\varphi_3(x)\,M_{\zeta_2}T_{z_2}\varphi_4(x)\,dz\,d\zeta, \tag{48}
$$

where $x\in\mathbb{R}^d$.

Notice that if the symbol $a\in\mathcal{S}'(\mathbb{R}^{4d})$ then the corresponding operator A_a maps $\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$.

In order to give a weak definition of the bilinear localization operator A_a, we first introduce the following time-frequency representation. For $\varphi_3,\varphi_4\in\mathcal{S}(\mathbb{R}^d)\setminus\{0\}$, $z=(z_1,z_2)$, $\zeta=(\zeta_1,\zeta_2)\in\mathbb{R}^{2d}$, we define $\mathcal{V}_{\varphi_3,\varphi_4}$ by

$$
\mathcal{V}_{\varphi_3,\varphi_4}g(z,\zeta)=\int_{\mathbb{R}^d} g(t)\,\overline{M_{\zeta_1}T_{z_1}\varphi_3(t)M_{\zeta_2}T_{z_2}\varphi_4(t)}\,dt,\quad g\in\mathcal{S}(\mathbb{R}^d). \tag{49}
$$

Thus, for $f_1,f_2,g\in\mathcal{S}(\mathbb{R}^d)$ the weak definition of (48) is given by

$$
\langle A_a(f_1,f_2),g\rangle=\langle a,\overline{V_{\varphi_1\otimes\varphi_2}(f_1\otimes f_2)}\mathcal{V}_{\varphi_3,\varphi_4}g\rangle. \tag{50}
$$

Multilinear localization operators. Without any further work — just some extra notation — it is straightforward to generalize the above definition of multilinear localization operators and relate it to a multilinear pseudodifferential operator.

Thus we are led to make the following definition. Fix $m \in \mathbb{N}$. For every symbol $a \in \mathcal{S}'(\mathbb{R}^{2md})$ and windows φ_i, $i = 1, \dots, 2m$, in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, we introduce the analysis, synthesis window functions $\phi_1, \phi_2 : \mathbb{R}^{md} \to \mathbb{C}$, defined respectively as tensor products of the m analysis, and m synthesis windows, i.e.,

$$\phi_1(t_1, \dots, t_m) := \varphi_1(t_1) \cdots \varphi_m(t_m), \tag{51}$$

and

$$\phi_2(t_1, \dots, t_m) := \varphi_{m+1}(t_1) \cdots \varphi_{2m}(t_m). \tag{52}$$

Let $\mathcal{R}$ be the trace mapping that assigns to each function defined on $\mathbb{R}^{md}$ a function defined on $\mathbb{R}^d$ by the following formula:

$$\mathcal{R}F(t) := F_{|_{\{t_1=t_2=\cdots=t_m=t\}}}(t_1, \dots, t_m) = F(t, \dots, t), \tag{53}$$

for any $t \in \mathbb{R}^d$.

Definition 7.2. *The multilinear localization operator A_a with symbol $a \in \mathcal{S}'(\mathbb{R}^{2md})$ and windows $\varphi_j \in \mathcal{S}(\mathbb{R}^d)$, $j = 1, \dots, 2m$ is the multilinear mapping defined on the m-fold product of $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ by*

$$\begin{aligned} A_a(\overrightarrow{f})(x) : &= \int_{\mathbb{R}^{2md}} a(z,\zeta)\, V_{\bigotimes_{j=1}^m \varphi_j} \left(\otimes_{j=1}^m f_j\right)(z,\zeta) \prod_{j=1}^m M_{\zeta_j} T_{z_j} \varphi_{m+j}(x)\, d\zeta\, dz \\ &= \int_{\mathbb{R}^{2md}} a(z,\zeta)\, V_{\phi_1} \left(\otimes_{j=1}^m f_j\right)(z,\zeta)\; \mathcal{R} M_\zeta T_z \phi_2(x)\, d\zeta\, dz, \end{aligned} \tag{54}$$

where $(z,\zeta) \in \mathbb{R}^{md} \times \mathbb{R}^{md}$, $x \in \mathbb{R}^d$, and $\overrightarrow{f} = (f_1, \dots, f_m) \in \mathcal{S}(\mathbb{R}^d) \times \cdots \times \mathcal{S}(\mathbb{R}^d)$.

If $m = 1$ we are back to the linear localization operator $A_a^{\varphi_1,\varphi_2}$, whereas the case $m = 2$ gives the bilinear localization operator introduced in (48).

One of the results of [13] is related to the boundedness properties of multilinear localization operators on products of modulations spaces. To this end, these operators are represented as bilinear (or, in general, as multilinear) pseudodifferential operators and known results on boundedness of multilinear pseudodifferential operators on products of modulation spaces ([2, 3]) lead to boundedness results of these multilinear localization operators. In analogy to the linear case, it is worth detailing their connection with multilinear pseudodifferential operators.

Proposition 7.3. *Let $a \in \mathcal{S}'(\mathbb{R}^{2md})$ and $\varphi_j \in \mathcal{S}(\mathbb{R}^d)$, $j = 1, \dots, 2m$. Then the multilinear localization operator A_a is the multilinear pseudodifferential operator T_τ defined on $\overrightarrow{f} = (f_j)_{j=1}^m \in \mathcal{S}(\mathbb{R}^d) \times \ldots \times \mathcal{S}(\mathbb{R}^d)$ by*

$$A_a(\overrightarrow{f})(x) = T_\tau(\overrightarrow{f})(x) := \int_{\mathbb{R}^{md}} \tau(x,\xi) \prod_{j=1}^m \hat{f}_j(\xi_j)\, e^{2\pi i x \sum_{j=1}^m \xi_j}\, d\xi\,. \tag{55}$$

The symbol τ is given as

$$\tau(x,\xi) = a * \Phi(X,\xi) \tag{56}$$

with $x \in \mathbb{R}^d$, $X = (x, \dots, x)$, $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^{md}$, *and*

$$\Phi(z, \xi) = \prod_{j=1}^{m} \mathcal{U}\mathcal{F}(V_{\varphi_j} \varphi_{j+m})(z_j, \xi_j), \tag{57}$$

for $z = (z_1, \dots, z_m) \in \mathbb{R}^{md}$.

According to what happens for linear localization operators, we shall provide both sufficient and necessary conditions for boundedness on products of modulation spaces.

Theorem 7.4. *(a)* **Sufficient conditions.** *Let* $m \in \mathbb{N}$, *a symbol* $a \in M^\infty(\mathbb{R}^{2md})$, *and window functions* $\varphi_j \in M^1(\mathbb{R}^d)$, $j = 1, \dots, 2m$, *be given. Then the* m*-linear localization operator* A_a *defined by* (54) *extends to a bounded operator from* $M^{p_1,q_1}(\mathbb{R}^d) \times \cdots \times M^{p_m,q_m}(\mathbb{R}^d)$ *into* $M^{p_0,q_0}(\mathbb{R}^d)$, *when*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p_0}, \ \frac{1}{q_1} + \cdots + \frac{1}{q_m} = m - 1 + \frac{1}{q_0},$$

and $1 \leq p_j, q_j \leq \infty$, *for* $j = 0, \dots, m$. *Moreover, we have the following norm estimate*

$$\|A_a\| \leq C \|a\|_{M^\infty(\mathbb{R}^{2md})} \prod_{i=1}^{2m} \|\varphi_i\|_{M^1(\mathbb{R}^d)}, \tag{58}$$

where the positive constant C *is independent of* a *and of* φ_j, $j = 1, \dots, 2m$.
(b) **Necessary conditions.** *Let* $m \in \mathbb{N}$, *and* $a \in \mathcal{S}'(\mathbb{R}^{2md})$ *be given. Assume that (i) the* m*-linear localization operator* A_a *is bounded from* $M^{p_1,q_1}(\mathbb{R}^d) \times \cdots \times M^{p_m,q_m}(\mathbb{R}^d)$ *into* $M^{p_0,q_0}(\mathbb{R}^d)$, *where*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p_0}, \ \frac{1}{q_1} + \cdots + \frac{1}{q_m} = m - 1 + \frac{1}{q_0},$$

and $1 \leq p_j, q_j \leq \infty$, *for* $j = 0, \dots, m$, *and moreover that*
(ii) A_a *satisfies the following norm estimate*

$$\|A_a\| \leq C(a) \prod_{i=1}^{2m} \|\varphi_i\|_{M^1(\mathbb{R}^d)}, \quad \forall \varphi_i \in \mathcal{S}(\mathbb{R}^d),\ i = 1, \dots, 2m, \tag{59}$$

with a positive constant $C(a)$ *depending only on* a. *Then the symbol* a *belongs necessarily to* $M^\infty(\mathbb{R}^{2md})$.

An application of this theory is that it provides symbols for multilinear bounded Kohn-Nirenberg operators. Two steps are needed to construct symbols in $M^{\infty,1}$: first, suitable windows φ_i, $i = 1, \dots, d$ are chosen for computing the function Φ defined in (57). Secondly, symbols a are provided explicitly, the convolution with Φ in (56) is computed, yielding the Kohn-Nirenberg symbols τ desired. We refer to [13, Section 7] for concrete examples.

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