ON SPECTRALITY OF THE ALGEBRA OF CONVOLUTION DOMINATED OPERATORS

G. FENDLER, K. GRÖCHENIG, AND M. LEINERT

This note is about work in progress. Complete details and more will be given in [3]. If G is a discrete group, the algebra $CD(G)$ of convolution dominated operators on $l^2(G)$ is canonically isomorphic to a twisted L^1-algebra $l^1(G, l^\infty(G), T)$. Using this, we show that $CD(G)$ is spectral in the algebra of all bounded operators, if G is amenable and rigidly symmetric. For G commutative, this result is known (see [1], [6]), for G noncommutative discrete it appears to be new.

Let G be a discrete group. For $x \in G$ we denote by $\lambda(x)$ the operator of left translation on $l^1(G)$ and on $l^2(G)$, i.e. $\lambda(x)f\,(y) = f(x^{-1}y)$ for $f \in l^1(G)$ or $f \in l^2(G)$, $\ x, y \in G$. By $B(l^2(G))$ we denote the algebra of bounded operators on $l^2(G)$.

For an operator $A : l^2(G) \to l^2(G)$ let $A(x, y) =< A\delta_y, \delta_x >,\ x, y \in G$ be its matrix, where $< \, , \, >$ is the scalar product of the Hilbert space $l^2(G)$.

Definition 1. *The operator A is called convolution dominated if there exists some $a \in l^1(G)$ such that*

$$|A(x, y)| \leq a(xy^{-1}), \quad \forall x, y \in G.$$

We define its norm as

$$\| A \|_1 := \inf\{\| a \|_{l^1} : a \in l^1(G),\ |A(x, y)| \leq a(xy^{-1})\ \forall x, y \in G\}.$$

We remark that $A \in B(l^2(G))$ is convolution dominated if the supremum norms of the side diagonals of its matrix are summable, i.e. if

$$\sum_{z \in G} \sup_{\{x,y:\, xy^{-1}=z\}} |A(x, y)| \ < \ \infty.$$

Moreover this quantity just equals its norm $\| A \|_1$.

Since $l^1(G)$ is a convolution algebra it follows that the space of subconvolutive operators is an algebra under composition of operators. Moreover it is not hard to see that it becomes a Banach $*$-algebra (containing an identity) with respect to the usual involution of operators in $B(l^2(G))$. We denote this Banach $*$-algebra by $CD(G)$.

$l^\infty(G)$ is a C^*-algebra (really a von Neumann algebra) with respect to pointwise multiplication and complex conjugation as involution. It is isometrically represented as multiplication operators on $l^2(G)$:

$$D^m f(x) = m(x)f(x), \quad \text{where } x \in G, f \in l^2(G), m \in l^\infty(G).$$

We have the covariance relation $\lambda(x^{-1})D^m\lambda(x) = D^{T_x^{-1}m}$, where $T_x : l^\infty(G) \to l^\infty(G)$ denotes the C^* automorphism of the algebra $l^\infty(G)$ given by left translation $T_x n(z) = n(x^{-1}z)$, $n \in l^\infty(G)$, so from $\lambda : G \to B(l^2(G))$ and $D : l^\infty(G) \to B(l^2(G))$ we obtain a representation R of the twisted L^1-algebra, in the sense of Leptin[7, 8, 9]: $\mathcal{L} = l^1((G), l^\infty(G), T)$, on $l^2(G)$. An element $f \in \mathcal{L}$ may be uniquely written as

$$f = \sum_{z \in G} m_z \delta_z,$$

where $m_z = f(z) \in l^\infty(G)$. The representation $R : l^1(G, l^\infty(G), T) \to B(l^2(G))$ is given by the prescription

$$Rf = \sum_z D_z^{m_z}.$$

Now it is not hard to see:

Proposition 2. *The map $R : l^1(G, l^\infty(G), T) \to CD(G)$ is an isometric $*$-isomorphism.*

Recall that a Banach algebra A with involution is called symmetric if every positive element has its spectrum contained in the non-negative reals, i. e. $sp(a^*a) \subset [0, \infty)$ $\forall a \in A$. Accordingly, a locally compact group G is called symmetric if its convolution algebra $L^1(G)$ is symmetric. Various classes of groups are known to be symmetric, e. g. Abelian locally compact groups, compact groups, finite extensions of discrete nilpotent groups, compactly generated groups of polynomial growth.

Leptin and Poguntke[10] showed that the groups of the first three classes satisfy the stronger property of rigid symmetry. Namely for any C^*-algebra C the projective tensor product $L^1(G)\hat{\otimes}C$ is symmetric. Later Poguntke [11] showed that all nilpotent locally compact groups are rigidly symmetric.

Define a map

$$Q : l^1(G, l^\infty(G), T) \to l^1(G)\hat{\otimes}B(l^2(G))$$

by

$$f = \sum_v m_v \delta_v \mapsto \sum_v \delta_v \otimes D_v^{m_v}.$$

Proposition 3. *The above defined map Q is an isometric $*$-isomorphism of $l^1(G, l^\infty(G), T)$ onto a closed subalgebra of $l^1(G)\hat{\otimes}B(l^2(G))$.*

Since symmetry passes to closed subalgebras we have

Corollary 4. *Let G be a discrete rigidly symmetric group then $l^1(G, l^\infty(G), T)$ and $CD(G)$ are symmetric Banach $*$-algebras.*

Recall that by $D : m \mapsto D^m$ the C^*-algebra $l^\infty(G)$ is faithfully represented by multiplication operators on $l^2(G)$. On the Hilbert space $l^2(G, l^2(G))$, the D-regular representation λ^D of $\mathcal{L} = l^1(G, l^\infty, T)$ is defined (see [9, §3]) by

$$\lambda^D(f)\xi(x) = \sum_y D^{T_y f(xy)} \xi(y^{-1}), \text{where } \xi \in l^2(G, l^2(G)),\ f \in \mathcal{L}.$$

On the other hand we see that $R : \mathcal{L} \to CD(G) \subset B(l^2(G))$ is a $*$-representation of $\mathcal{L}$ on $l^2(G)$. Call this representation the canonical representation of $\mathcal{L}$. We identify $l^2(G, l^2(G))$ with $l^2(G \times G)$ and define a multiple of the canonical representation by letting the operators $R(f) = \sum_y \lambda(y) \circ D^{f(y)}$, $f \in \mathcal{L}$, act in the first coordinate of the $l^2(G \times G)$-functions only. The unitary operator $S\xi(x, z) = \xi(xz, z)$, where $\xi \in l^2(G \times G)$ actually intertwines these two representations, so we have

Proposition 5. *The D-regular representation of $\mathcal{L}$ is equivalent to a multiple of the canonical representation.*

Corollary 6. *Let G be an amenable discrete group, then the greatest C^* norm on $\mathcal{L}$ equals the operator norm on $CD(G)$.*

Proof. It follows from [9, Satz 6] of Leptin that for the representation D of $l^\infty(G)$ the D-regular representation λ^D defines the greatest C^* norm on $\mathcal{L}$. Denoting $\| \ . \ \|_*$ the greatest C^* norm we have for $f \in \mathcal{L}$:

$$\| f \|_* = \| \lambda^D(f) \| = \| R(f) \|_{B(l^2(G))},$$

where the last equality follows from Proposition 5. □

For an element a of a normed algebra A we denote by $r_A(a)$ its spectral radius.

Proposition 7. *Let G be a discrete, amenable, and rigidly symmetric group. Then for $f \in \mathcal{L}$*

$$r_{\mathcal{L}}(f^* f) = r_{CD(G)}(R(f)^* Rf) = \| R(f) \|^2_{B(L^2(G))}.$$

Proof. By Corollary 4 we know that $\mathcal{L}$ and $CD(G)$ are symmetric. By a theorem of Pták [12] it follows that $\| f \|_*^2 = r_{\mathcal{L}}(f^* f) = r_{CD(G)}(R(f)^* R(f))$ (see e.g. [2, §41 Corollary 8]). Corollary 6 now proves the assertion. $\square$

Theorem 8. *Let G be a discrete, amenable, and rigidly symmetric group. If $f \in \mathcal{L}$ is such that $R(f) \in CD(G)$ has an inverse in $B(l^2(G))$ then f^{-1} exists in $\mathcal{L}$ and $R(f^{-1}) = R(f)^{-1}$ is in $CD(G)$.*

Proof. If $f \in \mathcal{L}$ is hermitian, i.e. $f = f^*$, then

$$r_{\mathcal{L}}(f)^2 = r_{\mathcal{L}}(f^* f) = \| R(f) \|_{B(L^2(G))}^2.$$

We apply Hulanickis Lemma [5, Prop. 2.5] and obtain that

$$sp_{\mathcal{L}}(f) = sp_{B(l^2(G))}(R(f)), \quad \forall f = f^* \in \mathcal{L}.$$

The Lemma [4, 3.7]now implies

$$sp_{\mathcal{L}}(f) = sp_{B(l^2(G))}(R(f)), \quad \forall f \in \mathcal{L}.$$

$\square$

References

[1] A.G.Baskakov. Abstract harmonic analysis and asymptotic estimates for elements of inverse matrices. *Mat. Zametki*, 55(2): 17–26,155,1992.

[2] F. Bonsall and J. Duncan. *Complete normed algebras.* Springer Verlag, 1973.

[3] G. Fendler, K. Groechenig, M. Leinert. Paper in preparation.

[4] G. Fendler, K. Groechenig, M. Leinert, J. Ludwig, and C. Molitor-Braun. Weighted group algebras on groups of polynomial growth. *Math. Z.*, 245:791–821, 2003.

[5] A. Hulanicki. On the spectrum of convolution operators on groups with polynomial growth. *Invent. Math.*, 17:135–142, 1972.

[6] V. Kurbatov. Functional Differential Operators and Equations. *Vol. 473 of Mathematics and its applications.* Kluwer Academic Publishers 1999.

[7] H. Leptin. Verallgemeinerte L^1-Algebren und projektive Darstellungen lokal kompakter Gruppen I. *Invent. Math.*, 3:257–281, 1967.

[8] H. Leptin. Verallgemeinerte L^1-Algebren und projektive Darstellungen lokal kompakter Gruppen II. *Inventiones math*, 4:68–86, 1967.

[9] H. Leptin. Darstellungen verallgemeinerter L^1-Algebren. *Inventiones math*, pages 192–215, 1968.

[10] H. Leptin and D. Poguntke. Symmetry and non-symmetry for locally compact groups. *J. Funct. anal.*, 33:119–134, 1979.

[11] D. Poguntke. Rigidly symmetric L^1-group algebras. *Seminar Sophus Lie*, 2:189–197, 1992.

[12] V. Pták. On the spectral radius in Banach algebras with involution. *Bull. London math. Soc.*, 2:327–334, 1970.

Finstertal 16, D-69514 Laudenbach, Germany
E-mail address: gero.fendler@t-online.de

Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 wien, Austria
E-mail address: karlheinz.groechenig@gsf.de

Institut für Angewandte Mathematik, Fakultät für Mathematik, Im Neuenheimer Feld 288, D-69120 Heidelberg, Germany
E-mail address: leinert@math.uni-heidelberg.de