Pseudodifferential Operators on Locally Compact Abelian Groups and Sjöstrand's Symbol Class

Karlheinz Gröchenig and Thomas Strohmer*

Abstract

We investigate pseudodifferential operators on arbitrary locally compact abelian groups. As symbol classes for the Kohn-Nirenberg calculus we introduce a version of Sjöstrand's class. Pseudodifferential operators with such symbols form a Banach algebra that is closed under inversion. Since "hard analysis" techniques are not available on locally compact abelian groups, a new time-frequency approach is used with the emphasis on modulation spaces, Gabor frames, and Banach algebras of matrices. Sjöstrand's original results are thus understood as a phenomenon of abstract harmonic analysis rather than "hard analysis" and are proved in their natural context and generality.

1 Introduction

Pseudodifferential operators are a generalization of partial differential operators, and the subject is usually treated with the arsenal of "hard analysis", such as differentiation and decomposition techniques, commutators etc. In this paper we develop a new theory for pseudodifferential operators on general locally compact groups instead of on $\mathbb{R}^d$. Our main goal is to show the validity of three subtle results of J. Sjöstrand [36, 37] on a Banach algebra of pseudodifferential operators in the new context of locally compact abelian groups. This is not a mere generalization, because the formulation and extension of Sjöstrand's results requires the development of completely new methods in which "hard analysis" is replaced by phase-space (time-frequency) analysis.

To put the issues into a bigger context, recall Wiener's Lemma: it states that a periodic function f which has an absolutely summable Fourier series and which

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vanishes nowhere has an inverse f^{-1} which also has an absolutely summable Fourier series [42].

This result can also be stated in the following way. Assume the sequence $\{a_k\}_{k\in\mathbb{Z}}$ satisfies $\sum_{k\in\mathbb{Z}} |a_k| < \infty$. Let A be a biinfinite Toeplitz matrix with entries $A_{k,l} = a_{k-l}$ for $k, l \in \mathbb{Z}$. If A is invertible on $\ell^2(\mathbb{Z})$ then its inverse $B := A^{-1}$ has entries $B_{k,l} = b_{k-l}$ which satisfy $\sum_{k\in\mathbb{Z}} |b_k| < \infty$. In this context $f(t) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi ikt}$ is called the *symbol* of A.

An intriguing generalization is due to Bochner and Philips [4] who have shown that Wiener's Lemma remains true if the a_k belong to a non-commutative Banach algebra instead of to $\mathbb{C}$.

In recent years several remarkable extensions of Wiener's Lemma have been published. Using results from [4], Gohberg, Kaashoek, and Woerdeman [15], and independently Baskakov [1] proved the following result. Consider the Banach algebra $\mathcal{C}$ of matrices A with norm

$$\|A\|_{\mathcal{C}} := \sum_{k\in\mathbb{Z}^d} \sup_{i-j=k} |A_{i,j}| < \infty. \tag{1}$$

If $A \in \mathcal{C}$ is invertible in $\ell^2(\mathbb{Z}^d)$ then its inverse A^{-1} also belongs to $\mathcal{C}$.

Another Wiener-type theorem, this time in the context of pseudodifferential operators, is due to Sjöstrand [37]. His striking result goes as follows. Let $g \in \mathcal{S}(\mathbb{R}^{2d})$ be a compactly supported C^∞-function satisfying the property $\sum_{k\in\mathbb{Z}^{2d}} g(t-k) = 1$ for all $t \in \mathbb{R}^{2d}$. Then a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ belongs to $M^{\infty,1}(\mathbb{R}^{2d})$ – the *Sjöstrand class* – if

$$\int_{\mathbb{R}^{2d}} \sup_{k\in\mathbb{Z}^{2d}} |(\sigma \cdot g(.-k))^\wedge(\zeta)|\, d\zeta < \infty. \tag{2}$$

Now let K_σ be a pseudodifferential operator with (Weyl or Kohn-Nirenberg) symbol $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$. (On $\mathbb{R}^d$, K_σ is usually written as $\sigma(x, D)$ or $\sigma(x, D)^w$). Sjöstrand proved that $M^{\infty,1}$ is an algebra with respect to the composition of pseudodifferential operators. Furthermore, if K_σ is invertible on $L^2(\mathbb{R}^d)$, then $K_\sigma^{-1} = K_\tau$ for some $\tau \in M^{\infty,1}(\mathbb{R}^{2d})$. This is the Wiener property of $M^{\infty,1}(\mathbb{R}^{2d})$. These results had a deep influence on recent work on new symbol classes for pseudodifferential operators, as exemplified in the work of Boulkhemair [5], Lerner [27], and Toft [40].

At first glance there is no relation between the two results on the matrix algebra $\mathcal{C}$ and the symbol class $M^{\infty,1}$. However, on inspection of Sjöstrand's proof, which is in the realm of "hard analysis", one sees that he uses the Wiener property of the Gohberg-Baskakov matrix algebra $\mathcal{C}$ as a tool. In fact, he found an independent proof of their result, again using a "hard analysis" approach with commutators and decomposition methods. On the other hand, Gohberg et al. and Baskakov use a "natural" approach in the context of harmonic analysis and prove their result with classical methods from Fourier analysis.

One of the main insights of this paper is the observation that both results are a manifestation of a more general result, namely a Wiener property for a certain class of pseudodifferential operators on locally compact abelian (LCA) groups.

The main results of this paper can be summarized as follows:

(i) For every LCA group $\mathcal{G}$ with dual group $\hat{\mathcal{G}}$ we introduce a symbol class $M^{\infty,1}(\mathcal{G} \times \widehat{\mathcal{G}})$. When $\mathcal{G} = \mathbb{R}^d$, this class reduces to the Sjöstrand class, for $\mathcal{G} = \mathbb{Z}^d$, this class coincides with the matrix algebra $\mathcal{C}$ defined in (1).

(ii) We show that $M^{\infty,1}(\mathcal{G}\times\widehat{\mathcal{G}})$ is a Banach algebra under a twisted product that corresponds to the composition of the corresponding pseudodifferential operators.

(iii) We show that the Wiener property for the general Sjöstrand class $M^{\infty,1}(\mathcal{G}\times \widehat{\mathcal{G}})$, i.e., if $\sigma \in M^{\infty,1}(\mathcal{G} \times \widehat{\mathcal{G}})$ and the pseudodifferential operator K_σ is invertible on $L^2(\mathcal{G})$, then the inverse operator $K_\sigma^{-1} = K_\tau$ possesses again a symbol $\tau \in M^{\infty,1}(\mathcal{G} \times \widehat{\mathcal{G}})$.

(iv) We consider weighted versions of $M^{\infty,1}(\mathcal{G} \times \widehat{\mathcal{G}})$ and characterize those weights for which the Wiener property holds.

The extension of Sjöstrand's original results to LCA groups is of interest for both theoretical and practical reasons.
(a) Sjöstrand's proof is based on commutator estimates and decomposition techniques ("hard analysis"). Such techniques are not available on general LCA groups, and it is by no means clear whether and how such a generalization is actually possible.
(b) Proofs presented in a setting of LCA groups show in some sense "what is really going on". The derivations are stripped off of lengthy analytic estimates and replaced by a time-frequency (phase space) approach based on the ideas from [19]. Admittedly our approach requires more conceptual effort.
(c) Sjöstrand's class and its weighted versions as well as the nonstationary Wiener algebra have turned out to be very useful in applications, in particular in the modeling of operators and transmission pulses in connection with mobile communications, cf. [39]. The multidimensional setting is potentially useful in applications such as spatially varying image or video (de)blurring. Furthermore, the numerical implementation of pseudodifferential operators requires a discrete finite setting, it is thus useful to know that the almost diagonalization properties are preserved under appropriate discretization.

(d) Pseudodifferential operators on the p-adic groups $\mathbb{Q}_p$ occur often in the construction of a p-adic quantum theory [33, 22, 41]. Our result hold in particular for operators on $\mathbb{Q}_p$ and provide a new type of a symbolic calculus.

The paper is organized as follows. In Section 2 we develop time-frequency methods on locally compact abelian groups. This section is somewhat lengthy, but we feel it necessary to explain the main concepts of time-frequency analysis, such as amalgam spaces, modulation spaces, Gabor frames, and matrix algebras. (By contrast, in a paper on standard pseudodifferential operators it would suffice to refer to the expositions of Hörmander [24] or Stein [38].) In Section 3 we explain the main formalism of pseudodifferential operators on locally compact groups. Section 4 contains the key result about the almost diagonalization of pseudodifferential operators in the Sjöstrand class, and in Section 5 we formulate and prove our main results, the Banach algebra property and the Wiener property of Sjöstrand's class on locally compact abelian groups. In the final Section 6 we discuss special groups and show that the matrix algebra of Gohberg and Baskakov and Sjöstrand's symbol class are examples of the same phenomenon.

2 Tools from Time-Frequency Analysis

We first present the main concepts for time-frequency analysis on locally compact abelian (LCA) groups. The constructions of time-frequency analysis are well-known for $\mathbb{R}^d$ and available in textbook form [17, 10]. It is less well known that time-frequency analysis works similarly on LCA groups. For some contributions in this directions, we refer to [9, 16].

In the following we focus on the details (weight functions on LCA groups, spaces of test functions) that require special attention. Whenever a result can be formulated and proved as on $\mathbb{R}^d$, we will only formulate the result and refer to the proof on $\mathbb{R}^d$.

Locally Compact Abelian Groups. Let $\mathcal{G}$ be a locally compact abelian group. We assume that $\mathcal{G}$ satisfies the second countability axiom and is metrizable which is equivalent to the assumption that $L^2(\mathcal{G})$ is a separable Hilbert space [23]. The elements of $\mathcal{G}$ will be denoted by italics $x, y, u, \dots$ and the group operation is written additively as $x + y$. The dual group $\widehat{\mathcal{G}}$ is the set of characters on $\mathcal{G}$. We usually denote characters by Greek letters $\xi, \eta, \omega, \dots$. The action of a character $\xi \in \widehat{\mathcal{G}}$ on an element $x \in \mathcal{G}$ is denoted by $\langle \xi, x\rangle$. Clearly the action of $-\xi$ on x is then given by $\langle -\xi, x\rangle = \overline{\langle \xi, x\rangle}$ where the overline denotes complex conjugation.

The phase-space or time-frequency plane is $\mathcal{G} \times \widehat{\mathcal{G}}$, its elements are denoted by boldface letters $\mathbf{x}, \mathbf{y}, \mathbf{u}, \dots$. By Pontrjagin's duality theorem [31] $\widehat{\widehat{\mathcal{G}}}$ is isomorphic to $\mathcal{G}$, henceforth we will identify $\widehat{\widehat{\mathcal{G}}}$ with $\mathcal{G}$. Consequently the dual group of $\mathcal{G} \times \widehat{\mathcal{G}}$ is $\widehat{\mathcal{G}} \times \mathcal{G}$. We denote its elements by boldface Greek letters $\boldsymbol{\xi}, \boldsymbol{\omega}, \dots$. Consistent with the previously introduced convention we also write e.g. $\mathbf{x} = (x, \xi) \in \mathcal{G} \times \widehat{\mathcal{G}}$ and $\boldsymbol{\xi} = (\xi, x) \in \widehat{\mathcal{G}} \times \mathcal{G}$.

By the structure theorem for locally compact abelian groups, $\mathcal{G}$ is isomorphic to a direct product $\mathcal{G} \simeq \mathbb{R}^d \times \mathcal{G}_0$, where the LCA group $\mathcal{G}_0$ contains a compact open subgroup $\mathcal{K}$ [31, 23]. Furthermore, if $\mathcal{G}_0$ contains the compact open subgroup $\mathcal{K}$, then $\widehat{\mathcal{G}}_0$ contains the compact open subgroup $\mathcal{K}^\perp$, cf. Example 4.4.9 in [31] or Lemma 6.2.3 in [16].

The "time-frequency plane" (phase space) of $\mathcal{G}$ is $\mathcal{G} \times \widehat{\mathcal{G}}$. As a consequence of the structure theorem, the phase-space is $\mathcal{G} \times \widehat{\mathcal{G}} \simeq \mathbb{R}^{2d} \times (\mathcal{G}_0 \times \widehat{\mathcal{G}}_0)$, and $\mathcal{G}_0 \times \widehat{\mathcal{G}}_0$ contains the compact-open group $\mathcal{K} \times \mathcal{K}^\perp$.

The Fourier transform of a function f on $\mathcal{G}$ is is defined by [31]

$$\mathcal{F}f(\omega) = \hat{f}(\xi) = \int_{\mathcal{G}} f(x)\overline{\langle \xi, x\rangle}\, dx, \qquad \text{for } \xi \in \widehat{\mathcal{G}}. \tag{3}$$

By Plancherel's Theorem $\mathcal{F}$ is unitary from $L^2(\mathcal{G})$ onto $L^2(\hat{\mathcal{G}})$ [31].

Time-Frequency Analysis. For a function f on $\mathcal{G}$, $x, y \in \mathcal{G}$, and $\xi \in \widehat{\mathcal{G}}$, we define the operators of *translation* T_x and *modulation* M_ξ by

$$T_y f(x) = f(x-y), \qquad M_\xi f(x) = \langle \xi, x\rangle f(x). \tag{4}$$

The operators T_x, M_ξ satisfy the commutation relations

$$T_x M_\xi = \overline{\langle \xi, x\rangle} M_\xi T_x. \tag{5}$$

The time-frequency shift operator $\pi(\mathbf{x})$ on $\mathcal{G} \times \widehat{\mathcal{G}}$ is defined by $\pi(\mathbf{x}) = M_\xi T_x, \mathbf{x} = (x, \xi) \in \mathcal{G} \times \widehat{\mathcal{G}}$.

Given an appropriate function ("window") g, the *short-time Fourier transform* (STFT) of $f \in L^2(\mathcal{G})$ is defined by

$$\mathcal{V}_g f(x,\xi) = \int_{\mathcal{G}} f(y)\overline{g(y-x)\langle \xi, y\rangle} dy, \qquad (x,\xi) \in \mathcal{G} \times \widehat{\mathcal{G}}. \tag{6}$$

We note that

$$\mathcal{V}_g f(u,\omega) = \langle f, M_\omega T_u g\rangle = \langle \hat{f}, T_\omega M_{-u}\hat{g}\rangle = \mathcal{V}_{\hat{g}}\hat{f}(\omega, -u)\overline{\langle \omega, u\rangle}. \tag{7}$$

Furthermore, for $f, g \in L^2(\mathcal{G})$ there holds

$$\mathcal{V}_{M_\eta T_y g} M_\xi T_x f(u,\omega) = T_{(x-y,\xi-\eta)} \mathcal{V}_g f(u,\omega)\overline{\langle \omega - \xi, x\rangle}\langle \eta, u - x\rangle\,, \tag{8}$$

which follows from the definition of the STFT and the commutation relations (5).

We will also make use of the following formula concerning the Fourier transform of a product of STFTs, which follows from an easy computation (carried out in [32] and in [21])

$$(V_{g_1} f_1 \overline{V_{g_2} f_2})^\wedge\,(\xi, x) = (V_{f_2} f_1 \overline{V_{g_2} g_1})(-x, \xi)\,. \tag{9}$$

Weight Functions.

Definition 2.1 *(a) A non-negative function v on $\mathcal{G} \times \widehat{\mathcal{G}}$ is called an* admissible weight *if it satisfies the following properties:*

(i) v is continuous, even in each coordinate, and normalized such that $v(0) = 1$.

(ii) v is submultiplicative, i.e., $v(\mathbf{x}+\mathbf{y}) \le v(\mathbf{x})v(\mathbf{y})$, $\mathbf{x}, \mathbf{y} \in \mathcal{G} \times \widehat{\mathcal{G}}$.

(iii) v satisfies the Gelfand-Raikov-Shilov (GRS) condition [14]

$$\lim_{n\to\infty} v(n\mathbf{x})^{\frac{1}{n}} = 1 \qquad \textit{for all } \mathbf{x} \in \mathcal{G} \times \widehat{\mathcal{G}}. \tag{10}$$

(b) Let v be an admissible weight. The class of v-moderate weights is

$$\mathcal{M}_v = \Big\{ m \geq 0 : \sup_{\mathbf{x}\in\mathcal{G}\times\widehat{\mathcal{G}}} \frac{m(\mathbf{x}+\mathbf{y})}{m(\mathbf{x})} \leq Cv(\mathbf{y}), \quad \forall\, \mathbf{y} \in \mathcal{G} \times \widehat{\mathcal{G}} \Big\}. \tag{11}$$

Examples: The standard weight functions on $\mathcal{G}$ are of the form

$$m(x) = e^{a\rho(x)^b}(1+\rho(x))^s\,,$$

where $\rho(x) = d(x,0)$ for some left-invariant metric d on $\mathcal{G}$. Such a weight is submultiplicative, when $a, s \geq 0$ and $0 \leq b \leq 1$, and m satisfies the GRS-condition, if and only if $0 \leq b < 1$. If $a, s \in \mathbb{R}$ are arbitrary, then m is $e^{|a|\rho(x)^b}(1+\rho(x))^{|s|}$-moderate.

Test Functions. For the treatment of weights of super-polynomial growth the standard space of test functions, the Schwartz-Bruhat space [30], is not suitable. We therefore introduce a *space of special test functions*, its construction is based on the structure theorem. Let $\mathcal{K}$ be a compact-open subgroup of $\mathcal{G}_0$, let $\varphi(x_1, x_2) = e^{-\pi x_1^2}\chi_{\mathcal{K}}(x_2) = \varphi_1(x_1)\varphi_2(x_2)$ for $x = (x_1, x_2) \in \mathbb{R}^d \times \mathcal{G}_0$ and

$$\mathcal{S}_{\mathcal{C}}(\mathcal{G}) = \mathrm{span}\{\pi(\mathbf{x})\varphi : \mathbf{x} \in \mathcal{G} \times \widehat{\mathcal{G}}\} \subseteq L^2(\mathcal{G})$$

be the linear space of all finite linear combinations of time-frequency shifts of the "Gaussian" $\varphi = \varphi_1 \otimes \varphi_2$. Then

$$V_\varphi\varphi(x,\xi) = V_{\varphi_1}\varphi_1(x_1,\xi_1)\, V_{\varphi_2}\varphi_2(x_2,\xi_2) = e^{-\pi(x_1^2+\xi_1^2)/2} V_{\varphi_2}\varphi_2(x_2,\xi_2)\,,$$

where $x_1, \xi_1 \in \mathbb{R}^d$ and $(x_2,\xi_2) \in \mathcal{G}_0 \times \widehat{\mathcal{G}_0}$. Using the calculation on p. 228 of [16] we find that

$$\begin{aligned} V_{\varphi_2}\varphi_2(x_2,\xi_2) &= \langle \chi_{\mathcal{K}}, M_{\xi_2}T_{x_2}\chi_{\mathcal{K}}\rangle \\ &= \begin{cases} 0 & \text{if } x_2 \notin \mathcal{K} \\ (\chi_{\mathcal{K}} \cdot \chi_{x_2\mathcal{K}})^\wedge(\xi_2) = \widehat{\chi_{\mathcal{K}}}(\xi_2) = c(K)\chi_{\mathcal{K}^\perp}(\xi_2) & \text{if } x_2 \in \mathcal{K}\,. \end{cases} \end{aligned}$$

Hence $V_{\varphi_2}\varphi_2 = c(\mathcal{K})\chi_{\mathcal{K}} \otimes \chi_{\mathcal{K}^\perp}$, where $c(\mathcal{K}) > 0$ is a constant depending on $\mathcal{K}$, and the support of $V_{\varphi_2}\varphi_2$ is thus compact. Since a submultiplicative weight v on $\mathbb{R}^d$ grows at most exponentially [6, Lemma VIII.1.4], we find that

$$\begin{aligned} &\int\!\!\int_{\mathcal{G}\times\widehat{\mathcal{G}}} |V_\varphi\varphi(x,\xi)| v(x,\xi)\, dxd\xi \leq \qquad\qquad (12) \\ &\leq \int_{\mathbb{R}^{2d}} e^{-\pi(x_1^2+\xi_1^2)/2}\, v(x_1,0,\xi_1,0)\, dx_1 d\xi_1 \int_{\mathcal{G}_0\times\widehat{\mathcal{G}_0}} \chi_{\mathcal{K}}(x_2)\chi_{\mathcal{K}^\perp}(\xi_2)\, v(0,x_2,0,\xi_2)\, dx_2 d\xi_2 < \infty\,. \end{aligned}$$

Consequently, $V_\varphi \varphi$ and hence every function in $\mathcal{S}_\mathcal{C}$ is integrable with respect to arbitrary moderate weight functions.

Modulation Spaces. Let m be a weight function. We define the $M^{p,q}_m$-norm of $f \in \mathcal{S}_\mathcal{C}(\mathcal{G})$ to be

$$\|f\|_{M^{p,q}_m} := \|V_\varphi f\, m\|_{L^{p,q}} = \left(\int_{\widehat{\mathcal{G}}} \left(\int_{\mathcal{G}} |V_\varphi f(x,\xi)|^p \, m(x,\xi)^p \, dx \right)^{q/p} d\xi \right)^{1/q} . \qquad (13)$$

Analogous to [19] we define the *modulation space* $M^{p,q}_m(\mathcal{G})$ as the completion of the space $\mathcal{S}_\mathcal{C}(\mathcal{G})$ with respect to the $M^{p,q}_m$-norm, when $p, q < \infty$ and the weak*-completion if $pq = \infty$. If $p = q$ we also write M^p_m instead of $M^{p,p}_m$. If $m \geq 1$, then $M^{p,q}_m$ is a subspace of M^∞ which in turn is a particular subspace of the space of tempered distributions $\mathcal{S}'$ on $\mathcal{G}$.

In the sequel we will distinguish between modulation spaces on $\mathcal{G}$ and on $\mathcal{G} \times \widehat{\mathcal{G}}$. The space $M^1_v(\mathcal{G})$ will serve as "window space" and can be considered a space of test functions. Pseudodifferential operators will act on the modulation spaces $M^{p,q}_M(\mathcal{G})$.

Modulation spaces on $\mathcal{G} \times \widehat{\mathcal{G}}$ will be used as symbol classes. In particular, $M^{\infty,1}_v(\mathcal{G} \times \widehat{\mathcal{G}})$ is the appropriate generalization of Sjöstrand's class to LCA groups. If we take the norm completion of $\mathcal{S}_\mathcal{C}$ in the $M^{\infty,1}_v$-norm, we obtain a class of symbols that leads to compact operators, see [3].

We will use the following standard properties of modulation spaces.

Proposition 2.2 (Duality) *(i) Let $1 \leq p, q < \infty$ and $p' = \frac{p}{p-1}$ be the conjugate index. Then the dual space of $M^{p,q}_m(\mathcal{G})$ is the modulation space $M^{p',q'}_{1/m}(\mathcal{G})$.*

(ii) M^1_m is the dual space of $M^{0,0}_{1/m} := \mathrm{clos}_{M^{\infty,\infty}_{1/m}}(\mathcal{S}_\mathcal{C})$ (the closure of the test functions with respect to the M^∞-norm), likewise $M^{\infty,1}_m(\mathcal{G})$ is the dual of $\mathrm{clos}_{M^{1,\infty}_{1/m}}(\mathcal{S}_\mathcal{C})$ (See [3]).

(iii) If $f \in M^{\infty,1}_m(\mathcal{G})$ and $g \in M^{1,\infty}_{1/m}$, then $\langle f, g\rangle := \int_{\mathcal{G}\times\widehat{\mathcal{G}}} V_\varphi f(\mathbf{x}) \, \overline{V_\varphi g(\mathbf{x})} \, d\mathbf{x}$ is well-defined and satisfies

$$|\langle f, g\rangle| \leq C \|f\|_{M^{\infty,1}_m(\mathcal{G})} \, \|g\|_{M^{1,\infty}_{1/m}} . \qquad (14)$$

Lemma 2.3 *If $g \in M^1_v$ and $m \in \mathcal{M}_v$, then $\|V_g f\, m\|_{L^{p,q}}$ is an equivalent norm on $M^{p,q}_m$ [17, Ch. 11].*

Amalgam Spaces. A *lattice* Λ of $\mathcal{G}$ is a discrete subgroup such that $\mathcal{G}/\Lambda$ is compact. Then there exists a relatively compact set $U \subseteq \mathcal{G}$, a *fundamental domain* for Λ, such that $\bigcup_{\lambda\in\Lambda}(\lambda + U) = \mathcal{G}$ and $(\lambda + U) \cap (\mu + U) = \emptyset$ for $\lambda \neq \mu \in \Lambda$.

If $\mathcal{G}$ does not have a lattice, as is the case for p-adic groups, we resort to the following construction. Recall the structure theorem $\mathcal{G} \simeq \mathbb{R}^d \times \mathcal{G}_0$, where $\mathcal{G}_0$ possesses the compact-open subgroup $\mathcal{K}$. Now choose an invertible, real-valued

$d \times d$-matrix A, and a set of coset representatives D of $\mathcal{G}_0/\mathcal{K}$ in $\mathcal{G}_0$, and let $U = A[0,1)^d \times \mathcal{K}$. Then $\mathcal{G} = \bigcup_{\lambda \in \Lambda}(\lambda + U)$ is a partition of $\mathcal{G}$. We call the discrete set $\Lambda := A\mathbb{Z}^d \times D$ a *quasi-lattice* with *fundamental domain* U.

Consequently a quasi-lattice in the time-frequency plane $\mathcal{G} \times \widehat{\mathcal{G}}$ will have the form $\Lambda = \Lambda_1 \times \Lambda_2 := (A\mathbb{R}^d \times D_1) \times (B\mathbb{R}^d \times D_2) \simeq \mathcal{A}\mathbb{R}^{2d} \times D_1 \times D_2$ (where A, B are $d \times d$ invertible matrices) with fundamental domain $U = U_1 \times U_2 = (A[0,1)^d \times \mathcal{K}) \times (B[0,1)^d \times \mathcal{K}^\perp)$.

Using this construction, we can now define amalgam spaces on $\mathcal{G} \times \widehat{\mathcal{G}}$, see [12] and [7] for a detailed theory.

Definition 2.4 *Let Λ be a quasi-lattice of $\mathcal{G} \times \widehat{\mathcal{G}}$ and U a relatively compact fundamental domain of Λ in $\mathcal{G} \times \widehat{\mathcal{G}}$. Let m be a weight function on $\mathcal{G} \times \widehat{\mathcal{G}}$. A continuous function F on $\mathcal{G} \times \widehat{\mathcal{G}}$ belongs to the* amalgam space $W(C, \ell^{p,q}_m)(\mathcal{G} \times \widehat{\mathcal{G}})$ *if the sequence $\{a(\mathbf{l})\}_{\mathbf{l} \in \Lambda}$ with*

$$a(\mathbf{l}) = a(l, \lambda) = \sup_{(u,\eta) \in U} |F(u + l, \eta + \lambda)| \tag{15}$$

belongs to $\ell^{p,q}_m(\Lambda)$, that is $\left(\sum_{\lambda \in \Lambda_2} \left(\sum_{l \in \Lambda_1} a(l,\lambda)^p m(l,\lambda)^p\right)^{q/p}\right)^{1/q} < \infty$, with the usual modifications if $p\, q = \infty$.

We note that the definition of the amalgam spaces is independent of the quasi-lattice Λ and the fundamental domain U, and different choices for Λ lead to equivalent norms [12].

Among others, amalgam spaces occur in time-frequency analysis in the description of the fine local properties of the STFT.

Theorem 2.5 *Assume that $g \in M^1_v(\mathcal{G})$, $f \in M^{p,q}_m(\mathcal{G})$ for $1 \le p, q \le \infty$, and m a v-moderate weight. Then $V_g f$ is in $W(C, \ell^{p,q}_m)(\mathcal{G} \times \widehat{\mathcal{G}})$. In particular, $V_g g \in W(C, \ell^1_v)(\mathcal{G} \times \widehat{\mathcal{G}})$.*

Proof: The statement is a special case of [8, Lemma 7.2, Thm. 8.1] (use the representation $(x, \xi, \tau) \to \tau T_x M_\xi$ on $L^2(\mathcal{G})$ of the Heisenberg-type group $\mathcal{G} \times \widehat{\mathcal{G}} \times \mathbb{T}$). A direct proof for $\mathbb{R}^d$ is given in [17, Thm. 12.2.1]. ❏

As an important consequence, we formulate this result for the general Sjöstrand class $M^{\infty,1}_v(\mathcal{G} \times \widehat{\mathcal{G}})$, where v is an admissible weight on $\widehat{\mathcal{G}} \times \mathcal{G}$. Note that $V_\Phi \sigma$ is a function on $(\mathcal{G} \times \widehat{\mathcal{G}}) \times (\widehat{\mathcal{G}} \times \mathcal{G})$.

Corollary 2.6 *Let $\tilde{\Lambda}$ be a quasi-lattice in $\widehat{\mathcal{G}} \times \mathcal{G}$ with fundamental domain $\tilde{U}$. If $\Phi \in M^1_{1 \otimes v}(\mathcal{G} \times \widehat{\mathcal{G}})$ and $\sigma \in M^{\infty,1}_v(\mathcal{G} \times \widehat{\mathcal{G}})$, then the sequence*

$$h(\boldsymbol{\lambda}) := \sup_{\boldsymbol{\eta} \in \tilde{U}} \sup_{\mathbf{x} \in \mathcal{G} \times \widehat{\mathcal{G}}} |V_\Phi \sigma(\mathbf{x}, \boldsymbol{\lambda} + \boldsymbol{\eta})|$$

is in $\ell^1_v(\tilde{\Lambda})$.

Gabor Frames. We assume familiarity with Gabor frames and refer to [17], Ch. 5 and 7, for details. Given a quasi-lattice $\Lambda \subset \mathcal{G} \times \widehat{\mathcal{G}}$ and a *window* $g \in L^2(\mathcal{G})$ the associated *Gabor system* $\{g_{m,\mu}\}_{(m,\mu)\in\Lambda}$ consists of functions of the form

$$g_{m,\mu} = M_\mu T_m g, \quad (m,\mu) \in \Lambda. \tag{16}$$

The analysis operator or coefficient operator $C_g : L^2(\mathcal{G}) \mapsto \ell^2(\Lambda)$ is defined as

$$C_g f = \{\langle f, M_\mu T_m g\rangle\}_{(m,\mu)\in\Lambda}. \tag{17}$$

The adjoint operator, which is also known as synthesis operator, can be expressed as

$$C_g^* \{c_{m,\mu}\}_{(m,\mu)\in\Lambda} = \sum_{(m,\mu)\in\Lambda} c_{m,\mu} M_\mu T_m g \qquad \text{for } \{c_{m,\mu}\}_{(m,\mu)\in\Lambda} \in \ell^2(\Lambda). \tag{18}$$

Associated to a Gabor system is the Gabor frame operator S defined as

$$Sf = \sum_{(m,\mu)\in\Lambda} \langle f, M_\mu T_m g\rangle M_\mu T_m g = C_g^* C_g f\,. \tag{19}$$

We say that $\{M_\mu T_m g\}_{(m,\mu)\in\Lambda}$ with $g \in L^2(\mathcal{G})$ is a Gabor frame for $L^2(\mathcal{G})$ if S is invertible on $L^2(\mathcal{G})$. Equivalently there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \le \sum_{(m,\mu)\in\Lambda} |\langle f, M_\mu T_m g\rangle|^2 = \langle Sf, f\rangle \le B\|f\|_2^2, \qquad \text{for all } f \in L^2(\mathcal{G}). \tag{20}$$

A Gabor system $\{M_\mu T_m g\}_{(m,\mu)\in\Lambda}$ is called a *tight* Gabor frame if $A = B$ in (20). In this case S is just (a multiple of) the identity operator on $L^2(\mathcal{G})$.

For our purposes we need tight Gabor frames generated by a window $g \in M_v^1(\mathcal{G})$. The existence and construction of Gabor frames are well understood on $\mathbb{R}^d$, but our knowledge of explicit Gabor frames on LCA groups is thin. Therefore the following existence theorem may be of independent interest.

Theorem 2.7 *Let v be an admissible weight on $\mathcal{G}\times\widehat{\mathcal{G}}$ satisfying the GRS-condition, and let $\Lambda := \alpha\mathrm{I} \times D$ be a quasi-lattice in $\mathcal{G} \times \widehat{\mathcal{G}}$ with $\alpha < 1$ and D a set of representatives of $\mathcal{G}/\mathcal{K}\times\widehat{\mathcal{G}}_0/\mathcal{K}^\perp$. Then there exists a $g \in M_v^1(\mathcal{G})$, such that $\{\pi(\lambda)g : \lambda \in \Lambda\}$ is a tight Gabor frame for $L^2(\mathcal{G})$.*

Proof: According to the structure theorem we distinguish several cases.

Case I: $\mathcal{G} = \mathbb{R}^d$. We choose $\Lambda = \alpha\mathbb{Z}^{2d}$ for $\alpha < 1$ and the Gaussian $\varphi(t) = e^{-\pi t\cdot t}$. It follows from the main result in [28, 35] that $\{M_\mu T_m \varphi\}_{(m,\mu)\in\Lambda}$ is a Gabor frame for $L^2(\mathbb{R}^d)$ with $\varphi \in M_v^1(\mathcal{G})$. To this Gabor frame we apply Cor. 4.5 of [20]: *Let v be an admissible weight satisfying the GRS-condition. If $\{M_\mu T_m g\}_{(m,\mu)\in\Lambda}$ is a Gabor frame for $L^2(\mathbb{R}^d)$ with $g \in M_v^1(\mathbb{R}^d)$ and associated frame operator S, then $\{M_\mu T_m S^{-1/2} g : (m,\mu) \in \Lambda\}$ is a tight frame and the window $\gamma^\circ = S^{-1/2} g$ also* *belongs to* $M^1_v(\mathcal{G})$. This construction provides an abundance of tight Gabor frames for $L^2(\mathbb{R}^d)$.

Case II: $\mathcal{G} = \mathcal{G}_0$, where $\mathcal{G}_0$ contains the compact-open subgroup $\mathcal{K}$. Let D_1 be a set of coset representatives of $\mathcal{G}_0/\mathcal{K}$ and D_2 be a set of coset representatives of $\widehat{\mathcal{G}}_0/\mathcal{K}^\perp$. Then $D = D_1 \times D_2$ is a quasi-lattice in $\mathcal{G} \times \widehat{\mathcal{G}}$, and the family $\{M_\delta T_d \chi_\mathcal{K} : (d,\delta) \in D\}$ is an orthonormal basis for $L^2(\mathcal{G}_0)$. To verify this claim, we note that $\{\delta : \delta \in \widehat{\mathcal{K}}\}$ is an orthonormal basis for $L^2(\mathcal{K})$, because $\mathcal{K}$ is compact. Since $\widehat{\mathcal{K}} \simeq \widehat{\mathcal{G}}_0/\mathcal{K}^\perp \simeq D_2$, the set $\{M_\delta \chi_\mathcal{K} : \delta \in D_2\}$ is an orthonormal basis for $L^2(\mathcal{K}) \subseteq L^2(\mathcal{G}_0)$. Furthermore, since the translates $T_{d_1}\chi_\mathcal{K}, T_{d_2}\chi_\mathcal{K}$ have disjoint support for $d_1, d_2 \in D_1 \simeq \widehat{\mathcal{G}}_0/\mathcal{K}, d_1 \neq d_2$ and since

$$L^2(\mathcal{G}_0) = \oplus_{d \in D_1} L^2(d\,\mathcal{K}),$$

$\{M_\omega T_d \chi_\mathcal{K}\}_{(d,\delta)\in D}$ is an orthonormal basis for $L^2(\mathcal{G}_0)$.

Furthermore, $\chi_\mathcal{K} \in M^1_v(\mathcal{G}_0)$ by Proposition 6.4.5 in [16] or as a consequence of (12).

Case III: $\mathcal{G} \simeq \mathbb{R}^d \times \mathcal{G}_0$ **is an arbitrary LCA group.** Let $\Lambda = \alpha\mathbb{Z}^{2d} \times D$, $\alpha < 1$ be a quasi-lattice in $\mathcal{G} \times \widehat{\mathcal{G}}$, let $\{M_\mu T_m \gamma^\circ : (m,\mu) \in \alpha\mathbb{Z}^{2d}\}$ be a tight frame for $L^2(\mathbb{R}^d)$, and $\{M_\delta T_d \chi_\mathcal{K} : (d,\delta) \in D\}$ be the orthonormal basis for $L^2(\mathcal{G}_0)$. Then the set $\{M_{(\omega,\delta)} T_{(u,d)}(\gamma^\circ \otimes \chi_\mathcal{K})\}$ is a tight frame for $L^2(\mathcal{G})$, because the tensor product of (tight) frames is again a (tight) frame. Finally, the window $g = \gamma^\circ \otimes \chi_\mathcal{K}$ is in $M^1_v(\mathcal{G})$, which is shown as in (12). ❏

A Banach Algebra of Matrices. The following matrix algebra is a natural generalization of Wiener's algebra and will play a central role in our investigations.

Definition 2.8 ([15, 1, 2]) *Let $\mathcal{D}$ be a countable discrete abelian subgroup, and let v be an admissible weight on $\mathcal{D}$. The nonstationary Wiener algebra $\mathcal{C}_v = \mathcal{C}_v(\mathcal{D})$ consists of all matrices $A = [A_{i,j}]_{i,j\in\mathcal{D}\times\mathcal{D}}$ on the index set $\mathcal{D}$, for which*

$$\|A\|_{\mathcal{C}_v(\mathcal{D})} := \sum_{j\in\mathcal{D}} \sup_{i\in\mathcal{D}} |A_{i,i-j}|\, v(j) \tag{21}$$

is finite.

It is easy to verify that

$$\sum_{j\in\mathcal{D}} \sup_i |A_{i,i-j}| v(j) = \inf_{a\in\ell^1_v(\mathcal{D})} \{\|a\|_{\ell^1_v} : |A_{i,j}| \le a(i-j), i,j \in \mathcal{D}\}. \tag{22}$$

The unweighted version of the following result was mentioned already in the introduction of this paper.

Theorem 2.9 *Let $A = [A_{i,j}]_{i,j\in\mathcal{D}}$ be a matrix in $\mathcal{C}_v$ where v is an admissible weight. If A is invertible on $\ell^2(\mathcal{D})$, then $A^{-1} \in \mathcal{C}_v$.*

This theorem was obtained by Gohberg et al. [15] and independently by Sjöstrand [37] for the case $v = 1$. The weighted case as well as quantitative versions were derived by Baskakov [1, 2].

We need the following generalization of this theorem. We recall that a matrix A possesses a pseudoinverse, if there exists a subspace $\mathcal{M} \subseteq \ell^2(\mathcal{D})$, such that the restriction of A to $\mathcal{M}$ is invertible and $\ker(A) = \mathcal{M}^\perp$. The trivial extension of this inverse on $\mathcal{M}$ to all of $\ell^2(\mathcal{D})$ is called the pseudoinverse and denoted by A^+.

Corollary 2.10 ([11]) *Let $A = [A_{i,j}]_{i,j\in\mathcal{D}}$ be a matrix in $\mathcal{C}_v$ where v is an admissible weight. If A has a pseudoinverse A^+, then $A^+ \in \mathcal{C}_v$.*

3 Pseudodifferential Operators on Locally Compact Abelian Groups

We turn to the investigation of pseudodifferential operators on LCA groups. The abstract formalism was developed in [9]. With proper notation, most formulas are almost identical to those for pseudodifferential operators on $\mathbb{R}^d$.

Definition 3.1 *Let σ be a function or distribution in $M^\infty(\mathcal{G} \times \widehat{\mathcal{G}})$. The pseudodifferential operator with Kohn-Nirenberg symbol σ is the operator K_σ given by*

$$(K_\sigma f)(x) = \int\limits_{\widehat{\mathcal{G}}} \sigma(x,\xi)\hat{f}(\xi)\langle \xi, x\rangle d\xi. \tag{23}$$

If F is some function space, we write $K_\sigma \in \mathrm{Op}(F)$ whenever $\sigma \in F$.

Alternatively we can write K_σ as a superposition of time-frequency shifts [9, 10, 17]:

$$K_\sigma f(x) = \int\limits_{\widehat{\mathcal{G}}\times\mathcal{G}} \hat{\sigma}(\omega, u) M_\omega T_{-u} f(x) d\omega du\,. \tag{24}$$

If $\hat{\sigma} \in L^1(\hat{\mathcal{G}} \times \mathcal{G})$, $f \in L^1(\mathcal{G})$, and $\hat{f} \in L^1(\widehat{\mathcal{G}})$, this follows from the computation

$$\begin{aligned} K_\sigma f(x) &= \int\limits_{\widehat{\mathcal{G}}} \sigma(x,\xi)\hat{f}(\xi)\langle \xi, x\rangle d\xi = \int\limits_{\mathcal{G}\times\widehat{\mathcal{G}}} \sigma(x,\xi) f(y)\overline{\langle \xi, y-x\rangle} dy d\xi \\ &= \int\limits_{\widehat{\mathcal{G}}\times\mathcal{G}} \hat{\sigma}(\omega, y-x) f(y)\langle \omega, x\rangle d\omega dy = \int\limits_{\widehat{\mathcal{G}}\times\mathcal{G}} \hat{\sigma}(\omega, u) f(x+u)\langle \omega, x\rangle d\omega du \\ &= \int\limits_{\widehat{\mathcal{G}}\times\mathcal{G}} \hat{\sigma}(\omega, u) M_\omega T_{-u} f(x) d\omega du. \end{aligned} \tag{25}$$

Expression (24) is called the *spreading representation* of K_σ and $\hat{\sigma}$ is the *spreading function*. For more general symbol classes the validity of the spreading representation follows by a routine density argument [9, 18]. Expression (24) represents pseudodifferential operators as linear combination of time-frequency shift operators, which suggests that methods from time-frequency analysis are a natural tool for the study of pseudodifferential operators.

We have the following formal symbol calculus for Kohn-Nirenberg pseudodifferential operators.

Lemma 3.2 *If $\hat{\sigma}, \hat{\tau} \in L^1(\hat{\mathcal{G}} \times \mathcal{G})$, then*

$$K_\sigma K_\tau = K_{\mathcal{F}^{-1}(\hat{\sigma} \natural \hat{\tau})}, \tag{26}$$

where the twisted convolution $\natural$ of $\hat{\sigma}, \hat{\tau}$ is defined by

$$\hat{\sigma} \natural \hat{\tau}(\xi, u) = \int_{\widehat{\mathcal{G}} \times \mathcal{G}} \hat{\sigma}(\zeta, y) \hat{\tau}(\xi - \zeta, u - y) \langle \xi - \zeta, y \rangle d\zeta dy. \tag{27}$$

Proof: Our hypothesis guarantees that the integrals below converges absolutely and thus Fubini's theorem permits to change the order of integration.

$$\begin{aligned} K_\sigma K_\tau f &= \int_{\widehat{\mathcal{G}} \times \mathcal{G}} \hat{\sigma}(\zeta, y) M_\zeta T_{-y} d\zeta dy \int_{\widehat{\mathcal{G}} \times \mathcal{G}} \hat{\tau}(\xi, u) M_\xi T_{-u} f d\xi du \\ &= \int_{\widehat{\mathcal{G}} \times \mathcal{G}} \int_{\widehat{\mathcal{G}} \times \mathcal{G}} \hat{\sigma}(\zeta, y) \hat{\tau}(\xi, u) \langle \xi, y \rangle M_{\zeta + \xi} T_{-(y+u)} f d\xi du d\zeta dy \\ &= \int_{\widehat{\mathcal{G}} \times \mathcal{G}} \Big(\int_{\widehat{\mathcal{G}} \times \mathcal{G}} \hat{\sigma}(\zeta, y) \hat{\tau}(\xi - \zeta, u - y) \langle \xi - \zeta, y \rangle d\zeta dy \Big) M_\xi T_{-u} f d\xi du \\ &= \int_{\widehat{\mathcal{G}} \times \mathcal{G}} (\hat{\sigma} \natural \hat{\tau})(\xi, u) M_\xi T_{-u} f d\xi du = K_{\mathcal{F}^{-1}(\hat{\sigma} \natural \hat{\tau})} f. \end{aligned}$$

❏

Definition 3.3 *Let v be an admissible weight on $\hat{\mathcal{G}} \times \mathcal{G}$. The weighted Sjöstrand class $\mathrm{Op}(M_v^{\infty,1}(\mathcal{G} \times \widehat{\mathcal{G}}))$ is the class of pseudodifferential operators K_σ whose symbol $\sigma \in M^\infty(\mathcal{G} \times \widehat{\mathcal{G}})$ satisfies*

$$\|\sigma\|_{M_v^{\infty,1}} = \int_{\widehat{\mathcal{G}} \times \mathcal{G}} \sup_{\mathbf{x} \in \mathcal{G} \times \widehat{\mathcal{G}}} |\mathcal{V}_\Psi \sigma(\mathbf{x}, \boldsymbol{\omega})| \, v(\boldsymbol{\omega}) d\boldsymbol{\omega} < \infty \tag{28}$$

with $\Psi \in \mathcal{S}_{\mathcal{C}}(\mathcal{G} \times \widehat{\mathcal{G}}) \setminus \{0\}$.

Note that the weight in (28) depends only on $\boldsymbol{\omega}$, so consistency with (13) would require the clumsier notation $M^{\infty,1}_{1\otimes v}(\mathcal{G}\times\widehat{\mathcal{G}})$.

Let $\{M_\mu T_m g\}_{(m,\mu)\in\Lambda}$ be a Gabor system for $L^2(\mathcal{G})$ with respect to a quasi-lattice $\Lambda\subseteq\mathcal{G}\times\widehat{\mathcal{G}}$. Using the notation of time-frequency shift operators $\pi(\mathbf{x})=M_\xi T_x$, we can write this system as $\{\pi(\mathbf{m})g\}_{\mathbf{m}\in\Lambda}$. For a given pseudodifferential operator K_σ we define the matrix $M(\sigma)$ by

$$[M(\sigma)]_{\mathbf{m},\mathbf{n}}=\langle K_\sigma\pi(\mathbf{n})g,\pi(\mathbf{m})g\rangle,\qquad \mathbf{m},\mathbf{n}\in\Lambda. \tag{29}$$

Since $M(\sigma)$ depends also on g and Λ, it would be more accurate to use the notation $M(\sigma,g,\Lambda)$. However, whenever there is no danger of confusion, we will simply write $M(\sigma)$.

Assume K_σ is bounded on $L^2(\mathcal{G})$ and that $\{\pi(\mathbf{m})g\}_{\mathbf{m}\in\Lambda}$ is a tight frame for $L^2(\mathcal{G})$ with (lower and upper) frame bound equal to 1. In this case $C^*C=I$, where C_g and C_g^* are defined as in (17) and (18). We can represent $f\in L^2(\mathcal{G})$ as $f=\sum_{\mathbf{n}\in\lambda}\langle f,\pi(\mathbf{n})g\rangle\pi(\mathbf{n})g$. For $\mathbf{m}\in\Lambda$ we compute

$$\begin{aligned}C_g(K_\sigma f)(\mathbf{m})&=\langle K_\sigma f,\pi(\mathbf{m})g\rangle\\&=\sum_{\mathbf{n}\in\Lambda}\langle f,\pi(\mathbf{n})g\rangle\langle K_\sigma\pi(\mathbf{n})g,\pi(\mathbf{m})g\rangle=(M(\sigma)C_gf)(\mathbf{m}).\end{aligned} \tag{30}$$

Since $C^*C=I$, equation (30) can be expressed equivalently as

$$K_\sigma f=C_g^*M(\sigma)C_gf. \tag{31}$$

The following lemma specifies the kernel and the range of $M(\sigma)$ and is taken from [19, Lemma 3.4] (the proof carries over almost verbatim to our setting by replacing the Weyl symbol by the Kohn-Nirenberg symbol and $\mathbb{R}^{2d}$ by $\mathcal{G}\times\widehat{\mathcal{G}}$).

Lemma 3.4 *Let $\{M_\mu T_m g\}_{(m,\mu)\in\Lambda}$ be a Gabor frame for $L^2(\mathcal{G})$. If K_σ is bounded on $L^2(\mathcal{G})$ then $M(\sigma)$ is bounded on $\ell^2(\Lambda)$ and maps* $\operatorname{ran}(C_g)$ *into* $\operatorname{ran}(C_g)$ *with* $\operatorname{ran}(C_g)^\perp=\ker(C_g^*)\subseteq\ker(M(\sigma))$.

4 Almost Diagonalization

We characterize the Sjöstrand class by its almost diagonalization with respect to Gabor frames. The corresponding results on $\mathbb{R}^d$ were obtained in [19] and in a slightly different version that is more suitable to applications in [39].

Definition 4.1 *The (cross) Rihaczek distribution of $f,g\in L^2(\mathcal{G})$ is defined as*

$$R(f,g)(x,\xi)=f(x)\overline{\hat{g}(\xi)}\;\overline{\langle\xi,x\rangle}. \tag{32}$$

The next lemma states two important properties of the Rihaczek distribution and clarifies its appearance in the analysis of the Kohn-Nirenberg pseudodifferential operators. Now define

$$\mathcal{J}(x,\xi) = (-\xi, x) \qquad (x,\xi) \in \mathcal{G} \times \widehat{\mathcal{G}}\,, \tag{33}$$

then $\mathcal{J}$ is an isomorphism from $\mathcal{G} \times \widehat{\mathcal{G}}$ onto $\widehat{\mathcal{G}} \times \mathcal{G}$, and $\mathcal{J}$ preserves the Haar measure.

Lemma 4.2 *Let $f, g \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})$. Then:*
(i)

$$R(\pi(\mathbf{x})g, \pi(\mathbf{y})f) = \langle \eta, x-y \rangle\, M_{\mathcal{J}(\mathbf{y}-\mathbf{x})} T_{(x,\eta)} R(g,f), \tag{34}$$

(ii)

$$\langle K_\sigma \pi(\mathbf{x}) f, \pi(\mathbf{y}) f \rangle = \overline{\langle \eta, x-y \rangle} \mathcal{V}_{R(g,f)} \sigma\Big((x,\eta), \mathcal{J}(\mathbf{y}-\mathbf{x})\Big). \tag{35}$$

(iii) If $f, g \in M^1_v(\mathcal{G})$ for an admissible weight v on $\mathcal{G} \times \widehat{\mathcal{G}}$, then
$R(g,f) \in M^1_{1\otimes v \circ \mathcal{J}^{-1}}(\mathcal{G} \times \widehat{\mathcal{G}})$.

Proof: (i) follows from the calculation

$$\begin{aligned}
R(\pi(\mathbf{x})g, \pi(\mathbf{y})f)(t,\tau) &= \pi(\mathbf{x})g(t)\, \overline{(\pi(\mathbf{y})f)\hat{}\,(\tau)}\; \overline{\langle \tau, t \rangle} \\
&= M_\xi T_x g(t)\, \overline{T_\eta M_{-y} \hat{f}(\tau)}\; \overline{\langle \tau, t \rangle} \\
&= \langle \xi, t \rangle g(t-x) \langle \tau - \eta, y \rangle \overline{\hat{f}(\tau - \eta)} \\
&\quad \cdot \overline{\langle \eta, t \rangle \langle \tau, x \rangle} \langle \eta, x \rangle \overline{\langle \tau - \eta, t - x \rangle} \\
&= \langle \eta, x - y \rangle\, M_{(\xi-\eta, y-x)} T_{(x,\eta)}\, R(g,f)(t,\tau)\,.
\end{aligned}$$

Using the definition for $\mathcal{J}$, we have obtained (34)

(ii) We first calculate the action of K_σ for functions $f, g \in \mathcal{S}_{\mathcal{C}}(\mathcal{G})$ in terms of the Rihaczek distribution:

$$\begin{aligned}
\langle K_\sigma f, g \rangle &= \int_{\mathcal{G}} \int_{\widehat{\mathcal{G}}} \sigma(x,\xi) \hat{f}(\xi) \langle \xi, x \rangle\, \overline{g(x)}\, d\xi dx \\
&= \langle \sigma, R(g,f) \rangle.
\end{aligned} \tag{36}$$

Using (36) and (34) we compute

$$\begin{aligned}
\langle K_\sigma \pi(\mathbf{y}) f, \pi(\mathbf{x}) g \rangle &= \langle \sigma, R(\pi(\mathbf{x})g, \pi(\mathbf{y})f)) \rangle \\
&= \overline{\langle \eta, x-y \rangle}\, \langle \sigma, M_{\mathcal{J}(\mathbf{y}-\mathbf{x})} T_{(x,\eta)} R(g,f) \rangle \\
&= \overline{\langle \eta, x-y \rangle}\, V_{R(g,f)} \sigma((x,\eta), \mathcal{J}(\mathbf{y}-\mathbf{x}))\,.
\end{aligned} \tag{37}$$

which is (35).

(iii) is proved similar to Lemma 5.1(ii) and therefore omitted. ❑

Now we state the key theorem about almost diagonalization of pseudodifferential operators K_σ for symbols in the generalized Sjöstrand class $M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$. First we formulate a version for LCA groups that contain a lattice Λ and a corresponding Gabor frame.

Theorem 4.3 *Let $g \in M^1_v(\mathcal{G})$ for some admissible weight v on $\mathcal{G}\times\widehat{\mathcal{G}}$, $\Lambda \subseteq \mathcal{G}\times\widehat{\mathcal{G}}$ be a lattice, and assume that $\{\pi(\mathbf{m})g\}_{\mathbf{m}\in\Lambda}$ is a tight Gabor frame for $L^2(\mathcal{G})$. Then for $\sigma \in M^\infty(\mathcal{G}\times\widehat{\mathcal{G}})$ the following properties are equivalent.*

(i) $\sigma \in M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$.

(ii) There exists a function $H \in L^1_v(\mathcal{G}\times\widehat{\mathcal{G}})$ such that

$$|\langle K_\sigma\pi(\mathbf{z})g, \pi(\mathbf{y})g\rangle| \le H(\mathbf{y}-\mathbf{z}), \qquad \textit{for all } \mathbf{y},\mathbf{z} \in \mathcal{G}\times\widehat{\mathcal{G}}. \tag{38}$$

(iii) There exists a sequence $h \in \ell^1_v(\Lambda)$ such that

$$|\langle K_\sigma\pi(\mathbf{n})g, \pi(\mathbf{m})g\rangle| \le h(\mathbf{m}-\mathbf{n}), \qquad \textit{for all } \mathbf{m},\mathbf{n} \in \Lambda. \tag{39}$$

Proof:
(i) $\Rightarrow$ *(ii)*. Let $\sigma \in M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$. Denote $\mathbf{x} = (x,\xi)$ and $\mathbf{y} = (y,\eta)$, and set $\Psi = R(g,g)$, which is in $M^1_{1\otimes v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$ by Lemma 4.2(iii). We use Lemma 4.2(ii) to compute

$$|\langle K_\sigma\pi(\mathbf{x})g, \pi(\mathbf{y})g\rangle| = |\mathcal{V}_\Psi\sigma\big((x,\eta), \mathcal{J}(\mathbf{y}-\mathbf{x})\big)| \le \sup_{\mathbf{z}\in\mathcal{G}\times\widehat{\mathcal{G}}} |\mathcal{V}_\Psi\sigma(\mathbf{z}, \mathcal{J}(\mathbf{y}-\mathbf{x}))|. \tag{40}$$

Now set $H(\mathbf{x}) := \sup_{\mathbf{z}\in\mathcal{G}\times\widehat{\mathcal{G}}} |\mathcal{V}_\Psi\sigma(\mathbf{z},\mathcal{J}\mathbf{x})|$. Then

$$\begin{aligned}\int_{\mathcal{G}\times\widehat{\mathcal{G}}} H(\mathbf{x})v(\mathbf{x})\,d\mathbf{x} &= \int_{\mathcal{G}\times\widehat{\mathcal{G}}} \sup_{\mathbf{z}\in\mathcal{G}\times\widehat{\mathcal{G}}} |\mathcal{V}_\Psi\sigma(\mathbf{z},\mathcal{J}\mathbf{x})| v(\mathcal{J}^{-1}\mathcal{J}\mathbf{x})\,d\mathbf{x} \\ &= \int_{\widehat{\mathcal{G}}\times\mathcal{G}} \sup_{\mathbf{z}\in\mathcal{G}\times\widehat{\mathcal{G}}} |\mathcal{V}_\Psi\sigma(\mathbf{z},\boldsymbol{\omega})|\, v(\mathcal{J}^{-1}\boldsymbol{\omega})\,d\boldsymbol{\omega} = \|\sigma\|_{M^{\infty,1}_{v\circ\mathcal{J}^{-1}}}.\end{aligned}$$

(ii) $\Rightarrow$ *(i)*. For the converse, we note that if $(\mathbf{z},\boldsymbol{\omega}) = \big((x,\eta), \mathcal{J}(\mathbf{y}-\mathbf{x})\big)$ for $\mathbf{z} = (z,\zeta)$ and $\boldsymbol{\omega} = (\omega, u)$, then $\mathbf{y} = (u+z,\zeta)$ and $\mathbf{x} = (z, \omega+\zeta)$. Thus

$$|\mathcal{V}_{R(g,g)}\sigma(\mathbf{z},\boldsymbol{\omega})| = |\langle K_\sigma\pi(z,\omega+\zeta)g, \pi(u+z,\zeta)g\rangle|\,.$$

If (ii) holds, then

$$\sup_{\mathbf{z}\in\mathcal{G}\times\widehat{\mathcal{G}}} |V_\Phi\sigma(\mathbf{z},\boldsymbol{\omega})| \le H(\mathcal{J}^{-1}\boldsymbol{\omega})\,,$$

and thus

$$\begin{aligned}
\|\sigma\|_{M^{\infty,1}_{v\circ\mathcal{J}^{-1}}} &= \int_{\widehat{\mathcal{G}}\times\mathcal{G}} \sup_{\mathbf{z}\in\mathcal{G}\times\widehat{\mathcal{G}}} |V_\Phi\sigma(\mathbf{z},\boldsymbol{\omega})|\, v(\mathcal{J}^{-1}\boldsymbol{\omega})\, d\boldsymbol{\omega} \\
&\leq \int_{\widehat{\mathcal{G}}\times\mathcal{G}} H(\mathcal{J}^{-1}\boldsymbol{\omega}) v(\mathcal{J}^{-1}\boldsymbol{\omega})\, d\boldsymbol{\omega} = \|H\|_{L^1_v(\mathcal{G}\times\widehat{\mathcal{G}})}\,.
\end{aligned}$$

$(i) \Rightarrow (iii)$: Let U be a fundamental domain of $\Lambda \subseteq \mathcal{G}\times\widehat{\mathcal{G}}$. Set $h(\mathbf{m}) := \sup_{\mathbf{u}\in U}\sup_{\mathbf{z}\in\mathcal{G}\times\widehat{\mathcal{G}}} |\mathcal{V}_\Psi\sigma(\mathbf{z},\mathcal{J}(\mathbf{m}+\mathbf{u})|$. Since $\sigma\in M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$ and $\Phi\in M^1_{1\otimes v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$, Theorem 2.5 and Corollary 2.6 apply and warrant that $h\in\ell^1_v(\Lambda)$.

Next we use Lemma 4.2(ii) and argue as in (40) to obtain

$$|\langle K_\sigma\pi(\mathbf{n})g, \pi(\mathbf{m})g\rangle| \leq \sup_{\mathbf{z}\in\mathcal{G}\times\widehat{\mathcal{G}}} |\mathcal{V}_\Psi\sigma(\mathbf{z},\mathcal{J}(\mathbf{m}-\mathbf{n}))| = h(\mathbf{m}-\mathbf{n}) \tag{41}$$

for $\mathbf{m},\mathbf{n}\in\Lambda$. Thus (iii) is proved.

$(iii) \Rightarrow (ii)$: Since $\{\pi(\mathbf{m})g\}_{\mathbf{m}\in\Lambda}$ is a tight frame for $L^2(\mathcal{G}\times\widehat{\mathcal{G}})$ we can express an arbitrary time-frequency shift $\pi(\mathbf{u})g$ as

$$\pi(\mathbf{u})g = \sum_{\mathbf{m}\in\Lambda} \langle \pi(\mathbf{u})g, \pi(\mathbf{m})g\rangle \pi(\mathbf{m})g. \tag{42}$$

By assumption $g\in M^1_v$ and therefore Theorem 2.5 implies that $\mathcal{V}_g g\in W(C,\ell^1_v)(\mathcal{G}\times\widehat{\mathcal{G}})$. This means that for every relatively compact fundamental domain U of Λ and

$$\alpha(\mathbf{n}) = \sup_{\mathbf{u}\in U} |\mathcal{V}_g g(\mathbf{n}-\mathbf{u})| = \sup_{\mathbf{u}\in U} |\langle\pi(\mathbf{u})g, \pi(\mathbf{n})g\rangle|, \quad \mathbf{n}\in\Lambda, \tag{43}$$

the sequence $\alpha = \{\alpha(\mathbf{n})\}_{\mathbf{n}\in\Lambda}$ belongs to $\ell^1_v(\Lambda)$.

Given $\mathbf{y},\mathbf{z}\in\mathcal{G}\times\widehat{\mathcal{G}}$ we can write them uniquely as $\mathbf{y} = \mathbf{n}+\mathbf{u}, \mathbf{z} = \mathbf{n}'+\mathbf{u}'$ for $\mathbf{n},\mathbf{n}'\in\Lambda$ and $\mathbf{u},\mathbf{u}'\in U$. Inserting the expansion (42) and the definition of α in the matrix entries we obtain

$$\begin{aligned}
&|\langle K_\sigma\pi(\mathbf{n}'+\mathbf{u}')g, \pi(\mathbf{n}+\mathbf{u})g\rangle| = |\langle K_\sigma\pi(\mathbf{n}')\pi(\mathbf{u}')g, \pi(\mathbf{n})\pi(\mathbf{u})g\rangle| \\
&\leq \sum_{\mathbf{m},\mathbf{m}'\in\Lambda} |\langle K_\sigma\pi(\mathbf{n}'+\mathbf{m}')g, \pi(\mathbf{n}+\mathbf{m})g\rangle||\langle\pi(\mathbf{u}')g,\pi(\mathbf{m}')g\rangle||\langle\pi(\mathbf{u})g,\pi(\mathbf{m})g\rangle| \\
&\leq \sum_{\mathbf{m},\mathbf{m}'\in\Lambda} h(\mathbf{n}+\mathbf{m}-\mathbf{n}'-\mathbf{m}')\alpha(\mathbf{m}')\alpha(\mathbf{m}) \\
&= (h*\alpha*\tilde{\alpha})(\mathbf{n}-\mathbf{n}')
\end{aligned}$$

with $\tilde{\alpha}(\mathbf{n}) = \alpha(-\mathbf{n})$. Since $h\in\ell^1_v(\Lambda)$ by hypothesis (iii) and $\alpha\in\ell^1_v(\Lambda)$ by construction, we also have $h*\alpha*\tilde{\alpha}\in\ell^1_v(\Lambda)$.

Now set

$$H(\mathbf{z}) = \sum_{\mathbf{n}\in\Lambda} (h*\alpha*\tilde{\alpha})(\mathbf{n})\chi_{U-U}(\mathbf{z}-\mathbf{v}) \qquad \mathbf{z}\in\mathcal{G}\times\widehat{\mathcal{G}}\,.$$

Since $\|T_{\mathbf{n}}\chi_{U-U}\|_{L^1_v} \le v(\mathbf{n})\|\chi_{U-U}\|_{L^1_v}$, we obtain that

$$\|H\|_{L^1_v} \le \sum_{\mathbf{n}\in\Lambda}(h*\alpha*\tilde{\alpha})(\mathbf{n})v(\mathbf{n})\|\chi_{U-U}\|_{L^1_v} = c\|h*\alpha*\tilde{\alpha}\|_{\ell^1_v} < \infty.$$

For $\mathbf{y},\mathbf{z} \in \mathcal{G}\times\widehat{\mathcal{G}}$ write $\mathbf{y} = \mathbf{n}+\mathbf{u}$ and $\mathbf{z} = \mathbf{n}'+\mathbf{u}'$ as before, then we have $\mathbf{y}-\mathbf{z} \in \mathbf{n}-\mathbf{n}'+U-U$ and $(h*\alpha*\tilde{\alpha})(\mathbf{n}-\mathbf{n}') \le H(\mathbf{y}-\mathbf{z})$. Combining these observations we have shown that

$$|\langle K_\sigma \pi(\mathbf{z})g, \pi(\mathbf{y})g\rangle| \le (h*\alpha*\tilde{\alpha})(\mathbf{n}-\mathbf{n}') \le H(\mathbf{y}-\mathbf{z}),$$

and this is (ii). ❑

Remark: We have proved a bit more. The equivalence $(i) \Leftrightarrow (ii)$ requires only that $g \in M^1_v$ without any restriction; the implication $(ii) \Rightarrow (iii)$ holds for arbitrary Gabor systems with $g \in M^1_v$. Only the implication $(iii) \Rightarrow (ii)$ requires that the Gabor system is a frame for $L^2(\mathcal{G})$.

Since the almost diagonalization of implication (i) ⇒ (iii) is important in several applications (e.g., cf. [39]), we state it explicitly.

Corollary 4.4 *If $\sigma \in M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$, $g \in M^1_v(\mathcal{G})$ and $\Lambda \subseteq \mathcal{G}\times\widehat{\mathcal{G}}$ a lattice in $\mathcal{G}\times\widehat{\mathcal{G}}$, then there exists a sequence $h \in \ell^1_v(\Lambda)$ such that*

$$|\langle K_\sigma \pi(\mathbf{m})g, \pi(\mathbf{n})g\rangle| \le h(\mathbf{m}-\mathbf{n}), \qquad \textit{for all } \mathbf{m},\mathbf{n} \in \Lambda.$$

Next we formulate a similar result on almost diagonalization for arbitrary LCA groups, even when they do not contain a lattice. Once more, we take recourse to structure theory. Recall that $\mathcal{G}\times\widehat{\mathcal{G}} \simeq \mathbb{R}^{2d}\times\mathcal{G}_0\times\widehat{\mathcal{G}_0}$ with $\mathcal{G}_0\times\widehat{\mathcal{G}_0}$ containing the compact-open subgroup $\mathcal{K}\times\mathcal{K}^\perp$, and let $\mathbf{x}\to\dot{\mathbf{x}}$ be the canonical projection from $\mathcal{G}\times\widehat{\mathcal{G}}$ onto $\mathcal{G}\times\widehat{\mathcal{G}}/(\{0\}\times\mathcal{K}\times\mathcal{K}^\perp) \simeq \mathbb{R}^{2d}\times\mathcal{G}_0/\mathcal{K}\times\widehat{\mathcal{G}_0}/\mathcal{K}^\perp$. Now let $\Lambda = A\mathbb{Z}^{2d}\times D_1\times D_2$ be a quasi-lattice in $\mathcal{G}\times\widehat{\mathcal{G}}$, where D_1 is a set of representatives of $\mathcal{G}_0/\mathcal{K}$ and D_2 a set of representatives of $\widehat{\mathcal{G}_0}/\mathcal{K}^\perp$. Then by definition the projection of Λ in $\mathcal{G}\times\widehat{\mathcal{G}}/(\{0\}\times\mathcal{K}\times\mathcal{K}^\perp)$ is exactly $\dot{\Lambda} = A\mathbb{Z}^{2d}\times\mathcal{G}_0/\mathcal{K}\times\hat{\mathcal{G}}_0/\mathcal{K}^\perp$. Thus $\mathcal{D} := \dot{\Lambda}$ is a discrete abelian group. This is the correct index set for the formulation of the almost diagonalization in general LCA groups.

Finally, if v is a submultiplicative weight on $\mathcal{G}\times\widehat{\mathcal{G}}$, then the weight $\tilde{v}(\dot{\mathbf{x}}) = \sup_{\mathbf{u}\in\{0\}\times\mathcal{K}\times\mathcal{K}^\perp} v(\mathbf{x}+\mathbf{u})$ is submultiplicative on the quotient $\mathcal{G}\times\widehat{\mathcal{G}}/(\{0\}\times\mathcal{K}\times\mathcal{K}^\perp)$, and $\tilde{v}$ satisfies the GRS-condition if and only if v does.

Theorem 4.5 *Let $g \in M^1_v(\mathcal{G})$ for some admissible weight v on $\mathcal{G}\times\widehat{\mathcal{G}}$, $\Lambda \subseteq \mathcal{G}\times\widehat{\mathcal{G}}$ be a quasi-lattice, and assume that $\{\pi(\mathbf{m})g\}_{\mathbf{m}\in\Lambda}$ is a tight Gabor frame for $L^2(\mathcal{G})$. Then for $\sigma \in M^\infty(\mathcal{G}\times\widehat{\mathcal{G}})$ the following properties are equivalent.*

(i) $\sigma \in M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$.

(ii) There exists a function $H \in L^1_v(\mathcal{G}\times\widehat{\mathcal{G}})$ such that

$$|\langle K_\sigma \pi(\mathbf{z})g, \pi(\mathbf{y})g\rangle| \le H(\mathbf{y}-\mathbf{z}), \qquad \textit{for all } \mathbf{y},\mathbf{z}\in\mathcal{G}\times\widehat{\mathcal{G}}. \tag{44}$$

(iii) There exists a sequence $h_0 \in \ell^1_v(\mathcal{D})$ such that

$$|\langle K_\sigma \pi(\mathbf{n})g, \pi(\mathbf{m})g\rangle| \le h_0(\dot{\mathbf{m}}-\dot{\mathbf{n}}), \qquad \textit{for all } \mathbf{m},\mathbf{n}\in\Lambda. \tag{45}$$

Proof: The equivalence (i) $\Leftrightarrow$ (ii) does not make reference to any Gabor frame, and we have proved it already in Theorem 4.3.
(i) $\Rightarrow$ *(iii)*: Let $U = A[0,1)^{2d}\times\mathcal{K}\times\mathcal{K}^\perp$ be a fundamental domain of $\Lambda\subseteq\widehat{\mathcal{G}}\times\mathcal{G}$. Set $h(\mathbf{m}) := \sup_{\mathbf{u}\in U}\sup_{\mathbf{z}\in\widehat{\mathcal{G}}\times\mathcal{G}}|\mathcal{V}_\Psi\sigma(\mathbf{z},\mathcal{J}(\mathbf{m}+\mathbf{u}))|$ for $\mathbf{m}\in\Lambda$. Since $\sigma\in M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$ and $\Phi\in M^1_{1\otimes v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$, Corollary 2.6 implies that $h\in\ell^1_v(\Lambda)$. Since $\{0\}\times\mathcal{K}\times\mathcal{K}^\perp$ is a subgroup of $\mathcal{G}\times\widehat{\mathcal{G}}$, we find that $h(\mathbf{m}+\mathbf{u}) = h(\mathbf{m})$ for all $\mathbf{u}\in\{0\}\times\mathcal{K}\times\mathcal{K}^\perp$. Consequently we may define a function h_0 on $\mathcal{D}=\dot{\Lambda}$ unambiguously by $h_0(\dot{\mathbf{m}}) = h(\mathbf{m})$ for $\dot{\mathbf{m}}\in\mathcal{D}$. Since $h\in\ell^1_v(\Lambda)$, we have $h_0\in\ell^1_{\tilde{v}}(\mathcal{D})$.

Now we argue as above and we use Lemma 4.2(ii) and (40) to obtain

$$|\langle K_\sigma\pi(\mathbf{n})g,\pi(\mathbf{m})g\rangle| \le \sup_{\mathbf{z}\in\mathcal{G}\times\widehat{\mathcal{G}}}|\mathcal{V}_\Psi\sigma(\mathbf{z},\mathcal{J}(\mathbf{m}-\mathbf{n}))| \le h(\mathbf{m}-\mathbf{n}) = h_0(\dot{\mathbf{m}}-\dot{\mathbf{n}}) \tag{46}$$

for $\mathbf{m},\mathbf{n}\in\Lambda$. Thus (iii) is proved.
(iii) $\Rightarrow$ *(ii)*: As in the proof of Theorem 4.3 we express an arbitrary time-frequency shift $\pi(\mathbf{u})g$ as

$$\pi(\mathbf{u})g = \sum_{\mathbf{m}\in\Lambda}\langle\pi(\mathbf{u})g,\pi(\mathbf{m})g\rangle\pi(\mathbf{m})g \tag{47}$$

with respect to a tight Gabor frame $\{\pi(\mathbf{m})g\}_{\mathbf{m}\in\Lambda}$. The assumption $g\in M^1_v$ and Theorem 2.5 imply that $\mathcal{V}_g g\in W(C,\ell^1_v)(\mathcal{G}\times\widehat{\mathcal{G}})$. This means that for the fundamental domain $U = A[0,1)^{2d}\times\mathcal{K}\times\mathcal{K}^\perp$ of Λ, the sequence with entries

$$\alpha(\mathbf{n}) = \sup_{\mathbf{u}\in U}|\mathcal{V}_g g(\mathbf{n}-\mathbf{u})| = \sup_{\mathbf{u}\in U}|\langle\pi(\mathbf{u})g,\pi(\mathbf{n})g\rangle|, \quad \mathbf{n}\in\Lambda, \tag{48}$$

belongs to $\ell^1_v(\Lambda)$. As above we note that $\alpha(\mathbf{n}+\mathbf{u}) = \alpha(\mathbf{n})$ for $\mathbf{n}\in\Lambda$ and $\mathbf{u}\in\{0\}\times\mathcal{K}\times\mathcal{K}^\perp$. Thus α can be identified with a sequence $\alpha_0(\dot{\mathbf{n}}) = \alpha(\mathbf{n})$ on $\mathcal{D}$, and $\alpha_0\in\ell^1_{\tilde{v}}(\mathcal{D})$.

Now we follow the proof of Theorem 4.3. We write $\mathbf{y},\mathbf{z}\in\mathcal{G}\times\widehat{\mathcal{G}}$ in a unique form as $\mathbf{y}=\mathbf{n}+\mathbf{u}, \mathbf{z}=\mathbf{n}'+\mathbf{u}'$ for $\mathbf{n},\mathbf{n}'\in\Lambda$ and $\mathbf{u},\mathbf{u}'\in U$. Inserting the expansion (47) and the definition of α_0 in the matrix entries we obtain

$$\begin{aligned}
&|\langle K_\sigma\pi(\mathbf{n}'+\mathbf{u}')g,\pi(\mathbf{n}+\mathbf{u})g\rangle| = |\langle K_\sigma\pi(\mathbf{n}')\pi(\mathbf{u}')g,\pi(\mathbf{n})\pi(\mathbf{u})g\rangle|\\
&\le \sum_{\mathbf{m},\mathbf{m}'\in\Lambda}|\langle K_\sigma\pi(\mathbf{n}'+\mathbf{m}')g,\pi(\mathbf{n}+\mathbf{m})g\rangle||\langle\pi(\mathbf{u}')g,\pi(\mathbf{m}')g\rangle||\langle\pi(\mathbf{u})g,\pi(\mathbf{m})g\rangle|\\
&\le \sum_{\mathbf{m},\mathbf{m}'\in\Lambda} h_0(\dot{\mathbf{n}}+\dot{\mathbf{m}}-\dot{\mathbf{n}}'-\dot{\mathbf{m}}')\alpha_0(\dot{\mathbf{m}}')\alpha_0(\dot{\mathbf{m}})\\
=&(h_0*\alpha_0*\widetilde{\alpha_0})(\dot{\mathbf{n}}-\dot{\mathbf{n}}')
\end{aligned}$$

with $\widetilde{\alpha_0}(\mathbf{n}) = \alpha_0(-\dot{\mathbf{n}})$. Here it is crucial that $\mathcal{D} = \dot{\Lambda}$ is a group. Since $h_0 \in \ell^1_{\tilde{v}}(\mathcal{D})$ by hypothesis (iii) and $\alpha_0 \in \ell^1_{\tilde{v}}(\mathcal{D})$ by construction, the sequence $h_0 * \alpha_0 * \tilde{\alpha_0}$ is also in $\ell^1_{\tilde{v}}(\mathcal{D})$.

Now set

$$H(\mathbf{z}) = \sum_{\mathbf{n}\in\Lambda} (h_0 * \alpha_0 * \tilde{\alpha_0})(\dot{\mathbf{n}})\chi_{U-U}(\mathbf{z} - \mathbf{n}).$$

Since $\|T_{\mathbf{n}}\chi_{U-U}\|_{L^1_v} \le v(\mathbf{n})\|\chi_{U-U}\|_{L^1_v}$ and $\tilde{v}(\dot{\mathbf{n}}) \le Cv(\mathbf{n})$, we obtain that

$$\|H\|_{L^1_v} \le \sum_{\mathbf{n}\in\Lambda} (h_0 * \alpha_0 * \tilde{\alpha_0})(\dot{\mathbf{n}})v(\mathbf{n})\|\chi_{U-U}\|_{L^1_v} = c\|h_0 * \alpha_0 * \tilde{\alpha_0}\|_{\ell^1_{\tilde{v}}} < \infty.$$

Arguing as before, we show that

$$|\langle K_\sigma \pi(\mathbf{z})g, \pi(\mathbf{y})g\rangle| \le (h_0 * \alpha_0 * \widetilde{\alpha_0})(\dot{\mathbf{n}} - \dot{\mathbf{n}'}) \le H(\mathbf{y} - \mathbf{z}),$$

and this is (ii). ❏

Remark: Despite the resemblance of the proofs, Theorem 4.3 is not a special case of Theorem 4.5 because in the former case we consider an arbitrary lattice in $\mathcal{G} \times \widehat{\mathcal{G}}$, if it exists, whereas in the latter case we consider a very special quasi-lattice that respects the factorization of $\mathcal{G} \times \widehat{\mathcal{G}}$ as $\mathbb{R}^{2d} \times \mathcal{G}_0 \times \hat{\mathcal{G}}_0$.

5 Sjöstrand's Results on Locally Compact Abelian Groups

The characterization of almost diagonalization through time-frequency properties of the symbol leads to the generalization of Sjöstrand's results to LCA groups. Note that in no place do we resort to typical arguments from pseudodifferential operator calculus.

Boundedness on $L^2(\mathcal{G})$: First we prove that any pseudodifferential operator K_σ with a symbol in the generalized Sjöstrand class is bounded on all modulation spaces with appropriate weight. As a preparation we need a lemma on the properties of the Rihaczek distribution $R(f,g)(x,\xi) = f(x)\overline{\hat{g}(\xi)}\ \overline{\langle \xi, x\rangle}$ that generalizes [18] to LCA groups and weighted modulation spaces.

Lemma 5.1 *(i) Let $\varphi, \psi, f, g \in L^2(\mathcal{G})$ and set $\Phi = R(\varphi,\psi) \in L^2(\mathcal{G} \times \widehat{\mathcal{G}})$. Then, with $\mathbf{x} = (x,\xi) \in \mathcal{G} \times \widehat{\mathcal{G}}$, $\boldsymbol{\omega} = (\omega, u) \in \widehat{\mathcal{G}} \times \mathcal{G}$, we have*

$$\mathcal{V}_\Phi\big(R(g,f)\big)(\mathbf{x},\boldsymbol{\omega}) = \overline{\langle \xi, u\rangle}\, V_\psi g(x, \xi+\omega)\, \overline{V_\psi f(x+u,\xi)}\,. \tag{49}$$

(ii) If $f \in M^{p,q}_m(\mathcal{G})$ and $g \in M^{p',q'}_{1/m}(\mathcal{G})$,then $R(g,f) \in M^{1,\infty}_{1/v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$ and

$$\|R(g,f)\|_{M^{1,\infty}_{1/v\circ\mathcal{J}^{-1}}} \le C\|f\|_{M^{p,q}_m}\, \|g\|_{M^{p',q'}_{1/m}}\,. \tag{50}$$

Proof: The proof is similar to [18]. We write the time-frequency shifts of the Rihaczek distribution explicitly as

$$M_{\boldsymbol{\omega}} T_{\mathbf{x}} R(\varphi,\psi)(t,\tau) = \langle \omega, t\rangle \langle \tau, u\rangle\, \varphi(t-x)\, \overline{\hat{\psi}(\tau-\xi)\,\langle \tau-\xi, t-x\rangle}\,.$$

Consequently, after a substitution,

$$\begin{aligned}
\mathcal{V}_\Phi R(g,f)(\mathbf{x},\boldsymbol{\omega}) &= \langle R(g,f), M_{\boldsymbol{\omega}} T_{\mathbf{x}} R(\varphi,\psi)\rangle \\
&= \iint_{\mathcal{G}\times\widehat{\mathcal{G}}} g(t)\overline{\hat{f}(\tau)}\,\overline{\langle \tau, t\rangle}\,\overline{\varphi(t-x)}\,\hat{\psi}(\tau-\xi)\,\langle \tau-\xi, t-x\rangle\,\overline{\langle \omega, t\rangle\langle \tau, u\rangle}\; dt d\tau \\
&= \langle \xi, x\rangle \int_{\mathcal{G}} g(t)\overline{\varphi(t-x)}\,\overline{\langle \xi+\omega, t\rangle}\, dt \;\cdot\; \int_{\widehat{\mathcal{G}}} \overline{\hat{f}(\tau)}\,\hat{\psi}(\tau-\xi)\,\langle \tau, -x-u\rangle\, d\tau \\
&= \langle \xi, x\rangle\, V_\varphi g(x, \xi+\omega)\, \overline{V_{\hat{\psi}}\hat{f}(\xi, -x-u)} \\
&= \overline{\langle \xi, u\rangle}\, V_\varphi g(x, \xi+\omega)\, \overline{V_\psi f(x+u, \xi)}\,.
\end{aligned}$$

In the last transformation we have used the fundamental formula (7). Since both $R(g,f)$ and $R(\varphi,\psi)$ are in $L^2(\mathcal{G}\times\widehat{\mathcal{G}})$, the integral defining $\mathcal{V}_\Phi R(g,f)$ is absolutely convergent on $\mathcal{G}\times\widehat{\mathcal{G}}$, and so the application of Fubini's theorem is justified.

(b) is a consequence of (a). For simplicity we use the window $\Phi = R(\varphi,\varphi)$ and use the fact that different windows in $M^1_v(\mathcal{G})$ yield equivalent norms on $M^{p,q}_m(\mathcal{G})$ (Lemma 2.3). Consequently,

$$\begin{aligned}
\|R(g,f)\|_{M^{1,\infty}_{1/v\circ\mathcal{J}^{-1}}} &= \sup_{\boldsymbol{\omega}\in\widehat{\mathcal{G}}\times\mathcal{G}} \frac{1}{v(\mathcal{J}^{-1}\boldsymbol{\omega})} \int_{\mathcal{G}\times\widehat{\mathcal{G}}} |V_\Phi(R(g,f)(\mathbf{x},\boldsymbol{\omega})|\, d\mathbf{x} \\
&= \sup_{(\omega,u)\in\widehat{\mathcal{G}}\times\mathcal{G}} \frac{1}{v(u,-\omega)} \iint_{\mathcal{G}\times\widehat{\mathcal{G}}} |V_\varphi f(x+u,\xi)|\,|V_\varphi g(x,\xi+\omega)|\, dx d\xi = (*)
\end{aligned}$$

Since $m(x,\xi+\omega) \le Cv(-u,\omega)m(x+u,\xi)$ by (11) and since v is even, we can continue the estimate by

$$\begin{aligned}
(*) &\le C \sup_{(\omega,u)\in\widehat{\mathcal{G}}\times\mathcal{G}} \iint_{\mathcal{G}\times\widehat{\mathcal{G}}} |V_\varphi f(x+u,\xi)| m(x+u,\xi)\, |V_\varphi g(x,\xi+\omega)| \frac{1}{m(x,\xi+\omega)}\, dx d\xi \\
&\le C \sup_{(\omega,u)\in\widehat{\mathcal{G}}\times\mathcal{G}} \|V_\varphi f\, m\|_{L^{p,q}}\, \|V_\varphi g\, m^{-1}\|_{L^{p',q'}} \\
&= C\|f\|_{M^{p,q}_m}\, \|g\|_{M^{p',q'}_{1/m}}\,,
\end{aligned}$$

where in the last step we have applied Hölder's inequality. ❏

We are now ready to prove that operators in the Sjöstrand class are bounded on modulation spaces.

Theorem 5.2 *Let v be an admissible weight on $\mathcal{G}\times\widehat{\mathcal{G}}$. If $\sigma \in M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}})$, then K_σ is bounded on all modulation spaces $M^{p,q}_m(\mathcal{G})$ for $1\le p,q\le\infty$ and every v-moderate weight m. In particular, K_σ is bounded on $L^2(\mathcal{G})$.*

Proof: We apply the duality (14) and Lemma 5.1(ii) to obtain

$$\begin{aligned}|\langle K_\sigma f, g\rangle| &= |\langle \sigma, R(g,f)\rangle| \\ &\leq C\,\|\sigma\|_{M^{\infty,1}_{v\circ\mathcal{J}^{-1}}}\,\|R(g,f)\|_{M^{1,\infty}_{1/v\circ\mathcal{J}^{-1}}} \\ &\leq C'\,\|\sigma\|_{M^{\infty,1}_{v\circ\mathcal{J}^{-1}}}\,\|f\|_{M^{p,q}_m}\,\|g\|_{M^{p',q'}_{1/m}}\,.\end{aligned}$$

Since this inequality holds for *all* $g \in M^{p',q'}_{1/m}(\mathcal{G}) = (M^{p,q}_m(\mathcal{G}))^*$, we have shown that

$$\|K_\sigma f\|_{M^{p,q}_m} \leq C\,\|\sigma\|_{M^{\infty,1}_{v\circ\mathcal{J}^{-1}}}\,\|f\|_{M^{p,q}_m}\,.$$

If $(p,q) = (1,\infty)$ or $(p,q) = (\infty,1)$, we observe that these spaces are also dual spaces of a modulation space [3], thus we have proved the boundedness of K_σ for all parameters $p,q \in [1,\infty]$. For $p = q = 2$ and $m \equiv 1$, we obtain the boundedness on $M^{2,2}(\mathcal{G}) = L^2(\mathcal{G})$. ❏

The Banach Algebra Property. Whereas the boundedness property uses typical arguments from time-frequency analysis, the Banach algebra property lies much deeper and requires the characterization of the generalized Sjöstrand class via almost diagonalization.

Recall that $\mathcal{C}_v(\mathcal{D})$ is the Banach algebra of all matrices on the index set $\mathcal{D}$ that are dominated by convolution operators in $\ell^1_v(\mathcal{D})$.

Theorem 5.3 *The space $\mathcal{A} = \mathrm{Op}(M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G}\times\widehat{\mathcal{G}}))$ is a Banach algebra with respect to the composition of operators and with the norm $\|K_\sigma\|_{\mathcal{A}} := \|M(\sigma)\|_{\mathcal{C}_v} \asymp \|\sigma\|_{M^{\infty,1}_{v\circ\mathcal{J}^{-1}}}$.*

Proof: Let $\{M_\mu T_m g\}_{(m,\mu)\in\Lambda}$ be a tight Gabor frame with $g \in M^1_v(\mathcal{G})$ with respect to a quasi-lattice $\Lambda \subseteq \mathcal{G}\times\widehat{\mathcal{G}}$. Recall that the projection of Λ into a quotient of $\mathcal{G}\times\widehat{\mathcal{G}}$ results in the discrete abelian group $\mathcal{D}$. We may assume without loss of generality that g is normalized such that its (lower and upper) frame bound is 1. Using Lemma 3.2 we compute

$$\begin{aligned}M(\mathcal{F}^{-1}(\hat\sigma\,\natural\,\hat\tau))C_g f &= C_g(K_{\mathcal{F}^{-1}(\hat\sigma\,\natural\,\hat\tau)}f) = C_g(K_\sigma K_\tau f) \\ &= M(\sigma)C_g(K_\tau f) = M(\sigma)M(\tau)C_g f.\end{aligned} \tag{51}$$

This means that $M(\mathcal{F}^{-1}(\hat\sigma\,\natural\,\hat\tau))$ and $M(\sigma)M(\tau)$ coincide on $\mathrm{ran}(C_g)$. Since K_σ, K_τ, and $K_{\mathcal{F}^{-1}(\hat\sigma\,\natural\,\hat\tau)}$ are all bounded on $L^2(\mathcal{G})$, it follows from Lemma 3.4 that $\ker(M(\mathcal{F}^{-1}(\hat\sigma\,\natural\,\hat\tau))) = \ker(M(\sigma)M(\tau))$. Hence

$$M(\mathcal{F}^{-1}(\hat\sigma\,\natural\,\hat\tau)) = M(\sigma)M(\tau) \tag{52}$$

on $\ell^2(\Lambda)$.

Now let $\sigma, \tau \in M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G} \times \widehat{\mathcal{G}})$. Then, by Theorem 4.5 $M(\sigma) \in \mathcal{C}_v(\mathcal{D})$ and $M(\tau) \in \mathcal{C}_v(\mathcal{D})$. Since $\mathcal{C}_v(\mathcal{D})$ is a Banach algebra, the product $M(\sigma)M(\tau)$ is also in $\mathcal{C}_v(\mathcal{D})$, and furthermore

$$\|K_{\mathcal{F}^{-1}(\hat{\sigma}\,\natural\,\hat{\tau})}\|_{\mathcal{A}} = \|M(\mathcal{F}^{-1}(\hat{\sigma}\,\natural\,\hat{\tau}))\|_{\mathcal{C}_v} \leq \|M(\sigma)\|_{\mathcal{C}_v} \|M(\tau)\|_{\mathcal{C}_v} = \|K_\sigma\|_{\mathcal{A}} \|K_\tau\|_{\mathcal{A}}\,.$$

By Theorem 4.5 we conclude that $\mathcal{F}^{-1}(\hat{\sigma}\,\natural\,\hat{\tau})) \in M^{\infty,1}_{v\circ\mathcal{J}^{-1}}$ and that $K_\sigma K_\tau \in \mathcal{A} = \mathrm{Op}(M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G} \times \widehat{\mathcal{G}}))$. ❑

The Wiener Property: We now state the main result of this paper, the Wiener property of the generalized Sjöstrand class. This is the deepest theorem of this paper and requires the combination of all methods developed so far, namely the almost diagonalization, the Wiener property of the matrix algebra $\mathcal{C}_v$, and the existence and properties of tight Gabor frames.

Theorem 5.4 *Let v be an admissible weight. If $\sigma \in M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G} \times \widehat{\mathcal{G}})$ and if K_σ is invertible on $L^2(\mathcal{G})$, then $(K_\sigma)^{-1} = K_\tau$ for some $\tau \in M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G} \times \widehat{\mathcal{G}})$.*

Proof: As in the proof of Theorem 5.3 we use a tight Gabor frame $\{M_\mu T_m g\}_{(m,\mu)\in\Lambda}$ with $g \in M^1_v$ and with (lower and upper) frame bounds equal to 1. Let τ be the unique distribution such that $(K_\sigma)^{-1} = K_\tau$. By Lemma 3.4 we have that the matrix $M(\tau, g, \Lambda) = M(\tau)$ is bounded on $\ell^2(\Lambda)$ and maps $\mathrm{ran}(C_g)$ into $\mathrm{ran}(C_g)$ with $\ker(C_g^*) \subseteq \ker(M(\tau))$.

We show that $M(\tau)$ is the pseudoinverse of $M(\sigma)$. Let $c = C_g f \in \mathrm{ran}(C_g)$, then

$$M(\tau)M(\sigma)C_g f = M(\tau)C_g(K_\sigma f) = C_g(K_\tau K_\sigma f) = C_g f, \tag{53}$$

where we have used (31) and the property that $\{M_\mu T_m g\}_{(m,\mu)\in\Lambda}$ is a tight frame. Relation (53) says that $M(\tau)M(\sigma) = I$ on $\mathrm{ran}(C_g)$. Furthermore, $\ker(M(\sigma))$, $\ker(M(\tau)) \supseteq \mathrm{ran}(C_g)^\perp$, thus we conclude that $M(\tau)$ is the pseudoinverse of $M(\sigma)$.

By Theorem 4.5 the property $\sigma \in M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G} \times \widehat{\mathcal{G}})$ implies that $M(\sigma) \in \mathcal{C}_v(\mathcal{D})$. Applying Corollary 2.10 to $M(\sigma)$, we deduce that $M(\tau) = M(\sigma)^+ \in \mathcal{C}_v(\mathcal{D})$. Using Theorem 4.5 once more, we have shown that $K_\tau \in M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G} \times \widehat{\mathcal{G}})$. ❑

Spectral invariance on modulation spaces: According to Theorem 5.4 the inverse K_σ^{-1} has again a symbol in $M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G} \times \widehat{\mathcal{G}})$. Consequently, by Theorem 5.2 K_σ^{-1} acts boundedly on a large class of modulation spaces depending only on the class of the weight v.

Corollary 5.5 *Let v be an admissible weight. If $\sigma \in M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathcal{G} \times \widehat{\mathcal{G}})$ and if K_σ is invertible on $L^2(\mathcal{G})$, then K_σ is invertible simultaneously on all modulation spaces $M^{p,q}_m(\mathcal{G})$ for $1 \leq p, q \leq \infty$ and v-moderate weight m.*

6 Special Groups

6.1 Sjöstrand's Class

Clearly, when we choose $\mathcal{G} = \mathbb{R}^d$ (whence $\widehat{\mathcal{G}} = \widehat{\mathbb{R}^d} = \mathbb{R}^d$) in the derivations of the previous sections, then we recover Sjöstrand's results [36, 37]. The additional insight gained from our approach is the identification of the cornerstones of Sjöstrand's result, namely modulation spaces and the corresponding time-frequency techniques, the striking appearance of certain matrix algebras and their spectral invariance, and the almost diagonalization by Gabor frames.

6.2 Discrete Pseudodifferential Operators

Let us consider the case $\mathcal{G} = \mathbb{Z}$, $\widehat{\mathcal{G}} = \mathbb{T}$. Thus K_σ, now acting on $\ell^2(\mathbb{Z})$, becomes a discrete pseudodifferential operator. We refer to [29] for a detailed review of discrete pseudodifferential operators. Of course, the action of K_σ can be described simply by a matrix. The next lemma elucidates the relation between the symbol class $M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathbb{Z}\times\mathbb{T})$, the "discrete Sjöstrand class", and the corresponding class of matrices. A similar calculation was made in [13].

Lemma 6.1 *Let K_σ be a pseudodifferential operator defined on $\ell^2(\mathbb{Z})$ and let v be a weight function on $\mathbb{Z}$. Then the matrix corresponding to K_σ is in $\mathcal{C}_v(\mathbb{Z})$ if and only if $\mathcal{T}_2\sigma \in M^{\infty,1}_{(v\otimes 1)\circ\mathcal{J}^{-1}}(\mathbb{Z}\times\mathbb{T})$, where $\mathcal{T}_2\sigma(x,\xi) := \sigma(x,-\xi)$.*

In this special case Theorem 5.4 coincides with Theorem 2.9 of Gohberg et al. and Baskakov. However, our proof of Theorem 5.4 does not give a new proof of Theorem 2.9, because we have used the Gohberg-Baskakov result in the proof of the general theorem.

Proof: We begin by calculating $\|\mathcal{T}_2\sigma\|_{M^{\infty,1}_{v\circ\mathcal{J}^{-1}}(\mathbb{Z}\times\mathbb{T})}$. We first compute $\mathcal{V}_\Psi\sigma$ for an appropriate window Ψ in $M^1_{v\otimes 1}(\mathbb{Z}\times\mathbb{T})$. We choose $\Psi = \delta\otimes\mathbf{1}$. Since Ψ is the characteristic function of the compact-open subgroup $\{0\}\times\mathbb{T}$ in $\mathbb{Z}\times\mathbb{T}$, its STFT $V_\Psi\Psi$ is integrable with respect to every submultiplicative weight v by (12), and thus $\Psi\in M^1_{v\otimes 1}(\mathbb{Z}\times\mathbb{T})$ for every submultiplicative v on $\mathbb{Z}$.

Let $\mathbf{x} = (x,\xi)\in\mathbb{Z}\times\mathbb{T}, \boldsymbol{\omega} = (\omega,u)\in\mathbb{T}\times\mathbb{Z}$. We compute

$$
\begin{aligned}
\mathcal{V}_\Psi\mathcal{T}_2\sigma(\mathbf{x},\boldsymbol{\omega}) &= \sum_{z\in\mathbb{Z}}\int_{\mathbb{T}} \mathcal{T}_2\sigma(z,\zeta)\overline{M_{\boldsymbol{\omega}}T_{\mathbf{x}}\Psi(z,\zeta)}\,d\zeta \\
&= \sum_{z\in\mathbb{Z}}\int_{\mathbb{T}} \sigma(z,-\zeta)\delta(z-x)\mathbf{1}(\zeta-\xi)e^{-2\pi i\zeta u}e^{-2\pi i z\omega}\,d\zeta \\
&= e^{-2\pi i x\omega}\int_{\mathbb{T}} \sigma(x,-\zeta)e^{-2\pi i\zeta u}\,d\zeta \\
&= e^{-2\pi i x\omega}\mathcal{F}_2^{-1}\sigma(x,u).
\end{aligned}
$$

Consequently, since $(v \otimes 1)(\mathcal{J}^{-1}(\omega, u)) = (v \otimes 1)(u, -\omega) = v(u)$, we obtain

$$\begin{aligned}
\|\mathcal{T}_2\sigma\|_{M^{\infty,1}_{(v\otimes 1)\circ\mathcal{J}^{-1}}} &= \sum_{u\in\mathbb{Z}} \int_{\mathbb{T}} \sup_{\mathbf{x}} |\mathcal{V}_\Psi \mathcal{T}_2\sigma(\mathbf{x}, \boldsymbol{\omega})| \, (v \otimes 1)(\mathcal{J}^{-1})(\omega, u)) \, d\omega \\
&= \sum_{u\in\mathbb{Z}} \int_{\mathbb{T}} \sup_{x,\xi} |e^{-2\pi i x\omega} \mathcal{F}_2^{-1}\sigma(x, u)| \, v(u) \, d\omega \\
&= \sum_{u\in\mathbb{Z}} \sup_{x\in\mathbb{Z}} |\mathcal{F}_2^{-1}\sigma(x, u)| \, v(u). \qquad (54)
\end{aligned}$$

Next we compute $K_\sigma f(x)$.

$$\begin{aligned}
K_\sigma f(x) &= \int_{\mathbb{T}} \sigma(x, \xi) \hat{f}(\xi) e^{2\pi i x\xi} \, d\xi \\
&= \sum_{u\in\mathbb{Z}} \Big(\int_{\mathbb{T}} \sigma(x, \xi) e^{-2\pi i(u-x)\xi} d\xi \Big) f(u) \\
&= \sum_{u\in\mathbb{Z}} \mathcal{F}_2\sigma(x, u - x) f(u) \\
&= \sum_{u\in\mathbb{Z}} \mathcal{F}_2^{-1}\sigma(x, x - u) f(u) \\
&= Af \, . \qquad (55)
\end{aligned}$$

This means that the matrix A corresponding to K_σ has the entries

$$A_{x,u} := \mathcal{F}_2^{-1}\sigma(x, x - u), \qquad x, u \in \mathbb{Z}. \qquad (56)$$

Comparing (54), (55) and (56), we find that the assumption $\mathcal{T}_2\sigma \in M^{\infty,1}_{(v\otimes 1)\circ\mathcal{J}^{-1}}(\mathbb{Z} \times \mathbb{T})$ yields

$$\sum_{u\in\mathbb{Z}} \sup_{x\in\mathbb{Z}} |\mathcal{F}_2\mathcal{T}_2\sigma(x, u)| v(u) < \infty \Longleftrightarrow \sum_{u\in\mathbb{Z}} \sup_{x\in\mathbb{Z}} |A_{x,x-u}| v(u) < \infty \, .$$

Comparing with Definition 2.8, we have shown that the matrix A corresponding to K_σ is in $\mathcal{C}_v(\mathbb{Z})$. ❏

6.3 Periodic Pseudodifferential Operators

We consider periodic pseudodifferential operators, cf. e.g. [25, 34]. In this case $\mathcal{G} = \mathbb{T}$, $\widehat{\mathcal{G}} = \mathbb{Z}$ and thus the symbol is defined on $\mathbb{T} \times \mathbb{Z}$. Analogous to the previous section we will first analyze $M^{\infty,1}_{(1\otimes v)\circ\mathcal{J}^{-1}}(\mathbb{T} \times \mathbb{Z})$.

We first compute $\mathcal{V}_\Psi\sigma$ for the window $\Psi = \mathbf{1} \otimes \delta$. As before $\Psi \in M^1_{1\otimes v}(\mathbb{T} \times \mathbb{Z})$ for any submultiplicative weight on $\mathbb{Z}$.

Let $\mathbf{x} = (x, \xi) \in \mathbb{T} \times \mathbb{Z}, \boldsymbol{\omega} = (\omega, u) \in \mathbb{Z} \times \mathbb{T}$. We compute

$$\mathcal{V}_\Psi \sigma(\mathbf{x}, \boldsymbol{\omega}) = \int_{\mathbb{T}} \sum_{\zeta \in \mathbb{Z}} \sigma(z, \zeta) \overline{M_{\boldsymbol{\omega}} T_{\mathbf{x}} \Psi(z, \zeta)} \, dz \tag{57}$$

$$= \int_{\mathbb{T}} \sum_{\zeta \in \mathbb{Z}} \sigma(z, \zeta) \mathbf{1}(z - x) \, \delta(\zeta - \xi) \, e^{-2\pi i z \omega} e^{-2\pi i \zeta u} \, dz \tag{58}$$

$$= e^{-2\pi i \xi u} \int_{\mathbb{T}} \sigma(z, \xi) e^{-2\pi i z \omega} \, dz \tag{59}$$

$$= e^{-2\pi i \xi u} \mathcal{F}_1 \sigma(\omega, \xi). \tag{60}$$

Consequently, since $(1 \otimes v)(\mathcal{J}^{-1}(\omega, u)) = (1 \otimes v)(u, -\omega) = v(-\omega) = v(\omega)$, we have

$$\|\sigma\|_{M^{\infty,1}_{(1\otimes v)\circ \mathcal{J}^{-1}}} = \int_{\mathbb{T}} \sum_{\omega \in \mathbb{Z}} \sup_{\mathbf{x}} |\mathcal{V}_\Psi \sigma(\mathbf{x}, \boldsymbol{\omega})| (1 \otimes v)(\mathcal{J}^{-1}(\boldsymbol{\omega})) \, du \tag{61}$$

$$= \sum_{\omega \in \mathbb{Z}} \sup_{\xi \in \mathbb{Z}} |\mathcal{F}_1 \sigma(\omega, \xi)| v(\omega). \tag{62}$$

Next we compute the Fourier transform of $K_\sigma f$.

$$\begin{aligned}(\mathcal{F} K_\sigma f)(\xi) &= \mathcal{F}\Big(\sum_{\omega \in \mathbb{Z}} \sigma(x, \omega) \hat{f}(\omega) e^{2\pi i x \omega} \Big)(\xi) \\ &= \int_{\mathbb{T}} \sum_{\omega \in \mathbb{Z}} \sigma(x, \omega) \hat{f}(\omega) e^{2\pi i x(\omega - \xi)} dx \\ &= \sum_{\omega \in \mathbb{Z}} \mathcal{F}_1 \sigma(\xi - \omega, \omega) \hat{f}(\omega) \,. \end{aligned} \tag{63}$$

Let A be the matrix with entries $A_{\xi,\omega} = \mathcal{F}_1 \sigma(\xi - \omega, \omega)$, $\xi, \omega \in \mathbb{Z}$. Then

$$A \hat{f} = \mathcal{F} K_\sigma \mathcal{F}^{-1} \hat{f} \,,$$

in other words, A describes the action of K_σ on the Fourier coefficients $\hat{f}$ of f. Using (62) and (63) we see that

$$\begin{aligned} \|A\|_{\mathcal{C}_v} &= \sum_{\omega \in \mathbb{Z}} \sup_{\xi \in \mathbb{Z}} |A_{\xi, \xi - \omega}| v(\omega) \\ &= \sum_{\omega \in \mathbb{Z}} \sup_{\xi \in \mathbb{Z}} |\mathcal{F}_1 \sigma(\omega, \xi - \omega)| \, v(\omega) \\ &= \sum_{\omega \in \mathbb{Z}} \sup_{\xi \in \mathbb{Z}} |\mathcal{F}_1 \sigma(\omega, \xi)| \, v(\omega) \\ &= \|\sigma\|_{M^{\infty,1}_{(1\otimes v)\circ \mathcal{J}^{-1}}} \,. \end{aligned}$$

So we have shown that $\mathcal{F} K_\sigma \mathcal{F}^{-1} \in \mathcal{C}_v(\mathbb{Z})$ if and only if $\sigma \in M^{\infty,1}_{(1\otimes v)\circ \mathcal{J}^{-1}}(\mathbb{T} \times \mathbb{Z})$.

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