On the Product of Localization Operators

Elena Cordero and Karlheinz Gröchenig

Abstract. We provide examples of the product of two localization operators. As a special case, we study the composition of Gabor multipliers. The results highlight the instability of this product and underline the necessity of expressing it in terms of asymptotic expansions.

We study the problem of calculating or estimating the product of two localization operators. The motivation comes either from signal analysis or pseudodifferential operator theory. On the one hand, in signal analysis the problem of finding a filter that has the same effect as two filters arranged in series amounts to the computation of the product of two localization operators, see [8, 9] and references therein.

On the other side, composition of pseudodifferential operators by means of symbolic calculus gives rise to asymptotic expansions, mainly employed in PDEs (see e.g., [14, 16]). Outcomes are regularity properties of partial differential operators and the construction of an approximate inverse (so-called parametrix).

Since localization operators are a sub-class of pseudodifferential operators, looking for asymptotic expansions of the localization operator product appears to be natural as well. Applications can be found in the framework of PDEs and Quantum Mechanics [1, 6, 7, 15].

In this paper, we survey the known approaches to this problem and provide concrete examples of the composition of localization operators. Indeed, very few cases allow the product to be written as a localization operator as well, consequently the class of localization operators is not closed under composition. Thus the product is unstable with respect to composition. This instability highlights the importance of a symbolic calculus for localization operators [6].

We present localization operators using language and tools from time-frequency analysis. First, the definition of the short-time Fourier transform is required.

Given a function f on $\mathbb{R}^d$ and a point (x,ω) of the phase space $\mathbb{R}^{2d}$, the operators of translation and modulation are defined to be

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i\omega t} f(t)\,. \tag{1}$$

1991 *Mathematics Subject Classification.* Primary 35S05; Secondary 47G30.

Key words and phrases. Localization operators, modulation spaces, time-frequency analysis, Weyl calculus, Wigner distribution, short-time Fourier transform.

K. G. was supported by the Marie-Curie Excellence Grant MEXT-CT 2004-517154.

We often combine translations and modulations into *time-frequency shift* (phase-space shifts in physical terminology). Set $z = (x, \omega) \in \mathbb{R}^{2d}$, then the general time-frequency shift is defined by

$$\pi(z) = M_\omega T_x. \tag{2}$$

Associated to time-frequency shifts is an important time-frequency representation, the *short-time Fourier transform* (STFT), also well-known as coherent state transform, Gabor transform and windowed Fourier transform. The STFT of a function or distribution f with respect to a fixed non-zero *window* function g is given by

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\,\overline{g(t-x)}\,e^{-2\pi i\omega t}\,dt = \langle f, M_\omega T_x g\rangle = \langle f, \pi(z) g\rangle\,, \tag{3}$$

whenever the integral or the brackets $\langle \cdot, \cdot \rangle$ (expressing a sesquilinear form) are well-defined. The short-time Fourier transform can be defined on many pairs of distribution spaces and test functions. For instance, $V_g f$ maps $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$ and $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^{2d})$. Furthermore, $V_g f$ can be extended to a map from $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^{2d})$ [12, p. 41]. The short-time Fourier transform is the appropriate tool for defining localization operators, as we shall see presently.

Let a be a *symbol* on the time-frequency plane $\mathbb{R}^{2d}$ and choose two windows φ_1, φ_2 on $\mathbb{R}^d$, then the localization operator $A_a^{\varphi_1,\varphi_2}$ is defined as

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2(t)\,dxd\omega\,. \tag{4}$$

Taking the inner product with a test function g, the definition of $A_a^{\varphi_1,\varphi_2}$ can be written in a weak sense, namely,

$$\langle A_a^{\varphi_1,\varphi_2} f, g\rangle = \langle a V_{\varphi_1} f, V_{\varphi_2} g\rangle = \langle a, \overline{V_{\varphi_1} f}\, V_{\varphi_2} g\rangle. \tag{5}$$

If $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then (5) is a well-defined continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. If $\varphi_1(t) = \varphi_2(t) = 2^{d/4} e^{-\pi t^2}$, then $A_a = A_a^{\varphi_1,\varphi_2}$ is well-known as (anti-)Wick operator and the mapping $a \to A_a^{\varphi_1,\varphi_2}$ is interpreted as a quantization rule [2, 8, 15, 16, 19].

Both the exact and the asymptotic product of localization operators rely upon the connection with the Weyl calculus. Namely, a localization operator $A_a^{\varphi_1,\varphi_2}$ can be represented as a Weyl transform. Here we need to refer to another time-frequency representation, the cross-Wigner distribution $W(g, f)$ of the functions g, f defined below (18).
The Weyl transform L_σ of $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ is then defined by

$$\langle L_\sigma f, g\rangle = \langle \sigma, W(g,f)\rangle, \qquad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{6}$$

Every continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ can be represented as a Weyl transform, and a calculation in [3, 11, 16] reveals that

$$A_a^{\varphi_1,\varphi_2} = L_{a * W(\varphi_2,\varphi_1)}, \tag{7}$$

so, the (Weyl) symbol of $A_a^{\varphi_1,\varphi_2}$ is given by

$$\sigma = a * W(\varphi_2, \varphi_1)\,. \tag{8}$$

The composition of the Weyl operators L_σ and L_τ, with symbols σ and τ belonging to suitable function spaces, can be expressed in the Weyl form [11, Chap. 2.3]

$$L_\sigma L_\tau = L_{\sigma\sharp\tau}, \tag{9}$$

where $\sigma\sharp\tau$, the twisted multiplication of the symbols σ and τ, is given by

$$\sigma\sharp\tau(\zeta) = 2^{2d} \iint_{\mathbb{R}^{2d}} \sigma(z)\tau(w) e^{4\pi i[\zeta-w,\zeta-z]}\, dzdw \tag{10}$$

and the brackets $[\cdot,\cdot]$ express the *symplectic form* on $\mathbb{R}^{2d}$

$$[(z_1,z_2),(\zeta_1,\zeta_2)] = z_1\zeta_2 - z_2\zeta_1, \qquad z=(z_1,z_2),\ \zeta=(\zeta_1,\zeta_2)\in\mathbb{R}^{2d}.$$

Thus, the composition of Weyl operators, whenever possible, defines a bilinear form (the twisted multiplication) on the corresponding symbols.

The product of localization operators will be studied along the following steps in Sections 3–5:
(i) Rewrite the two localization operators in terms of Weyl transforms (7);
(ii) Use the product formula for Weyl symbols (9), (10) to compute the Weyl symbol of their product;
(iii) Express, whenever possible, the resulting operator as a localization operator. In view of (8) this amounts to a deconvolution problem.

Notation. We define $t^2 = t\cdot t$, for $t\in\mathbb{R}^d$, and $xy = x\cdot y$ is the scalar product on $\mathbb{R}^d$. The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. We use the brackets $\langle f,g\rangle$ to denote the extension to $\mathcal{S}'(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)$ of the inner product $\langle f,g\rangle = \int f(t)\overline{g(t)}dt$ on $L^2(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\omega) = \mathcal{F}f(\omega) = \int f(t)e^{-2\pi i t\omega}dt$. Given a continuous positive function (so-called weight function) m and $1\le p\le\infty$, we define as $L^p_m(\mathbb{R}^d)$ the space of all (Lebesgue) measurable functions on $\mathbb{R}^d$ such that the norm $\|f\|_{L^p_m} := \left(\int_{\mathbb{R}^d}|f(x)|^p m(x)^p\,dx\right)^{1/p}$ is finite (obvious changes for $p=\infty$). For $1\le p,q\le\infty$, the mixed-norm space $L^{p,q}(\mathbb{R}^{2d})$ is the Banach space of all (Lebesgue) measurable functions on $\mathbb{R}^{2d}$ satisfying

$$\|F\|_{L^{p,q}} := \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|F(x,\omega)|^p\,dx\right)^{1/p} d\omega\right)^{1/q},$$

again with obvious modifications whether $p=\infty$ or $q=\infty$.

Throughout the paper, we shall use the notation $A \lesssim B$ to indicate $A \leq cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ if $A \leq cB$ and $B \leq kA$, for suitable $c, k > 0$.

1. Product Formulae

In this section we briefly review two approaches to handle the product of localization operators.

1.1. Exact Product

We reformulate the result of [8, 9] according to the notation of [11, 12].

We consider the window functions $\varphi_1(t) = \varphi_2(t) = \varphi(t) = 2^{d/4} e^{-\pi t^2}$, $t \in \mathbb{R}^d$. In this case, the Wigner distribution of the Gaussian φ is a Gaussian as well. Precisely, we have

$$W(\varphi, \varphi)(z) = 2^d e^{-2\pi z^2}, \quad z \in \mathbb{R}^{2d}. \tag{11}$$

If we compute the Weyl symbol σ of the operator $A_a^{\varphi,\varphi}$ we obtain $\sigma(\zeta) = 2^d (a * e^{-2\pi z^2})(\zeta)$, $z, \zeta \in \mathbb{R}^{2d}$. In order to express the product in the form of a localization operator, we rewrite the factors in a Weyl form, as detailed in the end of the previous section. Secondly, we come back to localization operators by means of relation (8). Given the Weyl symbols $\sigma, \tau \in \mathcal{S}(\mathbb{R}^{2d})$, we are interested in the Fourier transform of the twisted multiplication $\sigma \sharp \tau$, that is

$$\mathcal{F}(\sigma \sharp \tau)(\zeta) = \hat{\sigma} \natural \hat{\tau}(\zeta),$$

where the twisted convolution $\natural$ is given by

$$\hat{\sigma} \natural \hat{\tau}(\zeta) = \iint_{\mathbb{R}^{2d}} \hat{\sigma}(z) \hat{\tau}(\zeta - z) e^{\pi i [z, \zeta]} \, dz. \tag{12}$$

For any $f, g \in \mathcal{S}(\mathbb{R}^{2d})$, we define the $\natural^\flat$ product by

$$f \natural^\flat g(\zeta) = \iint_{\mathbb{R}^{2d}} f(z) g(z - \zeta) e^{\pi(z\zeta + i[z,\zeta])} \, e^{-\pi z^2} \, dz, \tag{13}$$

then the product of localization operators is given by the following formula.

Theorem 1.1. *Let $a, b \in \mathcal{S}(\mathbb{R}^{2d})$. If there exists a symbol $c \in \mathcal{S}'(\mathbb{R}^{2d})$ so that*

$$\hat{c} = 2^{-2d} \hat{a} \natural^\flat \hat{b}, \tag{14}$$

then we have

$$A_a^{\varphi,\varphi} A_b^{\varphi,\varphi} = A_c^{\varphi,\varphi}.$$

The proof is a straightforward consequence of relations (8) and (12). Indeed, one rewrites $A_a^{\varphi,\varphi} A_b^{\varphi,\varphi}$ in Weyl form and uses relation (12) for the product. The result is the Weyl operator L_μ, where the Fourier transform of μ is given by

$$\begin{aligned}\hat{\mu}(\zeta) &= [\mathcal{F}(a * W(\varphi,\varphi))\natural\mathcal{F}(b * W(\varphi,\varphi))](\zeta)\\ &= 2^{-2d}\iint_{\mathbb{R}^{2d}} \hat{a}(z)\hat{b}(z-\zeta)e^{-(\pi/2)z^2}e^{-(\pi/2)(\zeta-z)^2}e^{\pi i[z,\zeta]}\,dz\\ &= 2^{-2d}e^{-(\pi/2)\zeta^2}\iint_{\mathbb{R}^{2d}} \hat{a}(z)\hat{b}(z-\zeta)e^{\pi(z\zeta+i[z,\zeta])}\,e^{-\pi z^2}\,dz.\end{aligned}$$

Hence, we have

$$\hat{\mu}(\zeta) = \hat{c}(\zeta)(e^{-(\pi/2)\zeta^2}) = \mathcal{F}(c * W(\varphi,\varphi))(\zeta),$$

where $\hat{c}$ is given by relation (14).

The stability of the product when localization symbols live in the linear space spanned by the Gaussian functions is proven in [9, Thm. 2.1]. Here we reformulate the result using our terminology and we shall prove it using the Weyl connection and the twisted multiplication, instead of the $\natural^b$ one (Section 5).

Theorem 1.2. *Let $\varphi_i(t) = \varphi(t) = 2^{d/4}e^{-\pi t^2}$, $i = 1,\dots,4$, $t \in \mathbb{R}^d$. Consider the symbols*

$$a(z) = \sum_{k=1}^{m} C_k e^{-2\pi d_k z^2}, \quad b(z) = \sum_{j=1}^{l} C'_j e^{-2\pi d'_j z^2}, \quad z \in \mathbb{R}^{2d}, \tag{15}$$

where $C_1,\dots,C_m; C'_1,\dots,C'_l$ are complex numbers while $d_1,\dots,d_m; d'_1,\dots,d'_m$ are positive real numbers. Then $A_a^{\varphi,\varphi}A_b^{\varphi,\varphi} = A_c^{\varphi,\varphi}$, with

$$c(z) = \sum_{k=1}^{m}\sum_{j=1}^{l} C_k C'_j e^{-2\pi r_{k,j} z^2}, \quad z \in \mathbb{R}^{2d},$$

with $r_{k,j} = d_k + d'_j + 2d_k d'_j$.

1.2. Asymptotic Product

Asymptotic expansions realize the composition of two localization operators as a sum of localization operators plus a controllable remainder. Versions of such a symbolic calculus are developed in [1, 7, 15, 6]. Most of them, as we observed in the introduction, were mainly motivated by PDEs and energy estimates, and therefore used smooth symbols that are defined by differentiability properties, such as Hörmander or Shubin classes. For applications in quantum mechanics and signal analysis, alternative notions of smoothness — "smoothness in phase-space" or quantitative measures of "time-frequency concentration" — have turned out to be useful. This point of view is pursued in [6], and we shall present the corresponding results.

The starting point is the following composition formula for two localization operators derived in [7]:

$$A_a^{\varphi_1,\varphi_2} A_b^{\varphi_3,\varphi_4} = \sum_{|\alpha|=0}^{N-1} \frac{(-1)^{|\alpha|}}{\alpha!} A_{a\partial^\alpha b}^{\Phi_\alpha,\varphi_2} + E_N. \tag{16}$$

The essence of this formula is that the product of two localization operators can be written as a sum of localization operators, with new windows Φ_α suitably defined, and a remainder term E_N, which is "small".

In the spirit of the classical symbolic calculus, this formula was derived in [7, Thm. 1.1] for *smooth* symbols belonging to some Shubin class $S^m(\mathbb{R}^{2d})$ and for windows in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$.

In [6] the validity of (16) is established on the modulation spaces. The innovative features of this extension are highlighted below (we do not give here detailed statements and proofs).

(i) *Rough symbols.* While in (16) the symbol b must be N-times differentiable, the symbol a only needs to be locally bounded. The classical results in symbolic calculus require both symbols to be smooth.

(ii) *Growth conditions on symbols.* The symbolic calculus in (16) can handle symbols with ultra-rapid growth (as long as it is compensated by a decay of b or vice versa). For instance, a may grow subexponentially as $a(z) \sim e^{\alpha|z|^\beta}$ for $\alpha > 0$ and $0 < \beta < 1$. This goes far beyond the usual polynomial growth and decay conditions.

(iii) *General window classes.* A precise description of the admissible windows φ_j in (16) is provided. Usually only the Gaussian $e^{-\pi t^2}$ or Schwartz functions are considered as windows.

(iv) *Size of the remainder term.* Norm estimates for the size of the remainder term E_N are derived. They depend explicitly on the symbols a, b and the windows φ_j.

(v) *The Fredholm Property of Localization Operators.* By choosing $N = 1$, $\varphi_1 = \varphi_2 = \varphi$ with $\|\varphi\| = 1$, $a(z) \neq 0$, and $b = 1/a$, the composition formula (16) yields the following important special case:

$$A_a^{\varphi,\varphi} A_{1/a}^{\varphi,\varphi} = A_1^{\varphi,\varphi} + R = \mathrm{I} + R\,. \tag{17}$$

If the symbol a belongs to $L_m^\infty(\mathbb{R}^{2d})$ and $|a| \asymp 1/m$, and the first partial derivative satisfies $(\partial_j a)m \in L^\infty$ and vanishes at infinity for $j = 1, \dots, 2d$, then R is shown to be a compact operator. Besides, $A_a^{\varphi,\varphi}$ is proven to be a Fredholm operator between suitable modulation spaces. This result works even for ultra-rapidly growing symbols such as $a(z) = e^{\alpha|z|^\beta}$ for $\alpha > 0$ and $0 < \beta < 1$.

2. Concepts of Time-Frequency Analysis

We first present the tools and properties from time-frequency analysis that we shall use in the following sections.

2.1. The (Cross)-Wigner Distribution and the Weyl Calculus

The *cross-Wigner distribution* $W(f,g)$ of $f,g \in L^2(\mathbb{R}^d)$ is defined to be

$$W(f,g)(x,\omega) = \int f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}e^{-2\pi i\omega t}\,dt. \tag{18}$$

The quadratic expression $Wf = W(f,f)$ is usually called the Wigner distribution of f.

The Wigner distribution $W(f,g)$ is defined on many pairs of Banach or topological vector spaces. For instance, they both map $L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$ and $\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^{2d})$. Furthermore, they can be extended to a map from $\mathcal{S}'(\mathbb{R}^d)\times\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^{2d})$.

We first report a crucial property of the (cross-)Wigner distribution (for proofs, see [12, Ch. 4] and [11]).

Lemma 2.1. *For $\lambda=(u,\eta),\mu=(v,\gamma)\in\mathbb{R}^{2d}$ and $z=(x,\omega)\in\mathbb{R}^{2d}$ we have*

$$\begin{aligned} W(\pi(\lambda)f,\pi(\mu)g)(x,\omega) &= e^{\pi i(u+v)(\eta-\gamma)}e^{2\pi i x(\eta-\gamma)}e^{-2\pi i\omega(u-v)} \\ &\quad \times W(f,g)(z-\frac{\lambda+\mu}{2}). \end{aligned} \tag{19}$$

In particular, if $\lambda=\mu$, relation (19) *becomes*

$$W(\pi(\lambda)f,\pi(\lambda)g)(z) = W(f,g)(z-\lambda). \tag{20}$$

Since the (cross-)Wigner distribution of Schwartz functions $f,g\in\mathcal{S}(\mathbb{R}^d)$ is a Schwartz function on $\mathbb{R}^{2d}$, its partial derivatives are well-defined and may be expressed explicitly as linear combinations of (cross-)Wigner distributions of the functions $t^{\gamma_1}\partial^{\delta_1}f$ and $t^{\gamma_2}\partial^{\delta_2}g$.

Lemma 2.2. *Let $(x,\omega)\in\mathbb{R}^{2d}$, $f,g\in\mathcal{S}(\mathbb{R}^d)$, $\alpha=(\alpha_1,\alpha_2),\beta=(\beta_1,\beta_2)\in\mathbb{Z}^d_+\times\mathbb{Z}^d_+$, then*

$$\partial_x^{\alpha_1}\partial_\omega^{\alpha_2}W(f,g)(x,\omega)=(-2\pi i)^{|\alpha_2|}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}(-1)^{|\beta_2|}W(t^{\alpha_2-\beta_2}\partial^{\alpha_1-\beta_1}f,t^{\beta_2}\partial^{\beta_1}g)(x,\omega). \tag{21}$$

Proof. Using the product formula for derivatives, the first partial derivative with respect to the *time* variable x_j, with $j=1,\dots,d$, is given by

$$\partial_{x_j}W(f,g)(x,\omega) = W(\partial_{t_j}f,g)(x,\omega)+W(f,\partial_{t_j}g)(x,\omega),$$

and, by induction or by Leibniz' formula, we obtain

$$\partial_x^{\alpha}W(f,g)(x,\omega) = \sum_{\beta_1\le\alpha_1}\binom{\alpha_1}{\beta_1}W(\partial^{\alpha_1-\beta_1}f,\partial^{\beta_1}g)(x,\omega). \tag{22}$$

The first partial derivative with respect to the *frequency* variable ω_j, with $j = 1, \ldots, d$, is

$$\partial_{\omega_j} W(f,g)(x,\omega) = \int_{\mathbb{R}^d} (-2\pi i t_j) e^{-2\pi i t\omega} f\left(x+\frac{t}{2}\right) \overline{g\left(x-\frac{t}{2}\right)}\, dt. \tag{23}$$

The cross-Wigner distributions of the functions $t_j f, g$ and $f, t_j g$, respectively, are given by

$$W(t_j f,g)(x,\omega) = x_j W(f,g)(x,\omega) + \frac{1}{2}\int_{\mathbb{R}^d} t_j e^{-2\pi i t\omega} f\left(x+\frac{t}{2}\right) \overline{g\left(x-\frac{t}{2}\right)}\, dt\,,$$

$$W(f,t_j g)(x,\omega) = x_j W(f,g)(x,\omega) - \frac{1}{2}\int_{\mathbb{R}^d} t_j e^{-2\pi i t\omega} f\left(x+\frac{t}{2}\right) \overline{g\left(x-\frac{t}{2}\right)}\, dt.$$

Subtracting the latter from the former and using (23) we obtain

$$\partial_{\omega_j} W(f,g)(x,\omega) = (-2\pi i)[W(t_j f,g)(x,\omega) - W(f,t_j g)(x,\omega)].$$

As for the time case, the induction process yields

$$\partial_\omega^{\alpha_2} W(f,g)(x,\omega) = (-2\pi i)^{|\alpha_2|} \sum_{\beta_2 \le \alpha_2} \binom{\alpha_2}{\beta_2} (-1)^{\beta_2} W(t^{\alpha_2-\beta_2} f, t^{\beta_2} g)(x,\omega). \tag{24}$$

By combining (22) and (24) we obtain (21). □

2.2. Modulation Spaces

The modulation space norm measures the joint time-frequency distribution of a $f \in \mathcal{S}'$. For their basic properties we refer, for instance, to [12, Ch. 11-13] and the original literature quoted there. The introduction of the general theory of modulation spaces is beyond our scope, therefore we shall limit ourselves to draw on their unweighted version.

Given a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, and $1 \le p,q \le \infty$, the *modulation space* $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L^{p,q}(\mathbb{R}^{2d})$ (mixed-norm spaces). The norm on $M^{p,q}$ is

$$\|f\|_{M^{p,q}} = \|V_g f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p\, dx\right)^{q/p} d\omega\right)^{1/q}.$$

If $p = q$, we write M^p instead of $M^{p,p}$.

Modulation spaces $M^{p,q}$ are Banach spaces whose definition is independent of the choice of the window g. Moreover, if $g \in M^1 \setminus \{0\}$, then $\|V_g f\|_{L^{p,q}}$ is an equivalent norm for $M^{p,q}(\mathbb{R}^d)$ (see [12, Thm. 11.3.7]).

We recall that $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ and, among the weighted modulation spaces one can encounter Sobolev spaces and Shubin-Sobolev spaces. Furthermore, the space of tempered distribution $\mathcal{S}'$ is recovered as unions of suitable weighted modulation spaces.

2.3. Convolution Relations and Wigner Estimate

In view of the relation between the multiplier a and the Weyl symbol (8), we shall use convolution relations between modulation spaces and some properties of the Wigner distribution.

Convolution relations for modulation spaces, studied in [5, 18], yield the following unweighted version.

Proposition 2.3. *Let* $1 \leq p, q, r, s, t \leq \infty$. *If*

$$\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}, \qquad \textit{and} \qquad \frac{1}{t} + \frac{1}{t'} = 1\,,$$

then

$$M^{p,st}(\mathbb{R}^d) * M^{q,st'}(\mathbb{R}^d) \hookrightarrow M^{r,s}(\mathbb{R}^d) \tag{25}$$

with norm inequality $\|f * h\|_{M^{r,s}} \lesssim \|f\|_{M^{p,st}} \|h\|_{M^{q,st'}}$.

The modulation space norm of a cross-Wigner distribution may be controlled by the window norms, as expressed below (see [5]).

Proposition 2.4. *If* $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$ *we have* $W(\varphi_2, \varphi_1) \in M^1(\mathbb{R}^{2d})$, *with*

$$\|W(\varphi_2, \varphi_1)\|_{M^1} \lesssim \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}. \tag{26}$$

The modulation space $M^{\infty,1}$ is the so-called Sjöstrand class and deserves quite an attention when studying Weyl operators. In particular, Sjöstrand in [17] proved that, if the Weyl symbol σ belongs to $M^{\infty,1}$, the corresponding Weyl operator L_σ is bounded on $L^2(\mathbb{R}^d)$. Besides, if $\sigma, \tau \in M^{\infty,1}$, and $L_\mu = L_\sigma L_\tau$, then $\mu \in M^{\infty,1}$; thus $M^{\infty,1}$ is a Banach algebra of pseudodifferential operators. In [13] the previous result is recaptured by using time-frequency analysis techniques. In particular, the Banach algebra property follows from the continuity of the twisted multiplication (see [13, Thm. 4.2]):

Theorem 2.5. *The modulation space* $M^{\infty,1}$ *is a Banach* $*$*-algebra with respect to twisted multiplication* $\sharp$ *and the involution* $\sigma \to \bar{\sigma}$. *In particular, the* $M^{\infty,1}$*-norm with respect to a window Wigner distribution* $W(\varphi, \varphi)$, *with* $\varphi \in \mathcal{S}$, *is given by*

$$\|\sigma \sharp \tau\|_{M^{\infty,1}} \leq C_\varphi \|\sigma\|_{M^{\infty,1}} \|\tau\|_{M^{\infty,1}}, \quad \forall\, \sigma, \tau \in M^{\infty,1}. \tag{27}$$

The continuity of the twisted multiplication on $M^{\infty,1} \times M^{\infty,1}$ will be employed when composing Gabor multipliers (Section 4).

3. Examples of Well-Localized Products

We shall provide few examples of products of localization operators. To this aim, we first need some results on Weyl operators.

Every rank one linear operator acting on $L^2(\mathbb{R}^d)$ can be interpreted as Weyl operator. The characterization of its Weyl symbol is given in [12, Lemma 14.6.3]:

Lemma 3.1. *Given $h, k \in L^2(\mathbb{R}^d)$ and set $\sigma = W(h,k)$. Then L_σ is the rank one operator*

$$L_\sigma f = \langle f, k\rangle h, \quad f \in L^2(\mathbb{R}^d). \tag{28}$$

If we express the product of two rank one operators as a Weyl transform, we obtain immediately a formula for the twisted multiplication of cross-Wigner distributions.

Lemma 3.2. *Given $\varphi_1, \varphi_2, \psi_1, \psi_2 \in L^2(\mathbb{R}^d)$ we have*

$$W(\varphi_1,\varphi_2)\sharp W(\psi_1,\psi_2) = \langle \psi_1, \varphi_2\rangle W(\varphi_1,\psi_2) \tag{29}$$

Proof. Instead of the explicit formula (10), we use Lemma 3.1 and compute the product of the rank one operators $L_{W(\varphi_1,\varphi_2)}L_{W(\psi_1,\psi_2)}$. Let f be in $L^2(\mathbb{R}^d)$, then

$$\begin{aligned} L_{W(\varphi_1,\varphi_2)}L_{W(\psi_1,\psi_2)}f &= \langle L_{W(\psi_1,\psi_2)}f, \varphi_2\rangle \varphi_1 \\ &= \langle \langle f,\psi_2\rangle \psi_1, \varphi_2\rangle\varphi_1 \\ &= \langle \psi_1,\varphi_2\rangle\langle f,\psi_2\rangle\varphi_1 \\ &= L_{\langle\psi_1,\varphi_2\rangle W(\varphi_1,\psi_2)}f. \end{aligned}$$

Hence, the Weyl operator $L_{W(\varphi_1,\varphi_2)\sharp W(\psi_1,\psi_2)} = L_{W(\varphi_1,\varphi_2)}L_{W(\psi_1,\psi_2)}$ possesses the Weyl symbol claimed in (29). □

With these tools, we can now compute the product of two localization operators whose symbols have minimal support.

Proposition 3.3. *Let $1 \le j \le d$, and consider the distributions with support at the origin $a = \partial_{x_j}\delta$, $b = \delta$. For every $\varphi_k \in \mathcal{S}(\mathbb{R}^d)$, $k = 1, \dots, 4$, we have*

$$A^{\varphi_1,\varphi_2}_{\partial_{x_j}\delta} A^{\varphi_3,\varphi_4}_{\delta} = A^{\varphi_3,\partial_j\varphi_2}_{\langle\varphi_4,\varphi_1\rangle\delta} + A^{\varphi_3,\varphi_2}_{\langle\varphi_4,\partial_j\varphi_1\rangle\delta}. \tag{30}$$

Proof. Rewriting the composition of two localization operators as a Weyl transform (7), we reduce ourselves to compute the twisted multiplication of the corresponding Weyl symbols. Using Lemma 3.1 and (29), the desired result follows. Namely,

$$\begin{aligned} &[(\partial_{x_j}\delta) * W(\varphi_2,\varphi_1)]\,\sharp\,[\delta * W(\varphi_4,\varphi_3)] \\ &\quad= [\delta * \partial_{x_j} W(\varphi_2,\varphi_1)]\,\sharp\, W(\varphi_4,\varphi_3) \\ &\quad= [W(\partial_j\varphi_2,\varphi_1) + W(\varphi_2,\partial_j\varphi_1)]\,\sharp\, W(\varphi_4,\varphi_3) \\ &\quad= W(\partial_j\varphi_2,\varphi_1)\,\sharp\, W(\varphi_4,\varphi_3) + W(\varphi_2,\partial_j\varphi_1)\,\sharp\, W(\varphi_4,\varphi_3) \\ &\quad= \langle\varphi_4,\varphi_1\rangle W(\partial_j\varphi_2,\varphi_3) + \langle\varphi_4,\partial_j\varphi_1\rangle W(\varphi_2,\varphi_3), \\ &\quad= \delta * [\langle\varphi_4,\varphi_1\rangle W(\partial_j\varphi_2,\varphi_3)] + \delta * [\langle\varphi_4,\partial_j\varphi_1\rangle W(\varphi_2,\varphi_3)]. \end{aligned}$$

□

The product above is no longer a single localization operator, in this sense the composition is unstable. However, the product in (30) is still a sum of two localization operators, and both have symbols localized at the origin If we choose the window with the optimal time-frequency localization, i.e., the normalized Gaussian $\varphi(t) = 2^{d/4}e^{-\pi t^2}$ and set $\varphi_1 = \varphi_4 = \varphi$, then formula (30) reduces to

$$A^{\varphi,\varphi_2}_{\partial_{x_j}\delta} A^{\varphi_3,\varphi}_{\delta} = A^{\varphi_3,\partial_j\varphi_2}_{\delta}, \quad \forall\, \varphi_2, \varphi_3 \in \mathcal{S} \tag{31}$$

because $\langle \varphi_4, \partial_j \varphi_1 \rangle = 0$ and $\langle \varphi, \varphi \rangle = 1$.

Next, choosing the partial derivatives $\partial_{\omega_j}\delta$, $j = 1, \ldots, d$, as symbols, we obtain a similar formula.

Proposition 3.4. *Let $1 \le j \le d$, and consider the distributions with support at the origin $a = \partial_{\omega_j}\delta$, $b = \delta$. For every $\varphi_k \in \mathcal{S}(\mathbb{R}^d)$, $k = 1, \ldots, 4$, we have*

$$A^{\varphi_1,\varphi_2}_{\partial_{\omega_j}\delta} A^{\varphi_3,\varphi_4}_{\delta} = A^{\varphi_3,t_j\varphi_2}_{\langle \varphi_4,\varphi_1\rangle\delta} + A^{\varphi_3,\varphi_2}_{\langle \varphi_4,t_j\varphi_1\rangle\delta}. \tag{32}$$

The proof is similar to the one of Proposition 3.3. Again, if $\varphi_1 = \varphi_4 = \varphi = 2^{d/4}e^{-\pi t^2}$, then product is is stable, and

$$A^{\varphi,\varphi_2}_{\partial_{\omega_j}\delta} A^{\varphi_3,\varphi}_{\delta} = A^{\varphi_3,t_j\varphi_2}_{\delta}, \quad \forall\, \varphi_2, \varphi_3 \in \mathcal{S}.$$

If we increase the order of the derivative of the symbol, the product of two localization operators is never a single localization operator, and the stability of the product is definitely lost, as is shown the the following observation. Nevertheless, the supports of the symbols are all localized at the origin.

Proposition 3.5. *Let $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{Z}^d_+ \times \mathbb{Z}^d_+$, $\varphi_i \in \mathcal{S}(\mathbb{R}^d)$, $i = 1, \ldots, 4$. Then the product of localization operators whose symbols are derivatives of the delta distribution is given by*

$$A^{\varphi_1,\varphi_2}_{(\partial_x^{\alpha_1}\partial_\omega^{\alpha_2}\delta)} A^{\varphi_3,\varphi_4}_{(\partial_x^{\beta_1}\partial_\omega^{\beta_2}\delta)} = (-2\pi i)^{|\alpha_2+\beta_2|} \sum_{\substack{\gamma \le \alpha \\ \nu \le \beta}} A^{(t^{\nu_2}\partial^{\nu_1}\varphi_3),(t^{\alpha_2-\nu_2}\partial^{\alpha_1-\nu_1}\varphi_2)}_{c_{\alpha,\beta,\gamma,\nu}\delta}, \tag{33}$$

where $c_{\alpha,\beta,\gamma,\nu} = \binom{\alpha}{\gamma}\binom{b}{\nu}(-1)^{|\gamma_2+\nu_2|}\langle t^{\beta_2-\nu_2}\partial^{\beta_1-\nu_1}\varphi_4, t^{\gamma_2}\partial^{\gamma_1}\varphi_1\rangle.$

Again, the proof relies on the same tools as for Proposition 3.3 and therefore we shall omit it. If the derivative order is greater than one, we highlight that neither the Gaussian choice for the windows could help us to have a single localization operator in the right-hand side. In fact, the brackets $\langle t^{\beta_2-\nu_2}\partial^{\beta_1-\nu_1}\varphi, t^{\gamma_2}\partial^{\gamma_1}\varphi\rangle$ do not vanish if $|\beta - \nu + \gamma| \in 2\mathbb{N}$.

4. Product of Gabor Multipliers

In this section we shall study the product of Gabor multipliers, for a survey on the topic we refer to [10]. Consider a time-frequency lattice Λ in $\mathbb{R}^{2d}$, for instance, $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, $\alpha, \beta \in \mathbb{R}$ and set $\mathbf{a} = (a_\lambda)_{\lambda\in\Lambda}$; moreover, choose two non-zero window functions $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. Then, a *Gabor multiplier* $G_{\mathbf{a}}$ associated to the triple $(\varphi_1, \varphi_2, \Lambda)$ and with symbol $\mathbf{a}$ is given by

$$G_{\mathbf{a}} f = \sum_{\lambda \in \Lambda} a_\lambda \langle f, \pi(\lambda)\varphi_1 \rangle \pi(\lambda)\varphi_2, \quad f \in L^2(\mathbb{R}^d). \tag{34}$$

The domain of a Gabor multiplier can even be a subspace of tempered distribution rather than simply a space of functions. The distribution/function space of f in (34) depends on the decay and smoothness properties of the pair (φ_1, φ_2) of dual windows and on the decay of the symbol $\mathbf{a}$.

Gabor multipliers are special cases of localization operators, as we shall see presently.

Lemma 4.1. *Let Λ be a lattice in $\mathbb{R}^{2d}$, $(a_\lambda)_{\lambda\in\Lambda} \in \ell^\infty(\Lambda)$, $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$. Consider the mapping $i : \ell^\infty(\Lambda) \to M^\infty(\mathbb{R}^{2d})$ defined by $i(\mathbf{a}) = \sum_{\lambda\in\Lambda} a_\lambda \delta_\lambda$, then we obtain*

$$G_{\mathbf{a}} = A^{\varphi_1,\varphi_2}_{i(\mathbf{a})}\,. \tag{35}$$

Proof. An easy computation shows that $i(\mathbf{a}) \in M^\infty(\mathbb{R}^{2d})$. More precisely, choose a window $g \in \mathcal{S}(\mathbb{R}^d)$ with compact support such that, for some constants $A, B > 0$ we have $A \le \sum_{\mu\in\Lambda^\perp} |\hat{g}(\xi-\mu)|^2 \le B < \infty$ for all $\xi \in \mathbb{R}^d$ (as usual $\Lambda^\perp$ denotes the dual lattice of Λ). A result in approximation theory then implies that $\sup_{x\in\mathbb{R}^d} |\sum_{\lambda\in\Lambda} a_\lambda g(x-\lambda)| \asymp \|a\|_\infty$. Using this obversation, we obtain

$$\begin{aligned}
\|i(\mathbf{a})\|_{M^\infty} &= \sup_{x,\xi\in\mathbb{R}^d} |\langle \sum_{\lambda\in\Lambda} a_\lambda \delta_\lambda, M_\xi T_x g\rangle| \\
&= \sup_{x,\xi\in\mathbb{R}^d} |\sum_{\lambda\in\Lambda} a_\lambda e^{-2\pi i \lambda\xi} g(x-\lambda)| \\
&\asymp \sup_{\lambda\in\Lambda} |a_\lambda| = \|\mathbf{a}\|_\infty\,.
\end{aligned}$$

Consequently,

$$\|\sum_{\lambda\in\Lambda} a_\lambda \delta_\delta\|_{M^\infty} \asymp \|\mathbf{a}\|_\infty \qquad \forall \mathbf{a} \in \ell^\infty(\Lambda)\,. \tag{36}$$

By assumption $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$, we then appeal to [5, Theorem 3.2] to deduce the boundedness of the localization operator $A^{\varphi_1,\varphi_2}_{i(\mathbf{a})}$ on $L^2(\mathbb{R}^d)$. Using the weak definition (5), we observe that, for every $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}
\langle A^{\varphi_1,\varphi_2}_{i(\mathbf{a})} f, g\rangle &= \langle i(\mathbf{a}), \overline{V_{\varphi_1} f}\, V_{\varphi_2} g\rangle \\
&= \sum_{\lambda\in\Lambda} a_\lambda \langle \delta_\lambda, \overline{V_{\varphi_1} f}\, V_{\varphi_2} g\rangle \\
&= \sum_{\lambda\in\Lambda} a_\lambda V_{\varphi_1} f(\lambda) \overline{V_{\varphi_2} g(\lambda)} \\
&= \sum_{\lambda\in\Lambda} a_\lambda \langle f, \pi(\lambda)\varphi_1\rangle \langle \pi(\lambda)\varphi_2, g\rangle.
\end{aligned}$$

That is, the localization operator $A^{\varphi_1,\varphi_2}_{i(\mathbf{a})}$ coincides with the Gabor multiplier $\sum_{\lambda\in\Lambda} a_\lambda\langle\cdot,\pi(\lambda)\varphi_1\rangle\pi(\lambda)\varphi_2$.

Finally, we note that for $f,g\in L^2(\mathbb{R}^d)$ we have $(\langle f,\pi(\lambda)\varphi_1\rangle)\in\ell^2(\Lambda)$ with a norm estimate $\|\langle f,\pi(\lambda)\varphi_1\rangle_{\lambda\in\Lambda}\|_2\lesssim\|f\|_2$, and likewise $(\langle\pi(\lambda)\varphi_2,g\rangle)\in\ell^2(\Lambda)$, thus $(\langle f,\pi(\lambda)\varphi_1\rangle\langle\pi(\lambda)\varphi_2,g\rangle)\in\ell^1(\Lambda)$ and the ℓ^1-norm is bounded by $\|f\|_2\,\|g\|_2$. Consequently, the series defining $\langle A^{\varphi_1,\varphi_2}_{i(\mathbf{a})}f,g\rangle$ converges absolutely. This implies that the partial sums $S_N=\sum_{\lambda\in\Lambda:|\lambda|\le N} a_\lambda\langle\cdot,\pi(\lambda)\varphi_1\rangle\pi(\lambda)\varphi_2$ converge in the strong operator topology to $A^{\varphi_1,\varphi_2}_{i(\mathbf{a})}$. □

Since Gabor multipliers are a special case of localization operators, we may compute their product via Weyl calculus. We shall see that the result is a Gabor multiplier plus a remainder term. The remainder operator is no more expressible in terms of Gabor multipliers but it can be recaptured as series of suitable localization operators.

Proposition 4.2. *Let Λ be a time-frequency lattice in $\mathbb{R}^{2d}$. Consider $\varphi_i\in M^1$, $i=1,\dots,4$, two sequences $\mathbf{a},\mathbf{b}\in\ell^\infty(\Lambda)$ and the* Gabor multipliers *$G_\mathbf{a}$ and $G_\mathbf{b}$ associated to the triple $(\varphi_1,\varphi_2,\Lambda)$ and $(\varphi_3,\varphi_4,\Lambda)$. Set $\mathbf{c}=(a_\lambda b_\lambda)_\lambda$, and let $G_\mathbf{c}$ be the* Gabor multiplier *associated to the triple $(\varphi_3,\varphi_2,\Lambda)$. Then,*

$$G_\mathbf{a}G_\mathbf{b}=\langle\varphi_4,\varphi_1\rangle G_\mathbf{c}+\sum_{\substack{\lambda,\mu\in\Lambda\\ \lambda\neq\mu}} A^{\pi(\mu)\varphi_3,\pi(\lambda)\varphi_2}_{\langle\pi(\mu)\varphi_4,\pi(\lambda)\varphi_1\rangle a(\lambda)b(\mu)\delta}. \tag{37}$$

Proof. In virtue of (35), we rewrite the Gabor multipliers $G_\mathbf{a},G_\mathbf{b}$ as localization operators and then use the Weyl connection as in the preceding proofs. Precisely, if we name σ and τ the Weyl symbols of $G_\mathbf{a}$ and $G_\mathbf{b}$, respectively, their expression is given by

$$\sigma=\left(\sum_{\lambda\in\Lambda} a_\lambda\delta_\lambda\right)*W(\varphi_2,\varphi_1),\quad \tau=\left(\sum_{\lambda\in\Lambda} b_\lambda\delta_\lambda\right)*W(\varphi_4,\varphi_3).$$

Next, we need to compute $\sigma\,\sharp\,\tau$. In this framework, Proposition 2.4 guarantees $W(\varphi_2,\varphi_1),W(\varphi_4,\varphi_3)\in M^1(\mathbb{R}^{2d})$, whereas Proposition 2.3 provides the continuity of the convolution acting from $M^\infty\times M^1$ into $M^{\infty,1}$. Getting the preceding results all together, we observe that the Weyl symbols σ and τ belong to $M^{\infty,1}(\mathbb{R}^{2d})$. Thus, the corresponding operators L_σ and L_τ are bounded operators on $L^2(\mathbb{R}^d)$ and the same for their product $L_{\sigma\,\sharp\,\tau}$, with $\sigma\,\sharp\,\tau\in M^{\infty,1}$. In the following, we shall calculate $\sigma\,\sharp\,\tau$ explicitly, using the boundedness properties of the convolution and twisted multiplication listed above. Relation (20) will be repeatedly used in the sequel.

$$\begin{aligned}
\sigma\,\sharp\,\tau &= \left[\left(\sum_{\lambda\in\Lambda} a_\lambda\delta_\lambda\right) * W(\varphi_2,\varphi_1)\right]\,\sharp\,\left[\left(\sum_{\mu\in\Lambda} b_\mu\delta_\mu\right) * W(\varphi_4,\varphi_3)\right]\\
&= \left[\sum_{\lambda\in\Lambda}\left(a_\lambda\delta_\lambda * W(\varphi_2,\varphi_1)\right)\right]\,\sharp\,\left[\sum_{\mu\in\Lambda}\left(b_\mu\delta_\mu * W(\varphi_4,\varphi_3)\right)\right]\\
&= \left[\sum_{\lambda\in\Lambda} a_\lambda W(\pi(\lambda)\varphi_2,\pi(\lambda)\varphi_1)\right]\,\sharp\,\left[\sum_{\lambda\in\Lambda} b_\mu W(\pi(\mu)\varphi_4,\pi(\mu)\varphi_3)\right]\\
&= \sum_{\lambda,\mu\in\Lambda} a_\lambda b_\mu [W(\pi(\lambda)\varphi_2,\pi(\lambda)\varphi_1)\,\sharp\,W(\pi(\mu)\varphi_4,\pi(\mu)\varphi_3)]\\
&= \langle\varphi_4,\varphi_1\rangle\sum_{\lambda\in\Lambda} a_\lambda b_\lambda W(\pi(\lambda)\varphi_2,\pi(\lambda)\varphi_3)+\\
&\qquad+\sum_{\lambda,\,\mu\,\in\,\Lambda,\ \lambda\neq\mu} a_\lambda b_\mu\langle\pi(\mu)\varphi_4,\pi(\lambda)\varphi_1\rangle W(\pi(\lambda)\varphi_2,\pi(\mu)\varphi_3).\\
&= \langle\varphi_4,\varphi_1\rangle\sum_{\lambda\in\Lambda}(a_\lambda b_\lambda)\,\delta_\lambda * W(\varphi_2,\varphi_3)+\\
&\qquad+\sum_{\lambda,\,\mu\,\in\,\Lambda,\ \lambda\neq\mu} a_\lambda b_\mu\,\langle\pi(\mu)\varphi_4,\pi(\lambda)\varphi_1\rangle W(\pi(\lambda)\varphi_2,\pi(\mu)\varphi_3).
\end{aligned}$$

We notice that the convergence of the series in the line above is assumed to be in the $M^{\infty,1}$-norm. Passing to the corresponding Weyl operator we obtain the desired result. □

In the above argument we have assumed that all series converge in norm, in particular that $\sum_\lambda a_\lambda\delta_\lambda$ converges in the M^∞-norm. According to (36) this is the case if and only if $\mathbf{a}\in c_0(\Lambda)$, i.e., if $\lim_{|\lambda|\to\infty}|a_\lambda| = 0$. For arbitrary bounded sequences we now give an alternative argument.

Second proof of Proposition 4.2. Assume first that $\mathbf{a}$ and $\mathbf{b}$ have finite support, so that there are no issues about the convergence of the series involved. Then we calculate directly that

$$\begin{aligned} G_a G_b f &= \sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda} a_\lambda b_\mu \langle f, \pi(\mu)\varphi_3\rangle\langle \pi(\mu)\varphi_4, \pi(\lambda)\varphi_1\rangle \pi(\lambda)\varphi_2 \\ &= \sum_{\mu=\lambda} + \sum_{\mu\neq\lambda} \dots \\ &= \sum_{\lambda\in\Lambda} a_\lambda b_\lambda \langle \varphi_4, \varphi_1\rangle\langle f, \pi(\lambda)\varphi_3\rangle\pi(\lambda)\varphi_2 + \\ &+ \sum_{\lambda,\mu\in\Lambda,\mu\neq\lambda} a_\lambda b_\mu \langle f, \pi(\mu)\varphi_3\rangle\langle \pi(\mu)\varphi_4, \pi(\lambda)\varphi_1\rangle\pi(\lambda)\varphi_2 \\ &= \langle \varphi_4, \varphi_1\rangle G_c + \sum_{\lambda,\mu\in\Lambda,\lambda\neq\mu} A^{\pi(\mu)\varphi_3, \pi(\lambda)\varphi_2}_{\langle \pi(\mu)\varphi_4, \pi(\lambda)\varphi_1\rangle a(\lambda) b(\mu)\delta}. \end{aligned}$$

If $\mathbf{a}, \mathbf{b} \in \ell^\infty(\lambda)$, we consider the partial sums $G_{\mathbf{a}_M} f = \sum_{\lambda\in\Lambda, |\lambda|\leq M} a_\lambda \langle f, \pi(\lambda)\varphi_1\rangle\pi(\lambda)\varphi_2$ and $G_{\mathbf{b_N}} f = \sum_{\mu\in\Lambda, |\leq N} b_\mu \langle f, \pi(\mu)\varphi_1\rangle\pi(\mu)\varphi_2$. Since $G_{\mathbf{a}_M}$ and $G_{\mathbf{b}_N} f$ converge strongly to G_a and G_b, the above argument carries over to general bounded sequences $\mathbf{a}$ and $\mathbf{b}$. □

5. Gaussian Functions as Symbols

In this section we prove Theorem 1.2. Instead of using the product formula (12), as done in [9], we use the techniques of the preceding two sections. The result is a mere consequence of the Gaussian nice behavior under convolution and twisted products.

Lemma 5.1. *Let $a, b > 0$, then*
(i) Gaussian convolution:

$$[e^{-\pi a t^2} * e^{-\pi b t^2}](x) = (a+b)^{-d/2} e^{-\pi \frac{ab}{a+b} x^2}, \qquad t, x \in \mathbb{R}^d. \tag{38}$$

(ii) Gaussian twisted multiplication:

$$[e^{-2\pi a z^2} \sharp\, e^{-2\pi b z^2}](\zeta) = (1+ab)^{-d} e^{-2\pi \frac{a+b}{1+ab} \zeta^2}, \qquad z, \zeta \in \mathbb{R}^{2d}. \tag{39}$$

Proof. (i) The semigroup property of Gaussians is well known, see for instance [11], and follows by an easy calculation
(ii) We use the definition of $\sharp$ in (10) and make a direct computation using Gaussian integrals. All integrals converge absolutely and exchanging the order of integration is justified by Fubini's Theorem.

$$\begin{aligned} [e^{-2\pi a z^2} \sharp\, e^{-2\pi b z^2}](\zeta) &= 2^{2d} \iint_{\mathbb{R}^{2d}} e^{-2\pi a z^2} e^{-2\pi b w^2} e^{4\pi i[\zeta - w, \zeta - z]}\, dz dw \\ &= 2^{2d} \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} e^{-2\pi a z^2} e^{-4\pi i[\zeta - w, z]}\, dz \right) e^{-2\pi b w^2} e^{-4\pi i [w,\zeta]} dw. \end{aligned}$$

To begin with, we compute the interior integral with respect to the variable z. Namely, if $z=(z_1,z_2)$, $w=(w_1,w_2)$, $\zeta=(\zeta_1,\zeta_2)\in\mathbb{R}^{2d}$,

$$\begin{aligned}\int_{\mathbb{R}^{2d}} e^{-2\pi a z^2} e^{-4\pi i[\zeta-w,z]}\,dz &= \left(\int_{\mathbb{R}^d} e^{-2\pi a z_1^2} e^{-2\pi i(-2(\zeta_2-w_2)z_1)}\,dz_1\right)\\ &\quad\cdot\left(\int_{\mathbb{R}^d} e^{-2\pi a z_2^2} e^{-2\pi i(2(\zeta_1-w_1)z_2)}\,dz_2\right)\\ &= (2a)^{-d} e^{-\frac{2\pi}{a}(\zeta-w)^2}.\end{aligned}$$

Then, we have

$$\begin{aligned}[e^{-2\pi a z^2}\,\sharp\, e^{-2\pi b z^2}](\zeta) &= \left(\frac{2}{a}\right)^d \int_{\mathbb{R}^{2d}} e^{-\frac{2\pi}{a}(\zeta-w)^2} e^{-2\pi b w^2} e^{-4\pi i[w,\zeta]}\,dw\\ &= \left(\frac{2}{a}\right)^d e^{-\frac{2\pi}{a}\zeta^2}\int_{\mathbb{R}^{2d}} e^{-2\pi\frac{1+ab}{a}w^2} e^{\frac{4\pi i}{a}\zeta w} e^{-4\pi i[w,\zeta]} dw\\ &= \left(\frac{2}{a}\right)^d e^{-\frac{2\pi}{a}\left(1-\frac{1}{1+ab}\right)\zeta^2}\int_{\mathbb{R}^{2d}} e^{-2\pi\left[\left(\frac{1+ab}{a}\right)^{1/2} w-\frac{\zeta}{(a(1+ab))^{1/2}}\right]^2}\\ &\quad\cdot\ e^{-4\pi i[w,\zeta]}\,dw.\end{aligned}$$

Splitting up the variables again, the twisted multiplication reduces to

$$\begin{aligned}[e^{-2\pi a z^2}\,\sharp\, e^{-2\pi b z^2}](\zeta) &= \left(\frac{2}{a}\right)^d e^{-\frac{2\pi}{a}\left(1-\frac{1}{1+ab}\right)\zeta^2}\int_{\mathbb{R}^d} T_{\frac{\zeta_1}{(a(1+ab))^{1/2}}} e^{-2\pi\frac{(1+ab)}{a}w_1^2}\\ &\quad\cdot e^{-2\pi i(2\zeta_2)w_1}\,dw_1 \int_{\mathbb{R}^d} T_{\frac{\zeta_2}{(a(1+ab))^{1/2}}} e^{-2\pi\frac{(1+ab)}{a}w_2^2} e^{-2\pi i(-2\zeta_1)w_2} dw_2\\ &= \left(\frac{2}{a}\right)^d e^{-\frac{2\pi}{a}\left(1-\frac{1}{1+ab}\right)\zeta^2}\int_{\mathbb{R}^d} e^{-2\pi\frac{(1+ab)}{a}w_1^2} e^{-2\pi i(2\zeta_2)w_1}\,dw_1\\ &\quad\cdot\int_{\mathbb{R}^d} e^{-2\pi\frac{(1+ab)}{a}w_2^2} e^{-2\pi i(-2\zeta_1)w_2}\,dw_2\\ &= (1+ab)^{-d} e^{-2\pi\frac{a+b}{1+ab}\zeta^2},\end{aligned}$$

as desired. □

Corollary 5.2. *Let $\varphi(t)=2^{d/4}e^{-\pi t^2}$ anc $c>0$. Then,*

$$(e^{-2\pi c z^2} * W(\varphi,\varphi))(\zeta)=(c+1)^{-d}\ e^{-2\pi\frac{c}{c+1}\zeta^2},\quad z,\zeta\in\mathbb{R}^{2d}. \tag{40}$$

Proof. It is a straightforward consequence of the relations (11) and (38). □

Now we have all we need to prove the main result of this section.

Proof of Theorem 1.2. We use relation (7) and the bilinearity of both the convolution and twisted multiplication. This yields

$$[a * W(\varphi,\varphi)] \sharp [b * W(\varphi,\varphi)](\zeta) = \sum_{k=1}^{m}\sum_{j=1}^{l} C_k C'_j p_{k,j}(\zeta),$$

with $p_{k,j}(\zeta) := [e^{-2\pi d_k z^2} * W(\varphi,\varphi)] \sharp [e^{-2\pi d'_j z^2} * W(\varphi,\varphi)](\zeta)$. The convolution products are achieved by (40) and the outcomes are

$$p_{k,j}(\zeta) = (d_k+1)^{-d}(d'_j+1)^{-d}(e^{-2\pi \frac{d_k}{d_k+1} z^2} \sharp e^{-2\pi \frac{d'_j}{d'_j+1} z^2})(\zeta).$$

Finally, we compute the twisted multiplication using (39) and we get

$$\begin{aligned} p_{k,j}(\zeta) &= (d_k+1)^{-d}(d'_j+1)^{-d}(e^{-2\pi \frac{d_k}{d_k+1} z^2} \sharp e^{-2\pi \frac{d'_j}{d'_j+1} z^2})(\zeta) \\ &= (d_k + d'_j + 2d_k d'_j + 1)^{-d}\, e^{-2\pi \frac{d_k+d'_j+2d_k d'_j}{d_k+d'_j+2d_k d'_j+1} \zeta^2} \\ &= [e^{-2\pi(d_k+d'_j+2d_k d'_j) z^2} * W(\varphi,\varphi)](\zeta), \end{aligned}$$

where in the last equality we used relation (40) backwards. □

Acknowledgment

This work was started while both authors were visiting the Erwin Schrödinger Institute in Vienna. Its hospitality and great working conditions are gratefully acknowledged.

References

[1] H. Ando and Y. Morimoto. Wick calculus and the Cauthy problem for some dispersive equations. *Osaka J. Math.*, 39(1):123–147, 2002.

[2] F. A. Berezin. Wick and anti-Wick symbols of operators. *Mat. Sb. (N.S.)*, 86(128):578–610, 1971.

[3] P. Boggiatto, E. Cordero, and K. Gröchenig. Generalized Anti-Wick operators with symbols in distributional Sobolev spaces. *Integral Equations and Operator Theory*, 48:427–442, 2004.

[4] E. Cordero, Gelfand-Shilov windows for weighted modulation spaces. *Preprint*, 2005

[5] E. Cordero and K. Gröchenig. Time-frequency analysis of Localization operators. *J. Funct. Anal.*, 205(1):107–131, 2003.

[6] E. Cordero and K. Gröchenig. Symbolic calculus and Fredholm property for localization operators. *Preprint*, 2005.

[7] E. Cordero and L. Rodino. Wick calculus: a time-frequency approach. *Osaka J. Math.*, 42(1):43–63, 2005.

[8] J. Du and M. W. Wong. A product formula for localization operators. *Bull. Korean Math. Soc.*, 37(1):77–84, 2000.

[9] J. Du and M. W. Wong. Gaussian functions and Daubechies operators *Integral Equations and Operator Theory*, 38(1):1–8, 2000.

[10] H. G. Feichtinger and K. Nowak. A First Survey of Gabor Multipliers. In H. G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*. Birkhäuser, Boston, 2002.

[11] G. B. Folland. *Harmonic analysis in phase space*. Princeton Univ. Press, Princeton, NJ, 1989.

[12] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Birkhäuser, Boston, 2001.

[13] K. Gröchenig. Composition and spectral invariance of pseudodifferential operators on modulation spaces. *J. Anal. Math.*, to appear.

[14] L. Hörmander. *The analysis of linear partial differential operators. III*, volume 274 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1994.

[15] N. Lerner. The Wick calculus of pseudo-differential operators and energy estimates. In *New trends in microlocal analysis (Tokyo, 1995)*, pages 23–37. Springer, Tokyo, 1997.

[16] M. A. Shubin. *Pseudodifferential Operators and Spectral Theory*. Springer-Verlag, Berlin, second edition, 2001. Translated from the 1978 Russian original by Stig I. Andersson.

[17] J. Sjöstrand. An algebra of pseudodifferential operators. *Math. Res. Lett.*, 1(2):185–192, 1994.

[18] J. Toft. Continuity properties for modulation spaces, with applications to pseudo-differential calculus. I. *J. Funct. Anal.*, 207(2):399–429, 2004.

[19] M. W. Wong. *Wavelet transforms and localization operators*, volume 136 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2002.

Department of Mathematics, University of Torino, Italy

Faculty of Mathematics, University of Vienna, Austria
E-mail address: `elena.cordero@unito.it`
E-mail address: `karlheinz.groechenig@univie.ac.at`