Wiener's Lemma: Theme and Variations

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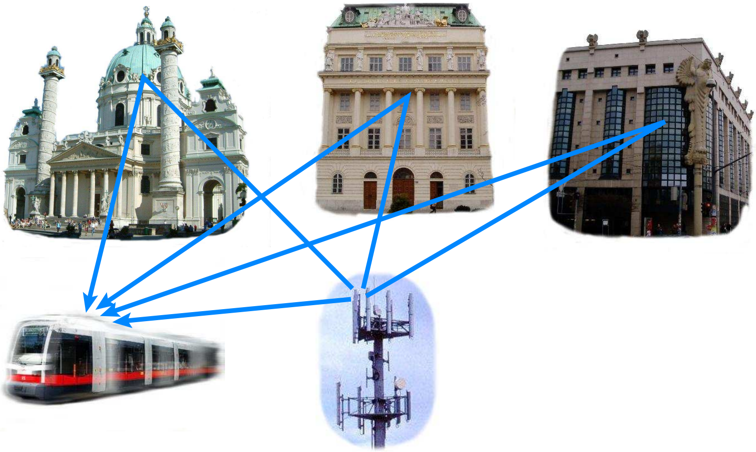
Outline

Time-Varying Systems



Quotient Rule

$C^k(\mathbb{T})\dots\dots$ k-times differentiable functions with period 1.

Product rule $(fg)' = f'g + fg'$ implies that $C^k(\mathbb{T})$ is an *algebra*.

Quotient rule $(1/f)' = -f'/f^2$ implies that $1/f$ is k-times differentiable, whenever $f(t) \neq 0$.

Lemma

If $f \in C^k(\mathbb{T})$ and $f(t) \neq 0$ for all $t \in \mathbb{T} \simeq [0,1)$, then $1/f \in C^k(\mathbb{T})$.

Absolutely Convergent Fourier Series

f possesses an absolutely convergent Fourier series, $\boxed{f \in \mathcal{A}(\mathbb{T})}$, if $f(t) = \sum_{k\in\mathbb{Z}} a_k\, e^{2\pi ikt}$ with coefficients in $\mathbf{a} \in \ell^1(\mathbb{Z})$.

Norm: $$\|f\|_{\mathcal{A}} = \|\mathbf{a}\|_1 = \sum_{k\in\mathbb{Z}} |a_k|$$

Lemma

$\mathcal{A}$ is a Banach algebra under pointwise multiplication.

Proof

$$
\begin{aligned}
f(t)\,g(t) &= \Big(\sum_{k\in\mathbb{Z}} a_k e^{2\pi ikt}\Big)\Big(\sum_{l\in\mathbb{Z}} b_l e^{2\pi ilt}\Big) \\
&= \sum_{k,l\in\mathbb{Z}} a_k b_l e^{2\pi i(k+l)t} \\
&= \sum_{n\in\mathbb{Z}} \underbrace{\Big(\sum_{k\in\mathbb{Z}} a_k b_{n-k}\Big)}_{(\mathbf{a}*\mathbf{b})(n)} e^{2\pi int}
\end{aligned}
$$

$$
\|fg\|_{\mathcal{A}} = \|\mathbf{a}*\mathbf{b}\|_1 \le \|\mathbf{a}\|_1\,\|\mathbf{b}\|_1 = \|f\|_{\mathcal{A}}\,\|g\|_{\mathcal{A}}
$$

Wiener's Lemma

Problem: if $f \in \mathcal{A}(\mathbb{T})$ and $f(t) \neq 0, \forall t$, what can we say about $1/f$?

Theorem (Classical Formulation)

If $f \in \mathcal{A}(\mathbb{T})$ and $f(t) \neq 0$ for all $t \in \mathbb{T}$, then also $1/f \in \mathcal{A}(\mathbb{T})$, i.e., $1/f(t) = \sum_{k\in\mathbb{Z}} b_k e^{2\pi ikt}$ with $\mathbf{b} \in \ell^1(\mathbb{Z})$.

Many proofs:

- Wiener 1932 (localization)
- Gelfand theory of commutative Banach algebras
- Levy, Zygmund
- Hulanicki 1970
- Newman 1975

Proof of Wiener's Lemma

Following Newman and Hulanicki.

Step 1. Reduction to special case. Since $\frac{1}{f} = \frac{\bar{f}}{|f|^2}$, it suffices to assume that f is non-negative.
W.l.o.g. $0 \le f(t) \le 1$.

Assumption $f(t) \neq 0, \forall t$ implies that $inf_t |f(t)| = \delta > 0$.

Step 2. Analyze invertibility in $C(\mathbb{T})$ by geometric series. Let $h = 1 - f$, then

$$0 \le h(t) = 1 - f(t) \le 1 - \delta\,.$$

and the geometric series $\sum_{n=0}^{\infty} h(t)^n$ converges in $C(\mathbb{T})$ to the limit

$$\sum_{n=0}^{\infty} h(t)^n = \frac{1}{1-h(t)} = \frac{1}{f(t)} \in C(\mathbb{T})\,.$$

Goal: show that $\sum h^n$ converges in $\mathcal{A}$.

Step 3. Approximate h by trigonometric polynomial. Given $\epsilon > 0$, choose a trigonometric polynomial $p(t) = \sum_{|k|\leq N} a_k e^{2\pi ikt}$, such that

$$\|h - p\|_{\mathcal{A}} = \sum_{|k|>N} |a_k| < \epsilon$$

Set $r = h - p$, then $h = p + r$ and $\|r\|_{\mathcal{A}} < \epsilon$.

Choice of ϵ: we will need $1 - \delta + 2\epsilon < 1$.

Step 4. Elementary estimates.

- If q is trigonometric polynomial of degree N, then

$$\|q\|_{\mathcal{A}} = \sum_{|k|\leq N} |b_k| \leq \|b\|_2 \Big(2N+1\Big)^{1/2} = \|q\|_2 \Big(2N+1\Big)^{1/2}.$$

- If p is trigonometric polynomial of degree N, then p^k is trigonometric polynomial of degree kN.

Step 5. Estimate $\mathcal{A}$-norm of h^n.

$$h^n = \sum_{k=0}^{n} \binom{n}{k} p^k r^{n-k}$$

So

$$\begin{aligned} \|h^n\|_{\mathcal{A}} &\leq \|r^n\|_{\mathcal{A}} + \sum_{k=1}^{n} \binom{n}{k} \|p^k r^{n-k}\|_{\mathcal{A}} \\ &\leq \epsilon^n + \sum_{k=1}^{n} \binom{n}{k} \|p^k\|_{\mathcal{A}} \|r^{n-k}\|_{\mathcal{A}} \end{aligned}$$

Now by Step 4,

$$\|p^k\|_{\mathcal{A}} \leq \|p^k\|_2 (2Nk+1)^{1/2} \leq \|p^k\|_\infty \frac{\|p\|_2}{\|p\|_\infty} (2Nk+1)^{1/2}.$$

Step 6. Complete estimate for $\mathcal{A}$-norm of h^n.

$$
\begin{aligned}
\|h^n\|_{\mathcal{A}} &\leq C(2Nn+1)^{1/2}\sum_{k=0}^{n}\binom{n}{k}\epsilon^{n-k}\|p\|_\infty^k \\
&= C(2Nn+1)^{1/2}\Big(\|p\|_\infty+\epsilon\Big)^n \\
&\leq C(2Nn+1)^{1/2}\Big(\|h-r\|_\infty+\epsilon\Big)^n \\
&\leq C(2Nn+1)^{1/2}\Big(\|h\|_\infty+2\epsilon\Big)^n \\
&\leq C(2Nn+1)^{1/2}\Big(\underbrace{1-\delta+2\epsilon}_{<1}\Big)^n .
\end{aligned}
$$

Step 7. Convergence of geometric series in $\mathcal{A}$-norm.

$$\sum_{n=0}^{\infty} \|h^n\|_{\mathcal{A}} \leq \sum_{n=0}^{\infty} (2Nn+1)^{1/2} \left(1-\delta+2\epsilon\right)^n < \infty\,.$$

Conclusion: $1/f = \sum_{n=0}^{\infty} h^n \in \mathcal{A}$. ■

REMARK: We have proved that

$$\lim_{n\to\infty} \|h^n\|_{\infty}^{1/n} = \lim_{n\to\infty} \|h^n\|_{\mathcal{A}}^{1/n}$$

(see below)

Wiener Pairs

Naimark's insight: Wiener's Lemma is a result about the relation between **two** Banach algebras, namely $\mathcal{A}(\mathbb{T})$ and $C(\mathbb{T})$.

Condition "$f(t) \neq 0, \forall t$" means that f is invertible in $C(\mathbb{T})$.

Definition

Let $\mathcal{A} \subseteq \mathcal{B}$ be two (involutive) Banach algebras with common identity. Then $\mathcal{A}$ is called *inverse-closed* in $\mathcal{B}$, if

$$a \in \mathcal{A} \text{ and } a^{-1} \in \mathcal{B} \quad \Longrightarrow \quad a^{-1} \in \mathcal{A}\,.$$

- Wiener's Lemma $\Leftrightarrow$ $\mathcal{A}(\mathbb{T})$ is inverse-closed in $C(\mathbb{T})$.
- in the large algebra there are more invertible elements and it may be easier to verify invertibility

Babylonian Confusion

- $\mathcal{A}$ *inverse-closed* in $\mathcal{B}$
- $(\mathcal{A}, \mathcal{B})$ is a *Wiener pair* (Naimark)
- $\mathcal{A}$ is a *spectral subalgebra* of $\mathcal{B}$ (Palmer)
- $\mathcal{A}$ is a *local subalgebra* of $\mathcal{B}$ (K-theory)
- $\mathcal{A}$ is a *full subalgebra* of $\mathcal{B}$
- $\mathcal{A}$ is invariant under holomorphic calculus in $\mathcal{B}$ (Connes)
- Spectral invariance, spectral permanence (Arveson)

Spectral Invariance

Spectrum in Banach algebra $\mathcal{A}$ (with unit e)

$$\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : a - \lambda e \text{ is not invertible in } \mathcal{A}\}$$

Spectral radius

$$r_{\mathcal{A}}(a) = \max\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(a)\} = \lim_{n\to\infty} \|a^n\|_{\mathcal{A}}^{1/n}$$

Lemma

$\mathcal{A} \subseteq \mathcal{B}$ with common unit e. TFAE:
(i) $\mathcal{A}$ is inverse-closed in $\mathcal{B}$.
(ii) $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$ for all $a \in \mathcal{A}$.
(iii) $r_{\mathcal{A}}(a) = r_{\mathcal{B}}(a)$ for all $a = a^* \in \mathcal{A}$.

Wiener's Lemma states that $\sigma_{\mathcal{A}(\mathbb{T})}(f) = \sigma_{C(\mathbb{T})}(f) = f(\mathbb{T})$.

Proof

(i) $\Rightarrow$ **(ii)** If $\lambda \notin \sigma_{\mathcal{A}}(a)$, then $(a - \lambda e)^{-1} \in \mathcal{A} \subseteq \mathcal{B}$, so $\lambda \notin \sigma_{\mathcal{B}}(a)$.

$$\sigma_{\mathcal{B}}(a) \subseteq \sigma_{\mathcal{A}}(a)$$

If $a \in \mathcal{A}$ and $\lambda \notin \sigma_{\mathcal{B}}(a)$, then $(a - \lambda e)^{-1} \in \mathcal{B}$. By inverse-closedness, $(a - \lambda e)^{-1} \in \mathcal{A}$, and so

$$\sigma_{\mathcal{A}}(a) \subseteq \sigma_{\mathcal{B}}(a)$$

(ii) $\Rightarrow$ **(i)** $a \in \mathcal{A}, a^{-1} \in \mathcal{B}$ means $0 \notin \sigma_{\mathcal{B}}(a)$, so $0 \notin \sigma_{\mathcal{A}}(a)$.

(ii) $\Rightarrow$ **(iii)** clear

Hulanicki's Lemma

(iii) $\Rightarrow$ **(ii)** is known as Hulanicki's Lemma.

Idea: Identity $r_{\mathcal{A}}(a) = r_{\mathcal{B}}(a)$ implies that power series have same radius of convergence in $\mathcal{A}$ and in $\mathcal{B}$.

Consider geometric series $c = \sum_{k=0}^{\infty} (e-a)^k$. Converges in $\mathcal{B}$, if and only if $r_{\mathcal{B}}(e-a) < 1$
Converges in $\mathcal{A}$, if and only if $r_{\mathcal{B}}(e-a) < 1$ if and only if $r_{\mathcal{A}}(e-a) < 1$

In case of convergence $c = a^{-1}$.
If $r_{\mathcal{B}}(e-a) = r_{\mathcal{B}}(e-a) < 1$, then convergence in both $\mathcal{B}$ and $\mathcal{A}$ and assumption $a^{-1} \in \mathcal{B}$ implies that also $a^{-1} \in \mathcal{A}$.
General case can be reduced to this case by looking at $b = a^*a/(2\|a^*a\|_{\mathcal{B}})$ instead of a.

Functional Calculus

Riesz functional calculus (holomorphic functional calculus)
f analytic on open neighborhood O of $\sigma_{\mathcal{B}}(a)$ and $\gamma \subseteq O$ is contour of $\sigma_{\mathcal{B}}(a)$. Define

$$f(a) = \frac{1}{2\pi i} \int_\gamma f(z)(ze - a)^{-1}\, dz$$

Corollary

If $\mathcal{A}$ is inverse-closed in $\mathcal{B}$, then the Riesz functional calculi for $\mathcal{A}$ and $\mathcal{B}$ coincide.

- $\Longrightarrow$ square roots, powers, pseudoinverse
- Theorem of Wiener-Levy

Time-Invariant Systems and Convolution Operators

$$(\mathbf{a} * \mathbf{b})(k) = \sum_{l \in \mathbb{Z}} a(l)b(k-l) \qquad k \in \mathbb{Z}$$

Convolution operator $C_{\mathbf{a}}\mathbf{b} = \mathbf{a} * \mathbf{b}$

- Commutes with translations $(T_m\mathbf{c})(k) = c(k-m)$:

$$\boxed{C_{\mathbf{a}} T_m = T_m C_{\mathbf{a}}}$$

- If $\mathbf{a} \in \ell^1(\mathbb{Z})$, then $C_{\mathbf{a}}$ is bounded on ℓ^p for $1 \leq p \leq \infty$.

$\sigma_{\ell^p}(C_{\mathbf{a}})$ ... spectrum of $C_{\mathbf{a}}$ as an operator acting on $\ell^p(\mathbb{Z})$.

Wiener's Lemma for Convolution Operators I

Theorem

If $\mathbf{a} \in \ell^1(\mathbb{Z})$ *and* $C_{\mathbf{a}}$ *is invertible on* $\ell^2(\mathbb{Z})$, *then* $C_{\mathbf{a}}^{-1} = C_{\mathbf{b}}$ *for some* $\mathbf{b} \in \ell^1(\mathbb{Z})$.

Proof.

- Fourier series $\widehat{\mathbf{a}}(t) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k t}$ for $\mathbf{a} \in \ell^2(\mathbb{Z})$ or $\ell^1(\mathbb{Z})$.
- $(C_{\mathbf{a}}\mathbf{b})\widehat{\ } = (\mathbf{a} * \mathbf{b})\widehat{\ } = \widehat{\mathbf{a}}\,\widehat{\mathbf{b}}$
- $C_{\mathbf{a}}$ corresponds to multiplication operator and is invertible if and only if $\inf_t \widehat{\mathbf{a}}(t) > 0$
- Let $\mathbf{b}$ be the sequence with Fourier series $\widehat{\mathbf{b}} = 1/\widehat{\mathbf{a}}$. By Wiener's Lemma $\mathbf{b} \in \ell^1(\mathbb{Z})$.
- Since $\mathbf{a} * \mathbf{b} = \mathbf{b} * \mathbf{a} = \delta$, we have $C_{\mathbf{b}} = C_{\mathbf{a}}^{-1} = \mathrm{Id}$. ∎

Wiener's Lemma for Convolution Operators II

Corollary

Assume that $\mathbf{a} \in \ell^1(\mathbb{Z})$*. TFAE:*
(i) $C_{\mathbf{a}}$ *is invertible on* $\ell^2(\mathbb{Z})$*.*
(ii) $C_{\mathbf{a}}$ *is invertible on* $\ell^p(\mathbb{Z})$ *for some* $p \in [1, \infty]$*.*
(iii) $C_{\mathbf{a}}$ *is invertible on* $\ell^p(\mathbb{Z})$ *for all* $p \in [1, \infty]$*.*

Corollary

If $\mathbf{a} \in \ell^1(\mathbb{Z})$*, then*

$$\sigma_{\ell^p}(C_{\mathbf{a}}) = \sigma_{\ell^2}(C_{\mathbf{a}}) = \widehat{\mathbf{a}}(\mathbb{T}) \qquad \forall p \in [1, \infty]\,.$$

Spectrum of convolution operator does NOT depend on the space ℓ^p

Symbolic Calculus

- parametrization of class of operators by “symbols”
- distinguished classes of "nice" symbols

Symbolic Calculus: *Symbol class is closed under inversion (and functional calculus). If an operator is parametrized by a “nice” symbol and invertible on Hilbert space, then its inverse is again parametrized by “nice” symbol.*

Wiener’s Lemma is the prototype of a symbolic calculus

♣ class of convolution operators $C_{\mathbf{a}}$

♣ “nice” symbols $\mathbf{a} \in \ell^1$

The inverse of convolution operator $C_{\mathbf{a}}$ is again a convolution operator $C_{\mathbf{a}}^{-1} = C_{\mathbf{b}}$. If $\mathbf{a} \in \ell^1$, then also $\mathbf{b} \in \ell^1$.

Outlook for Tomorrow

- Absolutely convergent Fourier series on compact abelian groups. Replace torus $\mathbb{T}$ by arbitrary compact abelian group K.
- Convolution operators on locally compact groups: Replace $\mathbb{Z}$ by lc group G
- weighted versions: $\ell^1 \longrightarrow \ell^1_v$.
- matrix algebras and off-diagonal decay
- time-varying systems, pseudodifferential operators
- rotation algebra, non-commutative tori

End of First Lecture

Wiener Pairs

Definition

Let $\mathcal{A} \subseteq \mathcal{B}$ be two (involutive) Banach algebras with common identity. Then $\mathcal{A}$ is called *inverse-closed* in $\mathcal{B}$, if

$$a \in \mathcal{A} \text{ and } a^{-1} \in \mathcal{B} \quad \Longrightarrow \quad a^{-1} \in \mathcal{A}\,.$$

Main questions:
1. How to determine when $\mathcal{A}$ is inverse-closed in $\mathcal{B}$.
Answer (in principle): check identity of spectral radii $r_{\mathcal{A}}(a) = r_{\mathcal{B}}(a)$ for all $a \in \mathcal{A}$.

2. How to construct inverse-closed subalgebras (in a systematic fashion)?
Answer: ??

Variations

- Absolutely convergent Fourier series on compact abelian groups. Replace torus $\mathbb{T}$ by arbitrary compact abelian group K.
- Convolution operators on locally compact groups: Replace $\mathbb{Z}$ by lc group G
- weighted versions: $\ell^1 \longrightarrow \ell^1_v$.
- matrix algebras and off-diagonal decay
- time-varying systems, pseudodifferential operators
- rotation algebra, non-commutative tori

Weights and Weighted ℓ^1

Weights quantify decay conditions

- submultiplicativity $v(k+l) \leq v(k)v(l)$ for $k, l \in \mathbb{Z}^d$
- symmetry $v(-k) = v(k)$

Weighted ℓ^1_v: $\|\mathbf{a}\|_{\ell^1_v} = \sum_{k \in \mathbb{Z}^d} |a_k| v(k)$

Weighted absolutely convergent Fourier series: $f \in \mathcal{A}_v(\mathbb{T}^d)$, if $f(t) = \sum_{k \in \mathbb{Z}^d} a_k \, e^{2\pi i k \cdot t}$ with norm

$$\|f\|_{\mathcal{A}_v} = \|\mathbf{a}\|_{\ell^1_v}$$

Lemma

If v is submultiplicative, then $\mathcal{A}_v$ is a Banach algebra with respect to pointwise convolution.

Examples

- Typical submultiplicative weight on $\mathbb{Z}^d$ or $\mathbb{R}^d$:

$$v(k) = e^{a|k|^b}(1+|k|)^s$$

for $a, b, s \geq 0$

Proof of Lemma.

$\|fg\|_{\mathcal{A}} = \|\mathbf{a} * \mathbf{b}\|_1 \leq \|\mathbf{a}\|_1 \|\mathbf{b}\|_1 = \|f\|_{\mathcal{A}} \|g\|_{\mathcal{A}}$ $\|fg\|_{\mathcal{A}_v} = \|\mathbf{a} * \mathbf{b}\|_{\ell^1_v} \leq \|\mathbf{a}\|_{\ell^1_v} \|\mathbf{b}\|_{\ell^1_v} = \|f\|_{\mathcal{A}_v} \|g\|_{\mathcal{A}_v}$

Wiener's Lemma — Weighted Version

Theorem

TFAE:
(i) Wiener's Lemma holds for $\mathcal{A}_v$, i.e. if $f \in \mathcal{A}_v$ and $f(t) \neq 0$ for all t, then $1/f \in \mathcal{A}_v$.
(ii) v satisfies the GRS-condition (Gelfand-Raikov-Shilov)

$$\lim_{n\to\infty} v(n\mathbf{k})^{1/n} = 1 \qquad \forall \mathbf{k} \in \mathbb{Z}^d\,.$$

- The weight $v(k) = e^{a|k|^b}(1+|k|)^s$ satisfies GRS, if and only if $0 \leq b < 1$.

Counter-Example

Proof $(ii) \Rightarrow (i)$.
If v violates GRS, then there is $\mathbf{k} \in \mathbb{Z}^d$ and $a > 0$ such that

$$v(n\mathbf{k}) \geq e^{an} \qquad \text{for } n \geq n_0 \,.$$

(A non-GRS weight grows exponentially on a subgroup)

Fix $\delta \in (0, a]$ and set

$$f(t) = 1 - e^{-\delta} e^{2\pi i \mathbf{k}\cdot t} \in \mathcal{A}_v(\mathbb{T}^d)$$

Then $f(t) \neq 0$ for all $t \in \mathbb{T}^d$ and

$$1/f(t) = (1 - e^{-\delta} e^{2\pi i \mathbf{k}\cdot t})^{-1} = \sum_{n=0}^{\infty} e^{-\delta n} e^{2\pi i n \mathbf{k} t} \,,$$

but

$$\|f\|_{\mathcal{A}_v} = \sum_{k=0}^{\infty} e^{-\delta n} v(n\mathbf{k}) \geq \sum_{k=n_0}^{\infty} e^{-\delta n} e^{an} = \infty$$

Proof $(i) \Rightarrow (ii)$. (in dimension $d = 1$) Use

$$\begin{aligned} \|q\|_{\mathcal{A}_v} &= \sum_{|k| \leq N} |b_k| v(k) \\ &\leq \|b\|_2 \left(2N+1\right)^{1/2} \max_{|k|\leq N} v(k) \\ &= \|q\|_2 \left(2N+1\right)^{1/2} \max_{|k|\leq N} v(k)\,. \end{aligned}$$

and adjust proof for unweighted case.

Convolution Operators on Groups

Postponed till later.

Time-Varying Channels (Discrete)

Time-invariant system corresponds to convolution operator

$$(C_{\mathbf{a}}\mathbf{c})(k) = \sum_{k \in \mathbb{Z}} a(k-l)c(l)$$

Corresponding matrix is $m_{kl} = a(k-l)$ is constant along diagonals (Toeplitz matrices).

Slowly time-varying systems — non-stationary matrices — small variation along diagonals.

Matrix norm

$$\|M\|_{\mathcal{C}_v} = \sum_{l \in \mathbb{Z}} \sup_{k \in \mathbb{Z}} |m_{k,k-l}| v(l)$$

Convolution Dominated Operators

$$d(l) = \sup_{k\in\mathbb{Z}} |m_{k,k-l}|$$

is supremum on ℓ-th diagonal $D(l)$ and

$$m_{kl} \leq d(k-l)$$

$$|(M\mathbf{c})(k)| = |\sum_{l\in\mathbb{Z}} m_{kl}c_l| \leq \sum_{l\in\mathbb{Z}} d(k-l)|c_l|$$

Action of M is dominated (pointwise) by convolution with $\mathbf{d}$

The Nonstationary Version of Wiener's Lemma

Theorem (Gohberg et al., Baskakov, Kurbatov, Sjöstrand)

If $M \in \mathcal{C}_1$ and M is invertible on $\ell^2(\mathbb{Z})$, then $M^{-1} \in \mathcal{C}_1$

Theorem (Baskakov)

Assume that v satisfies the GRS condition. If $M \in \mathcal{C}_v$ and M is invertible on $\ell^2(\mathbb{Z})$, then $M^{-1} \in \mathcal{C}_v$.

$\mathcal{C}_v$ is inverse-closed in $\mathcal{B}(\ell^2)$.

Spectral invariance

Corollary

Assume that v satisfies the GRS condition and $M \in \mathcal{C}_v$. Then

$$\sigma_{\ell^p_m}(M) = \sigma_{\ell^2}(M)$$

whenever $m(k+l) \leq Cv(k)m(l)$ (m is v-moderate).

Spectrum of M is independent of domain space.

Proof Idea

De Leeuw, Gohberg, Baskakov
Define *modulation* M_t by $(M_t \mathbf{c})(k) = e^{2\pi ikt} c(k)$ for $k \in \mathbb{Z}$.

Given matrix A, consider matrix-valued function

$$\mathbf{f}(t) = M_t A M_{-t}$$

- $\mathbf{f}(t)$ is periodic with period 1.

Lemma (Fourier coefficients of f)

The n-th Fourier coefficient is the n-th side-diagonal of A.

Lemma (Fourier coefficients of $\mathbf{f}$)

The n-th Fourier coefficient is the n-th side-diagonal of A.

Proof. $\Big(M_t A M_{-t}\mathbf{c}\Big)(k) = e^{2\pi ikt}\sum_{l\in\mathbb{Z}} a_{kl} e^{-2\pi ilt} c_l$

$$\begin{aligned}
\widehat{\mathbf{f}}(n)_{kl} &= \int_0^1 \mathbf{f}(t)_{kl} e^{-2\pi int} dt \\
&= \int_0^1 a_{kl} e^{2\pi i(k-l)t} e^{-2\pi int} dt \\
&= a_{kl}\delta_{k-l-n} = a_{k,k-n}\,\delta_{k-l-n}
\end{aligned}$$

So $\widehat{\mathbf{f}}(n) = D(n)$ and

$$\mathbf{f}(t) \asymp \sum_{n\in\mathbb{Z}} D(n) e^{2\pi int}$$

Operator-Valued Fourier Series

Recall that $\|A\|_{\mathcal{C}} = \sum_{n\in\mathbb{Z}} \sup_{k\in\mathbb{Z}} |m_{k,k-n}| = \sum_{k\in\mathbb{Z}} \|D(n)\|_{op}$, so **f** possesses an operator-valued absolutely convergent Fourier series.

We need Wiener's Lemma for operator-valued Fourier series (done by Bochner-Phillips 1946).

Off-Diagonal Decay of Matrices

Other conditions:

$$\|M\|_{\mathcal{A}_v} = \sup_{k,l\in\mathbb{Z}} |m_{kl}|v(k-l)$$

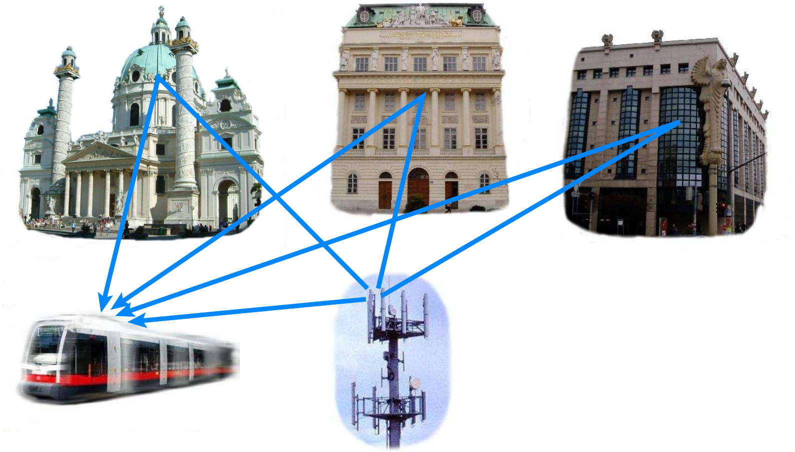
$$\|M\|_{\mathcal{A}_v^1} = \sup_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |m_{kl}|v(k-l) \qquad \forall M = M^*$$

Theorem (GL'06)

(a) If $v^{-1} \in \ell^1(\mathbb{Z})$, $v^{-1} * v^{-1} \leq Cv^{-1}$, and v satisfies the GRS condition, then $\mathcal{A}_v$ is inverse-closed in $\mathcal{B}(\ell^2)$.
(b) Assume that v is submultiplicative, $v(k) \geq (1+|k|)^\delta$ for some $\delta > 0$, and satisfies the GRS condition. Then $\mathcal{A}_v^1$ is inverse-closed in $\mathcal{B}(\ell^2)$.

Many open problems! Systematic constructions by A. Klotz, 2008

Time-Varying Systems



Time-Varying Channels — Continuous Case

Received signal $\tilde{f}$ is a superposition of time lags

$$\tilde{f}(t) = \int_{\mathbb{R}^d} V(u) \dots f(t+u)\, du$$

Received signal $\tilde{f}$ is a superposition of frequency shifts

$$\tilde{f}(t) = \int_{\mathbb{R}^d} W(\eta) \dots e^{2\pi i \eta t} f(t) d\eta$$

Thus received signal $\tilde{f} = K_\sigma f$ is a superposition of time-frequency shifts:

$$K_\sigma f(t) = \int_{\mathbb{R}^{2d}} \widehat{\sigma}(\eta, u) \underbrace{e^{2\pi i \eta \cdot t} f(t+u)} \, du d\eta$$

Modelling

Standard assumption of engineers: $\sigma \in L^2$ and $\hat{\sigma}$ has compact support.

Problem: Does not include distortion free channel and time-invariant channel.

So supp $\hat{\sigma}$ is compact, but $\hat{\sigma}$ is "nice" distribution. Then σ is bounded and an entire function.

Symbol Classes

Sjöstrand class $M^{\infty,1}(\mathbb{R}^{2d})$:

$$\|\sigma\|_{M^{\infty,1}} = \int_{\mathbb{R}^{2d}} \sup_{z\in\mathbb{R}^{2d}} |(\sigma \cdot T_z\Phi)\hat{}(\zeta)|\, d\zeta < \infty$$

$\Rightarrow \quad (\sigma \cdot T_z\Phi)\hat{} \in L^1_v$.
Locally σ is a Fourier transform of an L^1-function!

Weighted Sjöstrand class $M_v^{\infty,1}(\mathbb{R}^{2d})$.

$$\|\sigma\|_{M_v^{\infty,1}} = \int_{\mathbb{R}^{2d}} \sup_{z\in\mathbb{R}^{2d}} |(\sigma \cdot T_z\Phi)\hat{}(\zeta)|\, v(\zeta)\, d\zeta < \infty$$

Algebra Property of Sjöstrand Class

Proposition

If v is submultiplicative and $\sigma, \tau \in M_v^{\infty,1}$,then $K_\sigma K_\tau = K_{\sigma\circ\tau}$ with $\sigma \circ \tau \in M_v^{\infty,1}$. Thus $M_v^{\infty,1}$ is a Banach $*$-algebra with respect to $\circ$.

Transmission of Information by OFDM

Transmission of “digital word” $(c_k), c_k \in \mathbb{C}$ via pulse g
Transmitted signal is

$$f(t) = \sum_{k=0}^{\infty} c_k g(t-k)$$

with Fourier transform

$$\hat{f}(\omega) = \sum_{k=0}^{\infty} c_k \, e^{-2\pi i k \omega} \hat{g}(\omega)$$

$$\text{ess.supp}\, \hat{f} = \text{ess.supp}\, \hat{g}.$$

Multiplexing

Transmission of several “words” ($\Longleftrightarrow$ simultaneous transmission of a symbol group) by distribution to different frequency bands with modulation

Partial signal for ℓ-th word $\mathbf{c}^{(\ell)} = \left(c_{kl}\right)_{k\in\mathbb{Z}}$

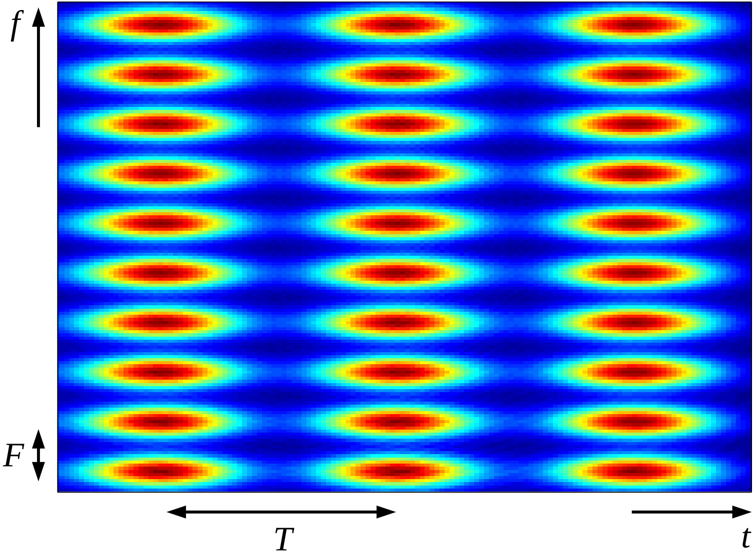
$$f_\ell = \sum_k c_{kl} T_k g$$

$$f_\ell = M_{\theta\ell}\left(\sum_k c_{kl} T_k g\right)$$

Total signal is a Gabor series (Gabor expansion)

$$f = \sum_{k,l} c_{kl} M_{\theta l} T_k g$$

If $M_\theta T_k g$ orthogonal, then OFDM (orthogonal frequency division multiplexing)


f
F
T
t

Decoding and the Channel Matrix

Received signal is

$$\widetilde{f} = K_\sigma\Big(\sum_{k',l'} c_{k'l'} M_{\theta l'} T_{k'} g\Big)$$

Standard procedure: take correlations

$$\langle \widetilde{f}, M_{\theta l} T_k g\rangle = \sum_{k',l'} c_{k'l'} \langle K_\sigma M_{\theta l'} T_{k'} g, M_{\theta l} T_k g\rangle$$

Solve the system of equations

$$\mathbf{y} = A\mathbf{c}$$

where $A_{kl,k'l'} = \langle K_\sigma(M_{\theta l'} T_{k'} g), M_{\theta l} T_k g\rangle$ is the channel matrix.

Almost Diagonalization of Time-Varying Channels

Standard assumption in wireless communications: A is diagonal. **!!**

Theorem

(A) If $g \in M^1_v$ and $\sigma \in M^{\infty,1}_v$, then there is $h \in \ell^1_v(\mathbb{Z}^{2d})$, such that

$$|\langle K_\sigma(M_{\theta l'} T_{k'} g), M_{\theta l} T_k g\rangle| \leq h(k-k', l-l') \tag{1}$$

(B) If $\{M_{\theta l} T_k g\}$ is a frame for $L^2(\mathbb{R}^d)$, then almost diagonalization (1) *implies that $\sigma \in M^{\infty,1}_v$.*

Short version:

Theorem

$\sigma \in M^{\infty,1}_v$ if and only if channel matrix $A \in \mathcal{C}_v$.

$M_v^{\infty,1}$ is Inverse-Closed

Theorem (Sjöstrand)

If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ and K_σ is invertible on $L^2(\mathbb{R}^d)$, then $K_\sigma^{-1} = K_\tau$ for some $\tau \in M^{\infty,1}$.

Theorem (KG)

Assume that v is submultiplicative and

$$\lim_{n\to\infty} v(nz)^{1/n} = 1, \quad \forall z \in \mathbb{R}^{2d}.$$

If $\sigma \in M_v^{\infty,1}(\mathbb{R}^{2d})$ and K_σ is invertible on $L^2(\mathbb{R}^d)$, then $K_\sigma^{-1} = K_\tau$ for some $\tau \in M_v^{\infty,1}$.

Intuition

Ingredients of Proof. Based on Wiener's Lemma for the matrix algebra $\mathcal{C}_v$ and several identities of time-frequency analysis.

Composition of pseudodifferential operators $\simeq$ matrix multiplication
$\Rightarrow$ algebra property
Inversion of pseudodifferential operators $\simeq$ matrix inversion
$\Rightarrow$ If matrix algebra is inverse-closed, then algebra of pseudodifferential operators is inverse-closed.

Hörmander's Class

Hörmander class: $\sigma \in S^0_{0,0}$ if and only if $\partial^\alpha \sigma \in L^\infty(\mathbb{R}^{2d})$, $\forall \alpha \geq 0$.

Observation: If $v_s(\zeta) = (1 + |\zeta|)^s$, then

$$S^0_{0,0} = \bigcap_{s \geq 0} M^{\infty,1}_{v_s}$$

Corollary (Beals '75)

If $\sigma \in S^0_{0,0}$ and K_σ is invertible on $L^2(\mathbb{R}^d)$, then $K_\sigma^{-1} = K_\tau$ for some $\tau \in S^0_{0,0}$.

Rotation Algebra

Time-frequency shifts

$$T_x f(t) = f(t-x) \qquad \text{and} \qquad M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \qquad x, \omega, t \in \mathbb{R}^d .$$

Commutation Relation (CCR)

$$T_x M_\omega = e^{-2\pi i x \cdot \omega} M_\omega T_x$$

The *rotation algebra* (non-commutative torus) $C^*(\theta), 0 \leq \theta < 1$, is the C^*-Algebra generated by T_k and $M_{\theta l}$, $k, l \in \mathbb{Z}^d$, i.e., $A \in \mathcal{B}(L^2)$ belongs to $C^*(\theta)$, if it can be approximated by finite series of time-frequency shifts $\sum_{|k|,|l| \leq N} c_{kl} M_{\theta l} T_k$

Interesting Subalgebras

Absolutely convergent series of time-frequency shifts

$$\mathcal{A}_v(\theta) = \{T \in \mathcal{B}(L^2(\mathbb{R}^d)) : T = \sum_{k,l\in\mathbb{Z}^d} a_{kl} T_k M_{\theta l}, \ \mathbf{a} \in \ell^1_v(\mathbb{Z}^{2d})\}$$

Norm $\|A\|_{\mathcal{A}_v} = \|\mathbf{a}\|_{1,v} = \sum_{k,l\in\mathbb{Z}^d} |a_{kl}|\, v(k,\theta l)$

Smooth non-commutative torus

$$\mathcal{A}_\infty(\theta) = \{T \in \mathcal{B}(L^2(\mathbb{R}^d)) : T = \sum_{k,l\in\mathbb{Z}^d} a_{kl} T_k M_{\theta l} \quad |a_{kl}| = \mathcal{O}((|k|+|l|)^{-N})\}$$

Wieners Lemma for the Rotation Algebra

Analogy: absolutely convergent Fourier series — absolutely convergent series of time-frequency shifts

Theorem

Assume that

- $A \in \mathcal{A}_v(\theta)$ *and*
- A *is invertible on* $L^2(\mathbb{R}^d)$,

then $A^{-1} \in \mathcal{A}_v(\theta)$.

Recall: If $f \in C^\infty(\mathbb{T})$ and $f(t) \neq 0, \forall t$, then $1/f \in C^\infty(\mathbb{T})$.

Corollary (Janssen '95, Connes '80, Rieffel '88)

If $A \in \mathcal{A}_\infty(\theta)$ *and is invertible on* $L^2(\mathbb{R}^d)$ $(|a_{kl}| = \mathcal{O}((1+|k|+|l|)^{-N}) \forall N \geq 0)$, *then* $A^{-1} \in \mathcal{A}_\infty(\theta)$, *i.e.* $A^{-1} = \sum_{k,l \in \mathbb{Z}^d} b_{kl} T_k M_{\theta l}$ *with rapidly decaying* **b**.

Twisted Convolution

$(\mathbf{a} \natural_\theta \mathbf{b})(k,l) = \sum_{k',l' \in \mathbb{Z}^d} a_{k'l'} b_{k-k',l-l'} e^{-2\pi i \theta k' \cdot (l-l')}$

1. $\theta \in \mathbb{Z}$, $\mathbf{a} \natural_\theta \mathbf{c} = \mathbf{a} * \mathbf{c}$
2. $\theta \notin \mathbb{Z} \implies \natural_\theta$ non-commutative.

(Twisted) convolution operator $T_\mathbf{b}\mathbf{c} = \mathbf{b} \natural_\theta \mathbf{c}$

Theorem (Wiener's lemma)

Assume that

- $\mathbf{a} \in \ell^1(\mathbb{Z}^{2d})$ *and*
- $T_\mathbf{a}$ *is invertible on* $\ell^2(\mathbb{Z}^{2d})$,

then $\mathbf{a}$ *is invertible on* $(\ell^1(\mathbb{Z}^{2d}), \natural_\theta)$ *and* $T_\mathbf{a}^{-1} = T_\mathbf{b}$ *for some* $\mathbf{b} \in \ell^1(\mathbb{Z}^{2d})$.

Convolution Operators on Locally Compact Groups I

G . . . locally compact group with Haar measure dx

convolution: $(f * g)(x) = \int_G f(y)g(y^{-1}x)\, dy$

convolution operator C_f: $C_f h = f * h$
If $f \in L^1(G)$, then C_f is bounded on $L^p(G)$.

$\sigma_{L_p}(C_f) = \{\lambda \in \mathbb{C} : C_f - \lambda \mathrm{I} \text{ not invertible on } L^p\}$

Barnes' Lemma

Lemma

Spectral invariance $\sigma_{L^p}(C_f) = \sigma_{L^2}(C_f)$ for $1 \leq p \leq \infty$ holds, if and only if G is amenable and symmetric.

- Amenability: existence of (translation) invariant mean on $L^\infty(G)$
- Symmetry: spectrum of positive elements is positive

$$\sigma_{L^1}(f^* * f) \subseteq [0, \infty) \quad \forall f \in L^1(G)$$

- Symmetry of $\mathcal{A} \Leftrightarrow \mathcal{A}$ is inverse-closed in its enveloping C^*-algebra.

Convolution Operators on Locally Compact Groups II

Polynomial growth: G has polynomial growth, if for some neighborhood $U = U^{-1}$ of identity $|U^n| \leq Cn^d$

Example: $G = \mathbb{R}^d$, $U = [-1, 1]^d$, $U^n = [-n, n]^d$, $|U^n| = (2n)^d$.

Theorem (Losert '01)

Every compactly generated group of polynomial growth is symmetric.

Corollary

Wiener's Lemma holds in all groups of polynomial growth, i.e., $\sigma_{L^p}(C_f) = \sigma_{L^2}(C_f)$ for all $f \in L^1(G)$ and all $1 \leq p \leq \infty$.

Weights on Groups

- submultiplicativity $v(xy) \leq v(x)v(y)$ for $x, y \in G$
- symmetry $v(x^{-1}) = v(x)$
- weighted L^1 with norm

$$\|f\|_{L^1_v} = \int_G |f(x)|v(x)\,dx$$

Then L^1_v is involutive Banach algebra contained in $L^1(G)$.

Weighted Versions

Theorem (FGLLM)

Assume that G is compactly generated and possesses polynomial growth. TFAE:
(i) $L^1_v(G)$ *is symmetric.* Spectrum of positive operators is positive.
(ii) Spectral invariance

$$\sigma_{L^1_v}(C_f) = \sigma_{L^2}(C_f) \qquad \forall f \in L^1_v(G)$$

(iii) The weight v satisfies the GRS condition (Gelfand, Raikov, Shilov)

$$\lim_{n\to\infty} v(x^n)^{1/n} = 1 \qquad \forall x \in G$$

More Applications in Mathematics

- Theory of localized frames: what can be said about the dual frame
- (Non-uniform) Sampling in shift-invariant spaces, local reconstructions
- Collaboration with F. Hlawatsch, G. Matz, G. Tauböck from TU Vienna on almost diagonalization of channel matrix and improved equalization

References

- Wiener's Lemma:

Banach algebra books by Rickart and Bonsall-Duncan

- Weights: Naimark "Normed Rings"

Gelfand-Raikov-Shilov
K. Gröchenig "Weight functions in time-frequency analysis" (Comm. Fields Institute)

- Matrix algebras:

Jaffard, 1990
Baskakov 1990, Gohberg, Kashoeck, Woerdeman (1990)
Gröchenig, Leinert, TAMS 2006

- Rotation algebra: Rieffel, Bull. Can. Math. Soc. 1988

Gröchenig, Leinert, "Wiener's Lemma for twisted convolution and Gabor frames", JAMS 2004

References Continued

- Pseudodifferential operators:

Sjöstrand, 1994, 1995
Gröchenig, J. Anal. Math. 2006, Rev. Math. Iberoam. 2006

- Mobile communication:

Strohmer, JFAA 1999, ACHA 2006

- Convolution Operators:

work of Leptin, Poguntke, Hulanicki from 1965-1980
Fendler, Gröchenig, Leinert, Ludwig, Molitor-Braun "Symmetry of weigthed L^1-algebras of groups of polynomial growth", Math. Z. 2003, Bull. LMS 2005.

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Summary

Thank you!