HANS GEORG FEICHTINGER
FROM ABSTRACT TO NUMERICAL HARMONIC ANALYSIS

KARLHEINZ GRÖCHENIG

Dedicated to Hans Feichtinger on the occasion of his 65th birthday.

1. Introduction

Hans Feichtinger has officially retired in December 2015 and has celebrated his 65th birthday on June 16, 2016. However, the "retirement" is a distortion of facts. Since Hans's official retirement he has been more active than ever, his traveling activity has increased exponentially. Between two short courses or colloquia he briefly visits us at NuHAG and brings us news.

The official date shall serve as an occasion to celebrate and honor Hans Feichtinger. This article pays homage to Hans and highlights a few accomplishments of his "official" career. It is also an account of our long personal and mathematical friendship.

2. Mathematical Origins — Abstract Harmonic Analysis (AHA)

Hans started his mathematical career at the peak of abstract harmonic analysis. His thesis "Subalgebras of $L^1(G)$" (1974) was written under the advisor Hans Reiter. Reiter's book "Classical Harmonic Analysis and Locally Compact Groups" [36] had a profound impact on the scientific orientation of the Department of Mathematics at the University of Vienna and must be considered the basis of the Viennese school of harmonic analysis.

Originally Hans had wanted to become a high school teacher, but he soon discovered his vocation for active research. After the Ph. D. thesis he became very productive, and within only five years he wrote his habilitation thesis on "Banach convolution algebras of functions" (1979). At that time the habilitation was equivalent to tenure, consequently at the age of 28, Hans had already a permanent position at the Department of Mathematics at the University of Vienna.

The topics of his first years in mathematical research covered convolution algebras, ideal theory of subalgebras, factorization theorems, and many other directions of abstract harmonic analysis.

Looking at his thesis and his habilitation one notices that Hans has remained faithful to his mathematical origins. Hans is still pursuing and refining his first mathematical love. The style, however, has changed radically. Abstract harmonic

K. G. was supported in part by the project P26273 - N25 of the Austrian Science Fund (FWF).

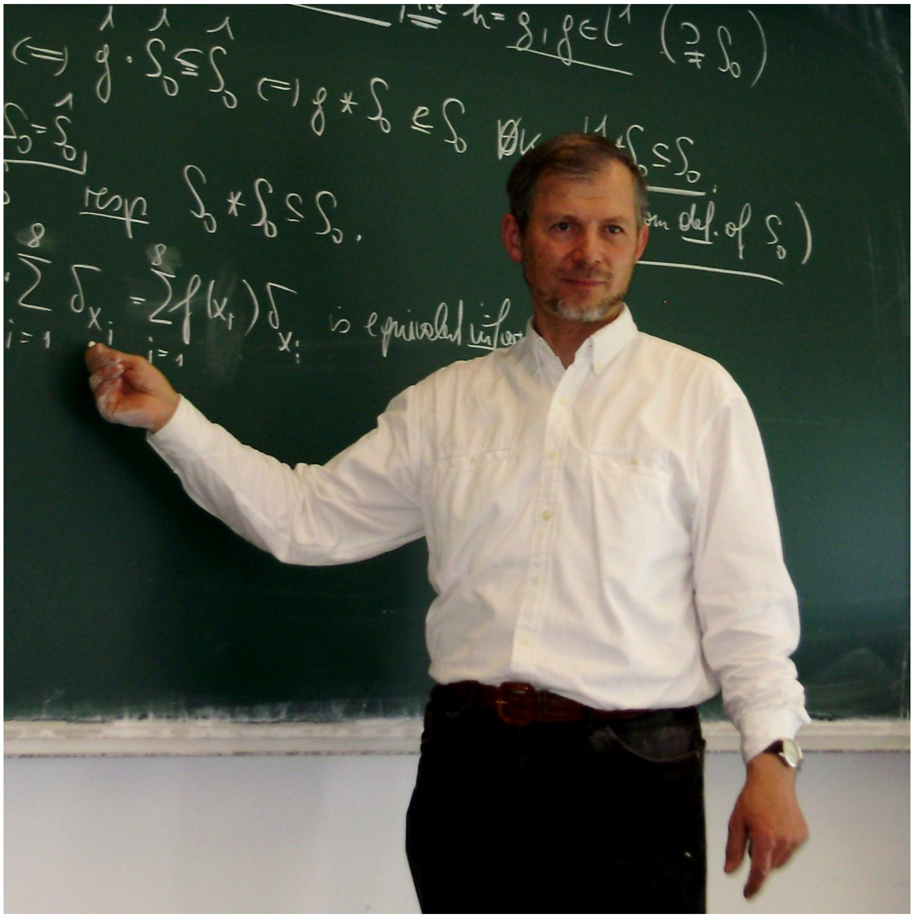


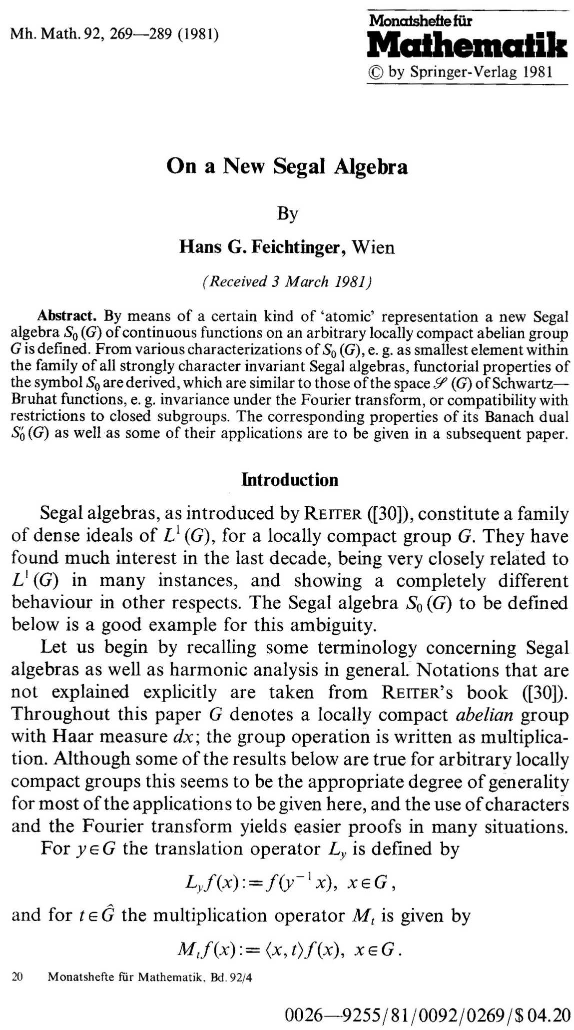
FIGURE 1.1. Hans Feichtinger explaining the Feichtinger algebra

analysis is no longer persued as an end in itself, but now serves as a tool for the elegant formulation of problems in signal analysis and as a framework for application-oriented mathematics.

3. Mathematical Visions

3.1. **The Feichtinger Algebra.** Hans first claim to fame is the discovery of what we now call the Feichtinger algebra. Starting as an abstract harmonic analyst, he investigated a class of Segal algebras and tried to find a minimal isometrically character invariant Segal algebra. In [10] he showed that there exists such a minimal Segal algebra and for obvious reasons called it S_0. The first version of S_0 can be formulated as a mathematical existence theorem.

Let $T_x f(t) = f(t-x)$ be the operator of translation and $M_\xi f(t) = e^{2\pi i \xi \cdot t}$ be the operator of modulation acting on $f \in L^2(\mathbb{R}^d)$. Hans found and characterized the *minimal character-invariant Segal algebra.* In less technical terms, the existence theorem can be formulated as follows.



Mh. Math. 92, 269—289 (1981)

Monatshefte für Mathematik

On a New Segal Algebra

By

Hans G. Feichtinger, Wien

(Received 3 March 1981)

Abstract. By means of a certain kind of 'atomic' representation a new Segal algebra $S_0(G)$ of continuous functions on an arbitrary locally compact abelian group G is defined. From various characterizations of $S_0(G)$, e. g. as smallest element within the family of all strongly character invariant Segal algebras, functorial properties of the symbol S_0 are derived, which are similar to those of the space $\mathscr{S}(G)$ of Schwartz—Bruhat functions, e. g. invariance under the Fourier transform, or compatibility with restrictions to closed subgroups. The corresponding properties of its Banach dual $S_0'(G)$ as well as some of their applications are to be given in a subsequent paper.

Introduction

Segal algebras, as introduced by Reiter ([30]), constitute a family of dense ideals of $L^1(G)$, for a locally compact group G. They have found much interest in the last decade, being very closely related to $L^1(G)$ in many instances, and showing a completely different behaviour in other respects. The Segal algebra $S_0(G)$ to be defined below is a good example for this ambiguity.

Let us begin by recalling some terminology concerning Segal algebras as well as harmonic analysis in general. Notations that are not explained explicitly are taken from Reiter's book ([30]). Throughout this paper G denotes a locally compact *abelian* group with Haar measure dx; the group operation is written as multiplication. Although some of the results below are true for arbitrary locally compact groups this seems to be the appropriate degree of generality for most of the applications to be given here, and the use of characters and the Fourier transform yields easier proofs in many situations.

For $y \in G$ the translation operator L_y is defined by

$$L_y f(x) := f(y^{-1}x), \quad x \in G,$$

and for $t \in \hat{G}$ the multiplication operator M_t is given by

$$M_t f(x) := \langle x, t\rangle f(x), \quad x \in G.$$

0026—9255/81/0092/0269/$ 04.20

FIGURE 3.1. The famous paper on S_0

Theorem 1. *There exists a unique minimal Banach space S_0 of functions on $\mathbb{R}^d$ with a norm $\|\cdot\|_{S_0}$ such that S_0 contains the Gaussian $e^{-\pi x\cdot x}$ and $\|T_x M_\xi f\|_{S_0} = \|f\|_{S_0}$ for all $f \in S_0$. In addition, S_0 is invariant under the Fourier transform and is a Banach algebra with respect to pointwise multiplication and convolution.*

Hans gave a detailed construction of S_0 and formulated this result for general locally compact Abelian groups. See Figure 3.1.

It soon turned out that this space has many beautiful properties that go far beyond its abstract characterization. The Feichtinger algebra now serves as a space of test functions in harmonic analysis and replaces the more complicated Schwartz space for most purposes.

Indeed, Hans soon developed a very original and personal approach to Fourier analysis that is based on the properties of S_0. This approach does not require the fine details of Lebesgue integration and only a minimum of basic functional

H.G. Feichtinger: A Banach space of Distributions with Applications to Harmonic Analysis

Let G be a locally compact, abelian group, and an open subset with compact closure. Then the space $S_o(G)$ is given by

$S_o(G) := \{ f \mid f = \sum_{n=1}^{\infty} L_{y_n} f_n,\ \mathrm{supp}\, f_n \subseteq Q,\ \Sigma \|\hat{f}_n\|_1 < \infty \}$.

For a suitable norm this space is a translation invariant Banach space of continuous, integrable functions on G. It does not depend on the choice of Q and contains the space $S(G)$ of Schwartz-Bruhat functions. Besides several other useful properties one has: Fourier transform maps $S_o(G)$ onto $S_o(\hat{G})$. Consequently $S'_o(G)$ is a Banach space of tempered distributions which is mapped onto $S'_o(\hat{G})$ by the F.T. . As most of the important properties of the space $S'(G)$ is shared by $S'_o(G)$ (e.g. the 'kernel-theorem'), but can be proved without making use of structure theory, this space appears as useful tool for Harmonic Analysis. Examples are given (multiplier-theory, transformable measures). There are also results for non-abelian groups G.

FIGURE 3.2. New developments on S_0. Abstract for an Oberwolfach conference.

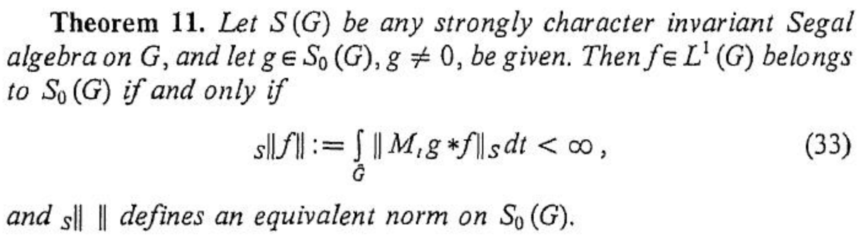
analysis. As a student I attended one of Hans's first courses in which he began to develop his personal style of Fourier analysis. Now, 35 years later, Hans has refined and perfected his approach to Fourier analysis. He can rightly claim that this point of view may be better suited to the needs of engineers than the classical presentations of Fourier analysis.

From 1995 on our systematic work on time-frequency analysis revealed that S_0 is the canonical space of test functions for time-frequency analysis.

The modern definition of S_0 is given in the context of time-frequency analysis and modulation spaces. Let

$$M_\xi T_x g(t) = e^{2\pi i \xi \cdot t} g(t-x) \qquad x, \xi, t \in \mathbb{R}^d \,, \tag{1}$$

be the time-frequency shift by $(x,\xi) \in \mathbb{R}^{2d}$ of a function $g \in L^2(\mathbb{R}^d)$ and define the short-time Fourier transform of f with respect to the fixed, non-zero window g to


Theorem 11. *Let $S(G)$ be any strongly character invariant Segal algebra on G, and let $g \in S_0(G)$, $g \neq 0$, be given. Then $f \in L^1(G)$ belongs to $S_0(G)$ if and only if*

$$_S\|f\| := \int_{\hat{G}} \|M_t g * f\|_S \, dt < \infty\,, \qquad (33)$$

and $_S\|\ \|$ defines an equivalent norm on $S_0(G)$.

FIGURE 3.3. Almost the modern definition of S_0 as a modulation space from [10] .

be

$$V_g f(x,\xi) = \langle f, M_\xi T_x g\rangle = (f * M_\xi \bar{g})(x)\,. \tag{2}$$

Then a function $f \in L^2(\mathbb{R}^d)$ belongs to S_0, if and only if

$$\|f\|_{S_0} := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |V_g f(x,\xi)|\, dx d\xi < \infty\,. \tag{3}$$

In contrast to the original construction, it is very easy to see that translations and modulations are isometries in the norm (3).

3.2. **Modulation Spaces.** The definition (3) lead to an immediate generalization. Replacing the L^1-norm on $V_g f$ by other norms, Hans introduced the class of modulation spaces. As so often, he immediately proceeded in full generality and defined and investigated the basic properties of modulation spaces on locally compact Abelian groups. His long manuscript from 1983 did not receive the approval of the referees and editors and circulated as an influential technical report [11], until it was finally published in 2003 in [14].

The original definition of the modulation space $M^{p,q}_m$ was through the norm

$$\|f\|^q_{M^{p,q}_m} = \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} |(f * M_\xi g)(x)|^p m(x,\xi)^p dx\Big)^{q/p} d\xi\,.$$

Here g is some non-zero test function, $1 \le p, q \le \infty$ and $m > 0$ a suitable weight function on $\mathbb{R}^{2d}$. In view of (2) we see that this is indeed a weighted mixed L^p-norm of the short-time Fourier transform, the Feichtinger algebra S_0 is the special modulation space $M^{1,1}$ with constant weight.

This definition should be compared with the analogous definition of a Besov space, which is defined by taking a mixed L^p-norm of the expression $f * D_t g$ where $D_t g(x) = t^{-d} g(x/t), t > 0$ is the dilation. Hans's original motivation for modulation spaces was not generality per se, but he tried to invent a new theory of functions spaces and to offer an interesting alternative to the class of Besov spaces. Thus, in the beginning, the investigation naturally emphasized the analogy between Besov spaces and modulation spaces and focussed on modulation spaces as an aspect of the theory of function spaces. It was only some years later that the full use and context of modulation spaces became apparent.

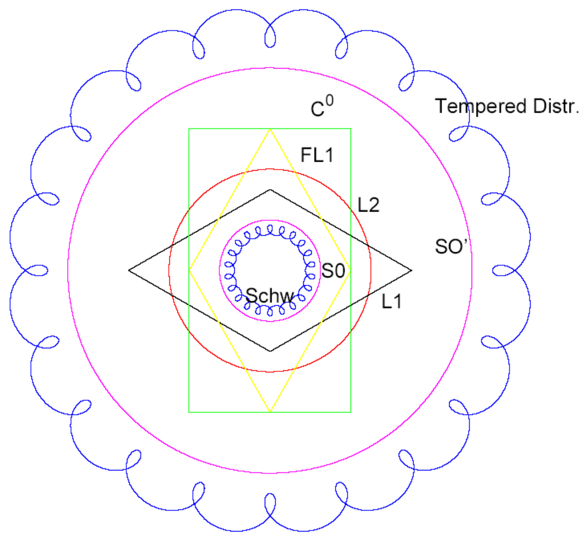


FIGURE 3.4. S_0 in relation to other spaces

From a modern perspective, $M^{p,q}_m$-norms measure the smoothness of a function by means of the time-frequency concentration (or phase-space concentration in terms of physics) rather than by differences and derivatives.

It is an irony of historical development that in recent years several mathematicians have rediscovered the similarity of modulation spaces with Besov spaces and applied corresponding techniques to the modulation spaces.

Some modulation spaces were known before the general definition, notably the Bessel potential spaces $H^2 = M^{2,2}_m$ with weight $m(x,\xi) = (1+|\xi|^2)^{s/2}$ and the Shubin classes $M^{2,2}_{v_s}$ with weight $v_s(x,\xi) = (1+|x|^2+|\xi|^2)^{s/2}$. The special modulation space $M^{\infty,1}_m$ with constant weight function $m \equiv 1$ was rediscovered by Sjöstrand [38]. Because of its importance in PDE is often called the Sjöstrand class. Similarly, the modulation space $M^{2,1}_m$ with $m(x,\xi) = (1+|\xi|)^s$ was rediscovered in PDE by B. Wang [2].

Figure 3.2 is Hans's favorite visualization of S_0 and its relation to other standard spaces in Fourier analysis.

3.3. **Contemporary Applications of Modulation Spaces.** Modulation spaces are useful and arise in many areas of mathematics and in many real applications. In fact, modulation spaces appear naturally and necessarily whenever time-frequency shifts $M_\xi T_x$ are used. Thus modulation spaces are unavoidable in time-frequency analysis. Here is a short list of such applications:

(1) Gabor expansions [19, 27]
(2) Nonlinear approximation with time-frequency shifts [31]
(3) Modulation spaces as symbol classes for pseudodifferential operators [28, 29, 38]
(4) Formulation of uncertainty principles [26]
(5) Mathematics and Music [1]

(6) Investigation of time-frequency localization operators [7, 25]
(7) Modulation spaces and the Schrödinger equation [4, 8]
(8) Wireless communications [35, 39]

As a particularly important application I mention the theory of Gabor expansions. These yield series expansions of arbitrary functions or distributions with respect ot a discrete set of time-frequency shifts $\{M_{\xi_k} T_{x_k} g\}$ of a fixed template g. A very general early theorem is already contained in [17].

Theorem 2 (Gabor Expansions)**.** *Let $g \in \mathcal{S}(\mathbb{R}^d), g \neq 0$ and $\mathcal{X} = \{(x_j, \xi_j)\} \subseteq \mathbb{R}^{2d}$ relatively separated, $1 \leq p \leq \infty$ and $m > 0$ a moderate weight function.*

(i) If $\mathbf{c} \in \ell^p_m(X)$, i.e., $\sum_j |c_j|^p \, m(x_j, \xi_j)^p < \infty$, then $f = \sum_{j \in J} c_j M_{\xi_j} T_{x_j} g$ is in $M^{p,p}_m(\mathbb{R}^d)$ and $\|f\|_{M^{p,p}_m} \leq C \|c\|_{\ell^p_m}$.

(ii) Conversely, if $\mathcal{X}$ is dense enough in $\mathbb{R}^{2d}$, then every $f \in M^{p,p}_m(\mathbb{R}^d)$ possesses a Gabor expansion with ℓ^p_m coefficients.

3.4. **Gabor Analysis.** While Hans sees himself mainly as a theory builder and not no much as a problem solver, he has several important and deep theorems to his credit.

In particular, I would like to mention the following results:

(i) *The duality theory of Gabor frames over general time-frequency lattices:* The duality theory was discovered by Janssen [33] and fully exhibited by Ron-Shen [37] and Daubechies-Landau-Landau [9]. Originally the duality was formulated only for rectangular lattices, and it took some time and Hans's insight to understand the role of the adjoint lattice and to formulate the duality theory for arbitary time-frequency lattices [22]. In particular Hans found an easy and transparent proof of the duality theorem that requires only the Poisson summation formula [23] (whereas the previous proofs has used fancy tools from von Neumann algebras or the fibrization technique).

(ii) *Perturbation of the lattice (deep) and other perturbation results:* While the standard perturbation theory of frames deals with local perturbations, Hans established a new line of research by studying global perturbations, which are better called deformations. To see what is at stake, let $\{\pi(\lambda)g : \lambda \in \Lambda\}$ be a Gabor frame on a lattice Λ. A perturbation of Λ is a set λ', such that the Hausdorff distance between Λ and Λ', $d(\Lambda, \Lambda') = \sup_{\lambda \in \Lambda} \inf_{\lambda' \in \Lambda'} |\lambda' - \lambda| \leq \delta$ for small $\delta > 0$. It is then easy and has been proved many times that the frame property is preserved under a sufficiently small perturbation. Hans and N. Kaiblinger considered how the frame property behaves under a change of the lattice from Λ to $\Lambda' = A\Lambda$ for some matrix A close to the identity matrix. In this case the Hausdorff distance between Λ and Λ' can never be small. Nevertheless, Hans proved the following fundamental result [21].

Theorem 3. *Assume that $g \in S_0$, Λ is a lattice in $\mathbb{R}^{2d}$, and $\{\pi(\lambda)g : \lambda \in \Lambda\}$ is a Gabor frame. Then there exists a $\delta > 0$ such that for every matrix A with $\|A - \mathrm{Id}\| < \delta$, the deformed Gabor system $\{\pi(\lambda')g : \lambda' \in A\Lambda\}$ is still a Gabor frame.*

Note again the crucial role of the assumption $g \in S_0$. The theorem is false without this hypothesis. This result has deeply inspired my own research [30] and will ultimately lead to a much bigger theory for the deformation of general frames. As it turns out, Hans's result is closely related to a deep result of Bellissart in non-commutative geometry [3].

3.5. **Coorbit Theory.** The biggest mathematical success was undoubtedly the creation of coorbit theory. In 1986 Hans read the papers of Daubechies, Grossmann, Morlet, Meyer and saw the connections between the emerging theory of wavelets and Besov spaces on the one hand, and between Gabor frames and modulation spaces on the other hand. He immediately understood that the natural framework would be that of square-integrable group representations and their reproducing formulas. Grossmann and Morlet [32] had formulated such reproducing formulas just for the underlying Hilbert space, but Hans saw how to define abstract function spaces in this setting.

I am a first-hand witness to Hans's mathematical vision. We attended an AMS-DMV seminar on harmonic analysis near Düsseldorf (with Elias Stein and Detlev Poguntke as lecturers, and I understood nothing at that time). On the train ride from Düsseldorf to Vienna Hans started to talk about his ideas, and he did not stop talking until we arrived eight hours later. With infinite energy he kept circling around his idea of a unifying theory of function spaces. In many iterations he kept refining it and sharpening his formulations. I cannot say that I fully understood at the time what he wanted, but occasionally I could keep him on track and prevent him from following thoughts that did not make sense. As I had finished my Ph. D. several months before on a topic in abstract harmonic analysis and representation theory and had some knowledge of representation theory. I got hooked on Hans's ideas, and so we started to collaborate for many years.

As so often in his career, Hans startet a new idea with a conference article. To this day I am stunned that the brandnew coorbit theory was first published as a conference proceedings [16] and was classified as a *survey* article! See Figure 3.5.

Coorbit theory is undoubtedly our most successful work. According to MathSciNet our paper [17] is the most cited papers in the MSC class 43 "Abstract Harmonic Analysis". However, coorbit theory was also one of our most frustrating experiences. Some of the initial referees' reports dismissed the theory as "soft analysis". From the onset coorbit theory was formulated in full generality and a high degree of abstraction. Perhaps for this reason coorbit theory was recognized and became more fashionable only after 2000.

A modern "publication strategy" would start with the treatment of classes of examples, say two papers for the coorbit theory of Besov spaces and two papers for modulation spaces, then try to generalize the pattern of the examples step by step, say, from L^p to weighted mixed norm L^p to arbitrary solid function spaces. Ultimately we should have proposed a unifying "theory of everything", and then have continued with detailed papers on special aspects of coorbit space theory. Instead we crammed the most important aspects of what we knew into three densely written papers and stopped. We certainly knew what else could be done, but we

CONTENTS.

FIGURE 3.5. The "survey" paper

moved to more successful topics, and later found some of our unpublished material in other papers.

What is coorbit theory? Hans and I developed coorbit theory in 1987-90 as a universal wavelet theory based on group representations. As a sample theorem that exhibits the ingredients of coorbit theory I formulate the result about the existence of coherent frames.

Theorem 4 (coherent frames)**.** *Let $(\pi, \mathcal{H})$ be (square) integrable, irreducible unitary representation of a locally compact group $\mathcal{G}$. If $\mathcal{X} = \{x_j : j \in J\} \subseteq \mathcal{G}$ is sufficiently dense in $\mathcal{G}$ and $g \in \mathcal{B}$ ("nice admissible vectors"), then*

$$\{\pi(x_j)g : j \in J\}$$

is a frame for $\mathcal{H}$

Comments:

Theorem 4. *Let G be a non-compact locally compact abelian group, B a Banach space of locally integrable functions with the properties stated above, $\Omega \subseteq \hat{G}$ a compact set, and $h \in L^1_w(G)$ a function such that $\hat{h}(t) = 1$ for $t \in \Omega$ and $\hat{h}(t) = 0$ outside an open neighborhood Ω_0 of Ω. Then there exists a neighborhood U of the identity such that any $f \in B^\Omega$ can be completely and stably reconstructed from any U-dense, discrete set $(x_i)_{i \in I}$ of G.*

The reconstruction can be carried out by the iteration procedure (53)–(55). *Alternatively there exist functions $e_i \in L^1_w(G)$, $\operatorname{supp} e_i \subseteq \Omega_0$ such that*

$$f = \sum_{i \in I} f(x_i) e_i$$

where the series converges in B and uniformly on compact sets.

Figure 3.6. Abstract sampling theorem.

(1) This theorem implies the existence of wavelet and Gabor frames, it implies sampling theorems for Bargmann and Bergman spaces, anisotropic wavelet frames etc. In fact, some of these special cases were derived much later by the specialists in their respective areas.
(2) The set $\mathcal{X}$ must be "dense enough", but is not required to have a particular structure. Thus at a time, when the experts tried to understand Gabor frames on rectangular lattices or wavelet frames with dyadic dilations and integer shifts, this theorem already offered a construction, albeit qualitative, of non-uniform wavelet and Gabor frames.
(3) Hans always considered it a very important aspect of the general coorbit theory that it treats families of function spaces and not just a single space. Thus the abstract coorbit space $\mathrm{Co}_\pi L^p_m$ is characterized by expansions of the form $f = \sum_{j \in J} c_j \pi(x_j) g$ with coefficients in ℓ^p_m. Whereas Theorem 4 is formulated for Hilbert space, the general version for coorbit spaces yields atomic decompositions and Banach frames for coorbit spaces.

3.6. **Non-Uniform Sampling of Bandlimited Functions.** While working on the coorbit theory, Hans saw also a vague connection to the sampling of bandlimited functions. In 1988 the state of the art was the important survey article of Butzer, Splettstößer, Stens [5]. This survey treated exclusively the uniform sampling in the line of Shannon's sampling theorem. According to Butzer nothing was known about non-uniform sampling (except for a few classical papers in complex analysis).

The methods of coorbit theory lead to a series of abstract "constructive sampling results" for bandlimited functions in $L^p_m(\mathbb{R}^d)$, and even on LCA groups. See Figure 3.6. The first papers on sampling theory amounted to a "proof of concept" (to use a fashionable word) and showed that a bandlimited function with given spectrum can be reconstructed by an explicit iterative algorithm from its nonuniform samples. As in the case of coorbit theory, the generality was too much for the time. Only nowadays it has become fashionable to reformulate results for $\mathbb{R}^d$ also for LCA groups.

From a technical point of view, the nonuniform sampling theory and coorbit theory share the same inspiration and the same mathematical sources: reproducing

Theorem 1 (and Algorithm). *Let M be the size of the spectrum and let $0 \le t_1 < \ldots < t_r < 1$ be an* arbitrary *sequence of sampling points with $r \ge 2M+1$. Set $t_0 = t_r - 1, t_{r+1} = t_1 + 1$ and $w_j = \frac{1}{2}(t_{j+1} - t_{j-1})$ and compute*

$$\gamma_k = \sum_{j=1}^{r} \mathrm{e}^{-2\pi i k t_j} w_j \quad \textit{for } k = 0, 1, \ldots, 2M \, .$$

The associated Toeplitz matrix has $(T_w)_{lk} = \gamma_{l-k}$ for $|l|, |k| \le M$.

To reconstruct a trigonometric polynomial $p \in \mathscr{P}_M$ from its samples $p(t_j)$, compute first

$$b_k = \sum_{j=1}^{r} p(t_j) w_j \mathrm{e}^{-2\pi i k t_j} \quad \textit{for } |k| \le M \, ,$$

and set $r_0 = q_0 = b \in \mathbb{C}^{2M+1}$, $a_0 = 0$. Compute iteratively for $n \ge 1$

$$a_n = a_{n-1} + \frac{\langle r_{n-1}, q_{n-1} \rangle}{\langle T_w q_{n-1}, q_{n-1} \rangle} \, q_{n-1}$$

$$r_n = r_{n-1} - \frac{\langle r_{n-1}, q_{n-1} \rangle}{\langle T_w q_{n-1}, q_{n-1} \rangle} \, T_w q_{n-1}$$

and

$$q_n = r_n - \frac{\langle r_n, T_w q_{n-1} \rangle}{\langle T_w q_{n-1}, q_{n-1} \rangle} \, q_{n-1} \, .$$

Then a_n converges in at most $2M+1$ steps to a vector $\mathbf{a} \in \mathbb{C}^{2M+1}$ solving $T_w \mathbf{a} = \mathbf{b}$. The reconstruction $p \in \mathscr{P}_M$ is then given by $p(t) = \sum_{k=-M}^{M} a_k \mathrm{e}^{2\pi i k t}$.

If in addition $\delta < \frac{1}{2M}$ and $p_n(t) = \sum_{k=-M}^{M} a_{n,k} \mathrm{e}^{2\pi i k t} \in \mathscr{P}_M$ denotes the approximating polynomial after n iterations, then

$$\left(\sum_{j=1}^{r} |p(t_j) - p_n(t_j)|^2 w_j \right)^{1/2} \le 2(2\delta M)^n \left(\sum_{j=1}^{r} |p(t_j)|^2 w_j \right)^{1/2} \tag{38}$$

Figure 3.7. The ACT algorithm

formulae, the theory of amalgam spaces, and convolution relations. Hans was and is a great virtuoso in the application of these tools.

3.7. **Towards Numerical Harmonic Analysis.** Our first papers on nonuniform sampling were again extremely general and derived iterative algorithms that work on weighted L^p-spaces and even on locally compact groups. However, simultaneously we thought about numerical implementations. I remember my first semester at the University of Connecticut in 1988; I had to learn everything simultaneously: English, teaching, LaTex, and Matlab. My first numerical experiment in Matlab was a ten line implementation of one of the iterative algorithms we had proposed. It worked, it converged, it demonstrated the expected behavior. The problem seemed settled, I lost interest, and moved to more theoretical questions.

No so Hans. He developed a contagious enthusiasm for the new algorithmic toy and continued to experiment with them and to refine the implementations. From the beginning he pushed for the development of numerical algorithms and emphased their implementation. What a complete change of perspective! After years of abstract harmonic analysis and important general ideas, Hans devoted himself with great passion to numerical questions and details of implementation

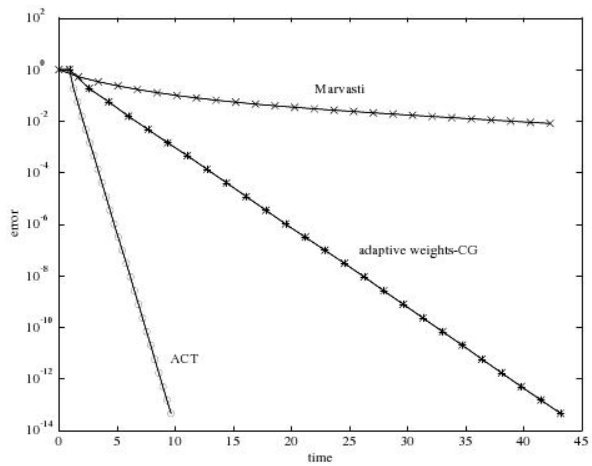


Fig. 1. Comparison of required time for 25 iterations

FIGURE 3.8. Convergence rates of several reconstruction algorithms

(without neglecting mathematics). The list of titles from that period documents this change of paradigm for Hans.

1990 "Iterative methods in irregular sampling theory: numerical results"

1991 "Iterative methods in irregular sampling: a first comparison"

1992 "IRSATOOL — Irregular Sampling of band-limited signals, TOOLBOX"

1990 — 1992 Project "Experimental signal analysis"

The final product of this period is the ACT algorithm from 1995 [20] with Thomas Strohmer. See Figure 3.7. This algorithm for nonuniform sampling combines "**A**daptive weights", the "**C**onjugate gradient acceleration", and the "**T**oeplitz structure" of the problem and yields an algorithm of order $n \log n$. To this date this seems to be most successful reconstruction algorithm for nonuniform sampling. Later versions and variations appeared under various names, such as "nonuniform FFT", on the numerical market. In computer tomography the adaptive weights have become a big deal under the name "density compensation factors".

In my view Hans was far ahead of his time and had a sixth sense of things to come. At a time when numerical analysis was a subject of its own and numerical simulation was practiced mainly in engineering departments, he developed his own brand of numerical harmonic analysis. Indeed, not long afterwards, scientific computating, computational sciences, and the foundations of computational mathematics became mainstream. In the wavelet community this trend led to the foundation of "Applied and Computational Harmonic Analysis" in 1993, the

FIGURE 3.9. Analysis of astronomical data.

"Journal of Fourier Analysis and Applications" in 1994, and the biannual conference series on "Sampling Theory and Applications" in 1995. Needless to say that Hans was involved in these enterprises from the very beginning.

3.8. **A Zoo of Applications.** Over the years, Hans initiated and guided many applied projects. The sources of data cover many different fields of sciences, such as astronomy, medical imaging, and music.

He went into the image processing of astronomical data. Astronomy is an excellent test ground for many data processing methods. Hans succeeded in applying both time-frequency analysis and nonuniform sampling (Figure 3.8). This application is still close to Hans's interests and he is currently working on an interdisciplinary project with the department of astronomy.

An earlier application was devoted to the analysis of electrocardiograms, specifically the ECG-analysis during ventricular fibrillation (Figure 3.8).

Finally I mention mathematics and music, a subject dear to many mathematicians. Hans used Gabor analysis for the music transcription problem. Roughly the goal is to generate a score from an acoustic recording. In this problem time-frequency methods are unavoidable, since spectrogram, short-time Fourier transform, and other time-frequency representations are considered the mathematical metaphors for a musical score. Hans organized a workshop and edited a book on this topic [1] (Figure 3.8).

3.9. **Mathematical Tools.** With all his activity in both numerical and abstract harmonic analysis one should not forget Hans's many other contributions. Another

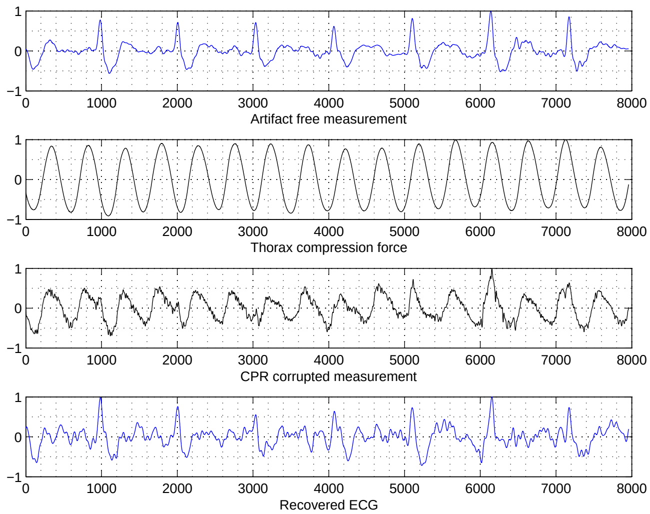


FIGURE 3.10. ECG-analysis during ventricular fibrillation

role of Hans is that of a mathematical tool maker. He likes the investigation of function spaces as an interesting topic in its own right, but he also considered function spaces as a tool for the appropriate formulation of mathematical problems.

Hans is probably the world's foremost expert on Wiener amalgam spaces which he uses with great virtuosity in many contexts. While amalgam spaces were invented earlier, Hans has pushed the theory furthest. He used convolution relations, multiplier theorems, and the interpolation of amalgam spaces [13] as a powerful tool for sampling theory, generalized harmonic analysis, Gabor frames, etc.

His work on *decomposition spaces* dates back to 1985 and 1987 [12, 15]. This time the chronological order was right. Modulation spaces, Besov spaces, and alpha-modulation spaces were already available and understood. Hans found a suitable unification that covers all these families of function spaces.

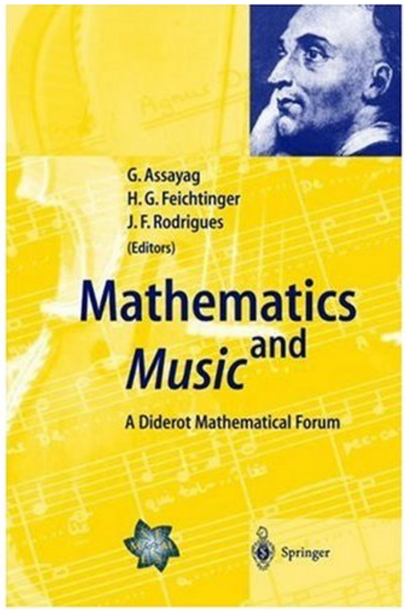


Figure 3.11. Mathematics and Music.

Decomposition spaces are based on suitable partitions of Fourier space. Choose partition $\mathcal{U} = (U_j)$ of $\mathbb{R}^d$ and a corresponding bounded admissible partition of unity (ψ_j), $\operatorname{supp}\psi_j \subseteq U_j$ (in Hans's language a BAPU). Then the decomposition space $B^{p,q}_{\mathcal{U},w}$ is defined by the norm

$$\|f\|_B = \Big(\sum_{j\in J} \|\hat{f}\psi_j\|_p^q w(j)^q\Big)^{1/q},$$

where $0 < p, q \leq \infty$ and w is a weight function. Hans's paper was almost unnoticed, but recently decomposition spaces became really fashionable and many people are now feeding on Hans's ideas.

In the last few years Hans has developed his enthusiasm for *Gelfand triples* [24]. A Gelfand triple $B \hookrightarrow H \hookrightarrow B$ consists of a Banach space B and a Hilbert space H with continuous embedding $\hookrightarrow$ between these spaces. Hans uses the Gelfand triple $S_0 \hookrightarrow L^2 \hookrightarrow S_0'$ as a fundamental object on which he builds his approach to the Fourier transform. See Figure 3.9 for an early example of Gelfand triples in Hans's work.

4. The Professional Mathematician

Hans loves to communicate, to interact, and to get involved. This personal character deeply shaped his life as a professional mathematician and has undoubtedly contributed to his impact on the mathematical community.

4.1. **Numerical Harmonic Analysis Group.** Hans's interest in the combination of problems in applied harmonic analysis with computational issues led also to a new emphasis in the structure of his work. From 1990 he began to direct his

Throughout this paper the following situation will be referred as STANDARD SITUATION ("the space B is in standard situation with respect to A"):

$(B, \| \|_B)$ is a Banach space, such that for some Banach algebra $(A, \| \|_A)$ satisfying the general assumptions above one has $A_0 \hookrightarrow B \hookrightarrow A_0'$ (for the $\sigma(A_0', A_0)$-topology). It is assumed that $(B, \| \|_B)$ is a left Banach module over some Beurling algebra $L^1_w(G)$ with respect to convolution, and that $(B, \| \|_B)$ is a Banach module over $(A, \| \|_A)$ with respect to pointwise multiplication, and that there exists a net $(\tau_\alpha)_{\alpha \in I}$ in A_0 of trapezoid functions of bounded action on B (i.e., satisfying the following two conditions:

$$\sup_{\alpha \in I} \sup_{h \in {}_0B} \|\tau_\alpha h\|_B =: C_B < \infty,$$

and for any compact subset $K \subseteq G$ there exists $\alpha_0 = \alpha(K)$ such that $\tau_\alpha(x) = 1$ for all $x \in K$, and $\alpha \geqslant \alpha_0$).

FIGURE 3.12. A very early example of Gelfand triples.

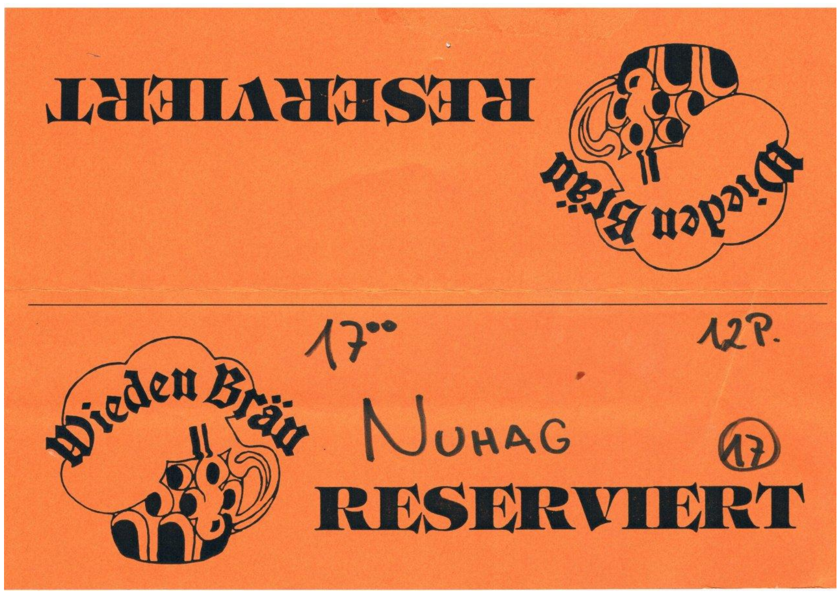


FIGURE 4.1. The Founding Session (1992)

students systematically to numerical questions. Informally he united his students under the umbrella of the "Numerical Harmonic Analysis Group", in short NuHAG.

I guess that this group was in the making for several years. The first official record of NuHAG is a reservation tag for a restaurant in 1992 (Figure 4.1). Thus we may take 1992 as the official birth of the "Numerical Harmonic Analysis Group". Under the acronym "NuHAG" it has become a brand name that has attained high visibility not only for Hans, but for the University of Vienna as a whole.

The importance of NuHAG for Hans is best expressed in his own words. On his homepage http://www.univie.ac.at/nuhag-php/home/feiinterests.php he mentions under the rubric "Private matters and interests" that we wants to

Figure 4.2. The spirit of NuHAG. SampTA 1997 in Aveiro. From left to right: Thomas Strohmer, Hans Feichtinger, Karlheinz Gröchenig.

Establish NuHAG as an international "long term, global player" within the European research landscape

After almost 25 years of NuHAG it is fair to say that he has reached this goal. At the time of his retirement the Numerical Harmonic Analysis Group is stronger than ever and has great potential for further growth.

Figures 4.1 and 4.1 express best how we all want to see NuHAG: visionary, dynamic, and creative.

4.2. **The Problem Poser.** Hans likes to talk about mathematics.

Talking to Hans is always inspiring. He developed a culture of getting people interested by asking stimulating questions. During a short discussion, he would ask dozens of mathematical questions. Amazingly he could ask questions on all levels of difficulty. Sometimes a question was doable by a Ph. D. student, got a student hooked, and, in the end, led to a Ph. D. thesis. This is probably one of the reasons why he attracted so many students.

The Feichtinger Conjecture. Hans's most famous question is a seemingly simple, but extremely deep question about general frames. Around 2000 Hans asked whether every every frame $\mathcal{F} = \{f_j : j \in J\}$ with $\inf \|f_j\| > 0$ can be decomposed into a finite union of Riesz sequences. This question was motivated by an experimental observation of the behavior of Gabor frames. Pete Casazza likes to tell the story that Hans posed this problem to him via email. Pete, enthusiastic as always, promised to answer within a few hours. Well, the answer took much longer, and the problem became known as the Feichtinger conjecture. Pete Casazza

FIGURE 4.3. The spirit of NuHAG: a far-sighted trio. SampTA 1997 in Aveiro. From left to right: Georg Zimmermann, Hans Feichtinger, Karlheinz Gröchenig.

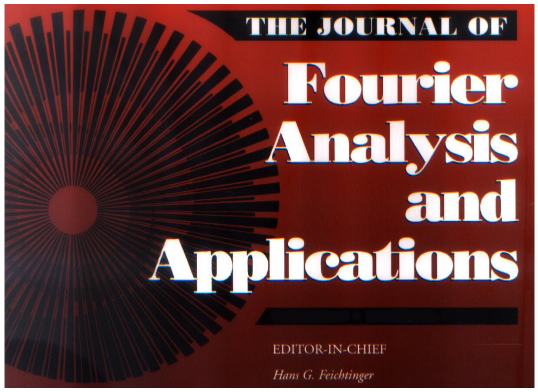
discovered successively that it implied, was implied, and finally, was equivalent to the famous Kadison-Singer conjecture [6]. Only recently in 2014, Hans's question was solved affirmatively along with the Kadison-Singer conjecture [34].

Yet this is not the full truth. Rereading the coorbit theory from 1989 in the light of the recent past, one easily sees Hans must have been thinking about similar questions for many years. Here is a special case of an abstract theorem in [18, Thm. 7.3].

Theorem 5. *Let* $(\pi, \mathcal{H})$ *be a (square-) integrable irreducible representation of a locally compact group* $\mathcal{G}$*,* $\mathcal{X} = \{x_j : j \in J\} \subseteq \mathcal{G}$ *relatively separated in* $\mathcal{G}$*, and* $g \in \mathcal{B}$ *("nice" admissible vector). Then the Bessel sequence* $\{\pi(x_j)g : j \in J\}$ *can be partioned into a finite union of Riesz sequences in* $\mathcal{H}$*.*

4.3. **The Editor.** The Journal of Fourier Analysis and Applications (JFAA) was founded in 1994 by John Benedetto. After John stepped down, Hans became the Editor-in-Chief of JFAA in 2000 (Figure 4.3). Since then Hans has led the journal and tried to raise its standards and impact factor. JFAA is flourishing and is, side by side with Appl. Comp. Harm. Anal. (ACHA), the leading journal of the large applied harmonic analysis community.

Being Editor-in-Chief is a challenge with diverging demands. On the one hand, the job requires a steadfast adherence to high scientific standards, on the other hand, it requires diplomacy, patience, and kindness towards the authors. If Hans



did not have these qualities, he would not have survived so long. Stated in simple terms: he is a great editor.

4.4. **The Networker.** Hans enjoys to work in a team and to lead a team. His own team was always the Numerical Harmonic Analysis Group. But for special tasks he really liked to create new teams. Funding is important, and it is a common place that Hans was always very successful in attracting funds and people. But for Hans this meant more. He was particularly eager to operate on the European level and initiated numerous European networks. Here is a short, by no means exhaustive list of networks and European research projects that he initiated:

- HASSIP (harmonic analysis and statistics for signal and image processing)
- EUCETIFA (European Center of Time-Frequency Analysis)
- UNLOcX
- ESO (European Southern Observatory — astronomical data)
- ECG — Analysis during ventricular fibrillation
- NetAGES (Networks for automated extraction from seismic data)
- Gabor analysis for the music transcription problem

This is just a small part of his funded projects. A full list can be found on his homepage.

4.5. **The Organizer.** Hans loves to bring together people. On a large scale this means that he was always involved in the organization of workshops and conferences. He has organized workshops in most major research centers in Europe. To mention some highlights: Hans organized a workshop on time-frequency analysis at the Newton Institute in Cambridge, a wonderful workshop in Oberwolfach, he was Jean Morlet Professor at the Centre International de Rencontres Mathmatiques in Marseille-Luminy in 2014, where he organized a whole semester with several workshops and a big conference. Together we organized two semester programs on modern time-frequency analysis at the Erwin Schrödinger Institute in Vienna in 2005 and 2012.

Perhaps the most important conference for us all is the biannual Strobl conference on topics in "Modern Time-Frequency Analysis and Related Topics." The first Strobl conference in 2003 was one of the early conferences of the series "Sampling Theory and Applications" (SampTA) and was so nice and successful that Hans wanted to continue and immediately reserved a week for 2005. Since then we have had the Strobl conferences in 2005, 2007, 2009, 2011 (for Hans's 60th birthday), 2014, and2016.

Strobl is a great place in a beautiful touristic region near Salzburg with an excellent conference center that usually serves as a federal institute for continuing education. As in Oberwolfach, all participants are accommodated in the same place and spend the entire week together. Many of us have spent their evenings outdoors under a starry sky and discussed math over one or several beers.

Over time, the list of plenary speaker included Ingrid Daubechies, John J. Benedetto, Emanuel Candes, Kristian Seip, Pete Casazza, Albert Cohen, to mention just a few.

The organization of a conference is usually a full time job. It starts with the reservation of the conference site and the invitation of plenary speakers two years in advance (just when the previous conference is finished) and gets really intense in the three months before the conference. For myself the organization of one conference would be enough for a career, but Hans does never seems to get tired and in fact, seems to enjoy the organization. He always feels rewarded by the large attendance and by the happy participants.

4.6. **The Teacher.** Hans loves to talk about mathematics and to share his vision of mathematics. He loves to talk about mathematics on all levels. This is the innermost prerequisite for every teacher. Originally Hans wanted to become a high school teacher, but his mathematic talent guided him to a different destiny. Nevertheless, teaching was and is a pillar of his activities, and he has become a dedicated and generous teacher who has attracted myriads of students. Hans never gets tired to explain his research, he must have explained his results on S_0 (Figure 1), on amalgam spaces and convolution relations, or Gelfand triples thousands of times — and he still enjoys it!

Hans's greatest strength is his openness, his patience, and his ability to meet his students at their own level. This allows them to develop according to their own pace and abilities. Hans has had some really outstanding students who are now among the leaders in applied harmonic analysis: Ole Christensen (1993), Thomas Strohmer (1993), Massimo Fornasier (2003). Many of his students have moved to academia, and many are now successful in industry.

According to the MathGenealogy project he has had 29 Ph. D. students (with some more to come). Thus the *NuHAG school of applied harmonic analysis* is a real legacy!

I really do not have the words to fully do justice to the teacher Hans Feichtinger.

5. Outlook

Retirement at age 65 is mandatory in Austria, where most people want to retire earlier (the average retirement age is below 60 in Austria). However, mandatory retirement is not the end of the professional life. It is clear that Hans will not retire as a mathematician. Indeed, the formal retirement gives him a boost of energy to continue to do research and to teach. He no longer needs to deal with petty administrative formalities at the Faculty of Mathematics, he is free to travel at any time without consideration of a teaching schedule, he is in great physical and mental shape to contribute to the world of mathematics for many years to come!

Perhaps the best way to conclude this article is the song by Ingrid Daubechies composed on the occasion of Hans's 60th birthday in 2011 in Strobl (Figure 5). We all celebrated his birthday in Strobl. I do not remember the melody that was supposed to be sung to Ingrid's text, but certainly this text describes Hans as he is.

5.1. **Resources.** Hans's homepage

`http://www.univie.ac.at/nuhag-php/home/fei.php`

NuHAG homepage:

`http://www.univie.ac.at/nuhag-php/home/index.php`

List of publications:

`http://www.univie.ac.at/nuhag-php/home/feipub_db.php`

References

[1] G. Assayag, H. G. Feichtinger, and J. F. Rodrigues, editors. *Mathematics and music.* Springer-Verlag, Berlin, 2002. A Diderot Mathematical Forum, Papers from the forum held simultaneously in Lisbon ("Historical Aspects"), Paris ("Mathematical Logic and Music Logic in the XX Century") and Vienna ("Mathematical and Computational Methods in Music"), December 3–4, 1999.

[2] W. Baoxiang, Z. Lifeng, and G. Boling. Isometric decomposition operators, function spaces $E^{\lambda}_{p,q}$ and applications to nonlinear evolution equations. *J. Funct. Anal.*, 233(1):1–39, 2006.

[3] J. Bellissard. Lipshitz continuity of gap boundaries for Hofstadter-like spectra. *Comm. Math. Phys.*, 160(3):599–613, 1994.

[4] Á. Bényi, K. Gröchenig, K. A. Okoudjou, and L. G. Rogers. Unimodular Fourier multipliers for modulation spaces. *J. Funct. Anal.*, 246(2):366–384, 2007.

[5] P. L. Butzer, W. Splettstösser, and R. L. Stens. The sampling theorem and linear prediction in signal analysis. *Jahresber. Deutsch. Math.-Verein.*, 90(1):70, 1988.

[6] P. G. Casazza and D. Edidin. Equivalents of the Kadison-Singer problem. In *Function spaces*, volume 435 of *Contemp. Math.*, pages 123–142. Amer. Math. Soc., Providence, RI, 2007.

[7] E. Cordero and K. Gröchenig. Time-frequency analysis of localization operators. *J. Funct. Anal.*, 205(1):107–131, 2003.

[8] E. Cordero, K. Gröchenig, F. Nicola, and L. Rodino. Generalized metaplectic operators and the Schrödinger equation with a potential in the Sjöstrand class. *J. Math. Phys.*, 55(8):081506, 17, 2014.

[9] I. Daubechies, H. J. Landau, and Z. Landau. Gabor time-frequency lattices and the Wexler-Raz identity. *J. Fourier Anal. Appl.*, 1(4):437–478, 1995.

[10] H. G. Feichtinger. On a new Segal algebra. *Monatsh. Math.*, 92(4):269–289, 1981.

[11] H. G. Feichtinger. Modulation spaces on locally compact abelian groups. Technical report, University of Vienna, 1983.

[12] H. G. Feichtinger. Banach spaces of distributions defined by decomposition methods. II. *Math. Nachr.*, 132:207–237, 1987.

We have come to celebrate dear Hans

Should you have some funding wants
This may be your lucky chance
Because he knows where to find grants

For NUHAG he has many plans
It seems as if it ever expands
Something or other always warrants
A new effort, or a few transplants

And if, perchance
You have a problem and ask him to glance
At it, you'd better expect a fast ans
-wer

A visit to him always enchants
Such a wonderful group with no clans
He does everything to enhance
Your time here; you are in the best hands
It's such a joy, you could just dance

[13] H. G. Feichtinger. Generalized amalgams, with applications to Fourier transform. *Canad. J. Math.*, 42(3):395–409, 1990.

[14] H. G. Feichtinger. Modulation spaces on locally compact abelian groups. In *Proceedings of "International Conference on Wavelets and Applications" 2002*, pages 99–140, Chennai, India, 2003. Updated version of a technical report, University of Vienna, 1983.

[15] H. G. Feichtinger and P. Gröbner. Banach spaces of distributions defined by decomposition methods. I. *Math. Nachr.*, 123:97–120, 1985.

[16] H. G. Feichtinger and K. Gröchenig. A unified approach to atomic decompositions via integrable group representations. In *Function spaces and applications (Lund, 1986)*, pages 52–73. Springer, Berlin, 1988.

[17] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions. I. *J. Functional Anal.*, 86(2):307–340, 1989.

[18] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions. II. *Monatsh. Math.*, 108(2-3):129–148, 1989.

[19] H. G. Feichtinger and K. Gröchenig. Gabor frames and time-frequency analysis of distributions. *J. Functional Anal.*, 146(2):464–495, 1997.

[20] H. G. Feichtinger, K. Gröchenig, and T. Strohmer. Efficient numerical methods in non-uniform sampling theory. *Numer. Math.*, 69(4):423–440, 1995.

[21] H. G. Feichtinger and N. Kaiblinger. Varying the time-frequency lattice of Gabor frames. *Trans. Amer. Math. Soc.*, 356(5):2001–2023 (electronic), 2004.

[22] H. G. Feichtinger and W. Kozek. Quantization of TF lattice-invariant operators on elementary LCA groups. In *Gabor analysis and algorithms*, pages 233–266. Birkhäuser Boston, Boston, MA, 1998.

[23] H. G. Feichtinger and F. Luef. Wiener amalgam spaces for the fundamental identity of Gabor analysis. *Collect. Math.*, (Vol. Extra):233–253, 2006.

[24] H. Feichtinger, F. Luef, and E. Cordero. Banach Gelfand triples for Gabor analysis. In *Pseudo-differential operators*, volume 1949 of *Lecture Notes in Math.*, pages 1–33. Springer, Berlin, 2008.

[25] H. G. Feichtinger and K. Nowak. A first survey of Gabor multipliers. In *Advances in Gabor analysis*, Appl. Numer. Harmon. Anal., pages 99–128. Birkhäuser Boston, Boston, MA, 2003.

[26] K. Gröchenig. An uncertainty principle related to the Poisson summation formula. *Studia Math.*, 121(1):87–104, 1996.

[27] K. Gröchenig. *Foundations of time-frequency analysis*. Birkhäuser Boston Inc., Boston, MA, 2001.

[28] K. Gröchenig. Time-frequency analysis of Sjöstrand's class. *Revista Mat. Iberoam.*, 22(2):703–724, 2006.

[29] K. Gröchenig and C. Heil. Modulation spaces and pseudodifferential operators. *Integral Equations Operator Theory*, 34(4):439–457, 1999.

[30] K. Gröchenig, J. Ortega-Cerdà, and J. L. Romero. Deformation of Gabor systems. *Adv. Math.*, 277:388–425, 2015.

[31] K. Gröchenig and S. Samarah. Non-linear approximation with local Fourier bases. *Constr. Approx.*, 16(3):317–331, 2000.

[32] A. Grossmann and J. Morlet. Decomposition of Hardy functions into square integrable wavelets of constant shape. *SIAM J. Math. Anal.*, 15(4):723–736, 1984.

[33] A. J. E. M. Janssen. Duality and biorthogonality for Weyl-Heisenberg frames. *J. Fourier Anal. Appl.*, 1(4):403–436, 1995.

[34] A. W. Marcus, D. A. Spielman, and N. Srivastava. Interlacing families II: Mixed characteristic polynomials and the Kadison-Singer problem. *Ann. of Math. (2)*, 182(1):327–350, 2015.

[35] G. Matz, D. Schafhuber, K. Gröchenig, M. Hartmann, and F. Hlawatsch. Analysis, optimization, and implementation of low-interference wireless multicarrier systems. *IEEE Transactions on Wireless Communications*, 6(5):1921–1931, 2007.

[36] H. Reiter. *Classical harmonic analysis and locally compact groups*. Clarendon Press, Oxford, 1968.

[37] A. Ron and Z. Shen. Weyl–Heisenberg frames and Riesz bases in $L_2(\mathbb{R}^d)$. *Duke Math. J.*, 89(2):237–282, 1997.

[38] J. Sjöstrand. An algebra of pseudodifferential operators. *Math. Res. Lett.*, 1(2):185–192, 1994.

[39] T. Strohmer. Approximation of dual Gabor frames, window decay, and wireless communications. *Appl. Comput. Harmon. Anal.*, 11(2):243–262, 2001.

Faculty of Mathematics, University of Vienna, Nordbergstrasse 15, A-1090 Vienna, Austria

E-mail address: karlheinz.groechenig@univie.ac.at