TIME AVERAGES FOR UNPREDICTABLE ORBITS OF DETERMINISTIC SYSTEMS

Karl SIGMUND
Institut für Mathematik der Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

Abstract

In many cases, the orbits of deterministic systems displaying highly irregular oscillations yield smoothly converging time averages. It may happen, however, that these time averages do not converge and themselves display wild oscillations. This is analyzed for heteroclinic attractors and hyperbolic strange attractors.

1. Introduction

Whether one plots stock exchange prices (e.g. [1,2]), population densities [3], medical indicators [4], chemical concentrations [5], atmospheric variables [6] or what not, one regularly encounters irregular fluctuations. Mankind has long been accustomed to unpredictable oscillating time series, and recently modellers too are getting used to them (see e.g. [7–9]). In fact, these are nowadays enjoyed by most system analysts. But when it comes to tasks of evaluation and decision making, the recipe is usually to wait for the fluctuations to die down and, if they show no sign of doing this of their own accord, to average them out. In most cases, this is indeed a sensible thing to do, validated by some law of large numbers or by an ergodic theorem (see e.g. [10–12]). To quote Ruelle [10]: ... *there are many situations in which a geometric description [of the attractor] is no longer feasible, due to the extreme complications of the dynamics ... These cases are those in which statistical analysis becomes really relevant ... If we analyse a sufficiently long record of a chaotic signal generated by a deterministic time evolution, we may find that the signal amplitudes are within a definite range for a well defined fraction of the time ... More precisely, this means that the expected time average does exist for suitable initial points, providing a well defined invariant probability measure (in the sense that the time average of the observable is equal to its space avarage).*

Nevertheless, there are situations where the time average refuses to converge, no matter how long one waits. The purpose of this article is to highlight such situations by some simple examples of deterministic dynamics. What we have in

J.C. Baltzer AG, Scientific Publishing Company

mind are models describing the repeated interactions of players in some social, economic or biological game, or some hypothetical dynamics underlying the fluctuations of prices or population numbers. In the first part, we deal with heteroclinic attractors: such cycles exemplify unpredictable behaviour where almost all orbits fail to converge. In the second part, we turn to strange attractors of hyperbolic type. There is an interesting contrast, here, between the probabilistic point of view (where almost all initial data lead to converging time averages) and the topological point of view (generically, the time averages do not converge: in fact, they exhibit an extreme form of misbehaviour). If the initial conditions are only known statistically, i.e. as probability distributions, the former point of view becomes meaningless and the latter one gains interest.

Thus the two types of deterministic but unpredictable behaviour which we consider here lead to very different asymptotic behaviour of the time averages. For heteroclinic attractors, Lebesgue-almost all initial conditions lead to time averages which diverge, but they all diverge in precisely the same way. For hyperbolic strange attractors, Lebesgue-almost all initial conditions lead to converging time averages, but there exists nevertheless a huge variety of different divergent behaviours, and for each possible type of divergence there is a dense set of initial conditions leading to it.

2. Heteroclinic attractors

Let us consider a dynamical system on a subset X of $\mathbb{R}^n$. If the state at time 0 is given by $x \in X$, then it will be given by $x(t) \in X$ at time t. We shall consider both continuous time ($t \in \mathbb{R}$) and discrete time ($t \in \mathbb{Z}$) dynamical systems. A fixed point x satisfies $x(t) = x$ for all t. A heteroclinic cycle consists of several fixed points $y^j (j = 1, \dots, k; k \geq 2)$ and orbits $t \to z^j(t)$ connecting them, in the sense that $z^j(t) \to y^j$ for $t \to -\infty$ and $z^j(t) \to y^{j+1}$ for $t \to +\infty$. (We set $y^{k+1} = y^1$ so that z^k connects y^k to y^1 and thereby closes the cycle.) The fixed points are saddle points (since some orbits approach them for positive, some for negative time) and the connecting orbits are saddle connections.

We note in passing that dynamical systems with heteroclinic cycles are not structurally stable. The saddle connections can be broken up by arbitrarily small perturbations (where "small" can be defined, here, in a variety of ways). This seems to be the reason why heteroclinic cycles have been neglected by the mainline theory of dynamical systems (see, however, Guckenheimer and Holmes [13]). Nevertheless, they occur robustly in a variety of models, in the sense that if the perturbation respects some essential feature or symmetry, it also preserves the saddle connections and hence the heteroclinic cycles.

As an instance of a system featuring heteroclinic cycles we mention the hypercycle which models the evolution of concentrations of several types of self-replicating molecules in a flow reactor, each type acting catalytically upon the growth of the next type, thereby forming a closed loop of catalytic actions [14]. This mechanism from chemical kinetics has also been used in economic or social contexts to describe the self-organization of cooperating units. Other examples of heteroclinic cycles occur in game dynamics, where the variables denote the frequencies of certain strategies, and the successful strategies spread (see e.g. [15]). Whenever there exists a cyclic structure (like for the stone–scissors–paper game), heteroclinic cycles occur. This happens for strategies in the repeated Prisoner's Dilemma [16]; it also occurs whenever there are two players with two strategies each, if no pure pair of strategies is a Nash equilibrium [17,18]. Further examples of heteroclinic cycles can be found in population ecology, as when (for instance) two competing species 1 and 2 are beset by two strains a and b of predators (or parasites): if the four equilibria $(1,a)$, $(1,b)$, $(2,b)$ and $(2,a)$ (each with only two species present) are cyclically connected, in the sense that predator b outcompetes a on a regime of prey 1, that prey 2 replaces prey 1 if only predator b is around, and so on ..., this yields a heteroclinic cycle [19]. For discrete dynamics there are analogous cycles: in a sense, they occur even more readily [17].

Now let us assume that the heteroclinic cycle Γ is a limit in the sense that there exists an orbit $\boldsymbol{x}(t)$ such that the points of Γ are the accumulation points of sequences $\boldsymbol{x}(t_k)$, with $t_k \to +\infty$, i.e. that Γ is the ω-limit of $\boldsymbol{x}$. The continuity of the dynamical system implies that the motion of $\boldsymbol{x}(t)$ proceeds in fits and starts. If it is near one of the fixed points of Γ, it remains there for a long time; then it follows the outgoing saddle connection to the next fixed point, where it hovers for a much longer time; then it proceeds along the next saddle connection to the following fixed point, remains there for some still longer period of time, etc. Since $\boldsymbol{x}(t)$ converges to Γ, it comes closer and closer to the fixed points and consequently remains there for longer and longer times. If the fixed points are hyperbolic (the generic case, which we shall assume from now on), the periods of near-stationarity increase exponentially. On the other hand, the lengths of the time intervals for switching from a neighbourhood of one saddle to a neighbourhood of the next one do not change much: asymptotically, they are given by the time needed by the saddle connection to cross from one neighbourhood to the other. Compared with the amount of time spent within the neighbourhoods, the duration of the journeys in between can be neglected.

The dynamics is highly unpredictable: close-by orbits may leave the neighbourhoods of the fixed points at very different times, and evolve quite out of phase. It is difficult to tell in advance, for some given large T, near which fixed point $\boldsymbol{x}(T)$ will be lingering.

Averaging will not help with such highly erratic oscillations: the time averages

$$\frac{1}{T}\int_0^T \boldsymbol{x}(t)\mathrm{d}t \quad \text{or} \quad \frac{1}{N}\sum_{t\neq 0}^{N-1} \boldsymbol{x}(t)$$

(depending on whether the dynamics is continuous or discrete) will fail to converge.

As an example, let us consider a game with three strategies such that strategy 1 beats 2, 2 beats 3 and 3 beats 1. The payoff a_{ij} for a player using strategy i against a j-player is given, without restricting generality, by

$$\begin{pmatrix} 0 & a_3 & -b_2 \\ -b_1 & 0 & a_2 \\ a_1 & -b_3 & 0 \end{pmatrix},$$

with $a_i, b_i > 0$. (For the rock–scissors–paper game, we can use $a_i = b_i = 1$.) Let us consider a population consisting of three types, each playing one of the strategies. Let $x_i(t)$ be the frequency of the i-players at time t. The state of the population at any time is given by a vector $\boldsymbol{x} = (x_1, x_2, x_3)$ on the simplex S_3 (since $x_i \geq 0$ and $\Sigma x_i = 1$). If we assume random encounters, the expected payoff for strategy i is $(A\boldsymbol{x})_i = \Sigma a_{ij} x_j$ and the average payoff for the population is $\boldsymbol{x} \cdot A\boldsymbol{x} = \Sigma x_i (A\boldsymbol{x})_i$. In game dynamics, one assumes that (due to some effect of inheritance or learning) the rate of increase $\dot{x}_i / x_i$ of strategy i is given by $(A\boldsymbol{x})_i - \boldsymbol{x} \cdot A\boldsymbol{x}$, i.e. by the difference between its payoff and the average payoff. Hence, the state evolves according to

$$\dot{x}_i = x_i [(A\boldsymbol{x})_i - \boldsymbol{x} \cdot A\boldsymbol{x}]. \tag{1}$$

It is easy to see that the simplex S_3 and the boundary faces are invariant. The corners $\boldsymbol{e}_i$ are saddle points, the edges are saddle connections (see fig. 1). Thus the

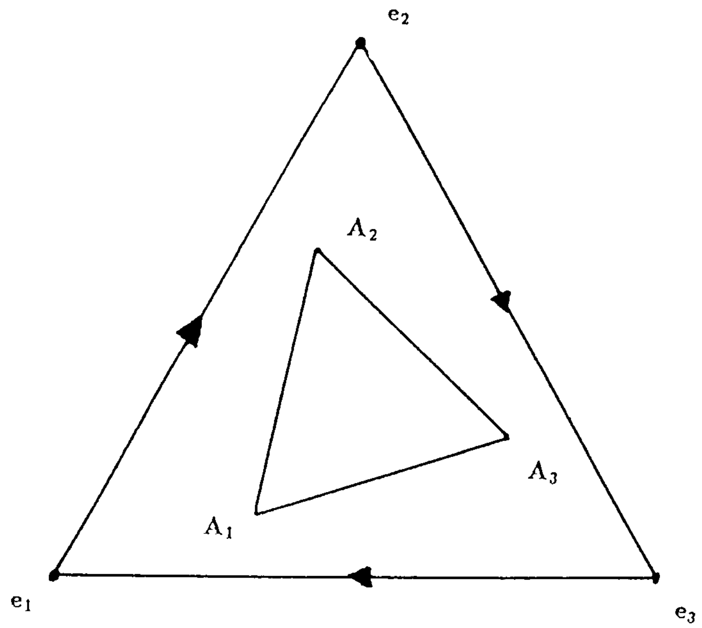

Fig. 1.

boundary is a heteroclinic cycle. In [15] it is shown that if $b_1b_2b_3 > a_1a_2a_3$, then all orbits $t \mapsto \boldsymbol{x}(t)$ in the interior of S_3 (with the exception of the centre $\boldsymbol{m} = \frac{1}{3}(1,1,1)$) converge to the boundary: it is a heteroclinic attractor.

In [20] Gaunersdorfer shows that the corresponding time averages $z(T)$, with coordinates

$$z_i(T) = \frac{1}{T}\int x(t)\mathrm{d}t$$

fail to converge. Instead, they approach the boundary of the triangle spanned by

$$A_1 = \frac{1}{a_1a_3 + a_1b_2 + b_2b_3}(b_2b_3, a_1b_2, a_1a_3),$$

and the corresponding points $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ defined similarly. We note that $\boldsymbol{A}_3$, $\boldsymbol{A}_1$ and $\boldsymbol{e}_1$ are colinear. Asymptotically, during the period where $\boldsymbol{x}(t)$ is bogged down in a vicinity of $\boldsymbol{e}_1$, the time average, which was close to $\boldsymbol{A}_3$, moves straight towards $\boldsymbol{e}_1$. When $\boldsymbol{x}(t)$ jumps over to $\boldsymbol{e}_2$, the motion of time average changes its direction towards $\boldsymbol{e}_2$; this happens at the corner $\boldsymbol{A}_1$ of the triangle. (This behaviour seems to have been first noticed by E.C. Zeeman; see also [15].)

One way of proving this uses a Poincaré-section type of argument, applied to the heteroclinic cycle instead of a periodic orbit. The saddle $\boldsymbol{e}_i$ has a_i as positive and $-b_i$ as negative eigenvalue. An orbit coming in at a distance x from the orbit converging to the saddle leaves at a distance proportional to x^{b_i/a_i} from the orbit issuing from the saddle. For the time average, one can neglect the time spent on switching from one saddle to the next. The time spent by $\boldsymbol{x}(t)$ near $\boldsymbol{e}_i$ is approximately the b_{i-1}/a_ith of the time spent near the previous corner and hence the time spent by $\boldsymbol{x}(t)$ on the nth round grows like ρ^n, with $\rho = \prod(b_i/a_i)$. This allows to compute the asymptotic behaviour of the time average.

The same method can be used quite generally for heteroclinic attractors, with $-b_i$ and a_i as corresponding negative and positive eigenvalues of the ith saddle point. For example, we may consider the Lotka–Volterra competition equation

$$\begin{aligned}
\dot{x}_1 &= x_1(1 - x_1 - \alpha x_2 - \beta x_3),\\
\dot{x}_2 &= x_2(1 - \beta x_1 - x_2 - \alpha x_3),\\
\dot{x}_3 &= x_3(1 - \alpha x_1 - \beta x_2 - x_3),
\end{aligned} \tag{2}$$

where the x_i are population densities of three competing species. This equation was introduced by May and Leonard [21] and is at the origin of much recent work on competition (see e.g. [22,23]). We assume $0 < b < 1 < a$ and $a + b > 2$. There is a heteroclinic cycle on the boundary of the state space $\mathbb{R}^3_+$: it consists of the saddle points $\boldsymbol{e}_i$ and connecting orbits. With the exception of the orbits on the diagonal $x_1 = x_2 = x_3$, all orbits $\boldsymbol{x}(t)$ in the interior of $\mathbb{R}^3_+$ converge to the heteroclinic cycle. Again, the time average spirals closer and closer to a triangle $\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3$, slowing down but never converging. One has

$$A_1 = \frac{1}{(1-\beta)^2 + (1-\beta)(\alpha-1) + (\alpha-1)^2}((\alpha-1)^2, (1-\beta)^2, (\alpha-1)(1-\beta))$$

and similar expressions for A_2 and A_3.

It follows that both for the game dynamical eq. (1) and the competition eq. (2), almost all orbits have time averages which do not converge. This seems at first glance somewhat at odds with the ergodic theorem, according to which almost all the time averages converge. But of course *almost all* is used with two different meanings. In the former case, it is with respect to Lebesgue measure (on S_3 or on $\mathbb{R}^3_+$), and in the latter case, with respect to any invariant measure. It is easy to see that an invariant measure for the dynamical system (1) has its support contained in the set of fixed points $\{y_1, y_2, y_3, \boldsymbol{m}\}$: for such measures the statement of the ergodic theorem is obviously trivial.

Similar behaviour is found for a great variety of dynamical systems. For example, the fixed points of the heteroclinic cycle can be replaced by periodic orbits (see [24] for a seven-dimensional example) or by a strange attractor on some lower-dimensional submanifold. The orbit approaches the attractor for some time, then moves off to visit another attractor, but returns after a while for a much longer time etc. The oscillatory regime of such an orbit seems to switch, at exponentially increasing times, into different modes (similar to distinct climates in meteorology): the time averages, again, do not converge. Analogous results also hold for the corresponding discrete systems.

3. Strange attractors

There is no commonly agreed definition for a strange attractor yet, but one certainly expects oscillations there to exhibit a livelier behaviour than for heteroclinic attractors: instead of bogging down for long periods of near-immobility, they should be restlessly wiggling around. It is not paradoxical to expect that these wilder oscillations in turn should make for tamer time averages.

The simplest example is the discrete dynamical system on the unit interval $[0,1]$ given by the map

$$x \mapsto 4x(1-x). \tag{3}$$

This map plays a star role in theoretical population ecology [3]; if x denotes the size of the population (rescaled so that $x \in [0,1]$) and if the population grows for small x, but decreases for large x, then the behaviour can be modelled by the logistic map $x \mapsto ax(1-x)$. For small values of a, there is a unique equilibrium $(a-1)/a$ attracting all orbits in $(0,1)$, but if one increases the value of a, a sequence of period doubling occurs, and ultimately chaotic motion. Erratic behaviour caused by some very simple regulatory mechanism is found not only in population ecology, but in economics and other fields as well. It also underlies many continuous dynamical systems.

Other examples of chaotic motion occur for hyperbolic attractors, whose ergodic behaviour is well understood. They are the prototypes of strange attractors (in spite of the fact that no differential equations arising in applications have been shown to belong to this class). Hyperbolicity means that transversal sections to the orbits decompose into exponentially contracting and expanding directions in a smooth uniform way (for a more precise definition we refer to [10,11,26]). The Bowen–Ruelle–Sinai theorem [26] implies, as we shall see, that almost every orbit in the neighbourhood of such an attractor has a converging time average. In spite of this, there remain plenty of orbits without converging time averages.

Let us first fill in some background, starting with a continuously dynamical system on a compact metric set X (see e.g. [27]). A subset A is transformed, after a time t, into a subset $A(t) = \{x(t) : x \in A\}$. A probability measure μ is said to be *invariant* if $\mu(A(t)) = \mu(A)$ holds for all measurable $A \subset X$ and all $t \in \mathbb{R}$. The ergodic theorem of Birkhoff implies that if $f: X \to \mathbb{R}$ is integrable with respect to μ, there exists a set $Q_f \subset X$ with $\mu(Q_f) = 1$ such that the time average

$$\frac{1}{T}\int_0^T f(x(t))\mathrm{d}t \tag{4}$$

converges for $T \to +\infty$ whenever $x \in Q_f$. It does *not* hold, in general, that $Q_f = X$ if f is continuous, i.e. that all time averages of continuous functions converge. The example in the previous section shows how wrong this can be (as continuous function f, we can use in this case the ith coordinate function $x \mapsto x_i$). On the other hand, since the set of continuous functions is separable, there exists a set $Q \subset X$ such that

(a) for all $x \in Q$ and all continuous f the time average (4) converges for $T \to +\infty$;

(b) $\mu(Q) = 1$ for all invariant probability measures μ.

The points of Q are called quasiregular [28] or *statistically regular* [10]. The results we mentioned so far show that points are almost surely statistically regular, both with respect to every invariant probability measure and (in the neighbourhood of a hyperbolic attractor) with respect to Lebesgue measure. There can be many points, nevertheless, which are not statistically regular.

Let us denote by $M(X)$ the space of probability measures on X (we always consider Borel measures, i.e. measures defined on the smallest σ-algebra containing all open sets). If $\mu \in M(X)$, the integral

$$f \mapsto \int_X f\mathrm{d}\mu$$

is a linear functional on the space of continuous functions on X which maps the positive function $\mathbf{1}_X$ to 1. In fact the probability measures correspond precisely to these functionals.

The space $M(X)$ is obviously convex, and X can be imbedded into it: to each $x \in X$ we associate the *point measure* δ_x (where $\delta_x(A)$ is 1 if $x \in A$, and 0 if not). The point measures are, incidentally, just the extremal points of $M(X)$. Now the topology of X extends in a natural way to the so-called *weak topology* in $M(X)$. We say that a sequence of probability measures μ_n converges to the probability measure μ if

$$\int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu$$

holds for all continuous functions $f : X \to \mathbb{R}$. The imbedding $x \mapsto \delta_x$ of X into $M(X)$ is then a continuous mapping onto the closed set of all point measures. Since X is compact, $M(X)$ is also compact. The measures with finite support, i.e. the finite convex combinations of point measures, are dense in $M(X)$.

Just as the linear functional

$$f \mapsto f(x)$$

on the space of continuous functions corresponds to the point measure δ_x, so the linear functional

$$f \mapsto \frac{1}{T}\int_0^T f(x(t))\mathrm{d}t$$

also yields a probability measure: it corresponds to the time average of the value of f along the portion of the orbit of x from time 0 to time $T > 0$. We denote this average by $\delta_x(T)$ and define $\delta_x(0) = \delta_x$.

If x is statistically regular then the limit

$$f \mapsto \lim_{T \to +\infty} \frac{1}{T}\int_0^T f(x(t))\mathrm{d}t$$

is also a well-defined linear functional on the space of continuous functions and thus a probability measure. It is easy to see that it has the property of being invariant, which translates into the property that the continuous functions $x \mapsto f(x)$ and $x \mapsto f(x(t))$ have the same integral for all t. Since there always exist statistically regular points, there always exist invariant measures. Hence, the space $M_I(X)$ of invariant probability measures is nonempty. It is also compact and convex.

There are dynamical systems such that every x is statistically regular: in particular, there exist systems admitting a unique invariant measure μ. In this case all averages $\delta_x(T)$ converge to μ, for $T \to +\infty$. But in general, all one can say is that the set

$$\rho(x) = \{\mu \in M(X) : \exists T_k \to +\infty \quad \text{such that} \quad \delta_x(T_k) \to \mu\} \tag{5}$$

is a nonempty subset of the set $M_I(X)$ of invariant measures which is compact and connected [27]. This latter property is a simple consequence of the fact that $T \mapsto \delta_x(T)$ is continuous.

In the game dynamics example (1), if the orbit of x has the boundary of S_3 as limit set, then $\rho(x)$ consists of three measures

$$\boldsymbol{a}_1 = \frac{1}{a_1a_3 + a_1b_2 + b_2b_3}(b_2b_3\delta_{e_1} + a_1b_2\delta_{e_2} + a_1a_3\delta_{e_3})$$

and the analogously defined $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ as well as the line segments joining them in the convex set $M(X)$. This set is connected (but not convex!).

If Λ is a hyperbolic attractor, then the theorem of Bowen–Ruelle–Sinai implies that there exists an invariant probability measure μ on Λ such that for Lebesgue-almost all x in a neighbourhood of Λ, the set $\rho(x)$ reduces to $\{\mu\}$. However, there exist points $x \in \Lambda$ without converging time average. In fact if R is *any* nonempty compact connected subset of the set $M_I(\Lambda)$ of invariant measures, there exist points x such that $\rho(x) = R$, and the set of these x is actually dense in Λ, see [27]. There exists a staggering variety of compact connected subsets of $M_I(\Lambda)$: even if we look only at the two-dimensional convex subset spanned by three distinct invariant measures on Λ, we find a huge diversity of such subsets. This shows that there is an enormous repertoire of possible asymptotic behaviours for the time averages.

In particular, if we take $R = M_I(\Lambda)$, we find that the set L of all $x \in \Lambda$ with $\rho(x) = M_I(\Lambda)$ is dense in Λ. Since a neighbourhood U of Λ is the union of stable manifolds of points of Λ (see e.g. [12, p. 262]), we can find arbitrarily close to any $y \in U$ a point $z \in U$ such that $d(z_t, x_t) \to 0$ for some $x \in L$, where d is the distance and $t \to +\infty$. Hence, the set

$$\{x \in U : \rho(x) = M_I(\Lambda)\}$$

(the set of points whose time-averages display *maximal oscillation*) is dense in U. Moreover, it is a G_δ-set, i.e. the countable intersection of open sets: this follows exactly as in [27, p. 207]. Thus, it is not only dense but large, and even prevalent, from the topological point of view: its complement is a meagre set (of first category). Topologically speaking, then, the typical initial condition x leads to maximal oscillations of the time averages: every invariant measure occurs as a limit point of $\delta_x(T_k)$ for some $T_k \to +\infty$. It should be emphasized that the set $M_I(\Lambda)$ is very large. In particular, it has the interesting property that its extremal points are dense (see [27]) – something that evidently can only happen if the convex set is infinite-dimensional.

Hence the question about the typical averaging behaviour receives different answers, depending on whether "typical" is meant measure-theoretically or topologically. (In no case is there an efficient classification for the initial conditions.) The option for one or the other point of view is a matter of taste (see Oxtoby [29] for an interesting discussion). Probably most (maybe we should say almost all) mathematicians would favour the measure-thoretic answer. Nevertheless, it should be mentioned that it is in some way limited by the assumption that the initial condition is perfectly determined–i.e. given by a point x. In many cases of practical relevance the initial condition is not perfectly known, and corresponds to a probability distribution which does not reduce to a point measure. This corresponds to models where the dynamics is perfectly deterministic but the state stochastic. There seems to be no natural measure on the space $M(X)$ of probability measures–but, as we have seen, a very sensible topology, namely the weak one. Just as we can consider orbits of points, we can also consider orbits of statistical states, i.e. of probability measures. The dynamical system sends a probability measure $\mu \in M(X)$ into the probability measure μ_t at time t, where μ_t is given by

$$\mu_t(A) = \mu_t(A_t)$$

for measureable $A \subset X$. Thus the dynamical system on X induces a dynamical system on the extension $M(X)$. The time average of the orbit $t \to \mu_t$ in $M(X)$ is given by

$$\mu(T) := \frac{1}{T}\int_0^T \mu_t \mathrm{d}t,$$

which corresponds to the linear functional

$$f \mapsto \frac{1}{T}\int_0^T f\,\mathrm{d}\mu_t = \frac{1}{T}\int_0^T\int_X f(x(t))\mathrm{d}\mu\,\mathrm{d}t.$$

Again, one defines

$$\rho(\nu) = \{\mu \in M(X) : \exists T_k \to +\infty \text{ such that } \nu(T_k) \to \mu\}$$

for $\nu \in M(X)$, and obtains a nonempty compact connected subset of $M_I(X)$. If the dynamics converges to some hyperbolic attractor Λ, then for every nonempty $R \subset M_I(\Lambda)$ which is compact and connected, the set $\{\nu : \rho(\nu) = R\}$ is dense in $M(\Lambda)$ [30]. Again, the set $\{\nu : \rho(\nu) = M_I(\Lambda)\}$ is a countable intersection of open dense sets. Hence the typical probability distribution will have a time average which oscillates maximally (where *typical*, here, is understood from the topological point of view, while the measure-theoretic point of view does not enter into consideration).

In order to verify the extreme misbehaviour of the time averages, one uses properties of the dynamics on the basic set Λ (see e.g. [12] or [27]):

(a) *topological mixing*: for every two nonempty open subsets $U, V \subset \Lambda$, there is a time T such that for all $t \geq T$, one finds an $x \in U$ with $x(t) \in V$. Thus we can move from every neighbourhood to every other one, provided we allow enough time for it;

(b) a very weak form of the *shadowing property*: for every ε, there is a β with the property that whenever the points y and z in Λ satisfy $d(y,z) < \beta$ then there exists an $x \in \Lambda$ such that $d(x(t),y(t)) < \varepsilon$ for all $t < 0$ and $d(x(t),z(t)) < \varepsilon$ for all $t > 0$. The orbit of x shadows that of y for negative time and that of z for positive time.

In [33], it has been convincingly argued that a stronger form of this shadowing property is fundamental for what is meant by chaotic dynamics. All chaotic attractors of this type have orbits whose time averages do not converge, and even generically diverge in some maximal sense. There are other dynamical systems with only a weaker form of the shadowing property for which this oscillatory behaviour of the time averages still holds (see e.g [34]). In particular this is the case for the discrete system $x \mapsto 4x(1-x)$ on $[0,1]$. The measure μ corresponding to the Bowen–Ruelle–Sinai theorem has the density

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}}.$$

Lebesgue-almost all points in $[0,1]$ have this measure as time-average: but again, they form a set of first category, while the set of points whose time average displays maximal oscillation (i.e. has all invariant measures as limit-points) is generic.

To summarize, there are important classes of orbits whose time averages do not converge. It should be interesting to have a method which, by evaluating time averages in concrete situations, allows to characterize the type of underlying attractor. This does not seem to exist yet. There can be combinations of the asymptotic regimes described in sections 2 and 3, with heterocycles visiting strange attractors; there could be altogether different types of behaviour. Furthermore, the transient motion can subsist for a very long while, thus affecting the time averages. This occurs, e.g. in non-linear networks, where the attractors are fixed points and the irregular oscillations of the approaching trajectories are due to the complicated geometry of the basins of attraction.

Acknowledgements

Part of this work was supported by the Austrian Forschungsförderungsfonds P8043. I would like to thank J. Hofbauer, A. Gaunersdorfer, M. Sievcking and E.C. Zeeman for helpful discussions.

References

[1] M. Frank and T. Stengos, Chaotic dynamics in economic time series, J. Econ. Surveys 2(1988) 103–133.

[2] W. Brock, Chaos and complexity in economic and financial science, in: *Acting under Uncertainty: Multidisciplinary Conceptions,* ed. G. von Fürstenberg (Kluwer, Boston, 1990).

[3] R.M. May, Simple mathematical models with very complicated dynamics, Nature 261(1976) 459–467.

[4] L. Glass and M.C. Mackey, *From Clocks to Chaos* (Princeton University Press, 1988).

[5] O.E. Roessler, Chaos and chemistry, in: *Nonlinear Phenomena in Chemical Dynamics,* ed. C. Vidal and A. Pacault (Springer, Berlin, 1981).

[6] E.N. Lorenz, Deterministic non-periodic flow, J. Atmos. Sci. 20(1963)130–141.

[7] P. Cvitanovic (ed.), *Universality in Chaos* (Adam Higler, Bristol, 1984).

[8] Hao Bai-Lin (ed.), *Chaos* (World Scientific, Singapore, 1984).
[9] H. Haken (ed.), *Chaos and Order in Nature,* Springer Series in Synergetics (Springer, Berlin, 1981).
[10] D. Ruelle, *Chaotic Evolution and Strange Attractors* (Cambridge University Press, 1989).
[11] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms,* Lecture Notes in Mathematics 470 (Springer, Berlin, 1975).
[12] J. Guckenheimer and Ph. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer, Berlin, 1983).
[13] J. Guckenheimer and Ph. Holmes, Structurally stable heteroclinic cycles, Math. Proc. Cambridge Phil. Soc. 103(1988)189–192.
[14] M. Eigen and P. Schuster, *The Hypercycle: A Principle of Natural Self-organization* (Springer, Berlin, 1979).
[15] J. Hofbauer and K. Sigmund, *The Theory of Evolution and Dynamical Systems* (Cambridge University Press, 1988).
[16] M. Nowak and K. Sigmund, Oscillations in the evolution of reciprocity, J. Theor. Biol. 137(1989) 21–26.
[17] J. Maynard Smith, *Evolution and the Theory of Games* (Cambridge University Press, 1982).
[18] A. Gaunsdorfer, J. Hofbauer and K. Sigmund, On the dynamics of asymmetric games, to appear in Theor. Pop. Biol. (1991)345–357.
[19] G. Kirlinger, Permanence in Lotka–Volterra equations, Math. Biosci. 82(1986)165–191.
[20] A. Gaunsdorfer, Time averages for heteroclinic attractors, SIAM J. Appl. Math., to appear.
[21] R.M. May and W. Leonard, Nonlinear aspects of competition between three species, SIAM J. Appl. Math. 29(1975)243–252.
[22] A. Arneodo, P. Coullet and C. Tresser, Occurrence of strange attractors in three dimensional Volterra equations, Phys. Lett. A79(1980)259–263.
[23] P. Schuster, K. Sigmund and R. Wolff, On ω-limits for competition between three species, SIAM J. Appl. Math. 37(1979)49–54.
[24] K. Sigmund, Permanence and heteroclinic cycles for ecological equations, *Proc. Conf. on Nonlinear Oscillations,* ed. M. Farkas, Budapest (1987).
[25] S. Smale, *The Mathematics of Time* (Springer, Berlin, 1980).
[26] J.P. Eckmann and D. Ruelle, Ergodic theory of chaos and strange attractors, Rev. Mod. Phys. 57(1985).
[27] M. Denker, Ch. Grillenberger and K. Sigmund, *Ergodic Theory on Compact Spaces,* Springer Lectures in Mathematics 527 (Springer, Berlin, 1976).
[28] J.C. Oxtoby, Ergodic sets, Bull. Amer. Math. Soc. 58(1952)116–136.
[29] J.C. Oxtoby, *Measure and Category* (Springer, New York, 1971).
[30] K. Sigmund, On the space of invariant measures for hyperbolic flows, Amer. J. Math. 94(1972) 31–37.
[31] K. Sigmund, Generic properties for invariant measures of Axiom A diffeomorphisms, Inventiones Math. 11(1970)9–109.
[32] K. Sigmund, On the time evolution of statistical states for Anosov systems, Math. Z. 138(1974) 183–189.
[33] U. Kirchgraber and D. Stoffer, On the definition of chaos, Z. angew. Math. 69(1989)175–185.
[34] F. Hofbauer, Generic properties of invariant measures for continuous piecewise monotonic transformations, Mh. Math. 106(1988)301–312.