How to Play Out of Equilibrium: Beating the Average (*First Draft*)*

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Abstract

We propose a new concept for how to make choices in games without assuming an equilibrium. To beat the average means to obtain a higher payoff against the others than the others obtain amongst themselves, for any way in which the game might be played. Only Nash equilibrium strategies can beat the average. Beating the average is possible in many symmetric games, including Cournot competition with convex demand. In many other games, including Betrand competition, there are strategies that "almost" beat the average. The methodology is easy to implement and extremely versatile, for instance it can incorporate incomplete information.

1 Introduction

Equilibrium analysis is pervasive whenever there are models with multiple agents who influence each other by their choices. A good choice seems to be one that correctly anticipates the choices of the others. Equilibria seem the only way to predict behavior. Most of game theory, the mathematical formalism to analyze such environments, is identified with the study of Nash equilibria. Yet there are numerous objections to reducing game theory to an equilibrium analysis. At a theoretical level, the type of

*This is a first draft that certainly contains many missing references and mistakes. It was written as suplement to be able to gather information from friends and colleagues about what they think and as first source for citations. Please send me your suggestions and comments.

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environment and objectives of others has to be known by all. In particular, all have to have the same understanding of how uncertainty is resolved, which on a conceptual level seems to be a contradiction in itself (see also the Wilson (1987) critique that blames common knowledge for the disparity between theory and reality). A major obstacle for a consultant is the difficulty to elicit the necessary mathematical details from the person who needs to make a choice. From an empirical standpoint, the experimental literature is not shy in constantly revealing behavior in laboratory and field experiments that do not conform to Nash equilibrium behavior. Last but not least, increasingly complex, sophisticated and lengthy papers are believed to be able to describe how real people behave and are thought to help decision making in real environments. Equilibrium analysis may also be so prominent as there are only few alternatives.

In this paper we present a novel solution concept for how to play in games when one does not expect that an equilibrium will be played. It applies when there are several players in the same role as oneself, as in a symmetric game or when there are types of players. It also applies when one wishes to compare performance to that of a set of other players. The new methodology is meant to aid in decision making in real environments where own and others' choices influence each other.

Imagine that you wish to choose a strategy in a game but do not know what others will be choosing. They may be colluding, they may have different objectives than oneself. We will not change how we evaluate our own success when we know what all others are doing, we continue to measure performance in terms of expected utility when we know all details. However we will not posit some distribution if we do not know how others will be playing the game. So we do not reduce uncertainty to risk. The idea is to change the benchmark. One cannot best respond to something that one does not know but one might be able to better respond to what one does not know by better responding to all possible situations. Better than what? We propose to take the average payoff of those that are like you as benchmark. How does this work? For the moment consider an n player symmetric game where all players are exante indistinguishable, in particular all have the same action set and the same utility function. The idea is to postulate how people play this game when you are not present, to take their average payoff as benchmark and to then think what happens if you were one of the players. Both times it is the same n player game, when you are participating in the game you randomly face $n-1$ of these players. Your wish is to get a payoff when playing with them that is higher than what they obtain on average payoff when you are not part of the game. It is as if you replace one of the players and compare your own average payoff after this replacement to the average payoff before the replacement. You better respond. As you do not know what others are doing, you wonder if you can perform better than the average for all possible configurations of play. If this is possible, then in our terminology we say that you can *beat the average.* It is as if one is computing the empirical performance of a strategy given data of how people play the game and, as you do not know this data when designing your strategy, you try to do better than the average for any possible data set.[1] Can you beat the average? The answer is "yes" in some important examples, in other cases the answer is "no". In any case this approach leads to a new solution concept. When it is not possible to beat the average then the idea is to try to almost beat the average, formally to minimize the maximal amount that one might end up below the average. The term "almost" makes sense when this shortcoming is small. In astonishingly many salient examples it turns out that one can almost beat the average. Our concept generalizes readily to asymmetric games and can be nicely extended to include incomplete information and uncertainty.

We insert a numerical illustration.

> Consider Cournot competition with two firms facing linear demand $P(q) = \max\{1-q,0\}$ and no costs. If firm 1 chooses $q_1 = \frac{1}{4}$ and firm 2 best responds with $q_2 = \frac{3}{8}$ then they obtain profits $\frac{1}{4}\left(1-\frac{1}{4}-\frac{3}{8}\right) \approx 0.09$ and $\frac{3}{8}\left(1-\frac{1}{4}-\frac{3}{8}\right) \approx 0.141$ respectively, their average profit equals 0.117. Now if we choose the symmetric NE quantity $\frac{1}{3}$, then in a market with firm 1 we get $\frac{1}{3}\left(1-\frac{1}{4}-\frac{1}{3}\right) \approx 0.138$, with firm 2 we get $\frac{1}{3}\left(1-\frac{1}{3}-\frac{3}{8}\right) \approx 0.097$, so on average we get 0.118. This is more than 0.117, than what they got on average when we did not participate. This is not a coincidence. In this game the symmetric NE quantity beats the average. Now consider instead Bertrand competition. The symmetric NE price 0 will not beat the average as it obtains profit 0 in any market while the two firms can get strictly positive profits when we do not participate. Under Bertrand competition we select the price 0.092. It never attains a payoff that is more than 0.042 below the average.

In any game, even when all players are exante identical, it is clear that some players will perform better while others will perform worse. Clearly one cannot always expect to be better than the previously best. To focus on the previously lowest payoff seems too meek of an objective. We propose to use the average payoff as benchmark where

[1] This concept appeared on June 3 2017 when empirically investigating how different strategies, in particular, how the Nash equilibrium quantity would perform in Cournot experiments.

it is important that own play does not influence the benchmark. One compares the average when one was not part of the game to own payoff when one is playing the game. One does not compare own payoff to the payoff of others that one is playing with. Note that the average payoff in the game where one is not present is a theoretical construct that one cannot observe after playing the game. Only when not participating and observing play of others then the average payoff is observable.

Mean performance is the natural statistic, the most important statistic when analyzing random variables. It works well mathematically. It plays an important role in evolutionary models.[2] It has been used in financial incentives in an attempt to control for other factors that influence performance. It has been used to increase contributions to a public good (Falkinger et al., 2000). Salary comparisons for new hirings as well as for promotions have become an important tool in human resource management. One of their measures is the so-called compa-ratio which is the ratio of own pay to market rate. Note that we compare own to average by using the difference.[3]

How to successfully propose a new solution concept? We compare it to existing ones and then argue its usefulness and versatility in a plethora of examples. The richness of possible applications makes it clear that many interesting open questions have to be left for future research, also be it for space reasons.

In this paper we present a novel concept that can be used as an aid or benchmark when thinking about how to play a game (normative). It is less clear at this point as to whether it also is good as a prediction of how people actually play games (positive or descriptive).[4]

The prominent existing solution concept is that of a Nash equilibrium for which the following four critique points have been voiced. (i) Nash equilibrium applies when one believes that everyone is in equilibrium. Its foundation rests on players knowing a lot about each other and the environment. However, one might not believe that players are playing some equilibrium, an educated belief that emerges after teaching game theory for many years. One might wish for an alternative as a complement. (ii) It is hard to use the Nash equilibrium concept as a recommendation when there

[2]In the replicator dynamics (Taylor and Jonker, 1978), the canonical model of evolutionary dynamics, strategies increase in proportion if and only if they perform better than average.

[3]While there are many ways to evaluate relative performance based on own and average payoff, differences and ratios are the main contenders. Ratios are useful for comparisons across games. We need a measure of comparison within the same game. Differences capture a sense that utilities and payoffs are measured in absolute terms and most importantly, taking differences avoids caring about ratios of arbitrarily small payoffs.

[4]In the experiments analyzed in this paper our concept fails to predict what subjects are doing.

are multiple Nash equilibria, in particular as there is no commonly accepted equilibrium selection criterion. (iii) Sometimes Nash equilibria involve mixed strategies, and mixed strategies are difficult to implement in practical contexts. (iv) Nash equilibria are hard to compute in large games (e.g. see Gilboa and Zemel, 1989).

We turn now to our proposed concept and point out that while it will have its own limitations and critique points, it is not subject to any of the four critique points listed above. Ad (i), it is designed to deal with out of equilibrium behavior. Ad (ii), there may be multiple solutions but all these perform equally well according to our criterion. Ad (iii), our solution concept is well defined if one limits attention to pure strategies while the Nash equilibrium paradigm does not leave room to recommend a pure strategy when there is no Nash equilibrium in pure strategies. Ad (iv), our solution concept is easy to compute (it is the solution to a zero sum game, so following (Khachiyan, 1979) it can be solved efficiently using a linear programming problem).

Other solution concepts and approaches to analyze out of equilibrium behavior already exist. Rationalizability and level k thinking make a lot of assumptions about how players think about how others make choices. They often suffer from multiplicity. They cannot deal with uncertainty that is not reduced to risk. Recently resurged models of ambiguity can be used but none have been suggested as a general solution concept. Maximin utility (Wald, 1950, Gilboa and Schmeidler, 1989) as the most popular suffers from generating useful predictions in simplest economics games.[5] Exante minimax regret as considered in Schlag and Zapechelnyuk (2016, 2017) and Kasberger and Schlag (2016) has been insightful in specific applications but suffers from its more intricate mathematical structure for general applicability. Expost minimax regret as considered in (Renou and Schlag, 2011) is further from the Nash equilibrium concept as regret in that paper is evaluated expost after the state of nature is realized. In the context of learning in decision problems there is a related concept called improving (Schlag, 1998). Accordingly, it is as if in an unknown decision problem the person wishes to beat the average after observing the choice and payoff of two random individuals who face this decision problem. In that context, the objective is to investigate how much better than the average one can perform as beating the average alone is not difficult (one can beat the average by choosing the action of the first individual observed). In the context of this paper, the person does not observe choices of others before making a choice.

In Section 2 we give a preview of the results. Section 3 introduces the methodology.

[5]For instance, in Cournot and Bertrand competition all actions are equally good under the maximin utility approach.

In Section 4 we apply the methodology to Cournot games, Bertrand games, the Nash demand game and to a coordination game. In Section 5 we show three different ways of applying the methodology to asymmetric games. Under subset comparison the comparison is only with a subset of the players, as example we consider a model with buyers and sellers. Under ignorance one compares to the payoff one would have achieved with the actions of the others, ignoring their utility functions. First price auction and double auction are among the examples. Under role comparison one compares payoffs of players with different utilities, anticipating these utilities when computing averages. Section 6 extends the framework to allow for incomplete information, uncertainty about parameters of the game (as example we consider a seller with limited information about demand), beliefs about how others play the game and mutual understanding that several players are attempting to beat the average. In Section 7 we conclude.

2 Preview of Results

Interestingly, only strategies that are played in a symmetric Nash equilibrium are candidates for a strategy that can beat the average. The reason is as follows. Assume that ξ is not a symmetric NE strategy and consider a configuration in which all but one player choose ξ and one player chooses a best response to the others. To beat the average is like replacing some player and doing on average better than this player performed prior to the replacement. In this particular configuration the person using ξ does as well when replacing a player with strategy ξ but does strictly worse when replacing the one player that is choosing a best response to the others.

It turns out that one can beat the average in a general class of Cournot games with homogeneous and heterogeneous goods. In these games the Nash equilibrium strategy becomes a recommendation even when one does not expect others to be playing an equilibrium. Probing this result on experimental data we obtain that the NE strategy generates a payoff that is far above the average payoff and is very close to the empirical best response. When goods are homogeneous the proof is simple. One shows that the symmetric NE strategy beats the average in a neighborhood of the NE and uses convexity to extend this to all possible profiles. The proof for the heterogenous good case is different, first showing that it is hardest to beat the average on the diagonal when all firms choose the same quantity and then proving the statement on the diagonal.

The analysis of Bertrand competition with heterogenous goods is intricate due to the treatment of prices that drive quantities to zero. For this model we only present the duopoly case with linear demand and show that the symmetric NE can beat the average if goods are not too close substitutes.

In Bertrand competition with homogeneous goods it is not possible to beat the average, but one can almost beat the average. If there is a single consumer with unit demand and no production costs then one prices at $1/(n+1)$ of the consumer's willingness to pay. In general, the recommended price is above marginal cost and decreases in the number of firms. So the Bertrand paradox does not arise when one does not believe that the market is in equilibrium. The maximum that one can fall below the average is equal to $\frac{1}{n(n+1)}$ of the monopoly profits M, so one is within $5\% \cdot M$ of the average payoff or better if there are at least four firms in the market. To put this in context, note that any strategy falls at most $\frac{1}{n}M$ below the average payoffs. The proofs are extremely simple as one readily identifies the worst cases. In experimental data on Bertrand competition with at most four players we find that the methodology to beat the average yields payoffs well above the theoretical lower bound and typically near or above the average.

Of course, in games that require a great deal of coordination (such as pure coordination games) it is hard to even come close to beating the average as own performance is extremely sensitive to what others are doing.

We then move to asymmetric games for which we present three approaches to deal with them. We derive bids in a first price auction when values of other bidders are not known. The model easily extends to incomplete information. We also investigate uncertainty, like how to price in markets with only minimal information about the demand.

3 Methodology

Let ΔB be the set of all distributions that have support in B. Let Γ be a symmetric normal form game with the following ingredients. There are n players. Each player has the same set of actions A. Let a_i denote the action of player i, $a = (a_i)_{i=1}^n$, $a_{-i} = (a_j)_{j \neq i}$, and $A^k = \times_{i=1}^k A$. For $y \in A$ let $y^k \in A^k$ be such that $\left(y^k\right)_i = y$ for all $i \leq k$. Let $u(a_1, a_{-1})$ be the payoff to a player choosing a_1 when the others play $a_2, .., a_n$ where payoffs do not depend on which other player plays which action. So $u : A^n \to \mathbb{R}$. We call $y \in \Delta A$ a symmetric Nash equilibrium (short, NE) strategy if y^n is a Nash equilibrium of Γ.

In the rest of the paper we use the term "person" to identify the individual who wishes to beat the average. We suggest how a person should play a game when she does not know which actions the other players are choosing. Accordingly, the person considers all possible configurations in which the game may be played by others, as specified by the profile of actions $a \in A^n$. The person now imagines joining this game by being matched against a random sample of $n-1$ actions drawn from this profile. So if the person chooses ξ, then he expects a payoff $\frac{1}{n}\sum_{i=1}^{n} u(\xi, a_{-i})$. Ideally the person would like to choose a best response to his opponents, this would be an element of $\arg\max_{x \in \Delta A} \frac{1}{n}\sum_{i=1}^{n} u(x, a_{-i})$. However this is not possible as he does not know a. Instead he considers as a benchmark how players play this game without him, specifically he considers as benchmark the average payoff in the game if he had not entered, namely $\frac{1}{n}\sum_{i=1}^{n} u(a_i, a_{-i})$. We then say that ξ *beats the average* if the person expects a payoff that is at least as large as the average in the game if he had not participated in the game.

Note that it is as if you randomly replace a player and either choose an action ξ or act as if you would be able to adapt the action of the player replaced. Then the action ξ beats the average if it is better to choose ξ than to take the action of the player replaced. This is particularly valuable as the option to take the action of the player replaced is only a fiction. If ξ beats the average then it is a recommendation for playing a simultaneous move game in which all others have not made their choice yet, in particular there will be no replacement of any player.

The way in which ξ is evaluated emerges naturally when one has empirical data of how a game has been played. As a first step one plots the distribution of payoffs obtained by the players and identifies the better actions, for instance those that achieved above average performance. This identifies a benchmark for how well one can perform. Of course, actions might lie in this desirable region because of a lucky match they realized. So one attempts to identify the better actions in a way that is not subject to lucky matches within this sample. Hence the fiction of random replacement. The benchmark remains, to compare performance to the average payoff in the data set. An action that ends up under this evaluation in the desirable region for any data set is said to beat the average.

Note that it is important to emphasize that one does not compare payoffs to players one is playing with. This would have a flavor of spite and envy. The idea in this paper is to model better responding to others. The comparison to the average shapes the benchmark where this benchmark should not be influenced by your own choice.

Under the above methodology one can evaluate any action by the maximal amount that it can perform below average. We call this amount the *shortcoming of an action.* Consequently, in games where it is not possible to beat the average, we propose to choose a rule with the smallest shortcoming. This is a rule that minimizes the maximum amount that one can perform below the average. Mixing across actions is often useful to reduce the shortcoming. Yet, as mixing tends to be infeasible for practical purposes we often select among pure actions.

The formal definitions follow.

Definition 1 *(i) $\xi \in \Delta A$ beats the average if*

$$f_{\xi}(a) := \frac{1}{n}\sum_{i=1}^{n}\left(u(\xi, a_{-i}) - u(a_i, a_{-i})\right) \geq 0$$

holds for all $a \in A^n$.

(ii) $\xi \in \Delta A$ beats the average up to (a shortcoming) ε if $f_{\xi}(a) \geq -\varepsilon$ for all $a \in A^n$, where $\varepsilon \geq 0$.

(iii) $\xi \in \Delta A$ is a best attempt to beat the average if

$$\xi \in \arg\max_{\bar{\xi} \in \Delta A} \inf_{a \in A^n} f_{\bar{\xi}}(a).$$

(iv) $\xi \in A$ is a best attempt to beat the average with a pure action if

$$\xi \in \arg\max_{\bar{\xi} \in A} \inf_{a \in A^n} f_{\bar{\xi}}(a).$$

We provide some comments on the above definitions.

Remark 1 *(i) The restriction to $a \in A^n$ instead of $a \in (\Delta A)^n$ in the definitions above is without loss of generality.*

(ii) To better understand the maximum operator in the definition of best attempt to beat the average, note the following. $\xi \in \Delta A$ is a best attempt to beat the average if and only if there exists no $\bar{\xi} \in \Delta A$ such that $\inf_{a \in A^n} f_{\bar{\xi}}(a) > \inf_{a \in A^n} f_{\xi}(a)$.

(iii) $\xi \in \Delta A$ beats the average if and only if $\xi^n \in \arg\min_a f_{\xi}(a)$.

(iv) ξ is a best attempt to beat the average (with a pure action) if and only if and only if ξ beats the average up to $\varepsilon^* = \max_{\bar{\xi} \in \Delta A} \inf_{a \in A^n} f_{\bar{\xi}}(a)$ ($\varepsilon^* = \max_{\bar{\xi} \in A} \inf_{a \in A^n} f_{\bar{\xi}}(a)$). ε^* is called the associated shortcoming of ξ. In particular, ξ beats the average if and only if it is a best attempt to beat the average and its associated shortcoming is equal to zero.

(v) Often it makes sense to limit the set of possible allocations where one wishes to beat the average, these may be symmetric allocations as suggested in Section 4.4 or small neighborhoods as discussed in Section 6.3.

Next we show some relationships to dominance and rationality.

Proposition 1 *(i) If ξ is a best attempt to beat the average and χ weakly dominates ξ then χ is also a best attempt to beat the average.*

(ii) If ξ is strictly dominated by χ then ξ is not a best attempt to beat the average.

(iii) If ξ is a best attempt to beat the average, $A \subset \mathbb{R}^n$ is compact and u is continuous then ξ is admissible in the sense that it is a best response to some distribution of actions.

The proofs of parts (i) and (ii) are straightforward. We defer the proof of part (iii) until after the next proposition.

Next we provide some necessary and sufficient conditions.

Proposition 2 *(i) If ξ beats the average then ξ is a symmetric NE strategy.*

(ii) If ξ is a symmetric NE strategy, $A \subseteq \mathbb{R}^K$, A is convex, u is differentiable in ξ^n and f_ξ is convex in a then ξ beats the average.

(iii) Assume that $A \subseteq \mathbb{R}^K$, A is convex and u is continuous and twice differentiable. Then ξ beats the average if

$$\frac{\partial}{\partial a_i}\frac{\partial}{\partial a_i} f_\xi(a) - 2\frac{\partial}{\partial a_i}\frac{\partial}{\partial a_j} f_\xi(a) + \frac{\partial}{\partial a_j}\frac{\partial}{\partial a_j} f_\xi(a) \geq 0 \tag{1}$$

for all $i \neq j$ and $a \in A^n$, and

$$u\left(\xi, y^{n-1}\right) \geq u\left(y, y^{n-1}\right) \text{ holds for all } y \in A. \tag{2}$$

(iv) Consider the two player zero sum game between a person choosing an action in A and nature choosing an action profile in A^n with payoffs of the person given by $f_\xi(a)$. Assume that this game has a NE. Then ξ is a best attempt to beat the average with the associated shortcoming ε if and only if there exists $\alpha \in \Delta(A^n)$ such that (ξ, α) is a NE of this game and $f_\xi(\alpha) = -\varepsilon$.

Proof. (i) Assume that $u\left(z, \xi^{n-1}\right) > u(\xi^n)$. Then

$$f_\xi\left(z, \xi^{n-1}\right) = -\frac{1}{n}\left(u(\xi^n) - u\left(z, \xi^{n-1}\right)\right) < 0$$

and hence ξ does not beat the average.

(ii) Note that $f_\xi(\xi^n) = 0$. We aim to show that ξ^n is a global minimum. As f_ξ is convex in a it is sufficient to show that ξ^n is a local minimum.

As u is differentiable at ξ^n we have that f_ξ is differentiable at ξ^n. As ξ is a symmetric NE strategy, we have that $f_\xi(\xi^n + e_i x) \geq 0$ for all $x \geq -\xi$.

Assume $\xi > 0$. Then $\nabla f_\xi (\xi^n) = 0$. As $f_\xi (\xi^n) = 0$ we have by convexity of f_ξ that $f_\xi (a) \geq f_\xi (\xi^n) + \nabla f_\xi (\xi^n) (a - \xi^n)$ and hence $f_\xi (a) \geq 0$.

Assume $\xi = 0$. Then $\nabla f_\xi (0^n) \geq 0$ and by the same argument we obtain $f_\xi (a) \geq \nabla f_\xi (0^n) a \geq 0$ as $a \geq 0$.

(iii) We first show that (1) implies that $f_\xi (a) \geq f_\xi \left(\left(\frac{s(a)}{n} \right)^n \right)$. Fix some i and j with $i \neq j$. Let $h(w) = f_\xi (a + w (e_i - e_j))$ where e_k is the k-th unit vector. Then (1) implies that $h''(w) \geq 0$ for all w and hence that h is convex. Together with the fact that the symmetry of f_ξ implies that $h\left(\frac{a_j - a_i}{2} + y \right) = h\left(\frac{a_j - a_i}{2} - y \right)$ we obtain that h is minimized at $w = \frac{a_j - a_i}{2}$. Thus, replacing a_i and a_j each by $\frac{a_i + a_j}{2}$ reduces f_ξ. Note that it also strictly reduces the variance of the actions provided $a_i \neq a_j$.[6] As i and j were arbitrary indices with $i \neq j$ we obtain that $f_\xi (a) \geq f_\xi \left(\left(\frac{s(a)}{n} \right)^n \right)$.

Next we investigate $f_\xi (y^n)$. Note that (2) implies that $f_\xi (y^n) \geq f_\xi (\xi^n)$. The proof now follows from the fact that $f_\xi (\xi^n) = 0$.

(iv) Following the minimax theorem, $f_\xi (\alpha) = \max_{\bar{\xi}} \min_{\bar{\alpha}} f_{\bar{\xi}} (\bar{\alpha})$ holds for any NE (ξ, α). Note that the objective of the person in this zero sum game is to choose $\xi \in \arg\max \sum_i u (\xi, \alpha_{-i})$. In particular, ξ is a best response to this particular distribution of joint actions. ■

Corollary 1 *If ξ beats the average and $a^* \neq \xi^n$ is a NE then ξ is a best response for each player in the profile a^*, in particular, a^* is not a strict NE.*

Proof of Proposition 1 (iii). Following Proposition 2(iv), the statement is proven if we show that this zero sum game has a NE. Existence is ensured by Glicksberg (1952). ■

Beating the average as defined in Definition 1 is the central concept. For clarity of exposition it has now only been defined for symmetric games without any additional restriction on how the game is being played. In the subsequent presentation, when need be, it will be minorly adapted to the specific application or extension. In fact, the examples that follow have also been selected to demonstrate the versatility of this concept. We preview how it will be adapted. Section 4.4 incorporates beliefs that actions are chosen independently. In Section 5.1 the average payoff benchmark is only computed among a subset of the players. In Section 5.2 the comparison is not to the payoffs of others but to own payoffs that would have realized when using the actions chosen by the others. A variation is included that continues to allow that others choose dominated strategies but rules out that payoffs of dominated strategies

[6] Recall that the variance of $\{a_i\}_{i=1}^n$ is given by $\frac{1}{n} \sum_{i=1}^n a_i^2 - \left(\frac{1}{n} \sum_{i=1}^n a_i \right)^2$.

are used in the benchmark. In Section 5.3 the game is played among players with different utility functions and the person anticipates their different objectives. In Section 6.1 the definition is extended to incomplete information where actions are conditioned on types. Section 6.2 allows not only for uncertainty about play of others but also for uncertainty about parameters of the game. In Section 6.3 beliefs about how others play the game are included. In Section 6.4 a solution concept is introduced for the case where several players are simultaneously trying to beat the average and where this is common knowledge among them.

4 Applications

4.1 Special Aggregative Games

We start with a class of games where own payoff only depends on the actions of others through an interaction term that is linear in one's own action.

Proposition 3 *Let $A \subseteq \mathbb{R}$ be such that A is convex. Assume that $u(a) = g_0(a_1) + a_1 g_1(s(a))$ is such that $g_0(z)$ and $z g_1(z)$ are concave in z while $g_1(z)$ is convex in z. Let $\xi \in A$ be a symmetric NE strategy and assume that u is differentiable at ξ^n. Then ξ beats the average.*

Proof. Note that

$$\begin{aligned} f_\xi(a) &= \frac{1}{n}\sum_{i=1}^{n}\left(g_0(\xi) + \xi g_1(\xi + s(a_{-i}))\right) - \frac{1}{n}\sum_{i=1}^{n}\left(g_0(a_i) + a_i g_1(s(a))\right) \\ &= g_0(\xi) + \frac{1}{n}\sum_{i=1}^{n}\xi g_1(\xi + s(a_{-i})) - \left(\frac{1}{n}\sum_{i=1}^{n} g_0(a_i) + \frac{1}{n} s(a) g_1(s(a))\right). \end{aligned}$$

The assumptions on g imply that f_ξ is convex. Hence the result follows from Proposition 2 (ii). ■

Examples include private provision of a public good (see Acemoglu and Jensen, 2013). The result above also applies to Cournot competition with homogeneous goods where $u(q) = q_1 P(s(q)) - c(q_1)$ with convex costs c and convex inverse demand P such that $zP(z)$ is concave. For instance, this holds when $P(z) = z^\alpha$ for some $\alpha > 1$. Note that the assumption that $zP(z)$ is concave implies that $P(z) > 0$ for all z if $P(z) > 0$ for some $z > 0$. So this rules out linear demand where $P(z) = \max\{1 - z, 0\}$ which we deal with below.

4.2 Cournot Competition

Consider quantity competition with homogenous goods where the price is a function of the aggregate demand. In the following we deal with kinks in the inverse demand. Note that concavity of $zP(z)$ together with differentiability implies that $2P'(z) + zP''(z) \leq 0$ which is weaker than strategic substitutes (see (3) below).

Proposition 4 *Let $A \subseteq \mathbb{R}_+$ with A convex and let $\xi \in A$. Assume $u(q) = q_1 P(s(q)) - c(q_1)$ where (i) c is increasing, convex and c is differentiable at ξ and (ii) P is non-negative, decreasing, convex, P is differentiable at $n\xi$, and $zP(z)$ is concave when $P > 0$. If ξ is a symmetric NE strategy then ξ beats the average.*

Proof. As P is decreasing, $\{q : P(s(q)) > 0\}$ is a convex set. The proof of Proposition 3 then shows that $f_\xi(q) \geq 0$ when $P(s(q)) > 0$.

Assume that $P(s(q)) = 0$. If $P(n\xi) = 0$ then the fact that ξ is a symmetric NE strategy implies that $c(0) = c(\xi)$, so

$$f_\xi(q) \geq -c(0) + \frac{1}{n}\sum_{i=1}^{n} c(q_i) \geq 0$$

as c is increasing. If $P(n\xi) > 0$ then $s(q) > n\xi$, so using the fact that c is convex and increasing we obtain

$$f_\xi(q) \geq -c(\xi) + c\left(\frac{s(q)}{n}\right) \geq 0.$$

■

Given the above result it seems natural to compute expost how the symmetric NE strategy would perform in laboratory experiments. Interestingly we did not find any such computations in this literature that focusses on how the experimental aggregate demand differs from the equilibrium aggregate demand. We revisit the experiments of Huck et al. (2004) in which $u(q) = q_1 \max\{(99 - s(q)), 0\}$ and $A = \{0, 0.01, 0.02, .., 99.98, 99.99, 100\}$ where $n \in \{2, .., 5\}$. In the Figure 1 below we show on the left hand side for different values of n (separate figure for each n) the difference to the average payoff within the lab, from the best response to the true empirical distribution and from the NE strategy (that beats the average) across the 25 rounds. Each observation aggregates information from 6 different groups playing the game. On the right hand side we show the average strategy changes between rounds. Our proposition above predicts on the left hand side that the dots are above the dotted line. As it is impossible to outperform the best response to the true empirical distribution the dots must be below the reversed triangles. So we know that each dot will be somewhere between the dotted line and the reversed triangle. However, in this data set we see that the symmetric NE strategy performs astonishingly close to the best response. We briefly investigate this phenomenon further.

We are interested in how payoff of the best response $p^*(q) := \max_x \left(\frac{1}{n}\sum_{i=1}^n u(x, q_{-i})\right)$ compares to the payoff $p_b(q) := \frac{1}{n}\sum_{i=1}^n u(\xi, q_{-i})$ of the symmetric NE strategy ξ and to the average payoff $p_a(q) := \frac{1}{n}\sum_{i=1}^n u(q_i, q_{-i})$. We focus on the case of linear demand which we normalize to $P(z) = \max\{1-z, 0\}$. So $\xi = \frac{1}{n+1}$. Assume that q is sufficiently close to ξ^n such that $1 - s(q_{-i}) - \xi \geq 0$ holds for all i. Then it is easily verified that

$$\begin{aligned}
p_a &= \frac{1}{n} s(q)(1 - s(q)) \\
p_b(q) &= \xi\left(1 - \xi - s(q)\left(1 - \frac{1}{n}\right)\right) \\
p^*(q) &= b\left(1 - b - s(q)\left(1 - \frac{1}{n}\right)\right) \text{ with } b = \frac{1}{n+1} - \frac{n-1}{2}\left(\frac{s(q)}{n} - \xi\right).
\end{aligned}$$

So all three expressions depend on q only via $s(q)$. Note that the NE strategy ξ is a first order approximation of the best response, hence remains close to the best response when q is in the neighborhood of ξ^n. We calculate how close p_b is to p^* relative to $p_b - p_a$, so search for λ such that $\lambda p^* + (1-\lambda) p_a = p_b$, so $\lambda = \frac{p_b - p_a}{p^* - p_a}$. And we find that $\lambda = \frac{4n}{(n+1)^2}$. Due to the linear structure, λ is independent of $s(q)$. The values of λ for $n = 2, 3, 4$ and 5 are $\frac{8}{9}$, $\frac{3}{4}$, $\frac{16}{25}$ and $\frac{5}{9}$ respectively. Moreover, starting $n = 6$ we find that p_b is closer to p_a and as n tends to infinity that λ tends to 0 which means that p_b is arbitrarily close to p_a relative to its distance to p^*. So if each dot were a single market then for markets q in which $99 - s(q_{-i}) - \xi \geq 0$ holds for all q we should see that the dots are located exactly at the fraction λ of the reversed triangles. However, each observation is derived from 6 groups of n players playing the game. While p_b and p^* can be applied to the average quantity, p_a is concave in $s(q)$, hence an average over the quantities leaves the average payoff below the payoff of the average quantity. This means that each dot should be above the fraction λ of the reversed triangle whenever $99 - s(q_{-i}) - \xi \geq 0$ holds for all i and all q underlying the same observation. This condition is satisfied for 53 of the 100 observations. In the other markets similar mechanisms seem to be at work, a formal analysis of these cases is left for future research.

We summarize the above findings, referring to the definitions above. We include constant marginal costs as these translate directly into a scaling of the payoffs.

Remark 2 *Consider Cournot competition with constant marginal costs* $c(q_i) = c_0 q_i$

and linear demand where $P(z) = \max\{1-z, 0\}$. *If* $1 - s(q_{-i}) - \xi \geq 0$ *for all* i *then*

$$\frac{p_b(q) - p_a(q)}{p^*(q) - p_a(q)} = \frac{4n}{(n+1)^2}(1 - c_0).$$

So for this particular but salient demand function, the symmetric NE strategy does not only beat the average, it splits the range between average payoffs and best response in a constant fraction when quantities are sufficiently close to ξ such that ξ achieves positive profits against any sample of $n-1$ of the others.

We provide two counterexamples to Proposition 4. The first shows that convexity of inverse demand is needed as an assumption. Assume that $c = 0$ and $P(z) = \max\{0, 1 - z - \alpha \cdot z^2\}$ for $\alpha > 0$. Then $\xi = \frac{1}{16\alpha}\left(-3 + \sqrt{9 + 32\alpha}\right)$ is a symmetric NE strategy but ξ does not beat the average as $f_\xi\left(\frac{1}{2\alpha}\left(-1 + \sqrt{1 + 4\alpha}\right) - \xi, 0\right) < 0$ for all $\alpha > 0$.[7]

The second shows that concavity of $zP(z)$ is needed. Let $P(z) = 26 + c - 28z + 8z^2$ for $z \in \left[0, \frac{3}{2}\right]$ and $P(z) = \frac{1}{z-1} + c$ if $z > \frac{3}{2}$. Then P is twice continuously differentiable, strictly decreasing and convex. However $zP(z)$ is not concave for $z \geq \frac{3}{2}$. The symmetric NE strategy is given by $\xi = 1$. We find $\lim_{x\to\infty} f_\xi(x, x) = -\frac{1}{2}$.

We now consider heterogeneous goods where there exists $\gamma \in (0,1)$ such that $u(q) = q_1 P(q_1 + \gamma s(q_{-1}))$. A firm may have an excessive quantity and hence face price zero, and nevertheless influence with this quantity the positive profit of some other firm. This slightly complicates the presentation and analysis. As we are considering out of equilibrium quantities we need to make sure that the corresponding prices make sense. The price faced by firm i depends on the quantities chosen by the other firms. This only makes sense if the quantities chosen by the others are also sold. However, it is not clear that all quantities chosen will always be sold. Afterall, the fact the price falls to zero indicates that more cannot be sold. Clearly, the formula makes sense when $q \in \{\bar{q} : P(\bar{q}_i + \gamma s(\bar{q}_{-i})) > 0 \forall i\}$ as the willingness to pay a strictly positive price indicates that there is demand. When q lies outside the closure of this set, not all firms will sell the quantities they planned on selling. We will not specify the explicit rule for which firm sells how much. We will only posit some general properties of the vector q^L of quantities that are actually sold as a function of the intended quantities q, so $q^L = q^L(q)$. The fact that quantities q^L are actually sold, given our above intuition above, means that $q^L \in cl(\{\bar{q} : P(\bar{q}_i + \gamma s(\bar{q}_{-i})) > 0 \forall i\})$.[8] Moreover,

[7] Nevertheless, the symmetric NE strategy does not perform that bad in this example, its maximal shortcoming is bounded above by 0.0087 for any $\alpha > 0$.

[8] Let $cl(B)$ be the closure of the set B.

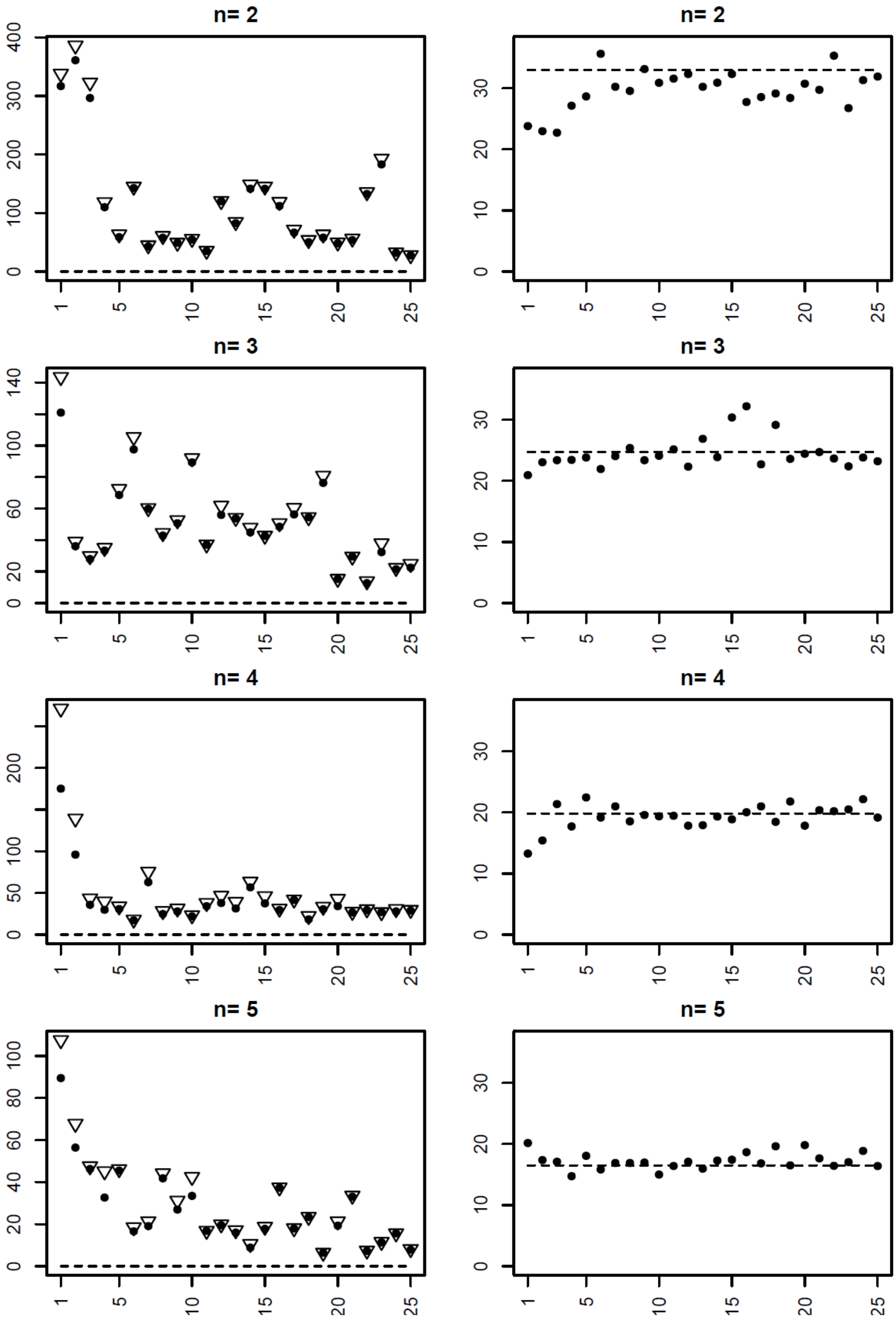

Figure 1: Cournot experiment data. The lefthand side panels show the difference to average payoff for the different rounds. Dots show difference from NE strategy, reversed triangle show difference from empirical best response. The righthand side panels show average quantities.

no firm sells more than planned, so $q_i^L \leq q_i$. As P is decreasing and c is increasing we obtain that $u\left(q^L\right) \geq u\left(q\right)$. This property allows us to get around specifying an explicit formula for q^L, see statement and proof below.

In the following we assume strategic substitutes whenever prices are strictly positive, so

$$\gamma\left(P'\left(q_i+\gamma s\left(q_{-i}\right)\right)+q_i P''\left(q_i+\gamma s\left(q_{-i}\right)\right)\right) \leq 0 \text{ whenever } P\left(q_j+\gamma s\left(q_{-j}\right)\right)>0 \text{ for all } j \tag{3}$$

which holds when $P'' \geq 0$ and $\gamma>0$ if and only if

$$P'\left(z\right)+zP''\left(z\right) \leq 0 \text{ whenever } P\left(z\right)>0.$$

Given sufficient differentiability, this is a stronger assumption than concavity of $zP\left(z\right)$ as assumed in Proposition 4 above.

Proposition 5 *Let $A \subseteq \mathbb{R}_+$ with A convex and $\gamma \in (0,1)$. Assume $u\left(q\right)=q_1^L P\left(q_1^L+\gamma s\left(q_{-1}^L\right)\right)-c\left(q_1^L\right)$ with $P\left(0\right)>c\left(0\right)$. Assume that c is increasing, twice differentiable and convex. Assume that P is convex and decreasing, and for $P>0$ that P is twice differentiable and goods are strategic substitutes. Then the symmetric NE strategy ξ beats the average.*

Note that the proof of Proposition 4 was particularly simple as we were able to verify that f_ξ was convex. There we used the fact that average payoffs in the market were only a function of the total quantity which is no longer true here as $\gamma<1$. Now the transitions between regions due to prices turning zero calls for a different method of proof.

Proof. By continuity of f_ξ it is enough to prove the statement for $q \in \{\bar{q}: P\left(\bar{q}_i+\gamma s\left(\bar{q}_{-i}\right)\right)>0 \forall i\}$. Thus $q_i^L\left(q\right)=q_i$ for all i. Assume that $u\left(\xi, q_{-1}\right)=\xi P\left(\xi+\gamma s\left(q_{-1}\right)\right)-c\left(\xi\right)$. By the assumptions on q^L note that this decreases $u\left(\xi, q_{-1}\right)$ which makes it more difficult to show that ξ beats the average.

Consider some $j \neq i$. We start by showing that $f_\xi\left(q+x\left(e_j-e_i\right)\right)$ is convex in x. For this we consider

$$h_i\left(q\right):=\left(\xi P\left(\xi+\gamma s\left(q_{-i}\right)\right)-c\left(\xi\right)\right)-\left(q_i P\left(q_i+\gamma s\left(q_{-i}\right)\right)-c\left(q_i\right)\right)$$

where $f_\xi(q) = \sum_{i=1}^n h_i(q)$. We will show that $h_i(q + x(e_j - e_i))$ is convex in x where h_i is differentiable with respect to q_i and to q_j. We calculate

$$\begin{aligned}
\frac{d}{dq_i} h_i(q) &= -P(q_i + \gamma s(q_{-i})) - q_i P'(q_i + \gamma s(q_{-i})) + c_i'(q_i) \\
\frac{d}{dq_i}\frac{d}{dq_i} h_i(q) &= -2P'(q_i + \gamma s(q_{-i})) - q_i P''(q_i + \gamma s(q_{-i})) + c_i''(q_i) \\
\frac{d}{dq_i}\frac{d}{dq_j} h_i(q) &= -\gamma P'(q_i + \gamma s(q_{-i})) - \gamma q_i P''(q_i + \gamma s(q_{-i})) \\
\frac{d}{dq_j} h_i(q) &= \gamma \xi P'(\xi + \gamma s(q_{-i})) - \gamma q_i P'(q_i + \gamma s(q_{-i})) \\
\frac{d}{dq_j}\frac{d}{dq_j} h_i(q) &= \gamma^2 \xi P''(\xi + \gamma s(q_{-i})) - \gamma^2 q_i P''(q_i + \gamma s(q_{-i})).
\end{aligned}$$

As

$$\left.\frac{d}{dx}\frac{d}{dx} h_i(q + x(e_j - e_i))\right|_{x=0} = \frac{d}{dq_i}\frac{d}{dq_i} h_i(q) + \frac{d}{dq_j}\frac{d}{dq_j} h_i(q) - 2\frac{d}{dq_i}\frac{d}{dq_j} h_i(q)$$

we find

$$\begin{aligned}
&\left.\frac{d}{dx}\frac{d}{dx} h_i(q + x(e_j - e_i))\right|_{x=0} \\
= \; & -2P'(q_i + \gamma s(q_{-i})) - q_i P''(q_i + \gamma s(q_{-i})) + \gamma^2 \xi P''(\xi + \gamma s(q_{-i})) - \gamma^2 q_i P''(q_i + \gamma s(q_{-i})) \\
& -2\gamma(-P'(q_i + \gamma s(q_{-i})) - q_i P''(q_i + \gamma s(q_{-i}))) + c''(q_i) \\
= \; & -2(1-\gamma) P'(q_i + \gamma s(q_{-i})) - (1-\gamma)^2 q_i P''(q_i + \gamma s(q_{-i})) + \gamma^2 \xi P''(\xi + \gamma s(q_{-i})) + c''(q_i) \\
\geq \; & -(1-\gamma)^2 (P'(q_i + \gamma s(q_{-i})) + q_i P''(q_i + \gamma s(q_{-i}))) + \gamma^2 \xi P''(\xi + \gamma s(q_{-i})) \\
\geq \; & 0
\end{aligned}$$

where we used the fact that $P' \leq 0$, $2 \geq 1 - \gamma$, goods are strategic substitutes and $P'' \geq 0$. As this holds for each i, j with $i \neq j$ we have verified that $f_\xi(q + x(e_j - e_i))$ is locally convex in x wherever f_ξ is differentiable.

We now consider changes at points of discontinuity of f_ξ. Assume that $q_j < q_i$. We look at $f_\xi(q + x(e_j - e_i))$ for $x \geq 0$. We aim to show that f_ξ is convex in x, so that its derivative with respect to x is increasing in x. We calculate

$$\begin{aligned}
\frac{d}{dq_j} h_i - \frac{d}{dq_i} h_i &= \gamma \xi P'(\xi + \gamma s(q_{-i})) + P(q_i + \gamma s(q_{-i})) + (1-\gamma) q_i P'(q_i + \gamma s(q_{-i})) - c'(q_i) \\
\frac{d}{dq_j} h_j - \frac{d}{dq_i} h_j &= -\gamma \xi P'(\xi + \gamma s(q_{-j})) - P(q_j + \gamma s(q_{-j})) - (1-\gamma) q_j P'(q_j + \gamma s(q_{-j})) + c'(q_j)
\end{aligned}$$

so

$$\begin{aligned}
& n \left.\frac{d}{dx} f_\xi(q + x(e_j - e_i))\right|_{x=0} = \frac{d}{dq_j} h_i - \frac{d}{dq_i} h_i + \frac{d}{dq_j} h_j - \frac{d}{dq_i} h_j \\
= \; & \gamma \xi P'(\xi + \gamma s(q_{-i})) + P(q_i + \gamma s(q_{-i})) + (1-\gamma) q_i P'(q_i + \gamma s(q_{-i})) - c'(q_i) \\
& -\gamma \xi P'(\xi + \gamma s(q_{-j})) - P(q_j + \gamma s(q_{-j})) - (1-\gamma) q_j P'(q_j + \gamma s(q_{-j})) + c'(q_j).
\end{aligned}$$

Let's first consider jumps in the derivative due to changes in the terms without ξ. Assume that $P(q_i + \gamma s(q_{-i})) = 0$. Then a marginal increase in x makes the term $(1-\gamma) q_i P'(q_i + \gamma s(q_{-i}))$ drop which makes the derivative increase.

Assume that $P(q_j + \gamma s(q_{-j})) = 0$. Then $P(q_i + \gamma s(q_{-i})) = 0$ as $q_j + \gamma s(q_{-j}) < q_i + \gamma s(q_{-i})$. But this cannot happen for $q \in \{\bar{q} : P(\bar{q}_i + \gamma s(\bar{q}_{-i})) > 0 \forall i\}$. Here we use the fact that $P' \leq 0$ when $P > 0$.

Consider now the terms with ξ. Increase of x will increase $\xi + \gamma s(q_{-i})$ and decrease $\xi + \gamma s(q_{-j})$. So a discontinuous jump in $\frac{d}{dx} f_\xi(q + x(e_j - e_i))$ can only occur when $\xi + \gamma s(q_{-i}) = 0$. In this case the term $\gamma \xi P'(\xi + s(q_{-i}))$ drops, which means that $\frac{d}{dx} f_\xi(q + x(e_j - e_i))$ increases.

Thus we have verified that $f_\xi(q + x(e_j - e_i))$ is convex for $q \in$
$\{\bar{q} : P(\bar{q}_i + \gamma s(\bar{q}_{-i})) > 0 \forall i\}$.

Next we investigate f_ξ on the diagonal, see (2). Let

$$g(x,y) = (xP(x + \gamma(n-1)y) - c(x)) - (yP(y + \gamma(n-1)y) - c(y)).$$

To show that $g(\xi, y) \geq 0$. For $P(x + \gamma(n-1)y) > 0$ we calculate

$$\begin{aligned}
\frac{\partial}{\partial x} g(x,y) &= P(x + \gamma(n-1)y) + xP'(x + \gamma(n-1)y) - c'(x) \\
\frac{\partial}{\partial x}\frac{\partial}{\partial y} g(x,y) &= \gamma(n-1)(P'(x + \gamma(n-1)y) + xP''(x + \gamma(n-1)y)) \\
\frac{\partial}{\partial x}\frac{\partial}{\partial x} g(x,y) &= 2P'(x + \gamma(n-1)y) + xP''(x + \gamma(n-1)y) - c''(x).
\end{aligned}$$

Note that $\frac{\partial}{\partial x}\frac{\partial}{\partial y} g(x,y) \leq 0$ by strategic substitutes and $\gamma > 0$.

Note that $P(0) > c(0)$ implies that $u(\xi^n) > 0$ and hence $\xi > 0$ and
$P(\xi + \gamma(n-1)\xi) > 0$. Consequently, $P(x + \gamma(n-1)y) > 0$ if $x, y \leq \xi$.

Assume $y \leq z \leq \xi$. Note that $g(z, \xi)$ is concave in z as $\frac{\partial}{\partial x}\frac{\partial}{\partial x} g(x,y) \leq 0$ which follows from $P' \leq 0$, by strategic substitutes and since c is convex. Note that $g(x, \xi) \leq 0$ as ξ is a symmetric NE strategy. By definition of g we have that $g(\xi, \xi) = 0$. Thus, $\xi \in \arg\max_x g(x, \xi)$. As $z < \xi$ we obtain that $\frac{\partial}{\partial x} g(z, \xi) \geq 0$. As $\frac{\partial}{\partial x}\frac{\partial}{\partial y} g(x,y) \leq 0$ and $y < \xi$ we obtain that $\frac{\partial}{\partial x} g(z,y) \geq \frac{\partial}{\partial x} g(z, \xi)$. Together this means $\frac{\partial}{\partial x} g(z,y) \geq \frac{\partial}{\partial x} g(z, \xi) \geq 0$. As $g(y,y) = 0$ and $\frac{\partial}{\partial x} g(z,y) \geq 0$ we obtain that $g(\xi, y) \geq 0$.

Now assume $y \geq \xi$. As $P(y + \gamma(n-1)y) > 0$ by assumption, for all $x, z \leq y$ we have $P(x + \gamma(n-1)z) > 0$.

Assume $z \in [\xi, y]$. Then $\frac{\partial}{\partial x} g(z, \xi) \leq 0$, $\frac{\partial}{\partial x} g(z,y) \leq \frac{\partial}{\partial x} g(z, \xi)$ and hence $\frac{\partial}{\partial x} g(z,y) \leq 0$ which then implies that $g(\xi, y) \geq g(y,y) = 0$. ■

We present an example that shows that this proof technique does not work when goods fail to be strategic substitutes. To see this, consider $P(q) = h(q_1 + \gamma q_2)$ for where $h(z) = \beta z - \ln z$ for $0 < z \leq -\frac{\text{LambertW}(-\beta)}{\beta}$ and $h(z) = 0$ otherwise, for $\beta \in \left(0, \frac{1}{e}\right)$, where $h' + zh''(z) = \beta$ while $h > 0$.[9] Then $f_\xi(z(1-x), zx)$ is no longer convex in x for $z = 1.13$ when $\beta = 0.01$ and $\gamma = 0.8$, it is only piecewise convex. Yet graphs indicate that ξ beats the average in this example for any $\beta \in \left(0, \frac{1}{e}\right)$ and any $\gamma \in (0, 1]$.

4.3 Bertrand Competition

We now consider price competition. To connect better to the previous section we first consider the case where goods are imperfect substitutes. We consider constant marginal costs, so $u(p) = (p_1 - c) Q(p_1 + \gamma s(p_{-1}))$ with $\gamma \leq 0$. Again we need to make some adjustments when actions are excessive, here if some prices are too high. So we will consider $u(p) = (p_1 - c) Q\left(p_1^L + \gamma s\left(p_{-1}^L\right)\right)$ for some $p^L = p^L(p) \in cl\{p : Q(p_i + \gamma s(p_{-i})) > 0 \forall i\}$ with the following properties. The prices used to determine profits are lower, so $p^L \leq p$. However, a strict cut in price is only used to influence demand of others, it should not change own demand, hence we posit $p_i^L < p_i$ implies $Q\left(p_i^L + \gamma s\left(p_{-i}^L\right)\right) = 0$. The fact that goods are now strategic complements means that the utility will be adjusted downward when moving from p to p^L. In particular we now need to know how big this adjustment is when we wish to prove that ξ beats the average. The adjustment is simple in the case of $n = 2$. Interestingly, our previous analysis (see proof of Proposition 5 and Proposition 14 below) carries over in the neighborhood of ξ. The difficulties arise when moving so far away that some firm faces zero demand. Hence we present here only the analysis for linear demand. We find that the symmetric NE strategy beats the average when goods are not too close substitutes.

Proposition 6 *Assume $n = 2$, $\gamma \in \left[-\frac{4}{5}, 0\right)$, $c \in \left[0, \frac{1}{1+\gamma}\right]$, $A = \mathbb{R}_+$ and*

$$u(p) = \begin{cases} (p_1 - c)(1 - (p_1 + \gamma p_2)) & \textit{if } \max\{p_1 + \gamma p_2, p_2 + \gamma p_1\} \leq 1 \\ (p_1 - c)(1 - (p_1 + \gamma(1 - \gamma p_1))) & \textit{if } [p_1 + \gamma p_2 \leq 1 < p_2 + \gamma p_1] \wedge \left[p_1 < \frac{1}{1+\gamma}\right] \\ 0 & \textit{otherwise.} \end{cases}$$

Then the symmetric NE strategy ξ beats the average.

The bound on γ given above is tight as $f_\xi\left(\frac{1}{2(1+\gamma)}, \frac{1}{2(1+\gamma)}\right) < 0$ when $\gamma \in \left(-1, -\frac{4}{5}\right)$ and $c = 0$.

[9] $\xi = -\frac{1}{(\gamma+2)\beta} \text{LambertW}\left(-\frac{\beta(\gamma+2)}{1+\gamma} e^{-\frac{1}{1+\gamma}}\right)$

Proof. Note that $\xi = \frac{1+c}{2+\gamma}$ where $\xi \geq c$ and $\xi(1+\gamma) \leq 1$ as $c \leq \frac{1}{1+\gamma}$. Hence, $u(\xi, x) \geq 0$ for all x.

Without loss of generality, assume $p_1 \leq p_2$. So $p_1 + \gamma p_2 \leq p_2 + \gamma p_1$. Note that $\xi = \frac{1+c}{2+\gamma} < 1$, hence $\xi + \gamma p_i \leq 1$. If $p_1 + \gamma p_2 > 1$ then $u(p_1, p_2) = u(p_2, p_1) = 0$ and $f_\xi \geq 0$. So assume from now on that $p_1 + \gamma p_2 \leq 1$.

(i) Assume that $p_2 + \gamma\xi \leq 1 < p_2 + \gamma p_1$. In particular, $p_1 < \xi$. Then $f_\xi(p_1, p_2) = \frac{1}{2}u(\xi, p_1) + \frac{1}{2}(u(\xi, p_2) - u(p_1, 1 - \gamma p_1))$, $\frac{d}{dp_1}\frac{d}{dp_1}f_\xi(p_1, p_2) = 1 - \gamma^2 > 0$ and $\frac{d}{dp_1}f_\xi(p_1, p_2) = \frac{1}{2}\gamma^2\frac{-1+c+c\gamma}{\gamma+2} < 0$ if $p_1 = \xi$. So f_ξ attains its minimum when $p_1 = \xi$ so in a different region.

(ii) Assume $\min\{p_2 + \gamma\xi, p_2 + \gamma p_1\} > 1$. Then

$$
\begin{aligned}
&f_\xi(p_1, p_2) = \frac{1}{2}u(\xi, p_1) + \frac{1}{2}(u(\xi, 1 - \gamma\xi) - u(p_1, 1 - \gamma p_1)) \\
&\frac{d}{dp_1}\frac{d}{dp_1}f_\xi(p_1, p_2) = 1 - \gamma^2 > 0 \\
&\frac{d}{dp_1}f_\xi(p_1, p_2) = 0 \text{ if } p_1 = \bar{p}_1 := \frac{1}{2}\frac{3c\gamma^2 - 2 + \gamma^2 - 2c + c\gamma^3}{(\gamma+2)(-1+\gamma^2)}.
\end{aligned}
$$

It is easily verified for $\gamma \geq -\frac{4}{5}$ that $\frac{d}{dc}\frac{d}{dc}f_\xi(\bar{p}_1, p_2) > 0$ and when $c = 0$ that $\frac{d}{dc}f_\xi(\bar{p}_1, p_2) > 0$ and $f_1(\bar{p}_1, p_2) > 0$. Hence $f_\xi \geq 0$ in this region.

(iii) Assume $p_2 + \gamma p_1 \leq 1 < p_2 + \gamma\xi$. Then $f_\xi(p_1, p_2) = \frac{1}{2}(u(\xi, p_1) - u(p_2, p_1)) + \frac{1}{2}(u(\xi, 1 - \gamma\xi) - u(p_1, p_2))$, $\frac{d}{dp_1}\frac{d}{dp_1}f_\xi(p_1, p_2) = 1$ and $\frac{d}{dp_1}f_\xi(p_1, p_2) = 0$ if $p_1 = \bar{p}_1 = -\frac{-\gamma - c\gamma + \gamma^2 p_2 + 2\gamma p_2 - 1 - c}{\gamma+2}$. Moreover, $\frac{d}{dp_2}\frac{d}{dp_2}f_\xi(\bar{p}_1, p_2) = 1 - \gamma^2$ and

$$
\frac{d}{dp_2}f_\xi(\bar{p}_1, p_2) = \left(2\gamma^2 - \gamma - 2\right)\frac{-1 + c + c\gamma}{2(\gamma+2)}
$$

if $p_2 = 1 - \xi\gamma$. So if $\gamma < \frac{1}{4} - \frac{1}{4}\sqrt{17} \approx -0.780\,78$ then $\frac{d}{dp_2}f_\xi(\bar{p}_1, 1 - \xi\gamma) \leq 0$ and f_ξ attains its minimum in a different region. If however $\gamma \geq \frac{1}{4} - \frac{1}{4}\sqrt{17}$ then

$$
\frac{d}{dp_2}f_\xi(\bar{p}_1, p_2) = 0 \text{ if } p_2 = \bar{p}_2 = \frac{1}{2}\frac{2\gamma^2 + c\gamma^2 + \gamma - 2 - c\gamma - 2c}{(\gamma+2)(-1+\gamma^2)}
$$

which implies that

$$
p_1 = p^\circ{}_1 := \frac{1}{2}\frac{3c\gamma^2 - 2 + \gamma^2 - 2c + c\gamma^3}{(\gamma+2)(\gamma-1)(1+\gamma)}.
$$

We find that $f_\xi(p^\circ{}_1, \bar{p}_2)$ attains its minimum when $c = \frac{1}{1+\gamma}$ where $f_\xi = 0$.

(iv) Assume $p_2 + \gamma p_1 \leq 1$ and $p_2 + \gamma\xi \leq 1$. Copying the arguments from the proof of Proposition 5 we see that the minimum is attained on the diagonal.

So consider f_ξ on the diagonal, so $f_\xi(x, x) = u(\xi, x) - u(x, x)$.

Assume $x \leq 1 - \gamma\xi$ we have $\frac{d}{dx}f_\xi(x, x) = 0$ if $x = \xi$ and $\frac{d}{dx}\frac{d}{dx}f_\xi(x, x) = 2(1+\gamma) > 0$, so f_ξ attains its minimum at $x = \xi$.

Assume $x > 1 - \gamma\xi$. Then $f_\xi(x,x) = u(\xi, 1-\gamma\xi) - u(x,x)$, $\frac{d}{dx}\frac{d}{dx}f_\xi(x,x) = 2(1+\gamma) > 0$ where $\frac{d}{dx}u(x,x) = 0$ if $x = \bar{x} := \frac{1}{2}\frac{1+c+c\gamma}{1+\gamma}$ such that $f_\xi(\bar{x},\bar{x})$ attains its minimum when $c = \frac{1}{1+\gamma}$ in which case $f_\xi(\bar{x},\bar{x}) = 0$. ■

We now move to the centerpiece of price competition where goods are perfect substitutes. We maintain that production takes place after sale and assume constant marginal costs to keep the exposition simple. We find that one can almost beat the average, but not by using the symmetric NE strategy. The symmetric NE strategy does worse by a factor of $n+1$.

Proposition 7 *Let $A = \mathbb{R}_+$ and $c \in \mathbb{R}_+$. Let Q be nonnegative, continuous and decreasing such that $\pi(z) := (z-c)Q(z)$ is single peaked with $\{z^*\} = \arg\max_{z\geq 0}\pi(z)$ and $z^* > 0$. Assume*

$$u(p) = \begin{cases} p_1 \frac{1}{|\{i:p_i=p_1\}|}Q(p_1) - c\left(\frac{1}{|\{i:p_i=p_1\}|}Q(p_1)\right) & \textit{if } p_1 = \min_i\{p_i\} \\ 0 \textit{ if } p_1 > \min_i\{p_i\}. \end{cases}$$

(i) It is not possible to beat the average.

(ii) The symmetric NE strategy $\xi = c$ beats the average up to $\frac{1}{n}\pi(z^*)$.

(iii) The unique best attempt to beat the average with a pure actions is the action $\xi \in (c, z^*)$ that solves $(\xi - c)Q(\xi) = \frac{1}{n+1}\pi(z^*)$. It beats the average up to $\frac{1}{n(n+1)}\pi(z^*)$.

Note that $\xi > c$ and that ξ is strictly decreasing with n, approaching c as $n \to \infty$. In particular, the Bertrand paradox does not arise when one anticipates that others may be playing out of equilibrium.

We illustrate with two salient examples. For unit demand and willingness to pay equal to v with $v > c$, the best attempt to beat the average ξ solves $\xi - c = \frac{1}{n+1}(v-c)$ so $\xi = c + \frac{v-c}{n+1}$. The associated shortcoming is equal to $\frac{v-c}{n(n+1)}$. For linear demand $Q(z) = \max\{v - bz, 0\}$ with $b > 0$ and $v > bc$ we solve $(\xi - c)(v - b\xi) = \frac{(v-bc)^2}{4(n+1)b}$ to find $\xi = c + \frac{1}{2b}\left(1 - \sqrt{\frac{n}{n+1}}\right)(v - bc)$. The associated shortcoming is equal to $\frac{(v-bc)^2}{4bn(n+1)}$.

Proof. (ii) If $\xi = c$ then

$$\begin{aligned} f_\xi(p) &= -\frac{1}{n}\sum_{i=1}^{n} u(p_i, p_{-i}) = -\frac{1}{n}\left((\min\{p_i\} - c)\cdot Q(\min\{p_i\})\right) \\ &\geq -\frac{1}{n}\pi(z^*) \end{aligned}$$

where equality is obtained if $p_1 = z^*$ and $p_i > p_1$ for all $i > 1$.

(i) As a strategy that beats the average has to be a symmetric NE strategy and as c is the unique symmetric NE strategy, the proof of (i) follows from (ii).

(iii) Consider only the i-th term $u(\xi, p_i) - u(p_i, p_{-i})$ of $f_\xi(p)$. There are only two potentially worst cases. One is where $p_j = x$ for all j and x is slightly below ξ, in which case $u(\xi, p_i) - u(p_i, p_{-i}) \approx -\frac{1}{n}(\xi - c) Q(\xi)$. In the other case, $p_j = z^*$ for all j, in which case $u(\xi, p_i) - u(p_i, p_{-i}) = (\xi - c) Q(\xi) - \frac{1}{n}(z^* - c) Q(z^*)$. Both worst cases are independent of i. Hence we choose $\xi < z^*$ to solve

$$(\xi - c) Q(\xi) - \frac{1}{n}\pi(z^*) = -\frac{1}{n}(\xi - c) Q(\xi)$$

which proves the desired claim. Note that a solution to this equation exists as Q is continuous. ■

Next we show how mixed pricing can improve performance.

Proposition 8 *Consider assumptions on the game specified in Proposition 7. Assume additionally that π is concave and Q is differentiable.*

(i) There is no deterministic price that is a best attempt to beat the average.

(ii) A best attempt to beat the average is attained by the mixed action ξ that has density $g(p) = \frac{\pi'(p)}{n(p-c)}$ for $p \in [z, z^*]$ where $\int_z^{z^*} g(p)\,dp = 1$. The associated shortcoming is equal to $\pi(z)$.

Proof. (ii) We derive a best attempt to beat the average. We construct this pricing strategy by solving the zero sum game between the person choosing ξ in an attempt to beat the average, and nature who chooses the price vector p of the firms. The payoff of the person is set equal to $f_\xi(p)$. Let $\pi(x) = (x - c) Q(x)$. Assume that the person chooses ξ from a distribution without point masses that has density g. Assume that nature chooses p such that $p_i = x$ and $p_j > x$ for all $j \neq i$. Then

$$\int f_\xi(p) g(\xi)\, d\xi = \int_0^x (y - c) g(y)\, dy - \frac{1}{n}(x - c) Q(x).$$

Differentiating with respect to x we obtain $(x - c) g(x) = \frac{1}{n}\pi'(x)$, hence we choose $g(x) = \frac{\pi'(x)}{n(x-c)}$.

The person will never choose ξ above z^*. Let $z > c$ be such that $\int_z^{z^*} g(p)\,dp = 1$. Note that z exists as $\pi'(x)$ is bounded away from 0 in the neighborhood of $x = c$ by concavity of π. Now we need to choose the distribution of x that makes the person indifferent on (z, z^*). We compute

$$\int f_\xi(x)\, dH(x) = P_H(x > \xi)\, \pi(\xi) - \frac{1}{n}\int (x - c) Q(x)\, dH(x)$$

and choose H such that $P_H(x \leq \xi) = 1 - \frac{\pi(z)}{\pi(\xi)}$, so H has point mass $1 - \frac{\pi(z)}{\pi^*}$ at $x = z^*$. It is easily verified that this is a Nash equilibrium of the zero sum game.

(i) As the zero sum game has a NE, any best attempt to beat the average must be a NE of this game. The only candidate for a pure action that is a best attempt to beat the average is presented in Proposition 7, as it is unique best attempt among the pure actions. However it is easily seen that this pure action is not part of a NE of the zero sum game. ■

We illustrate the improved performance of mixed pricing policies in the two examples presented after Proposition 7. For unit demand we obtain pricing density $g(x) = \frac{1}{n(x-c)}$ on $\left[c + \frac{v-c}{e^n}, 1\right]$, and cdf $G(x) = 1 - \frac{1}{n}\ln\frac{v-c}{x-c}$, the associated shortcoming equals $(v-c)\,e^{-n}$. The use of an appropriate mixed pricing policy yields an improvement as compared to best deterministic price by a factor of $\frac{e^n}{n(n+1)}$, which is equal to 1.23 if $n = 2$ and 4.95 if $n = 5$. For the case of linear demand with $c = 0$ and $b = v = 1$ we obtain pricing density $\frac{1-2x}{nx}$ on $\left[-\frac{1}{2}\,\mathrm{LambertW}\left(-e^{-(n+1)}\right), \frac{1}{2}\right]$ that beats the average up to $-\frac{1}{4}\,\mathrm{LambertW}\left(-e^{-n-1}\right)\left(2 + \mathrm{LambertW}\left(-e^{-n-1}\right)\right)$. The magnitude of improvement as compared to the case of pure actions is slightly larger than it was under unit demand, equal to 1.63 for $n = 2$ and 6.72 for $n = 5$.

Once again we consider the empirical performance of our solution in laboratory experiments. We consider the experiments by Dufwenberg and Gneezy (2000) on Bertrand competition with unit demand, willingness to pay equal to 100 and $A = \{2, .., 100\}$. We only use data from treatments 1a-f. In each round there are 10 data points which we randomly match into groups of size n. We show the results in the Figures $2-4$ below, now including both our deterministic and randomized solution.

In all cases the data points corresponding to the best attempts to beat the average lie well above the corresponding theoretical lower bounds. This is to be expected as these lower bounds are calculated based on very specific worst cases which then naturally do not occur in this data. In all cases except for round 2 under $n = 2$ the mixed pricing solution is closer to the average than the pure solution. This is intuitive as the mixed solution tries to equalize losses in many different configurations. We table the average difference to the average payoff over all rounds for the different models.

n	2	3	4
pure beat	2	-1.5	-0.6
mixed beat	-0.2	-0.9	-0.07
empirical BR	8.6	3.2	2.2

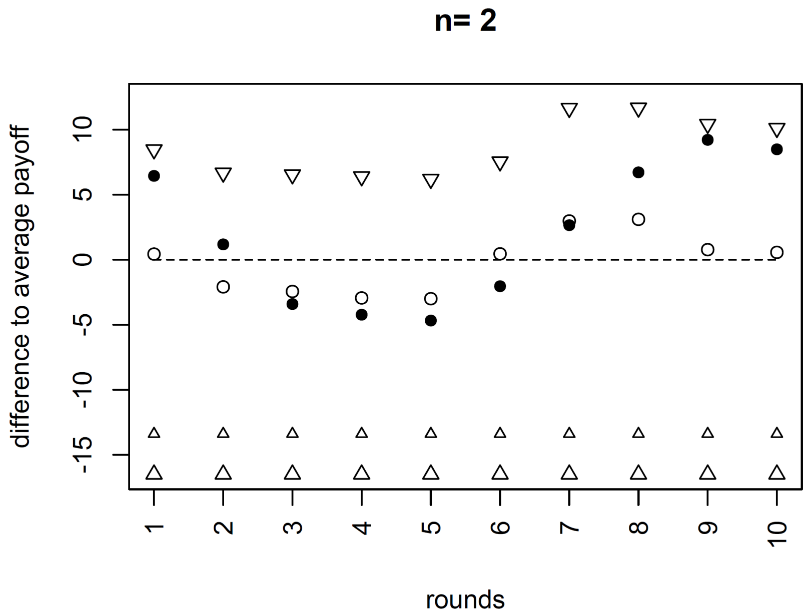

Figure 2: Bertrand competition with $n = 2$. Difference in payoffs to average payoffs of deterministic solution (dot), randomized solution (circle) and best response to empirical distribution (reversed triangle). Triangles show guaranteed lower bound of deterministic solution (small) and randomized solution (large).

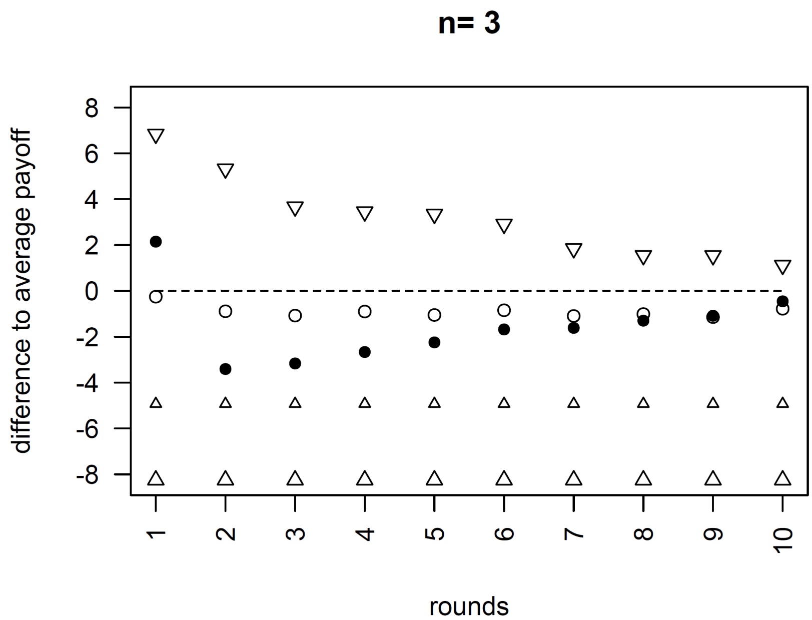

Figure 3: Bertrand competition with $n = 3$.

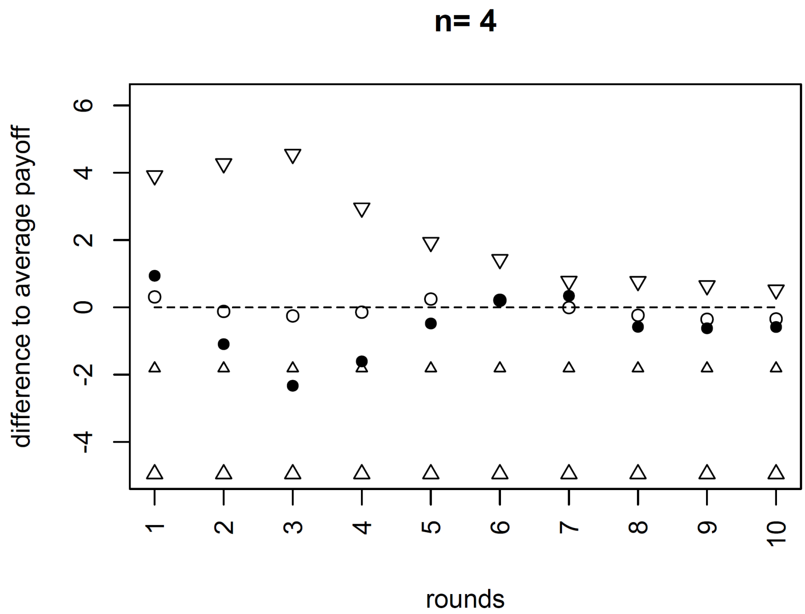

Figure 4: Bertrand competition with $n = 4$.

Substantial differences between pure and mixed pricing do not substantiate, at least not in this data set.

4.4 Nash Demand Game

We now consider the Nash demand game in its simplest form and make a methodological point.

Proposition 9 *Let $A = [0, 1]$ and $u(a) = a_i$ if $\sum_{i=1}^n a_i \leq 1$ and $u(a) = 0$ otherwise. The symmetric NE strategy $\frac{1}{n}$ is a best attempt to beat the average with a pure action, its shortcoming is equal to $\frac{n-1}{n^2}$.*

Proof. The proof is easy. If $\xi > \frac{1}{n}$ then $f_\xi(a) = -\frac{1}{n}$ when $a_i = \frac{1}{n}$ for all n. Assume $\xi \leq \frac{1}{n}$. If $a_i \leq \xi$ for all i then $f_\xi(a) = 0$. If $s(a) > 1$ then $f_\xi(a) \geq 0$. If $a_1 = 1$ and $a_i = 0$ for $i \geq 2$ then $f_\xi(a) = \xi\frac{1}{n} - \frac{1}{n}$. If $\xi = \frac{1}{n}$, $s(a) \leq 1$ and $a_i > \xi$ for some i then $f_\xi(a) \geq \frac{1}{n^2} - \frac{1}{n}$. Together this proves the statement. ■

The shortcoming equals 0.25, 0.16 and 0.09 for $n = 2$, 5 and 10 respectively. For large n it is not much better than the shortcoming of any other action as $f_{\bar{a}}(a) \geq -\frac{1}{n}$ holds for any $\bar{a} \in A$. Note that the shortcoming of our best choice is driven by coordinated play where one player grabs everything and all others concede. Only if the person replaces this greedy player she gets a strictly positive utility. It is hence natural to consider how to beat the average if such coordinated play is ruled out, where the person believes that players choose actions independently. We adjust our definition to incorporate such beliefs..

Definition 2 *ξ is a best attempt to beat the average with a pure action when actions are chosen independently if* $\xi^* \in \max_\xi \inf_{F\in\Delta A} \int f_\xi(a)\, dF^n(a_i)$.

How these beliefs influence behavior and performance in the Nash demand game is left for future research. The challenge is to get some understanding of the support of the worst case distribution F.

4.5 Coordination Game

It is clear that one cannot get close to beating the average in some games. This is the most obvious in the following simple (pure) coordination game.

Proposition 10 *Let $A = [0,1]$ and $u(a) = \prod_{i=1}^n a_i + \prod_{i=1}^n (1-a_i)$. Then there is no mixed action that yields a shortcoming strictly below $\frac{1}{2}$.*

Proof. We show that $\xi^* = \frac{1}{2}$ is a best attempt to beat the average by establishing an equilibrium of the zero sum game between the person and nature. When nature chooses $a_i = 1$ for all i or $a_i = 0$ for all i, then $Ef_\xi(a) = \frac{1}{2}(\xi-1) + \frac{1}{2}(1-\xi-1) = -\frac{1}{2}$. Moreover, it is easy to see for any $a \in A^n$ that $Ef_\xi(a) = \sum_{i=1}^n (\xi - a_i) \prod_{j\neq i} a_j \geq -\frac{1}{2}$. This completes the proof. ■

We return to this game in Section 6.3.

5 Asymmetric Games

We now extend our framework to asymmetric games. We present three ways to do this, termed subset comparison, ignorance, and role comparison.

5.1 Subset Comparison

For games where a subset of the players are like oneself one may choose to compare own payoff to the average payoff among this subset of players. A natural example that we present in more detail below is a market with buyers and sellers, where a seller tries to beat the average payoff of the existing sellers, regardless of the behavior of the buyers.

For the formal definition, consider an n person game in which player i chooses an action from a set A_i and gets payoff $u_i : A_i \times_{j\neq i} A_j \to \mathbb{R}$.

Definition 3 *Let $I \subseteq \{1,..,n\}$ with $|I| \geq 2$ such that $A_i = A_j$ and $u_i \equiv u_j$ when $\{i,j\} \subseteq I$. We call $\xi^* \in \Delta A_j$ for $j \in I$ a best attempt to beat the average payoff of the players belonging to I if*

$$\xi^* \in \max_{\xi \in A_j} \inf_{a \in \times A_i} \left\{ \frac{1}{|I|} \sum_{i \in I} \left(u_i \left(\xi, a_{-i} \right) - u_i \left(a_i, a_{-i} \right) \right) \right\}.$$

We illustrate how this applies under price competition. Assume that there are n sellers and m buyers. We formulate the game as a simultaneous move game in which each seller chooses a selling price and each buyer chooses a maximal price she is willing to pay for one unit of the homogeneous good. Buyers buy at the cheapest firm below her willingness to pay subject to availability. Some rationing rule determines how buyers are served when there is excess demand. Assume that no buyer is willing to pay more than v for the good. Then it follows easily that $c + \frac{v-c}{n+1}$ is a best attempt of a seller to beat the average among the sellers. The reason is that the worst cases are attained when all buyers are willing to pay any price $\leq v$. Hence our results from Proposition 7 apply when setting $Q = 1$. In particular, there is no need to explicitly specify the rationing rule.

5.2 Ignorance

Here we consider the setting where the person chooses to compare own payoffs to what she would have achieved with the actions used by others. In particular, no assumptions are made about what this person knows about the utility functions of the others. A natural example expanded below is a first price auction where own willingness to pay for the object is given and one compares own bids to what would have happened if one had adapted one of the bids made by the others.

Definition 4 *Let $I \subseteq \{1,..,n\}$ with $|I| \geq 2$ such that $A_i = A_j$ when $\{i,j\} \subseteq I$. We call $\xi^* \in \Delta A_j$ for $j \in I$ a best attempt to beat the average payoff of the actions used by the players belonging to I if*

$$\xi^* \in \max_{\xi \in \Delta A_j} \inf_{a \in \times A_i} \left\{ \frac{1}{|I|} \sum_{i \in I} \left(u_0 \left(\xi, a_{-i} \right) - u_0 \left(a_i, a_{-i} \right) \right) \right\}.$$

We illustrate this concept under Cournot competition. Assume that firm i has a cost function c_i, $i = 1,..,n$. Then a person acting as firm with convex cost function c_0 can beat the average payoff of the quantities chosen by the others by choosing the symmetric NE strategy of the game in which all firms have unit cost c_0. Note that the recommended strategy does not require making any assumptions about the cost functions of the other firms in the market. It of course may be that one is conceiving quantity profiles that will never emerge, however one does not know this as in this simple model one does not have information about costs of others.

Consider next a first price auction. This is an important application that will be used to motivate an adjustment to our definition. There are n bidders. An action of player i is a bid $b_i \in \mathbb{R}_+$ where the player who makes the highest bid wins the object, ties are broken at random. Consider a bidder with value v for the object, so $u_0 = v - \xi$ if he wins the object by bidding ξ and $u_0 = 0$ otherwise. Assume that we do not care about the values of the n players, but only wish to compare the empirical performance of a given bid to the payoff if one could adapt one of the bids used. So we set $I = \{1, .., n\}$. Then the best attempt to beat the average among the bids used by the others using a pure action is to bid $\xi = \frac{n}{n+1}v$. This result follows from our analysis of Bertrand price competition. Set $p_i = v - b_i$ and $c = 0$ to obtain from Proposition 7 the price $\frac{1}{n+1}v$ and hence $\xi_b = v - \frac{1}{n+1}v = \frac{n}{n+1}v$. This result holds for nonnegative prices, hence for markets in which $b_i \leq v$ for all i. However, the proof of Proposition 7 easily extends to allow for negative prices. So this is our recommendation based on Definition 4.

However it does not make much sense to compare performance of the bid ξ_b to bids that one would not have chosen, namely bids strictly larger than v. Of course one may be in a market in which other participants choose such high bids, possibly because they value the good more. The benchmark needs to take these concerns into account. To simply set $A_i = [0, v]$ in the first price auction does not make sense.

Assume that the person can choose from a set A_0 but will only choose actions from a subset B. Let g be a mapping from A_0 to ΔB with $g(a) = a$ if $a \in B$ where $g(a)$ is the action the player chooses if someone recommends a to him. In applications the set B as well as the mapping g are easily identified. For instance it is natural to include all actions that are strictly dominated by a pure action in $A_0 \backslash B$ in which case $g(a)$ is the action in B that strictly dominates a.

Definition 5 *Let $I \subseteq \{1, .., n\}$ with $|I| \geq 2$ such that $A_i = A_0$ for $i \in I$. Let $C \subset A_0$ and $g : A_0 \to C$ with $g(a) = a$ for $a \in C$. We call $\xi^* \in \Delta C$ a best attempt to beat the average payoff of the* adjusted *actions used by the players belonging to I if*

$$\xi^* \in \max_{\xi \in \Delta C} \inf_{a \in \times A_i} \left\{ \frac{1}{|I|} \sum_{i \in I} \left(u_0 (\xi, a_{-i}) - u_0 (g(a_i), a_{-i}) \right) \right\}.$$

Typically the shortcoming associated to the best attempt to beat the average is increased when adjusting actions, for instance this is the case when $g(a)$ weakly dominates a for all $a \in A_0 \backslash B$.

Returning to the first price auction, we set $g(b) = \min\{b, v\}$. The person does not wish to bid above his value and hence, when being recommended a higher bid higher than v he bids v. Clearly $\xi = \frac{n}{n+1}v$ remains our recommendation and the associated shortcoming remains unchanged. This is because in markets in which the highest bid is above v we have $f_\xi = 0$ as $u_0(\xi, a_{-i}) = u_0(g(a_i), a_{-i})$.

5.3 Role Comparison

In our third model of asymmetric games we consider a person who may adapt different roles in the game and chooses to evaluate his average performance across these roles, conditioning his action on the role he takes. He considers becoming a player that belongs to a given set of players I and conditions the action he chooses on the player whose role he adapts. So the person is looking for a profile of actions, one for each player whose role he adapts. We have complete information, so the person knows the utility function of each player in the game.

Let ΔA_*^I be such that $\xi \in \Delta A_*^I$ if $\xi : I \to \cup_{i \in I} \Delta A_i$ is such that $\xi_i \in \Delta A_i$ for $i \in I$.

Definition 6 *Let $I \subseteq \{1, .., n\}$ with $|I| \geq 2$. We call $\eta^* \in \Delta A_*^I$ a best attempt to beat the average when taking a role of a player belonging to I if*

$$\eta^* \in \max_{\eta \in \Delta A_*^I} \inf_{a \in \times A_i} \left\{ \frac{1}{|I|} \sum_{i \in I} \left(u_i(\eta_i, a_{-i}) - u_i(a_i, a_{-i}) \right) \right\}.$$

Remark 3 *If $I = \{1, .., n\}$ and $\eta^* \in \times A_i$ beats the average, then η^* is a pure strategy Nash equilibrium of the underlying game.*

To illustrate consider Cournot competition with homogeneous goods. This time, unlike in Proposition 4, we need to explicitly rule situations that do not make sense, namely we rule out sales where $q \notin cl\{q : P(s(q)) > 0\}$. This is because in our proof for the case where $P = 0$ we used the fact that all firms had the same cost function. Let $q^L = q^L(q)$ be such that $q^L = q$ when $P(s(q)) > 0$ and $q^L \in \partial\{q : P(s(q)) > 0\}$ when $P(s(q)) = 0$. In particular, note that $P\left(s\left(q^L\right)\right) = P(s(q))$.

Proposition 11 *Let $A \subseteq \mathbb{R}_+$ with A convex and let $\eta \in A^n$ such that $\eta \gg 0$. Assume $u_i(q_i, q_{-i}) = q_i^L P\left(s\left(q^L\right)\right) - c_i\left(q_i^L\right)$ where (i) c_i is increasing, convex and differentiable at η_i, (ii) P is nonnegative, decreasing, convex and differentiable at $s(\eta)$ such that $zP(z)$ is concave when $P > 0$. If η is a pure strategy NE then η beats the average.*

Proof. Given continuity of f_η and the definition of q^L it is enough to prove the proposition for all q such that $P(s(q)) > 0$. The proof is then a straightforward generalization of our proof of Proposition 2(ii). ■

Let's compare role comparison and ignorance within the context of Cournot competition. Under role competition, the person with cost function c thinks about what cost functions the other firms have and chooses his own quantity while anticipating the quantity he would choose if he had one of the cost functions of the other firms. His objective is to outperform the average payoff in the market. This may mean to take into account low payoffs when being in the role of an inefficient firm that is offset by high payoffs when in the role of an efficient firm. This leads the person to choose the quantity corresponding to the cost function c in the pure strategy NE. In the ignorance setting the person focuses on his own cost function c and tries to do better than what he would if he adapted a random quantity chosen by a firm in the market. This leads him to choose the symmetric NE quantity that would result if all had the same costs as him. So the solutions are very different and yet both beat the average, in their own context. In the ignorance setting, quantities chosen by others are evaluated with own costs, in the role comparison setting they are evaluated with the cost function of the firm choosing this quantity.

6 Other Extensions

6.1 Incomplete Information

We now introduce incomplete information and show how easy our model extends. There is a set of types T with typical representative denoted by θ. Types are drawn from some distribution F where each player only knows her own type. We consider this as a symmetric game. So we assume that F is symmetric in the sense that it is invariant to permutations of the player types. The utility of a player only depends on own and other actions as well as on his own type, but not on the label or index of this player. So $u : A^n \times T \to \mathbb{R}$. A strategy ρ of a player is now a mapping from T to A, so an element of A^T. In traditional game theory the concept of Bayesian NE replaces that of NE as a reminder that the description of the game contains probability distributions. All concepts introduced above can be immediately applied.

Proposition 12 *Let $A \subseteq \mathbb{R}$ be such that A is convex. Assume that $u(a, \theta) = g_0(a_1, \theta) + a_1 g_1(s(a), \theta)$ is such that $g_0(\cdot, \theta)$ is differentiable and concave, $g_1(\cdot, \theta)$ is differentiable and convex such that $z g_1(z, \theta)$ is concave for all $\theta \in T$. If $\rho \in A^T$ be a symmetric Bayesian NE strategy then ρ beats the average.*

Proof. The proof is a straightforward generalization of that of Proposition 3, verifying that f_ρ is convex and that ρ is a local minimum. ■

Note that the above result as stated does not apply to the typical Cournot competition game where prices are zero when quantities are excessive. More research is needed to deal with the kink in the inverse demand that now potentially interferes only with positive probability.

6.2 Uncertainty

In many real life situations it is natural that a player does not know everything about the game he will be playing. We formalize this by letting β be a parameter that describes the details of the game and assume that the person knows that β is in some set B. So $u(a, \beta)$ is the payoff when β is the true parameter. Let

$$f_\xi (a, \beta) := \frac{1}{n} \sum_{i=1}^{n} \left(u(\xi, a_{-i}, \beta) - u(a_i, a_{-i}, \beta) \right).$$

Then ξ is a best attempt to beat the average with a pure action for all β in B if

$$\xi \in \arg\max_{\bar{\xi} \in A} \inf_{a \in A^n, \beta \in B} f_{\bar{\xi}} (a, \beta).$$

We illustrate this concept in Bertrand competition with limited information about the demand. Uncertainty is given here by the player only knowing a maximal demand attainable for each price. It may be that a new firm does not know if it can attract customers, but knows the existing demand. We present the result for a more general specification and then illustrate the specific finding in two examples.

Proposition 13 *Consider Bertrand competition as defined in Proposition 7. Let $\bar{Q}(p)$ be the maximal demand at price p, so $Q(p) \le \bar{Q}(p)$ for all p, where $\bar{Q}$ is continuous and decreasing and $(p - c)\bar{Q}(p)$ is single peaked. Let c be the unit cost of production. Then ξ is a best attempt to beat the average with a pure action for all such Q if*

$$\xi \in \arg\max_{x \le p^*} \min_{p > x} \left\{ -\frac{1}{n}(x - c)\bar{Q}(x), \left((x - c) - \frac{1}{n}(p - c) \right) \bar{Q}(p) \right\}$$

where $\{p^*\} = \arg\max \left\{ (p - c)\bar{Q}(p) \right\}$. *The associated shortcoming equals* $\frac{1}{n}(\xi - c)\bar{Q}(\xi)$.

Proof. Let $\pi^* = (p^* - c)\bar{Q}(p^*)$. Choose some $\xi \le p^*$. If the market price p is such that $p \le \xi$ then

$$f_\xi \ge -\frac{1}{n}(p - c)Q(p) \ge -\frac{1}{n}(p - c)\bar{Q}(p) \ge -\frac{1}{n}(\xi - c)\bar{Q}(\xi),$$

where this lower bound is decreasing in ξ from 0 to $-\frac{1}{n}\pi^*$. If $p > \xi$ then

$$\begin{aligned} f_\xi &= (\xi - c) Q(\xi) - \frac{1}{n}(p - c) Q(p) \geq (\xi - c) Q(p) - \frac{1}{n}(p - c) Q(p) \\ &\geq \min\left\{0, \left(\xi - c - \frac{1}{n}(p - c)\right) \bar{Q}(p)\right\}, \end{aligned}$$

where this lower bound is increasing in ξ, from $-\frac{1}{n}\pi^*$ to $\frac{n-1}{n}\pi^*$. It follows that ξ is chosen to equalize these two bounds. ■

Note that by the definition of the respective problems, the shortcoming of the best attempt with a pure action is weakly larger when demand can only be bounded from above than it is when the upper bound is the true demand. If $\bar{Q}$ is strictly decreasing, then comparing Proposition 13 to Proposition 7 we see that in the present model with bounded demand the shortcoming is strictly larger, in fact also the best attempt is also strictly larger. We illustrate the result in the case with a linear upper bound, consider $\bar{Q}(p) = \max\{0, 1 - p\}$. Then it follows after some straightforward algebra that

$$\xi = \frac{2c + 2 + cn^2 + (1 - c) n - 2(1 - c)\sqrt{n}}{4 + n^2}$$

with an associated shortcoming given by

$$\frac{(2 + n - 2\sqrt{n})(2 + n^2 - n + 2\sqrt{n})}{n(4 + n^2)^2}(1 - c)^2,$$

which for $n \leq 20$ is approximately $\frac{3}{4n(n+4)}(1 - c)^2$. If we instead choose ξ as the solution to the case where $Q = \bar{Q}$ then we obtain shortcoming when $Q \leq \bar{Q}$ equal to

$$\frac{\left(-n + n\sqrt{\frac{n}{n+1}} + 2\right)^2}{16n}(1 - c)^2.$$

We plot shortcomings without the $(1 - c)^2$ term, so for the case where $c = 0$, associated to the best attempt to beat the average among pure actions when $Q = \bar{Q}$ (solid line), when $Q \leq \bar{Q}$ (dashed line), and when the solution from $Q = \bar{Q}$ is used for the setting where $Q \leq \bar{Q}$ (dotted line).

On the other hand, if $\bar{Q}$ is derived under unit demand where $\bar{Q}(z) = 1$ for $z \leq v$ and $\bar{Q}(z) = 0$ otherwise, then the solutions of Propositions 7 and 13 coincide.

6.3 Beliefs under Ambiguity and Robustness

In most of this paper we investigate best attempts to beat the average without narrowing down the set of environments which are believed to substantiate. Yet in many situations one may wish to first rule out some environments and then search for a

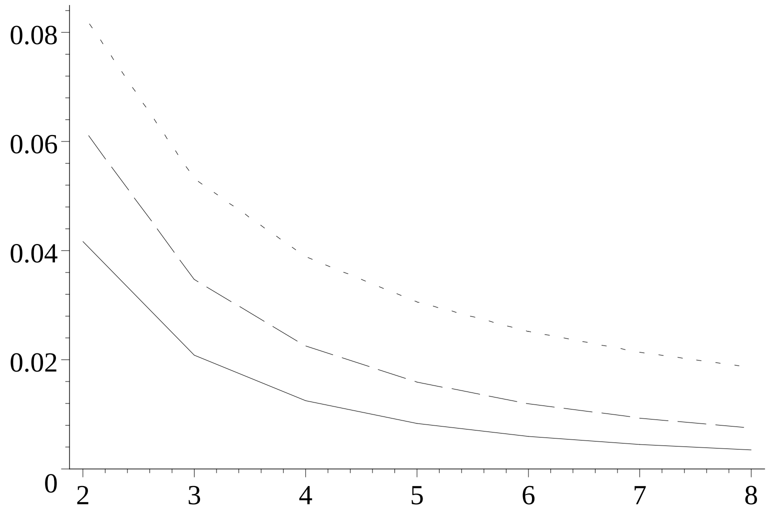

Figure 5: Shortcoming under Bertrand competition with demand equal to and bounded above by $\bar{Q}(z) = \max\{1 - z, 0\}$.

strategy that performs well for the remaining environments that one deems possible. One may also be interested in whether a specific symmetric NE is *robust to beating the average*, in the sense that it beats the average for all action profiles that are sufficiently close to the corresponding NE. Underlying this concept one imagines that some players have some doubt as to whether others will play this equilibrium and instead will play something a similar action. Formally, one specifies the set of (mixed) action profiles $U \subset \Delta A^n$ one wishes to investigate and then either tries to beat the average or searches for a best attempt to beat the average when facing some profile $a \in U$. The restriction to U can be considered a model of *beliefs under ambiguity* (cf. Kasberger and Schlag, 2016). A symmetric NE ξ^n is *robust to beating the average* if there exists some open neighborhood U of ξ^n such that ξ beats the average for all $a \in U$. A symmetric NE ξ^n is *robust to attempts to beat the average* if for each $\varepsilon > 0$ there exists a neighborhood U of ξ^n such that the symmetric NE strategy beats the average up to ε for all $a \in U$.

We illustrate and start by looking at Bertrand competition with heterogeneous goods. Considering only strategies sufficiently close to the NE substantially simplifies the analysis. In Proposition 6 we did not present a more general result given the intricacies that arise when some firms do not make sales. However, if we restrict attention to markets in which all prices without or without the person's choice are strictly positive then we obtain a nice result which we present a bit more general.

Proposition 14 *Let $A \subseteq \mathbb{R}$ be such that A is convex, $\gamma \in \left(-\frac{1}{n-1}, 0\right)$, $u(a) =$*

$g_0(a_1) + a_1 g_1(a_1 + \gamma s(a_{-1}))$ *and* ξ *be a symmetric NE strategy. Let* $U = \{a \in A^n : g_1(\xi + \gamma s(a_{-i})) \cdot g_1(a_i + \gamma s(a_{-i})) > 0 \forall i\}$. *If* g_0 *is concave with continuous second derivatives on* U *and* g_1 *is decreasing and convex with continuous second derivatives and*

$$g_1'(a_i + \gamma s(a_{-i})) + a_i g_1''(a_i + \gamma s(a_{-i})) < 0 \ \forall i$$

on U *then* ξ *beats the average for any* $a \in U$.

The central application for this result is Bertrand competition with heterogeneous goods, so $\gamma < 0$, in which costs are convex, inverse demand is convex and goods are (strict) strategic compliments whenever prices are strictly positive, so

$$\gamma\left(P'(q_i + \gamma s(q_{-i})) + q_i P''(q_i + \gamma s(q_{-i}))\right) > 0 \text{ whenever } P(q_j + \gamma s(q_{-j})) > 0 \text{ for all } j.$$

We find that the symmetric NE strategy beats the average whenever all market quantities are strictly positive, in particular we find that the symmetric NE is robust to beating the average.

Proof. Note that

$$f_\xi(a) = \frac{1}{n}\sum_{i=1}^{n}\left(g_0(\xi) + \xi g_1(\xi + \gamma s(a_{-i}))\right) - \frac{1}{n}\sum_{i=1}^{n}\left(g_0(a_i) + a_i g_1(a_1 + \gamma s(a_{-1}))\right).$$

Note also that $\{a : g_1(\xi + \gamma s(a_{-i})) \cdot g_1(a_i + \gamma s(a_{-i})) > 0 \forall i\}$ is a convex set.

Let $h_i(a) = a_i g_1(a_1 + \gamma s(a_{-1}))$. The proof is complete once we show that $h(a) = \sum_{i=1}^{n} h_i(a)$ is concave in a neighborhood of ξ^n. We consider directional second derivatives. Fix $y \in A^n$, let $r_i(\varepsilon) = h_i(\xi^n + \varepsilon(y - \xi^n))$ and $r(\varepsilon) = \sum_{i=1}^{n} r_i(\varepsilon)$. We aim to show that $r''(0) < 0$. Then

$$\begin{aligned} r_i''(0) &= 2(y_i - \xi) g_1'(\xi + \gamma(n-1)\xi) \cdot ((1-\gamma)(y_i - \xi) + \gamma(s(y) - n\xi)) \\ &\quad + \xi g_1''(\xi + \gamma(n-1)\xi) \cdot ((1-\gamma)(y_i - \xi) + \gamma(s(y) - n\xi))^2 \end{aligned}$$

and

$$\begin{aligned} r''(0) &= g_1'(\xi + \gamma(n-1)\xi) \sum_{i=1}^{n} 2(y_i - \xi)((1-\gamma)(y_i - \xi) + \gamma(s(y) - n\xi)) \\ &\quad + \xi g_1''(\xi + \gamma(n-1)\xi) \sum_{i=1}^{n} ((1-\gamma)(y_i - \xi) + \gamma(s(y) - n\xi))^2 \\ &= g_1'(\xi + \gamma(n-1)\xi) \cdot \left(2(1-\gamma)\sum_{i=1}^{n}(y_i - \xi)^2 + 2\gamma(s(y) - n\xi)^2\right) \\ &\quad + \xi g_1''(\xi + \gamma(n-1)\xi) \cdot \left((1-\gamma)^2 \sum_{i=1}^{n}(y_i - \xi)^2 + (n\gamma + 2(1-\gamma))\gamma(s(y) - n\xi)^2\right) \end{aligned}$$

As $\frac{1}{n}\sum_{i=1}^{n} z_i^2 \geq \left(\frac{1}{n}\sum_{i=1}^{n} z_i\right)^2$ and $\gamma > -\frac{1}{n-1}$ it follows that

$$2(1-\gamma)\sum_{i=1}^{n}(y_i-\xi)^2 + 2\gamma(s(y)-n\xi)^2 \geq 2\left((1-\gamma)\frac{1}{n}+\gamma\right)(s(y)-n\xi)^2 \geq 0$$

and hence, given

$$g_1'(\xi+\gamma(n-1)\xi) < -\xi g_1''(\xi+\gamma(n-1)\xi)$$

we obtain

$$\begin{aligned}
r''(0) &\leq -\xi g_1''(\xi+\gamma(n-1)\xi)\cdot\left(2(1-\gamma)\sum_{i=1}^{n}(y_i-\xi)^2+2\gamma(s(y)-n\xi)^2\right)\\
&\quad +\xi g_1''(\xi+\gamma(n-1)\xi)\cdot\left((1-\gamma)^2\sum_{i=1}^{n}(y_i-\xi)^2+(n\gamma+2(1-\gamma))\gamma(s(y)-n\xi)^2\right)\\
&= -\xi g_1''(\xi+\gamma(n-1)\xi)\cdot\left(\left(1-\gamma^2\right)\sum_{i=1}^{n}(y_i-\xi)^2-(n-2)\gamma^2(s(y)-n\xi)^2\right)\\
&\leq -\xi g_1''(\xi+\gamma(n-1)\xi)\cdot\left(\left(1-\gamma^2\right)\frac{1}{n}-(n-2)\gamma^2\right)(s(y)-n\xi)^2\\
&\leq 0
\end{aligned}$$

Moreover, as $|\gamma| < \frac{1}{n-1}$ and $g_1'(\xi+\gamma(n-1)\xi) < -\xi g_1''(\xi+\gamma(n-1)\xi)$ we find that $r''(0) < 0$ if $\frac{1}{n}\sum_{i=1}^{n}(y_i-\xi)^2 > \frac{1}{n^2}(s(y)-n\xi)^2$ or $s(y) \neq n\xi$, which holds if $y \neq \xi^n$. ■

Note that in the pure coordination game presented in Section 4.5 it is easy to see that both symmetric NE 0^n and 1^n are robust to beating the average. Yet a symmetric NE strategy need not necessarily beat the average when limiting attention to actions profiles that are sufficiently close to the corresponding NE. For instance, under Bertrand competition with symmetric firms, perfect substitutes and constant marginal cost, the symmetric NE strategy will not beat the average if all market prices are slightly above marginal cost. Thus, the NE is not robust to beating the average. However, it is robust to attempts to beat the average as the shortcoming of the symmetric NE strategy is arbitrarily small if all prices are sufficiently close to marginal cost.

6.4 Some Knowledge of Others

In this section we show how one can adjust the approach to the case where others are trying to beat the average too. Two different approaches come to mind. One could think that some anticipate behavior of others without this being common knowledge, like under rationalizability. One could think that there is common knowledge among a subset of the players that these players are attempting to beat the average. We illustrate the latter approach.

Consider a symmetric game and assume that there is common knowledge among k players that these are trying to beat the average, with $n \geq k \geq 2$. We refer to the other $n-k$ players as the unknown players. The benchmark for the person making the decision is the situation in which there are $(k-1)$ persons are attempting to beat the average and $n-(k-1)$ unknown players. The person anticipates playing against $(k-1)$ others trying to beat the average and a random subset of $n-k$ players drawn from the $n-(k-1)$ unknown players. The average among the $n-(k-1)$ unknown players with unknown is taken as benchmark. For simplicity we focus on a solution where players $i \leq k$ are choosing the same action. This implies that the players who are attempting to beat the average of the unknown players also beat the average of the other players attempting to beat the average.

This leads to the following formal definition.

Definition 7 *$\xi \in A$ beats the average when it is known that k players are attempting to beat the average if*

$$\frac{1}{n-k+1} \sum_{i=1}^{n-k+1} \left(u\left(\xi, \xi^{k-1}, a_{-i}\right) - u\left(a_i, \xi^{k-1}, a_{-i}\right) \right) \geq 0$$

for all $a \in A^{n-k+1}$.

Once again it follows that ξ is a symmetric NE strategy. Moreover, if $k = n$ then this is equivalent to the definition of a symmetric NE strategy. We illustrate for $k = 2$.

Proposition 15 *Consider Bertrand competition with homogeneous goods as defined in Proposition 7 with $c(0) = 0$ and c increasing and convex, Q continuous and decreasing and $n \geq 3$.*

(i) Each of the following conditions ensures that (the symmetric NE strategy) ξ beats the average when it is known that two players are attempting to beat the average: (a) $c(q) = c_0 \cdot q$ and $\xi = c_0$ for some $c_0 \geq 0$, and (b) $n = 3$ and $\xi \frac{1}{2} Q(\xi) - c\left(\frac{1}{2} Q(\xi)\right) = 0$.

(ii) It is not possible to beat the average when it is known that two players are attempting to beat the average if $n \geq 4$ and costs are strictly convex.

We conclude that common knowledge of several players following the proposed methodology may but need not lead to equilibrium play.

Proof. Assume that ξ beats the average when it is known that two players are attempting to beat the average. Consider the situation in which the unknown firms price higher than ξ. Hence, ξ needs to satisfy

$$\xi \frac{1}{2} Q(\xi) - c\left(\frac{1}{2} Q(\xi)\right) \geq 0$$

as the unknown firms are not making any sales. Consider now the situation in which all unknown firms price slightly below ξ. Hence, ξ needs to satisfy

$$0 \geq \xi \frac{1}{n-1} Q(\xi) - c\left(\frac{1}{n-1} Q(\xi)\right).$$

Here we use continuity of Q.

Note that

$$(n-1)\left(\xi \frac{1}{n-1} Q(\xi) - c\left(\frac{1}{n-1} Q(\xi)\right)\right) \geq 2\left(\xi \frac{1}{2} Q(\xi) - c\left(\frac{1}{2} Q(\xi)\right)\right)$$

where this inequality holds with strict inequality if c is strictly convex and $n \geq 4$. Hence,

$$\xi \frac{1}{2} Q(\xi) - c\left(\frac{1}{2} Q(\xi)\right) = \xi \frac{1}{n-1} Q(\xi) - c\left(\frac{1}{n-1} Q(\xi)\right) = 0,$$

and if c is strictly convex then $n = 3$.

Sufficiency is easily verified as the pricing scenarios considered for the unknown firms above are the only worst case scenarios. ■

7 Conclusion

Game theory is dominated by equilibrium analysis. In fact, equilibrium play seems the only natural prediction if players have a mutual understanding that they are trying to solve the game. Yet there are many obstacles to this simple and naive statement. Among the many we note that equilibria need not be unique and that information and objectives of others need not be known, let alone modellable in a satisfactory manor. Increasingly sophisticated, detailed and lengthy papers are believed to explain how economic actors behave.

We propose an alternative to NE that is less demanding on how play of others is modelled and that is easy to compute as demonstrated in this paper with many examples. The key is to benchmark to the average payoff of those playing the game in an unknown fashion, thereby not including the person who is deciding how to play the game. Restrictions on beliefs can be added. We can incorporate mutual understanding that this method is being used, which necessarily leads to Nash equilibrium if there is common knowledge that our methodology is used by all and that all ingredients to the game are commonly known. In this sense our methodology nicely complements and extends the NE concept. We designed our methodology to aid real life choices and to gain insights into incentives in games. As the methodology is under construction it cannot be used to explain how players are playing a game, though perhaps later it may serve as a benchmark for comparison to peoples' behavior and choices. The key ingredient, a comparison to average payoffs, should enhance the perspective of economic agents as opposed to postulating a desiderata. The methodology leads to novel analytic structures that call for new mathematical techniques. This paper offers a glimpse at the possible new insights by presenting many applications and refinements of the basic concept.

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