Elliptic and q-analogs of the Fibonomial numbers

Josef Küstner

Universität Wien

joint work with

Nantel Bergeron and Cesar Ceballos

Der Wissenschaftsfonds.

Fibonacci Numbers

Definition

The Fibonacci numbers are defined as $F_0 = 0$, $F_1 = 1$ and for $n \geq 2$

$$F_n := F_{n-1} + F_{n-2}.$$

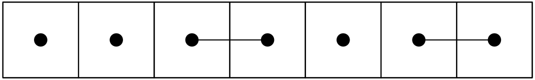
Fibonacci Numbers

Definition

The Fibonacci numbers are defined as $F_0 = 0$, $F_1 = 1$ and for $n \geq 2$

$$F_n := F_{n-1} + F_{n-2}.$$

F_n counts the number of tilings $T \in \mathcal{T}(n-1, 1)$ of an $n-1 \times 1$ rectangle with monominos and dominos.



q-number of the Fibonacci numbers

Definition

We define the q-number of the n-th Fibonacci number as $[F_0]_q = 1$, $[F_1]_q = 1$ and for $n \geq 2$

$$[F_n]_q := 1 + q + q^2 + \cdots + q^{F_n - 1}.$$

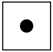
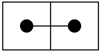
q-number of the Fibonacci numbers

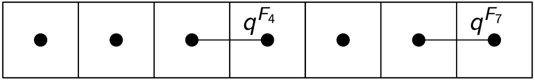
Definition

We define the q-number of the n-th Fibonacci number as $[F_0]_q = 1$, $[F_1]_q = 1$ and for $n \geq 2$

$$[F_n]_q := 1 + q + q^2 + \cdots + q^{F_n - 1}.$$

We assign weights for $T \in \mathcal{T}(n-1, 1)$:

$\omega\Big($  $\Big) = 1, \quad \omega\Big($  $\Big) = q^{F_i}.$



$$W(T) = q^{F_4} q^{F_7} = q^{16}$$

q-number of the Fibonacci numbers

For $[F_n]_q$ the recursion

$$[F_n]_q = [F_{n-1}]_q + q^{F_{n-1}}[F_{n-2}]_q$$

holds nor $n \geq 2$.

q-number of the Fibonacci numbers

For $[F_n]_q$ the recursion

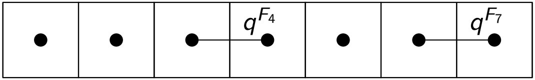
$$[F_n]_q = [F_{n-1}]_q + q^{F_{n-1}}[F_{n-2}]_q$$

holds nor $n \geq 2$.

Consequence

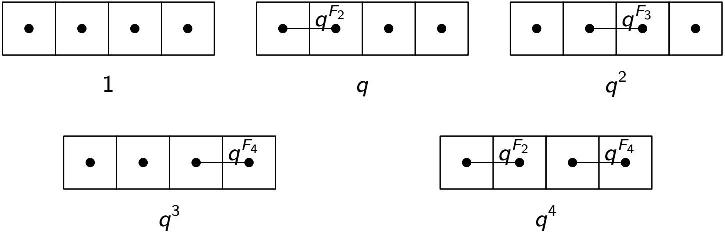
For the generating function of $\mathcal{T}(n-1,1)$ holds:

$$\sum_{T \in \mathcal{T}(n-1,1)} W(T) = [F_n]_q.$$



Example

$n = 5$



$$[F_5]_q = [5]_q = 1 + q + q^2 + q^3 + q^4$$

Elliptic number of the Fibonacci numbers

Definition

The elliptic number of n is defined as

$$[n]_{a,b;q,p} := \frac{\theta(q^n, aq^n, bq^2, a/b; p)}{\theta(q, aq, bq^{n+1}, aq^{n-1}/b; p)}.$$

Elliptic number of the Fibonacci numbers

Definition

The elliptic number of n is defined as

$$[n]_{a,b;q,p} := \frac{\theta(q^n, aq^n, bq^2, a/b; p)}{\theta(q, aq, bq^{n+1}, aq^{n-1}/b; p)}.$$

The elliptic analog of the q-number of the n-th Fibonacci number $[F_n]_q$ can be defined as the elliptic number of F_n:

$$[F_n]_{a,b;q,p} := \frac{\theta(q^{F_n}, aq^{F_n}, bq^2, a/b; p)}{\theta(q, aq, bq^{F_n+1}, aq^{F_n-1}/b; p)}.$$

Elliptic number of the Fibonacci numbers

Definition

The elliptic number of n is defined as

$$[n]_{a,b;q,p} := \frac{\theta(q^n, aq^n, bq^2, a/b; p)}{\theta(q, aq, bq^{n+1}, aq^{n-1}/b; p)}.$$

The elliptic analog of the q-number of the n-th Fibonacci number $[F_n]_q$ can be defined as the elliptic number of F_n:

$$[F_n]_{a,b;q,p} := \frac{\theta(q^{F_n}, aq^{F_n}, bq^2, a/b; p)}{\theta(q, aq, bq^{F_n+1}, aq^{F_n-1}/b; p)}.$$

This is an analog because it reduces to the q-analogue $[F_n]_q$ if we take the limit $p \to 0$, $a \to 0$ and $b \to 0$ in this order.

Elliptic number of the Fibonacci numbers

Lemma

For $m, n \in \mathbb{N}$, the following identities hold:

$$[m+n]_{a,b;q,p} = [m]_{a,b;q,p} + v_{a,b;q,p}(m,n)[n]_{a,b;q,p}$$
$$[m \cdot n]_{a,b;q,p} = [m]_{a,b;q,p}[n]_{a,bq^{1-m};q^m,p}$$

where $v_{a,b;q,p}(m,n) := \frac{\theta(aq^{2m+n}, bq, bq^{n+1}, \frac{a}{b}q^{n-1}, \frac{a}{bq}; p)}{\theta(aq^n, bq^{m+1}, bq^{m+n+1}, \frac{a}{b}q^{m-1}, \frac{a}{b}q^{m+n-1}; p)} q^m.$

Elliptic number of the Fibonacci numbers

Lemma

For $m, n \in \mathbb{N}$, the following identities hold:

$$[m+n]_{a,b;q,p} = [m]_{a,b;q,p} + v_{a,b;q,p}(m,n)[n]_{a,b;q,p}$$
$$[m \cdot n]_{a,b;q,p} = [m]_{a,b;q,p}[n]_{a,bq^{1-m};q^m,p}$$

where $v_{a,b;q,p}(m,n) := \frac{\theta(aq^{2m+n}, bq, bq^{n+1}, \frac{a}{b}q^{n-1}, \frac{a}{bq}; p)}{\theta(aq^n, bq^{m+1}, bq^{m+n+1}, \frac{a}{b}q^{m-1}, \frac{a}{b}q^{m+n-1}; p)} q^m.$

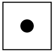
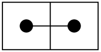
Corollary

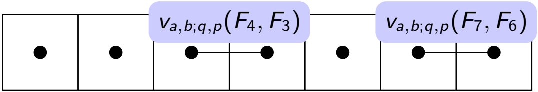
For $[F_n]_{a,b;q,p} = [F_{n-1} + F_{n-2}]_{a,b;q,p}$ *holds:*

$$[F_n]_{a,b;q,p} = [F_{n-1}]_{a,b;q,p} + v_{a,b;q,p}(F_{n-1}, F_{n-2})[F_{n-2}]_{a,b;q,p}$$

Elliptic number of the Fibonacci numbers

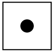
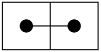
We assign weights for $T \in \mathcal{T}(n-1,1)$:

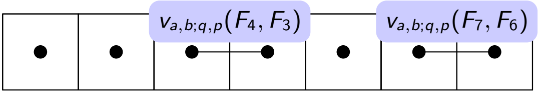
$\omega\Big($  $\Big) = 1, \quad \omega\Big($  $\Big) = v_{a,b;q,p}(F_i, F_{i-1}),$



Elliptic number of the Fibonacci numbers

We assign weights for $T \in \mathcal{T}(n-1,1)$:

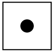
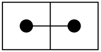
$\omega\Big($  $\Big) = 1, \quad \omega\Big($  $\Big) = v_{a,b;q,p}(F_i, F_{i-1}),$

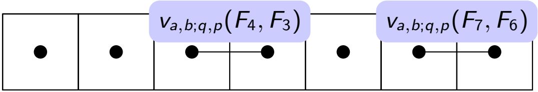


$$[F_n]_{a,b;q,p} = [F_{n-1}]_{a,b;q,p} + v_{a,b;q,p}(F_{n-1}, F_{n-2})[F_{n-2}]_{a,b;q,p}.$$

Elliptic number of the Fibonacci numbers

We assign weights for $T \in \mathcal{T}(n-1,1)$:

$\omega\Big($  $\Big) = 1, \quad \omega\Big($  $\Big) = v_{a,b;q,p}(F_i, F_{i-1}),$



$$[F_n]_{a,b;q,p} = [F_{n-1}]_{a,b;q,p} + v_{a,b;q,p}(F_{n-1}, F_{n-2})[F_{n-2}]_{a,b;q,p}.$$

Consequence

$$\sum_{T \in \mathcal{T}(n-1,1)} W(T) = [F_n]_{a,b;q,p}.$$

Fibonomial numbers

Definition

The Fibonomial numbers are defined as

$$\binom{m+n}{n}_F := \frac{F^!_{m+n}}{F^!_m F^!_n},$$

where $F^!_n := F_n F_{n-1} F_{n-2} \dots F_1$.

Fibonomial numbers

Definition

The Fibonomial numbers are defined as

$$\binom{m+n}{n}_F := \frac{F^!_{m+n}}{F^!_m F^!_n},$$

where $F^!_n := F_n F_{n-1} F_{n-2} \dots F_1$.

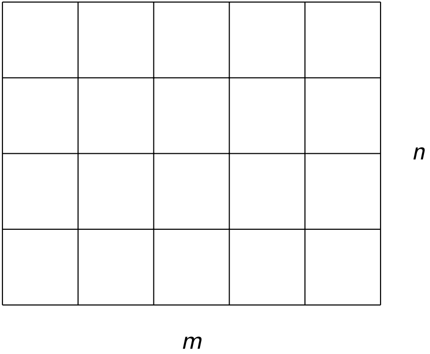
The Fibonomial numbers are positive integers for all $m, n \geq 1$:

$$\binom{1+n}{n}_F = F_{n+1}, \qquad \binom{m+1}{1}_F = F_{m+1},$$

$$\binom{m+n}{n}_F = F_{m+1}\binom{m+n-1}{n-1}_F + F_{n-1}\binom{m-1+n}{n}_F$$

Fibonomial numbers: Combinatorial model

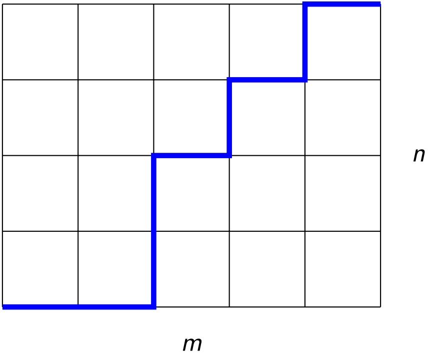
$$\binom{m+n}{n}_F = F_{m+1}\binom{m+n-1}{n-1}_F + F_{n-1}\binom{m-1+n}{n}_F$$



(Sagan and Savage 2011)

Fibonomial numbers: Combinatorial model

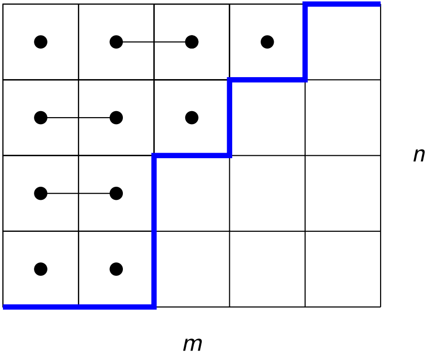
$$\binom{m+n}{n}_F = F_{m+1}\binom{m+n-1}{n-1}_F + F_{n-1}\binom{m-1+n}{n}_F$$



(Sagan and Savage 2011)

Fibonomial numbers: Combinatorial model

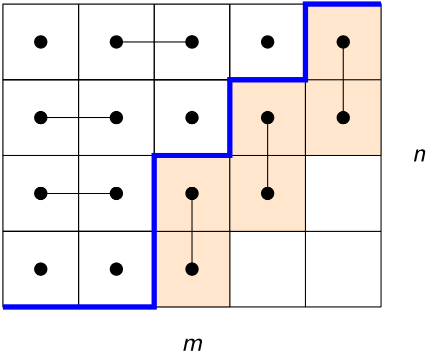
$$\binom{m+n}{n}_F = F_{m+1}\binom{m+n-1}{n-1}_F + F_{n-1}\binom{m-1+n}{n}_F$$



(Sagan and Savage 2011)

Fibonomial numbers: Combinatorial model

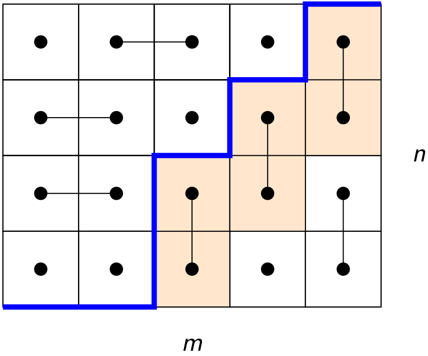
$$\binom{m+n}{n}_F = F_{m+1}\binom{m+n-1}{n-1}_F + F_{n-1}\binom{m-1+n}{n}_F$$



(Sagan and Savage 2011)

Fibonomial numbers: Combinatorial model

$$\binom{m+n}{n}_F = F_{m+1}\binom{m+n-1}{n-1}_F + F_{n-1}\binom{m-1+n}{n}_F$$



(Sagan and Savage 2011)

q-Fibonomial numbers

Definition

The q-Fibonomial numbers are defined as

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}} := \frac{[F_{m+n}]_q^!}{[F_m]_q^! [F_n]_q^!}$$

where $[F_n]_q^! := \prod_{k=1}^n [F_k]_q$.

q-Fibonomial numbers

Definition

The q-Fibonomial numbers are defined as

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}} := \frac{[F_{m+n}]_q^!}{[F_m]_q^![F_n]_q^!}$$

where $[F_n]_q^! := \prod_{k=1}^n [F_k]_q$.

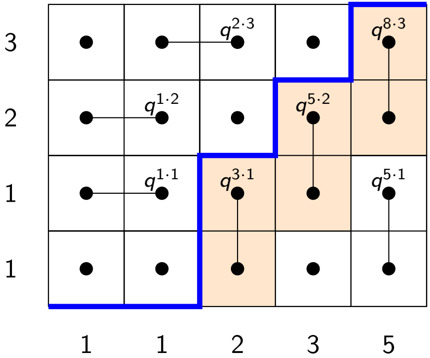
The q-Fibonomial numbers are polynomials with nonnegative integer coefficients for all $m, n \geq 1$:

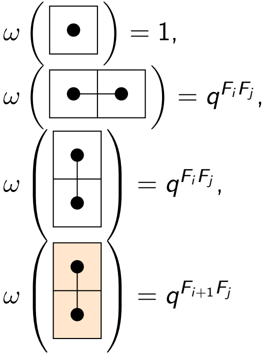
$$\begin{bmatrix} 1+n \\ n \end{bmatrix}_{\mathcal{F}} = [F_{n+1}]_q, \qquad \begin{bmatrix} m+1 \\ 1 \end{bmatrix}_{\mathcal{F}} = [F_{m+1}]_q,$$

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}} = [F_{m+1}]_{q^{F_n}} \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_{\mathcal{F}} + q^{F_{m+1}F_n}[F_{n-1}]_{q^{F_m}} \begin{bmatrix} m-1+n \\ n \end{bmatrix}_{\mathcal{F}}$$

q-Fibonomial numbers: Combinatorial model

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}} = [F_{m+1}]_{q^{F_n}} \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_{\mathcal{F}} + q^{F_{m+1}F_n}[F_{n-1}]_{q^{F_m}} \begin{bmatrix} m-1+n \\ n \end{bmatrix}_{\mathcal{F}}$$



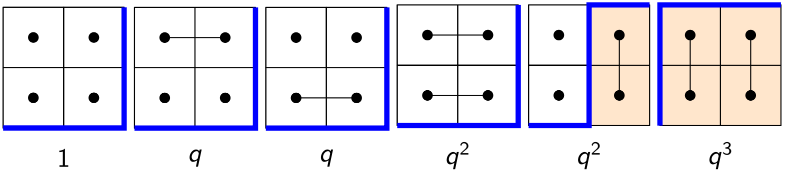


Example

For $m = n = 2$:

$$\begin{bmatrix} 2+2 \\ 2 \end{bmatrix}_{\mathcal{F}} = \frac{[F_4][F_3]}{[F_2][F_1]} = \frac{[3][2]}{[1][1]} = 1 + 2q + 2q^2 + q^3.$$

In a 2×2 rectangle there are 6 tilings:



Elliptic Fibonomial numbers

Definition

The elliptic Fibonomial numbers are defined as

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} := \frac{[F_{m+n}]^!_{a,b;q,p}}{[F_m]^!_{a,b;q,p} \cdot [F_n]^!_{a,b;q,p}}, \tag{1}$$

where $[F_n]^!_{a,b;q,p} := \prod_{k=1}^{n} [F_k]_{a,b;q,p}$.

Elliptic Fibonomial numbers

Definition

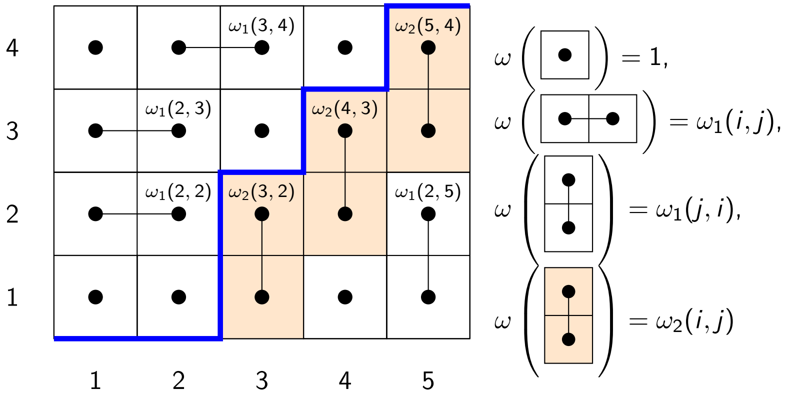
The elliptic Fibonomial numbers are defined as

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} := \frac{[F_{m+n}]^!_{a,b;q,p}}{[F_m]^!_{a,b;q,p} \cdot [F_n]^!_{a,b;q,p}}, \tag{1}$$

where $[F_n]^!_{a,b;q,p} := \prod_{k=1}^{n} [F_k]_{a,b;q,p}$.

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} = [F_{m+1}]_{a,bq^{1-F_n};q^{F_n},p} \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} + v_{a,b;q,p}(F_{m+1}F_n, F_m F_{n-1})[F_{n-1}]_{a,bq^{1-F_m};q^{F_m},p} \begin{bmatrix} m-1+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}}.$$

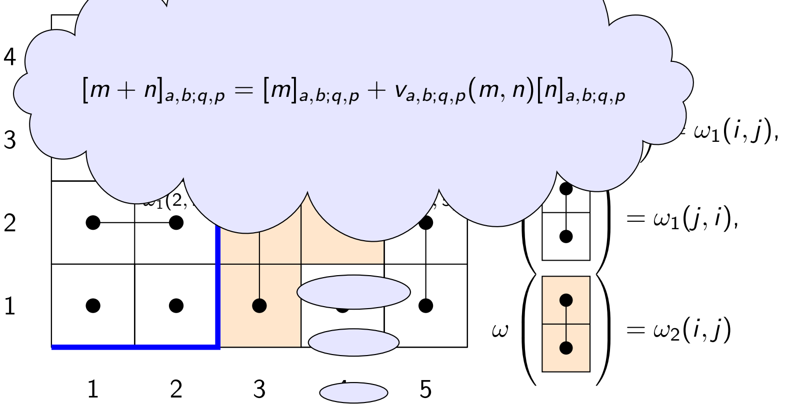
Elliptic Fibonomial numbers: Combinatorial model



$$\omega_1(i,j) := v_{a,bq^{1-F_j};q^{F_j},p}(F_i, F_{i-1}),$$
$$\omega_2(i,j) := v_{a,b;q,p}(F_{i+1}F_j, F_iF_{j-1})$$

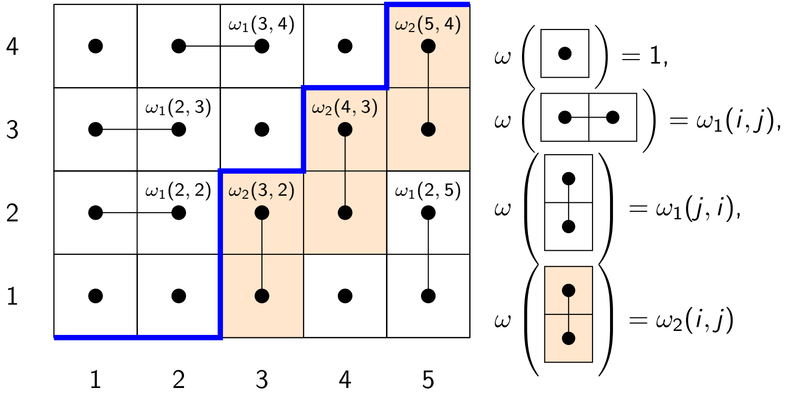
Elliptic Fibonomial numbers: Combinatorial model

$$[m+n]_{a,b;q,p} = [m]_{a,b;q,p} + v_{a,b;q,p}(m,n)[n]_{a,b;q,p}$$



$$\omega_1(i,j) := v_{a,bq^{1-F_j};q^{F_j},p}(F_i, F_{i-1}),$$
$$\omega_2(i,j) := v_{a,b;q,p}(F_{i+1}F_j, F_iF_{j-1})$$

Elliptic Fibonomial numbers: Combinatorial model



$$\omega_1(i,j) := v_{a,bq^{1-F_j};q^{F_j},p}(F_i, F_{i-1}),$$
$$\omega_2(i,j) := v_{a,b;q,p}(F_{i+1}F_j, F_iF_{j-1})$$

Open problem(s)

The q-Fibonacci analog of the m, n-Catalan number is defined as

$$\frac{1}{[F_{m+n}]_q}\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}}$$

and is proven to be a polynomial in q with integer coefficients for all $m, n \geq 1$ if $\gcd\{m, n\} = 1$.

Open problem(s)

The q-Fibonacci analog of the m,n-Catalan number is defined as

$$\frac{1}{[F_{m+n}]_q}\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}}$$

and is proven to be a polynomial in q with integer coefficients for all $m, n \geq 1$ if $\gcd\{m,n\} = 1$.

These polynomials seem to have only nonnegative integer coefficients.

Open problem(s)

The q-Fibonacci analog of the m,n-Catalan number is defined as

$$\frac{1}{[F_{m+n}]_q}\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}}$$

and is proven to be a polynomial in q with integer coefficients for all $m, n \geq 1$ if $\gcd\{m,n\} = 1$.

These polynomials seem to have only nonnegative integer coefficients.

Find a combinatorial interpretation!