Interpolation on L^p-spaces

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Chapter 1

Preface

The aim of this paper is to study some interpolation theorems of operators on L^p-spaces. The idea of interpolation is in broad outline the following: given numbers a and b such that $a < b$ and a certain condition holds with respect to a and b, then the same condition holds with every x in an interval between a and b. Thus, using interpolation, we are able to broaden our knowledge about certain conditions even to uncountable sets using only estimates proven in finite sets. Interpolation is widely used in mathematical analysis.

First we prove the Riesz-Thorin interpolation theorem for linear operators on L^p-spaces by using simple real analytic principles. We well see that the proof makes mainly use of Hölder's inequality and the density of simple functions in L^p-spaces. Especially we avoid completely complex analytical arguments, which the proof of this theorem is traditionally based on.

Then we prove the Marcinkiewicz interpolation theorem for subadditive operators on spaces consisting of functions which can be split into L^p-functions. Thus we can give up the assumption of linearity and consider operators that satisfy slightly weaker conditions.

As a prerequisite, the rudiments of real analysis, especially measure theory, as well as basic topology is assumed. A good understanding of linear algebra is also desirable. However, the finite-dimensional inner product spaces appearing in the theorems can always be replaced by Euclidean spaces.

In this paper the following source materials are used: The proofs of the interpolation theorems are based on [1]. Most of the definitions and proofs used in the second chapter are based on [2] and [3]. The proof of Lemma 2.15 can originally be found in [4].

Chapter 2

Definitions and prerequisites

First we define some basic concepts related to L^p-spaces.

Definition 2.1. Let (Ω, μ) be a measure space. Let $E \subset \Omega$ be an arbitrary zero-measurable set. Measure μ is *complete* if every $F \subset E$ is μ-measurable.

If the measure μ is complete, we also say that the corresponding measure space (Ω, μ) is complete. By defining this, we want to ensure that we have the concept *μ-measurable almost ewerywhere* by which we denote that some property is valid except for a zero-measurable set.

Definition 2.2. Let (Ω, μ) be a complete measure space, $\mathbb{V}$ a finite-dimensional inner product space and $1 \leq p < \infty$. We define

$$L^p(\Omega, \mu) = \{f : \Omega \to \mathbb{V} \mid f \text{ measurable and } \int_\Omega |f|^p d\mu < \infty\},$$

and denote

$$\|f\|_p = \left(\int_\Omega |f|^p d\mu \right)^{1/p}.$$

We would like to have normed vector space $(L^p(\Omega, \mu), \| \cdot \|_p)$. However, the condition $\|f\|_p = 0$ does not always lead to $f \equiv 0$. For example, if we have $\Omega = \mathbb{R}$ with Lebesgue measure $\mu = m$ and f is the characteristic function of zero, $f = \chi_{\{0\}}$, the condition $\|f\|_p = 0$ holds. Thus we define the following equivalence relation:

Definition 2.3. Functions $f, g \in L^p(\Omega, \mu)$ are equivalent, $f \sim g$, if $f = g$ almost everywhere.

We denote equivalence classes by

$$[f] = \tilde{f} = \{g \in L^p(\Omega, \mu) : g \sim f\}$$

and define

$$\tilde{L}^p(\Omega, \mu) = \{\tilde{f} : f \in L^p(\Omega, \mu)\}.$$

If $L^p(\Omega, \mu)$ is a real vector space, then is also $\tilde{L}^p(\Omega, \mu)$ with

$$[af + bg] = a[f] + b[g] = a\tilde{f} + b\tilde{g}, \; a, b \in \mathbb{R}, f, g \in L^p(\Omega, \mu).$$

We set

$$\|\tilde{f}\|_p = \|f\|_p,$$

which is well-defined by the definition of the equivalence relation $\sim$.

From now on we identify all functions which are equal almost everywhere. Hence we can write $L^p(\Omega, \mu) = \tilde{L}^p(\Omega, \mu)$. Now we are ready to prove that $L^p(\Omega, \mu)$ is a real vector space with the norm $\|\cdot\|_p$. For that we need various theorems.

Lemma 2.4. *(Young's inequality) If $a, b \geq 0, \; \alpha, \beta > 0$ and $\alpha + \beta = 1$, then*

$$a^\alpha b^\beta \leq \alpha a + \beta b.$$

Proof. As $a = 0$ or $b = 0$, the proposition trivially holds. Assume $a, b > 0$. The function $x \mapsto \ln x$ is concave for all $x > 0$, i.e. for all $x, y > 0$

$$\ln(tx + (1-t)y) \geq t \ln x + (1-t)\ln y$$

for any $t \in [0, 1]$. Thus

$$\ln(a^\alpha b^\beta) = \alpha \ln a + \beta \ln b \leq \ln(\alpha a + \beta b).$$

Because natural logarithm function is increasing, the claim follows. $\square$

Theorem 2.5. *(Hölder's inequality) If $p_1, p_2 > 1, \; \frac{1}{p_1} + \frac{1}{p_2} = 1, \; f \in L^{p_1}(\Omega, \mu)$ and $g \in L^{p_2}(\Omega, \mu)$, then*

$$fg \in L^1(\Omega, \mu) \quad \textit{and} \quad \|fg\|_1 \leq \|f\|_{p_1} \|g\|_{p_2}.$$

Proof. The cases $\|f\|_{p_1} = 0$ and $\|g\|_{p_2} = 0$ are clear. Hence we can assume $\|f\|_{p_1}, \|g\|_{p_2} > 0$. Now we fix $x \in \Omega$ and write

$$a = \frac{|f(x)|^{p_1}}{\|f\|_{p_1}^{p_1}}, \; b = \frac{|g(x)|^{p_2}}{\|g\|_{p_2}^{p_2}}, \; \alpha = \frac{1}{p_1} \text{ and } \beta = \frac{1}{p_2}.$$

We apply Young's inequality to get

$$\frac{|f(x)|}{\|f\|_{p_1}}\frac{|g(x)|}{\|g\|_{p_2}} \leq \frac{1}{p_1}\frac{|f(x)|^{p_1}}{\|f\|_{p_1}^{p_1}} + \frac{1}{p_2}\frac{|g(x)|^{p_2}}{\|g\|_{p_2}^{p_2}}.$$

Since f and g are measurable, $f \in L^{p_1}(\Omega, \mu)$, $g \in L^{p_2}(\Omega, \mu)$ and the integral of non-negative functions over Ω is monotonous, we get

$$\frac{\|fg\|_1}{\|f\|_{p_1}\|g\|_{p_2}} \leq \frac{1}{p_1}\frac{\|f\|_{p_1}^{p_1}}{\|f\|_{p_1}^{p_1}} + \frac{1}{p_2}\frac{\|g\|_{p_2}^{p_2}}{\|g\|_{p_2}^{p_2}} = \frac{1}{p_1} + \frac{1}{p_2} = 1.$$

Thus $fg \in L^1$ and

$$\|fg\|_1 \leq \|f\|_{p_1}\|g\|_{p_2}. \quad \square$$

Theorem 2.6. *(Minkowski's inequality) If $f, g \in L^p(\Omega, \mu)$, then $f + g \in L^p(\Omega, \mu)$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. If $p = 1$, we have

$$\|f + g\|_1 = \int_\Omega |f + g| d\mu \leq \int_\Omega (|f| + |g|) d\mu = \|f\|_1 + \|g\|_1.$$

Assume $p > 1$ and $q = \frac{p}{p-1}$. Then $\frac{1}{p} + \frac{1}{q} = 1$. If $a, b \geq 0$, we have the estimate

$$(a + b)^p \leq (2\max(a, b))^p \leq 2^p(a^p + b^p).$$

Using this estimate for every $x \in \Omega$ we get

$$|(f + g)(x)|^p \leq (|f(x)| + |g(x)|)^p \leq 2^p(|f(x)|^p + |g(x)|^p).$$

Thus

$$f + g \in L^p(\Omega, \mu).$$

Now using triangle inequality we have further estimate

$$|f + g|^p = |f + g||f + g|^{p-1} \leq |f||f + g|^{p-1} + |g||f + g|^{p-1}.$$

Because $f + g \in L^p(\Omega, \mu)$ and $(|f + g|^{p-1})^q = |f + g|^p$, we have

$$|f + g|^{p-1} \in L^q(\Omega, \mu).$$

We note that p and q are Hölder conjugates, so by previous observations and Hölder's inequality we get

$$\|f + g\|_p^p = \int_\Omega |f + g|^p d\mu \leq \int_\Omega |f||f + g|^{p-1} d\mu + \int_\Omega |g||f + g|^{p-1} d\mu$$

$$\leq \|f\|_p \left(\int_\Omega (|f+g|^{p-1})^q d\mu \right)^{1/q} + \|g\|_p \left(\int_\Omega (|f+g|^{p-1})^q d\mu \right)^{1/q}$$
$$= \|f\|_p \left(\int_\Omega |f+g|^p d\mu \right)^{1/q} + \|g\|_p \left(\int_\Omega |f+g|^p d\mu \right)^{1/q}$$
$$= (\|f\|_p + \|g\|_p)\|f+g\|_p^{p/q} = (\|f\|_p + \|g\|_p)\|f+g\|_p^{p-1}.$$

Hence

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p. \quad \square$$

By definition of $\|\cdot\|_p$, $\|f\|_p \geq 0$ for all $f \in L^p(\Omega,\mu)$. If $a \in \mathbb{R}$ and $f \in L^p(\Omega,\mu)$, we have

$$\|af\|_p = \left(\int_\Omega |af|^p d\mu \right)^{1/p} = (|a|^p)^{1/p} \left(\int_\Omega |f|^p d\mu \right)^{1/p} = |a|\|f\|_p.$$

Moreover, by the prior identification of equivalent L^p-functions, $\|f\|_p = 0$ if and only if $f \equiv 0$. Combining these observations with Minkowski's inequality we finally get:

Theorem 2.7. *The space $(L^p(\Omega,\mu), \|\cdot\|_p)$ is a normed vector space.* $\square$

Let (X,d) be a metric space. We say that a sequence (x_j) in X is a *Cauchy sequence* in X if for every $\epsilon > 0$ there is $j_\epsilon \in \mathbb{N}$ so that $d(x_i,x_j) < \epsilon$ for all $i,j \geq j_\epsilon$. The metric space X is *complete*, if every Cauchy sequence in X converges to some point of X. We use the name *Banach space* for a complete normed vector space.

Theorem 2.8. *(Riesz-Fischer) The space $L^p(\Omega,\mu)$ is a Banach space if $1 \leq p < \infty$.*

Proof. Let (f_j) be a Cauchy sequence in $L^p(\Omega,\mu)$. For each $k \in \mathbb{N}$ there is $j_k \in \mathbb{N}$ such that

$$\|f_i - f_j\|_p < \frac{1}{2^k} \quad \text{if} \quad i,j \geq j_k \quad \text{and} \quad j_1 < j_2 < \cdots .$$

We define an increasing sequence of real-valued functions g_k on Ω by setting

$$g_k = |f_{j_1}| + \sum_{l=1}^{k} |f_{j_{l+1}} - f_{j_l}|.$$

Applying Minkowski inequality k times we get

$$\|g_k\|_p = \||f_{j_1}| + \sum_{l=1}^{k} |f_{j_{l+1}} - f_{j_l}|\|_p \leq \|f_{j_1}\|_p + \sum_{l=1}^{k} \|f_{j_{l+1}} - f_{j_l}\|_p$$

$$\leq \|f_{j_1}\|_p + \sum_{l=1}^{k} \frac{1}{2^l} \leq \|f_{j_1}\|_p + \sum_{l=1}^{\infty} \frac{1}{2^l} = \|f_{j_1}\|_p + 1$$

for every $k \in \mathbb{N}$. Because the sequence (g_k) is increasing, there is a function $g = \lim_{k\to\infty} g_k$. Since every g_k is measurable, we deduce by monotone convergence theorem and the previous estimate

$$\int_\Omega g^p d\mu = \int_\Omega (\lim_{k\to\infty} g_k^p) d\mu = \lim_{k\to\infty} \int_\Omega g_k^p d\mu = \lim_{k\to\infty} \|g_k\|_p^p \leq (\|f_{j_1}\|_p + 1)^p < \infty.$$

Thus $g(x) < \infty$ almost everywhere in Ω. Hence the series $|f_{j_1}(x)| + \sum_{l=1}^{\infty} |f_{j_{l+1}}(x) - f_{j_l}(x)|$ is convergent for almost every $x \in \Omega$, which means that the series $f_{j_1}(x) + \sum_{l=1}^{\infty} (f_{j_{l+1}}(x) - f_{j_l}(x))$ converges as an absolutely convergent series. We denote the sum of this series pointwise by $f(x)$ and set $f(x) = 0$ in the set where the series is not convergent. Now we have a function $f : \Omega \to \mathbb{V}$ such that $f_{j_{k+1}} = f_{j_1} + \sum_{l=1}^{k} (f_{j_{l+1}}(x) - f_{j_l}(x)) \to f$ almost everywhere in Ω, as $k \to \infty$.

We prove that $f \in L^p(\Omega, \mu)$ and $\|f_k - f\|_p \to 0$ if $k \to \infty$. Let $\epsilon > 0$. Because (f_k) is a Cauchy sequence, there is $i_\epsilon \in \mathbb{N}$ such that $\|f_i - f_j\|_p < \epsilon$ if $i, j \geq i_\epsilon$. We will integrate non-negative measurable functions, and thus by Fatou's lemma we get

$$\int_\Omega |f_i - f|^p d\mu = \int_\Omega \lim_{k\to\infty} |f_i - f_{j_k}|^p d\mu = \int_\Omega \liminf_{k\to\infty} |f_i - f_{j_k}|^p d\mu$$

$$\leq \liminf_{k\to\infty} \int_\Omega |f_i - f_{j_k}|^p d\mu = \liminf_{k\to\infty} \|f_i - f_{j_k}\|_p^p \leq \epsilon^p$$

if $i \geq i_\epsilon$. This means that $f_i - f \in L^p(\Omega, \mu)$ and $\|f_i - f\|_p \leq \epsilon$ if $i \geq i_\epsilon$. Hence $f = f_i - (f_i - f) \in L^p(\Omega, \mu)$ and $f_i \to f$ in $L^p(\Omega, \mu)$. $\square$

Definition 2.9. A function $\phi : \Omega \to \mathbb{V}$ is simple if ϕ is measurable and gets only finite amount of different values.

If we decompose Ω into n mutually disjoint $\mu-$measurable sets, $\Omega = \bigcup_{k=1}^{n} \Omega_k$, we write

$$\phi = \sum_{k=1}^{n} \alpha_k \chi_{\Omega_k},$$

where $\alpha_k \in \mathbb{V}$. If $\mu(\Omega_k) = \infty$ for some $k \in \{1, .., n\}$, we assume that $\alpha_k = 0$ and say $0 \cdot \infty = 0$. For each decomposition, we denote the set of such simple functions by $\mathcal{L}(\Omega, \mu)$. Clearly $\mathcal{L}(\Omega, \mu) \subset L^p(\Omega, \mu)$, $\phi + \psi \in \mathcal{L}(\Omega, \mu)$ and $a\phi \in \mathcal{L}(\Omega, \mu)$ as well as $0 \in \mathcal{L}(\Omega, \mu)$ for all $\phi, \psi \in \mathcal{L}(\Omega, \mu)$, $a \in \mathbb{R}$ and $1 \leq p < \infty$. Therefore $\mathcal{L}(\Omega, \mu)$ is a finite-dimensional subspace of $L^p(\Omega, \mu)$, $(\chi_{\Omega_1}, \ldots, \chi_{\Omega_n})$ being a basis.

Definition 2.10. If X is a topological space and $A \subset X$, we say that A is dense in X if $\overline{A} = X$.

Suppose F is a subspace of a normed vector space $(E, \|\cdot\|)$. In that case, F is dense in E if for every $x \in E$ there is a sequence $(x_i)_{i\in\mathbb{N}}$ so that $\|x_i - x\| \to 0$ if $i \to \infty$.

Lemma 2.11. *The set of simple functions is dense in $L^p(\Omega, \mu)$.*

Proof. Let $f \in L^p(\Omega, \mu)$. We construct a sequence (ϕ_j) of simple functions $\phi_j : \Omega \to \mathbb{V}$ such that $\lim_{j\to\infty}\phi_j = f$. We assume that $\mathbb{V}$ is an n-dimensional inner product space. Thus, let $(v_1, \ldots, v_n)$ be an orthonormal basis of $\mathbb{V}$. If $x \in \Omega$, we can write $f(x) = f_1(x)v_1 + \cdots + f_n(x)v_n$, where $f_1, \ldots, f_n$ are real-valued component functions.

Let $A_i = \{x \in \Omega : f_i(x) \geq 0\}$ and $B_i = \{x \in \Omega : f_i(x) < 0\}$ for fixed $i \in \{1, \ldots n\}$. Now $f_i = f_i\chi_{A_i} + f_i\chi_{B_i}$. Since f_i is a measurable real-valued function (as a component function of measurable function f) for every $i \in \{1, \ldots, n\}$, by elementary measure theory, there is an increasing sequence $(\phi_j^{A_i})$ of non-negative simple functions such that $f_i\chi_{A_i} = \lim_{j\to\infty}\phi_j^{A_i}$. In the same way, there is a decreasing sequence $(\phi_j^{B_i})$ of non-positive simple functions such that $f_i\chi_{B_i} = \lim_{j\to\infty}\phi_j^{B_i}$. Now we define $\phi_j^i = \phi_j^{A_i} + \phi_j^{B_i}$, which is a well-defined simple function, since $\phi_j^{A_i} = 0$ in B_i and $\phi_j^{B_i} = 0$ in A_i. By the prior consideration, we have $f_i = \lim_{j\to\infty}\phi_j^i$. We define $\phi_j = \phi_j^1 v_1 + \cdots + \phi_j^n v_n$ for every $j \in \mathbb{N}$. Now $f = \lim_{j\to\infty}\phi_j$.

Since

$$|\phi_j(x)| = \langle\phi_j(x), \phi_j(x)\rangle^{1/2} = \langle\sum_{i=1}^n \phi_j^i(x)v_i, \sum_{i=1}^n \phi_j^i(x)v_i\rangle^{1/2}$$

$$= \sqrt{\sum_{i=1}^n \phi_j^i(x)^2} \leq \sqrt{\sum_{i=1}^n f_i(x)^2} = |f(x)|$$

for every $j \in \mathbb{N}$ and $x \in \Omega$, we have

$$|f(x) - \phi_j(x)|^p \leq (|f(x)| + |\phi_j(x)|)^p \leq 2^p|f(x)|^p,$$

where $2^p|f^p|$ is integrable because $f \in L^p(\Omega, \mu)$. In addition, the function $x \mapsto |f(x) - \phi_j(x)|$ is a measurable real-valued function since it is a composition of measurable function $x \mapsto f(x) - \phi_j(x)$ and the continuous norm function. Therefore we can apply Lebesgue's dominated convergence theorem to get

$$\|f - \phi_j\|_p^p = \int_\Omega |f(x) - \phi_j(x)|^p d\mu \longrightarrow 0$$

if $j \to \infty$. $\square$

Definition 2.12. A measure space (Ω, μ) is *sigma-finite*, if

$$\Omega = \bigcup_{j \in \mathbb{N}} \Omega_j \text{ and } \mu(\Omega_j) < \infty \text{ for all } j \in \mathbb{N},$$

where Ω_j is measurable for all $j \in \mathbb{N}$.

Definition 2.13. Let $g : I \to \mathbb{R}$ be a function defined on an open interval $I \subset \mathbb{R}$. The function g is *convex* on I if for any $x, y \in I$ and $t \in [0, 1]$

$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y).$$

Lemma 2.14. *If $I \subset \mathbb{R}$ is an open interval, $g : I \to \mathbb{R}$ convex and $f : \mathbb{R} \to \mathbb{R}$ convex and increasing, then $f \circ g$ is convex.*

Proof. Let $x, y \in I$ and $t \in [0, 1]$. Since g is convex on the interval I, we have $g(tx+(1-t)y) \leq tg(x)+(1-t)g(y)$. Hence, under the monotone and convexity assumptions of f,

$$\begin{aligned} (f \circ g)(tx + (1-t)y) &= f(g(tx + (1-t)y)) \leq f(tg(x) + (1-t)g(y)) \\ &\leq tf(g(x)) + (1-t)f(g(y)) = t(f \circ g)(x) + (1-t)(f \circ g)(y). \qquad \square \end{aligned}$$

Lemma 2.15. *If $g : I \to \mathbb{R}$ is convex, then g is Lipschitz-continuous in every closed interval $[a', b'] \subset I$.*

Proof. Let $\alpha, \beta, \gamma \in I$ such that $\alpha < \beta < \gamma$. Now, since $\beta \in]\alpha, \gamma[$, we can write $\beta = t\alpha + (1-t)\gamma$, where $t = (\gamma - \beta)/(\gamma - \alpha)$. As g is convex, we have

$$g(\beta) = g(t\alpha + (1-t)\gamma) \leq tg(\alpha) + (1-t)g(\gamma) = tg(\alpha) + g(\gamma) - tg(\gamma).$$

Using this inequality, we get

$$\frac{g(\beta) - g(\alpha)}{\beta - \alpha} \leq \frac{(1-t)(g(\gamma) - g(\alpha))}{\beta - \alpha} = \frac{\beta - \alpha}{\gamma - \alpha} \cdot \frac{g(\gamma) - g(\alpha)}{\beta - \alpha} = \frac{g(\gamma) - g(\alpha)}{\gamma - \alpha}.$$

Similarly,

$$\frac{g(\gamma) - g(\beta)}{\gamma - \beta} \geq \frac{t(g(\gamma) - g(\alpha))}{\gamma - \beta} = \frac{\gamma - \beta}{\gamma - \alpha} \cdot \frac{g(\gamma) - g(\alpha)}{\gamma - \beta} = \frac{g(\gamma) - g(\alpha)}{\gamma - \alpha}.$$

Hence

$$\frac{g(\beta) - g(\alpha)}{\beta - \alpha} \leq \frac{g(\gamma) - g(\alpha)}{\gamma - \alpha} \leq \frac{g(\gamma) - g(\beta)}{\gamma - \beta}. \tag{2.16}$$

Let $[a', b'] \subset I$ be a closed interval and $x, y \in [a', b']$. Let $a, b, c, d \in I$ be such that $a < b < x < y < c < d$. By applying (2.17) repeatedly we get

$$\frac{g(b) - g(a)}{b - a} \leq \frac{g(y) - g(a)}{y - a} \leq \frac{g(y) - g(x)}{y - x} \leq \frac{g(d) - g(x)}{d - x} \leq \frac{g(d) - g(c)}{d - c}.$$

Thus,

$$\frac{g(b) - g(a)}{b - a}(y - x) \leq g(y) - g(x) \leq \frac{g(d) - g(c)}{d - c}(y - x).$$

By choosing $M = \max\{|\frac{g(b)-g(a)}{b-a}|, |\frac{g(d)-g(c)}{d-c}|\}$ we get

$$|g(y) - g(x)| \leq M|y - x|.$$

Since $x, y \in [a', b']$ were arbitrary, g is Lipschitz on $[a', b']$. $\square$

Chapter 3

The Riesz-Thorin interpolation theorem

In this chapter we present first of the main theorems of this paper.

Theorem 3.1. *Let (Ω, μ) be a sigma-finite measure space. Let $p_1, p_2 \in \mathbb{R}$ be such that $1 \leq p_1 \leq p_2 < \infty$. Suppose that we have a linear operator*

$$\mathcal{T} : L^{p_1}(\Omega, \mu) \cap L^{p_2}(\Omega, \mu) \to L^{p_1}(\Omega, \mu) \cap L^{p_2}(\Omega, \mu)$$

with respect to L^{p_1}- and L^{p_2}-norms such that the conditions

$$\|\phi\|_{p_1} \leq \|\mathcal{T}\|_{p_1} \|\phi\|_{p_1} \quad \text{and} \quad \|\mathcal{T}\phi\|_{p_2} \leq \|\mathcal{T}\|_{p_2} \|\phi\|_{p_2}$$

hold for every $\phi \in L^{p_1}(\Omega, \mu) \cap L^{p_2}(\Omega, \mu)$. Then for every p, $p_1 \leq p \leq p_2$, $\mathcal{T}$ extends as a bounded linear operator

$$\mathcal{T} : L^p(\Omega, \mu) \to L^p(\Omega, \mu).$$

In addition, we have the uniform estimate

$$\|\mathcal{T}f\|_p \leq \|\mathcal{T}\|_{p_1}^{\alpha} \|\mathcal{T}\|_{p_2}^{\beta} \|f\|_p \tag{3.2}$$

for every $f \in L^p(\Omega, \mu)$, where α and β are determined from the relations

$$\frac{1}{p} = \frac{\alpha}{p_1} + \frac{\beta}{p_2}, \quad \alpha + \beta = 1.$$

First we prove that the intersection $L^{p_1}(\Omega\mu) \cap L^{p_2}(\Omega\mu)$ is contained in $L^p(\Omega, \mu)$ for $p \in [p_1, p_2]$.

Lemma 3.3. *If $1 \leq p_1 \leq p \leq p_2 < \infty$, then*

$$L^{p_1}(\Omega, \mu) \cap L^{p_2}(\Omega, \mu) \subset L^p(\Omega, \mu).$$

Proof. Suppose $f \in L^{p_1}(\Omega,\mu) \cap L^{p_2}(\Omega,\mu)$. We make the decomposition $\Omega = A \cup B$, where

$$A = \{x \in \Omega : |f(x)| > 1\} \text{ and } B = \{x \in \Omega : |f(x)| \le 1\}.$$

Now this union is disjoint, and we have $|f(x)|^p \le |f(x)|^{p_2}$ if $x \in A$ and $|f(x)|^p \le |f(x)|^{p_1}$ if $x \in B$. Thus

$$\int_\Omega |f|^p d\mu = \int_A |f|^p d\mu + \int_B |f|^p d\mu \le \int_A |f|^{p_2} d\mu + \int_B |f|^{p_1} d\mu \le \int_\Omega |f|^{p_2} d\mu + \int_\Omega |f|^{p_1} d\mu < \infty.$$

Hence $f \in L^p(\Omega,\mu)$. $\square$

Proof of the Riesz-Thorin theorem. For each disjoint decomposition $\Omega = \bigcup_{k=1}^n \Omega_k$ of measurable sets Ω_k we have the finite-dimensional subspace of simple functions

$$\mathcal{L}(\Omega,\mu) \subset L^{p_1}(\Omega,\mu) \cap L^{p_2}(\Omega,\mu) \subset L^p(\Omega,\mu).$$

By Lemma 2.11, the union $\cup\mathcal{L}(\Omega,\mu)$ is dense in $L^p(\Omega,\mu)$ if $p_1 \le p \le p_2$. We consider the operator

$$\mathcal{T}|(\cup\mathcal{L}(\Omega,\mu)) : \cup\mathcal{L}(\Omega,\mu) \to L^p(\Omega,\mu).$$

Since $L^p(\Omega,\mu)$ is complete as a range and $L^p(\Omega,\mu) = cl(\cup\mathcal{L}(\Omega,\mu))$, elementary metric topology tells us that if $\mathcal{T}|(\cup\ell(\Omega,\mu))$ is uniformly continuous, then it extends continuously to $L^p(\Omega,\mu)$. Hence we need only to show that the uniform estimate (3.2) holds for an arbitrary $\phi \in \cup\mathcal{L}(\Omega,\mu)$.

Now we fix the decomposition $\Omega = \bigcup_{k=1}^n \Omega_k$. Let $p \in [p_1,p_2]$ and $S(0,1) \subset \mathcal{L}(\Omega,\mu)$ be the unit sphere of $\mathcal{L}(\Omega,\mu)$. We assume that $\mathcal{T}$ is not a zero mapping and consider the function

$$\phi \mapsto \|\mathcal{T}\phi\|_p : S(0,1) \to \mathbb{R},$$

where

$$\phi = \sum_{k=1}^n a_k \chi_{\Omega_k}.$$

We define mappings

$$E : S(0,1) \to \mathbb{V}^n, \quad E\left(\sum_{k=1}^n a_k \chi_{\Omega_k}\right) = (a_1,\dots,a_n),$$

$$pr_j : \mathbb{V}^n \to \mathbb{V}, \quad pr_j(a_1,\dots,a_n) = a_j$$

and

$$|\cdot| : \mathbb{V} \to \mathbb{R}, \quad a_j \mapsto |a_j|$$

for every $j \in \{1, \dots, n\}$. Now the mappings $|pr_j \circ E|$ are continuous as compositions of continuous functions. Since $S(0,1)$ is compact, there is a maximum for every map $|pr_j \circ E|$. Thus, there is

$$a = \max\{\max_{\phi \in S(0,1)} |pr_j \circ E(\phi)| : j \in \{1, \dots, n\}\}.$$

Now, by writing $g_k = \mathcal{T}\chi_{\Omega_k} \in L^{p_1}(\Omega, \mu) \cap L^{p_2}(\Omega, \mu) \subset L^p(\Omega, \mu)$ and $M = \max_{k=1,\dots,n} \|g_k\|_p$, and using the linearity of $\mathcal{T}$, we deduce

$$\|\mathcal{T}\phi\|_p = \|\sum_{k=1}^{n} a_k g_k\|_p \leq \sum_{k=1}^{n} \|a_k g_k\|_p = \sum_{k=1}^{n} |a_k| \|g_k\|_p \leq naM < \infty$$

for every $\phi \in S(0,1)$. Hence the function $\phi \mapsto \|\mathcal{T}\phi\|_p$ restricted to $S(0,1)$ is continuous. Therefore, since $S(0,1)$ is compact, there is

$$A_p = \max\left\{\|\mathcal{T}\phi\|_p : \phi \in \mathcal{L}(\Omega, \mu) \text{ and } \|\phi\|_p = 1\right\}.$$

Now we have

$$\|\mathcal{T}\phi\|_p \leq A_p \quad \text{if } \phi \in \mathcal{L}(\Omega, \mu) \text{ and } \|\phi\|_p = 1,$$

which is equivalent to

$$\|\mathcal{T}\phi\|_p \leq A_p \|\phi\|_p \quad \text{if } \phi \in \mathcal{L}(\Omega, \mu).$$

This condition states that $\mathcal{T}|\mathcal{L}(\Omega, \mu)$ is a bounded linear map. Therefore we can set

$$A_p = \max\left\{\frac{\|\mathcal{T}\phi\|_p}{\|\phi\|_p} : \phi \in \mathcal{L}(\Omega, \mu)\right\}.$$

Now we have a well-defined function $p \mapsto A_p : [p_1, p_2] \to \mathbb{R}$. Let $q \in [p_1, p_2]$ and (p_j) be a sequence in $[p_1, p_2]$ such that $p_j \to q$ if $j \to \infty$. We denote $g_k = \mathcal{T}\chi_{\Omega_k}$ for $k \in \{1, \dots, n\}$. For a function $\phi \in \mathcal{L}(\Omega, \mu)$ we have

$$|\sum_{k=1}^{n} a_k g_k|^{p_j} = |\sum_{k=1}^{n} a_k g_k|^{p_j} \chi_{A_j} + |\sum_{k=1}^{n} a_k g_k|^{p_j} \chi_{B_j}$$

$$\leq |\sum_{k=1}^{n} a_k g_k|^{p_1} \chi_{A_j} + |\sum_{k=1}^{n} a_k g_k|^{p_2} \chi_{B_j} \leq |\sum_{k=1}^{n} a_k g_k|^{p_1} + |\sum_{k=1}^{n} a_k g_k|^{p_2}$$

for every $j \in \mathbb{N}$, where $A_j = \{x \in \Omega : |\sum_{k=1}^{n} a_k g_k|^{p_j} < 1\}$ and $B_j = \{x \in \Omega : |\sum_{k=1}^{n} a_k g_k|^{p_j} \geq 1\}$. We have found an integrable majorant of $|\phi|^{p_j}$ for every $j \in \mathbb{N}$. Moreover, by continuity of upcoming composition mappings,

$$|\sum_{k=1}^{n} (a_k)_{p_j} g_k|^{p_j} \to |\sum_{k=1}^{n} (a_k)_q g_k|^q \quad \text{if} \quad j \to \infty,$$

where the additional lower indice conventions p_j and q refer to the coefficients corresponding the maxima A_{p_j} and A_q. Hence, by Lebesgue's dominated convergence theorem,

$$\lim_{j\to\infty} A_{p_j}^{p_j} = \lim_{j\to\infty} \int_\Omega |\sum_{k=1}^n (a_k)_{p_j} g_k|^{p_j} = \int_\Omega |\sum_{k=1}^n (a_k)_q g_k|^q = A_q^q.$$

Thus the function $p \mapsto A_p$ is continuous.

Let $p \in [p_1, p_2]$ be fixed. Let $f \in \mathcal{L}(\Omega, \mu)$ be so that $\|\mathcal{T} f\|_p = A_p \|f\|_p$. Now for every $h \in \mathcal{L}(\Omega, \mu)$ and $t \in \mathbb{R}$ we have $\|\mathcal{T}(f + th)\|_p \le A_p \|f + th\|_p$, which leads to

$$A_p^p \int_\Omega |f + th|^p d\mu - \int_\Omega |\mathcal{T} f + t\mathcal{T} h|^p d\mu \ge 0 \tag{3.4}$$

by linearity of $\mathcal{T}$. The left-hand side of the inequality (3.4) defines a function $F : \mathbb{R} \to \mathbb{R}$ of the variable t for fixed $p > 1$. Since

$$\begin{aligned} F(t) &= A_p^p \int_\Omega |f + th|^p d\mu - \int_\Omega |\mathcal{T} f + t\mathcal{T} h|^p d\mu \\ &= A_p^p \int_\Omega (|f + th|^2)^{p/2} d\mu - \int_\Omega (|\mathcal{T} f + t\mathcal{T} h|^2)^{p/2} d\mu \\ &= A_p^p \int_\Omega \langle f + th, f + th\rangle^{p/2} d\mu - \int_\Omega \langle \mathcal{T} f + t\mathcal{T} h, \mathcal{T} f + t\mathcal{T} h\rangle^{p/2} d\mu, \end{aligned}$$

we deduce that F is continuously differentiable and

$$\frac{d}{dt} F(t) = A_p^p \int_\Omega \frac{d}{dt} \langle f + th, f + th\rangle^{p/2} d\mu - \int_\Omega \frac{d}{dt} \langle \mathcal{T} f + t\mathcal{T} h, \mathcal{T} f + t\mathcal{T} h\rangle^{p/2} d\mu$$

$$= A_p^p \int_\Omega \frac{p}{2} (|f + th|^2)^{p/2-1} \cdot 2\langle f + th, h\rangle d\mu - \int_\Omega \frac{p}{2} (|\mathcal{T} f + t\mathcal{T} h|^2)^{p/2-1} \cdot 2\langle \mathcal{T} f + t\mathcal{T} h, \mathcal{T} h\rangle d\mu$$

$$= A_p^p p \int_\Omega |f + th|^{p-2} \langle f + th, h\rangle d\mu - p \int_\Omega |\mathcal{T} f + t\mathcal{T} h|^{p-2} \langle \mathcal{T} f + t\mathcal{T} h, \mathcal{T} h\rangle d\mu.$$

We can do the derivation inside the integrals, because the integrands are simple functions and thus integrals can be considered as finite sums.

As F has its minimum at $t = 0$, the derivative must be zero at $t = 0$. Therefore

$$\begin{aligned} &A_p^p p \int_\Omega |f|^{p-2} \langle f, h\rangle d\mu - p \int_\Omega |\mathcal{T} f|^{p-2} \langle \mathcal{T} f, \mathcal{T} h\rangle d\mu \\ &= A_p^p p \int_\Omega \langle |f|^{p-2} f, h\rangle d\mu - p \int_\Omega \langle |\mathcal{T} f|^{p-2} \mathcal{T} f, \mathcal{T} h\rangle d\mu = 0. \end{aligned}$$

Dividing by p we get the identity

$$A_p^p \int_\Omega \langle |f|^{p-2} f, h\rangle d\mu = \int_\Omega \langle |\mathcal{T} f|^{p-2} \mathcal{T} f, \mathcal{T} h\rangle d\mu. \tag{3.5}$$

Let $r \in [p_1, p]$ be such that $r(p-1)/(r-1) \leq p_2$, and

$$h = |f|^{\frac{p-r}{r-1}} f \in \mathcal{L}(\Omega, \mu).$$

Now

$$\langle |f|^{p-2} f, h\rangle = \langle |f|^{p-2} f, |f|^{\frac{p-r}{r-1}} f\rangle = |f|^{p-2} |f|^{\frac{p-r}{r-1}} \langle f, f\rangle = |f|^p |f|^{\frac{p-r}{r-1}} = |f|^{\frac{pr-r}{r-1}}.$$

Moreover, we have

$$\mathcal{T} h = \mathcal{T}(|f|^{\frac{p-r}{r-1}} f) = |f|^{\frac{p-r}{r-1}} \mathcal{T} f.$$

This can be seen as follows: We write

$$h = |f|^{\frac{p-r}{r-1}} f = |\sum_{k=1}^n a_k \chi_{\Omega_k}|^{\frac{p-r}{r-1}} f$$

and decompose $h = \sum_{k=1}^n h\chi_{\Omega_k}$, where $h\chi_{\Omega_j} = |a_j|^{(p-r)/(r-1)} f\chi_{\Omega_j}$ for every $j \in \{1, \ldots, n\}$. Now, using the linearity of $\mathcal{T}$, we deduce

$$\mathcal{T} h = \sum_{k=1}^n \mathcal{T}(h\chi_{\Omega_k}) = \sum_{k=1}^n \mathcal{T}\left(|a_k|^{\frac{p-r}{r-1}} f\chi_{\Omega_k}\right) = \sum_{k=1}^n |a_k|^{\frac{p-r}{r-1}} \mathcal{T}(f\chi_{\Omega_k})$$

$$= \sum_{k=1}^n |a_k|^{\frac{p-r}{r-1}} (\mathcal{T} f)\chi_{\Omega_k} = \left|\sum_{k=1}^n a_k \chi_{\Omega_k}\right|^{\frac{p-r}{r-1}} \mathcal{T} f = |f|^{\frac{p-r}{r-1}} \mathcal{T} f,$$

since

$$\mathcal{T}(f\chi_{\Omega_j}) = \mathcal{T}((\sum_{k=1}^n a_k \chi_{\Omega_k})\chi_{\Omega_j}) = \mathcal{T}(a_j \chi_{\Omega_j}) = a_j \mathcal{T}\chi_{\Omega_j} = \left(\sum_{k=1}^n a_k \mathcal{T}\chi_{\Omega_k}\right)\chi_{\Omega_j} = (\mathcal{T} f)\chi_{\Omega_j}$$

for every $j \in \{1, \ldots, n\}$. Thus by (3.5) we get

$$A_p^p \int_\Omega |f|^{\frac{pr-r}{r-1}} d\mu = \int_\Omega \langle |\mathcal{T} f|^{p-2} \mathcal{T} f, |f|^{\frac{p-r}{r-1}} \mathcal{T} f\rangle d\mu = \int_\Omega |\mathcal{T} f|^p |f|^{\frac{p-r}{r-1}} d\mu = \| |\mathcal{T} f|^p |f|^{\frac{p-r}{r-1}} \|_1$$

$$= \| |f|^{\frac{p-r}{r-1}} |\mathcal{T} f| |\mathcal{T} f|^{p-1} \|_1 = \| |\mathcal{T} h| |\mathcal{T} f|^{p-1} \|_1.$$

Since $h \in \mathcal{L}(\Omega,\mu)$ and $r \in [p_1, p_2]$, we have $\|\mathcal{T}h\|_r \le A_r\|h\|_r < \infty$. Thus $|\mathcal{T}h| \in L^r(\Omega,\mu)$. Similarly, since $f \in \mathcal{L}(\Omega,\mu)$ and $p_1 \le r(p-1)/(r-1) \le p_2$, one can deduce $|\mathcal{T}f| \in L^{r(p-1)/(r-1)}(\Omega,\mu)$, which means $|\mathcal{T}f|^{p-1} \in L^{r/(r-1)}(\Omega,\mu)$. Moreover,

$$\frac{1}{r} + \frac{1}{\frac{r}{r-1}} = \frac{1}{r} + \frac{r-1}{r} = 1.$$

Hence we can apply Hölder's inequality to these functions to get

$$A_p^p \|f\|_{\frac{pr-r}{r-1}}^{\frac{pr-r}{r-1}} = A_p^p \int_\Omega |f|^{\frac{pr-r}{r-1}} d\mu$$

$$= \||\mathcal{T}h||\mathcal{T}f|^{p-1}\|_1 \le \|\mathcal{T}h\|_r \||\mathcal{T}f|^{p-1}\|_{\frac{r}{r-1}} = \|\mathcal{T}h\|_r \left(\int_\Omega |\mathcal{T}f|^{\frac{r(p-1)}{r-1}} d\mu\right)^{\frac{r-1}{r}}$$

$$= \|\mathcal{T}h\|_r \left(\left(\int_\Omega |\mathcal{T}f|^{\frac{pr-r}{r-1}} d\mu\right)^{\frac{r-1}{pr-r}}\right)^{p-1} = \|\mathcal{T}h\|_r \|\mathcal{T}f\|_{\frac{pr-r}{r-1}}^{p-1}$$

$$\le A_r \|h\|_r A_{\frac{pr-r}{r-1}}^{p-1} \|f\|_{\frac{pr-r}{r-1}}^{p-1} = A_r A_{\frac{pr-r}{r-1}}^{p-1} \left(\int_\Omega (|f|^{\frac{p-r}{r-1}}|f|)^r d\mu\right)^{\frac{1}{r}} \left(\int_\Omega |f|^{\frac{pr-r}{r-1}} d\mu\right)^{\frac{r-1}{r}}$$

$$= A_r A_{\frac{pr-r}{r-1}}^{p-1} \left(\int_\Omega |f|^{\frac{pr-r}{r-1}} d\mu\right)^{\frac{1}{r}} \left(\int_\Omega |f|^{\frac{pr-r}{r-1}} d\mu\right)^{\frac{r-1}{r}} = A_r A_{\frac{pr-r}{r-1}}^{p-1} \|f\|_{\frac{pr-r}{r-1}}^{\frac{pr-r}{r-1}}.$$

Therefore

$$A_p \le A_r^{1/p} A_{\frac{pr-r}{r-1}}^{1-1/p}, \tag{3.6}$$

which is a special case of the inequality (3.2), since with $\alpha = 1/p$ and $\beta = 1 - 1/p$ we have

$$\frac{\alpha}{r} + \frac{\beta(r-1)}{pr-r} = \frac{\alpha p - \alpha + \beta r - \beta}{pr-r} = \frac{1 - 1/p + r - r/p - 1 + 1/p}{pr-r} = \frac{1-1/p}{p-1} = \frac{1}{p}.$$

Let $I \subset [p_1, p_2]$ be the set consisting of those $p \in [p_1, p_2]$ such that

$$A_p \le A_{p_1}^\alpha A_{p_2}^\beta \text{ with } \frac{\alpha}{p_1} + \frac{\beta}{p_2} = \frac{1}{p} \text{ and } \alpha + \beta = 1. \tag{3.7}$$

First we note that $p_1, p_2 \in I$ by choosing either $\alpha = 1$ or $\beta = 1$. Now, let $a, b \in I$ and $c = b(1 - \frac{1}{a}) + 1$. We can assume that $a > 1$ and $b < a$. Now

$$p_1 \le b = b(1 - \frac{1}{b}) + 1 < b(1 - \frac{1}{a}) + 1 = c < a(1 - \frac{1}{a}) + 1 = a \le p_2.$$

Thus $c \in]b, a[\subset [p_1, p_2]$. By writing $b = (c-1)a/(a-1)$, we can apply (3.6) and (3.7) to get

$$A_c \le A_a^{1/c} A_b^{1-1/c} \le (A_{p_1}^{\alpha_a} A_{p_2}^{\beta_a})^{1/c} (A_{p_1}^{\alpha_b} A_{p_2}^{\beta_b})^{1-1/c} = A_{p_1}^{\frac{\alpha_a}{c}+\alpha_b-\frac{\alpha_b}{c}} A_{p_2}^{\frac{\beta_a}{c}+\beta_b-\frac{\beta_b}{c}} =: A_{p_1}^{\alpha_c} A_{p_2}^{\beta_c},$$

where α_a, β_a and α_b, β_b satisfy the condition of (3.8) for a and b. Now

$$\frac{\alpha_c}{p_1} + \frac{\beta_c}{p_2} = \frac{1}{c}\left(\frac{\alpha_a - \alpha_b + \alpha_b c}{p_1} + \frac{\beta_a - \beta_b + \beta_b c}{p_2}\right) = \frac{1}{c}\left(\frac{\alpha_a}{p_1} + \frac{\beta_a}{p_2} - \left(\frac{\alpha_b}{p_1} + \frac{\beta_b}{p_2}\right) + \left(\frac{\alpha_b}{p_1} + \frac{\beta_b}{p_2}\right)c\right)$$

$$= \frac{1}{c}\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{b}\left(b(1-\frac{1}{a}) + 1\right)\right) = \frac{1}{c}$$

and

$$\alpha_c + \beta_c = \left(\frac{\alpha_a}{c} + \alpha_b - \frac{\alpha_b}{c}\right) + \left(\frac{\beta_a}{c} + \beta_b - \frac{\beta_b}{c}\right) = \frac{\alpha_a + \beta_a}{c} - \frac{\alpha_b + \beta_b}{c} + \alpha_b + \beta_b = \frac{1}{p} - \frac{1}{p} + 1 = 1.$$

Hence $c \in I$.

The formula $c = b(1 - \frac{1}{a}) + 1$ defines a recursive sequence starting from the points p_1 and p_2. First we choose $b = p_1$ and $a = p_2$. The formula gives $c_1 \in]p_1, p_2[$. For given $t \in [p_1, p_2]$ we choose the successive elements a and b of the sequence such that $t \in [b, a]$. Because the point c divides the interval $[b, a]$ always in the same ratio and $b < c < a$, the chosen sequence converges to t. This means that I is dense in the interval $[p_1, p_2]$.

Since the function $p \mapsto A_p$ is continuous on the interval $[p_1, p_2]$, we can easily deduce $I = [p_1, p_2]$: Let $t \in [p_1, p_2]$ be arbitrary and $\epsilon > 0$. Since I is dense in $[p_1, p_2]$,, we can find elements of I arbitrarily close to t. Combining that to the continuity of the function $p \mapsto A_p$ we know that there is $\delta > 0$ such that

$$|A_t - A_p| < \epsilon \quad \text{as} \quad |t - p| < \delta \quad \text{and} \quad p \in I.$$

This yields

$$A_t < A_p + \epsilon \le A_{p_1}^{\alpha_p} A_{p_2}^{\beta_p} + \epsilon,$$

where $\alpha_p = \frac{p_1(p_2 - p)}{(p_2 - p_1)p}$ and $\beta_p = 1 - \alpha_p$. Now $\epsilon \to 0$ as $p \to t$, and hence

$$A_t \le A_{p_1}^{\alpha_t} A_{p_2}^{\beta_t},$$

where α_t and β_t are defined in the same way as α_p and β_p, and thus satisy the conditions

$$\frac{\alpha_t}{p_1} + \frac{\beta_t}{p_2} = \frac{1}{t} \quad \text{and} \quad \alpha_t + \beta_t = 1.$$

By the prior deduction, the condition (3.7) holds for every $p \in [p_1, p_2]$. By the definition of A_p, that means

$$\|\mathcal{T}\phi\|_p \leq A_{p_1}^{\alpha} A_{p_2}^{\beta} \|\phi\|_p$$

for $\phi \in \mathcal{L}(\Omega, \mu)$. This is the uniform estimate (3.2). $\square$

The Riesz-Thorin theorem states that the linear operator $\mathcal{T} : L^p(\Omega, \mu) \to L^p(\Omega, \mu)$ is bounded for every $p \in [p_1, p_2]$ with respect to the norm $\|\cdot\|_p$ as the conditions of the theorem hold. Therefore we can define the norm of the operator as follows:

Definition 3.8. If $\mathcal{T} : L^p(\Omega, \mu) \to L^p(\Omega, \mu)$ is a bounded linear operator, we define $\|\mathcal{T}\|_p := \sup\{\|\mathcal{T}f\|_p : f \in L^p(\Omega, \mu), \|f\|_p \leq 1\}$.

Especially, by homogeneity of the norm, we have the estimate

$$\|\mathcal{T}f\|_p \leq \|\mathcal{T}\|_p \|f\|_p$$

for every $f \in L^p(\Omega, \mu)$.

Corollary 3.9. *Under the assumptions of the Riesz-Thorin theorem, the function* $t \mapsto \ln\|\mathcal{T}\|_{1/t}$ *is convex on the interval* $[\frac{1}{p_2}, \frac{1}{p_1}]$.

Proof. Let $x_1, x_2 \in [\frac{1}{p_2}, \frac{1}{p_1}]$ and $s \in [0, 1]$. Now

$$\frac{1}{sx_1 + (1-s)x_2} \in [\frac{1}{x_2}, \frac{1}{x_1}] \subset [p_1, p_2] \quad \text{and} \quad \frac{1}{p} = \frac{\alpha}{\frac{1}{x_2}} + \frac{\beta}{\frac{1}{x_1}}$$

for $p = 1/(sx_1 + (1-s)x_2)$ and $\alpha = 1-s,\ \beta = s$. Thus the assumptions of the Riesz-Thorin theorem hold, and by (3.2) we get

$$\|\mathcal{T}\|_{\frac{1}{sx_1+(1-s)x_2}} \leq \|\mathcal{T}\|_{\frac{1}{x_1}}^{s} \|\mathcal{T}\|_{\frac{1}{x_2}}^{1-s}.$$

Since the natural logarithm function is increasing, we get

$$\ln\|\mathcal{T}\|_{\frac{1}{sx_1+(1-s)x_2}} \leq \ln(\|\mathcal{T}\|_{\frac{1}{x_1}}^{s} \|\mathcal{T}\|_{\frac{1}{x_2}}^{1-s}) = s\ln\|\mathcal{T}\|_{\frac{1}{x_1}} + (1-s)\ln\|\mathcal{T}\|_{\frac{1}{x_2}}. \qquad \square$$

Definition 3.10. Function $f : I \to \mathbb{R}$, defined on an interval $I \subset \mathbb{R}$, is *locally Lipschitz-continuous* if for every $x \in I$ there is an open interval $]a, b[\subset I$ such that $x \in]a, b[$ and $f_{|]a,b[}$ is Lipschitz-continuous.

Corollary 3.11. *Under the assumptions of the Riesz-Thorin theorem, the function* $t \mapsto \|\mathcal{T}\|_t$ *is locally Lipschitz-continuous on* $]p_1, p_2[$.

Proof. By Corollary 3.9, the function $t \mapsto \ln\|\mathcal{T}\|_{1/t}$ is convex on the interval $[\frac{1}{p_2}, \frac{1}{p_1}]$. Since the exponential map is increasing and convex, Lemma 2.14 tells us that also the function $t \mapsto e^{\ln\|\mathcal{T}\|_{1/t}} = \|\mathcal{T}\|_{1/t}$ is convex on that interval. Let $t_0 \in]p_1, p_2[$. We can choose a closed interval $[a', b'] \subset]p_1, p_2[$ such that $a' < t_0 < b'$. Now $\frac{1}{t_0} \in [\frac{1}{b'}, \frac{1}{a'}] \subset]\frac{1}{p_2}, \frac{1}{p_1}[$, and the function $t \mapsto \|\mathcal{T}\|_{1/t}$ is Lipschitz-continuous on the interval $[\frac{1}{b'}, \frac{1}{a'}]$ by Lemma 2.15. Since the function $t \mapsto \frac{1}{t}$ maps the interval $[a', b']$ onto the interval $[\frac{1}{b'}, \frac{1}{a'}]$ and is continuously differentiable on the closed interval $[a', b']$, its derivative is bounded and hence the function is Lipschitz-continuous. As a composition of these Lipschitz-functions, the function $t \mapsto \|\mathcal{T}\|_t$ is Lipschitz-continuous particularly on the open interval $]a', b'[\ni t_0$. Hence the claim holds. $\square$

The Riesz-Thorin theorem is a powerful tool: once we know that an operator is a bounded linear operator in $L^{p_j}(\Omega, \mu)$ for $j \in \{1, 2\}$, $1 \leq p_1 \leq p_2 < \infty$, we know also that it is bounded with respect to every p in the closed interval defined by p_1 and p_2. This means, if we are able to prove the boundedness of an operator in two certain points p_1 and p_2 relatively easily, then we get the boundedness in the interval for free. The theorem is widely used in Fourier analysis, for example.

However, linearity is quite a strict assumption. In addition, even though the linearity holds, the theorem is useless in many cases in practice: if the boundedness is clear with respect to two points p_1, p_2, it is that often with respect to any other value of p, too.

Chapter 4

The Marcinkiewicz interpolation theorem

In this chapter we present another interpolation theorem. Now we give up the requirement of linearity with respect to L^{p_1}- and L^{p_2}-norms and consider the case where the interpolating operator satisfies so called weak-type estimates.

Definition 4.1. If $1 \leq p_1 < p_2 < \infty$, we define the space $L^{p_1}(\Omega, \mu) + L^{p_2}(\Omega, \mu)$ consisting of functions f which can be split as $f = f_1 + f_2$, where $f_i \in L^{p_i}$ for $i \in \{1, 2\}$.

Definition 4.2. Let $\mathcal{M}$ be the space of measurable functions on Ω. An operator $\mathcal{T} : L^{p_1}(\Omega, \mu) + L^{p_2}(\Omega, \mu) \to \mathcal{M}$ is *subadditive* if $|\mathcal{T}(f + g)| \leq |\mathcal{T}f| + |\mathcal{T}g|$ pointwise.

Definition 4.3. Let $g : \Omega \to \mathbb{V}$ be measurable. The function

$$\lambda : [0, \infty[\to [0, \infty], \ \lambda(t) = \mu(\{z \in \Omega : |g(z)| > t\}),$$

is called the distribution function of g.

Clearly the distribution function is decreasing. Hence, as a monotone function, it is measurable.

Lemma 4.4. *Let $f : \Omega \to \mathbb{V}$ be measurable, $0 < p < \infty$ and λ be the distribution function of f. Then*

$$\int_\Omega |f|^p d\mu = p \int_0^\infty t^{p-1} \lambda(t) dt.$$

Proof. Since the integrands are measurable non-negative real valued functions, we can apply the Fubini theorem to change the integrating order. Hence

$$p \int_0^\infty t^{p-1} \lambda(t) dt = p \int_0^\infty t^{p-1} \left(\int_\Omega \chi_{\{x : |f(x)| > t\}}(x) d\mu \right) dt$$

$$= \int_\Omega \int_0^\infty pt^{p-1}\chi_{\{x:|f(x)|>t\}}(x)dtd\mu = \int_\Omega \int_0^{|f(x)|} pt^{p-1}dtd\mu = \int_\Omega |f(x)|^p d\mu. \qquad \square$$

We present yet another lemma, which will be used in further analysis of the Marcinkiewicz theorem.

Lemma 4.5. *(Chebyshev's inequality) Let $f \in L^p(\Omega,\mu)$. Then the estimate*

$$\lambda(t) \le \left(\frac{\|f\|_p}{t}\right)^p$$

holds for every $t > 0$.

Proof. Let $t > 0$ be fixed. Now

$$t^p\lambda(t) = t^p\mu(\{z \in \Omega : |f(z)| > t\}) = \int_{\{z:|f(z)|>t\}} t^p d\mu \le \int_\Omega |f(z)|^p d\mu = \|f\|_p^p.$$

Dividing by t^p, the claim follows. $\square$

Now we can present and proof the Marcinkiewicz interpolation theorem for subadditive operators on L^p-spaces.

Theorem 4.6. *Let $\mathcal{T}$ be a subadditive operator on $L^{p_1}(\Omega,\mu) + L^{p_2}(\Omega,\mu)$ for $1 \le p_1 < p_2 < \infty$ such that*

$$\lambda(t) \le \left(\frac{A_j}{t}\|g\|_{pj}\right)^{p_j} \tag{4.7}$$

holds for every $g \in L^{p_j}(\Omega,\mu)$, $j \in \{1,2\}$, where λ is the distribution function of $\mathcal{T}g$ and A_j is a constant for $j \in \{1,2\}$. Then for all $p \in \,]p_1,p_2[$

$$\|\mathcal{T}f\|_p \le A_p\|f\|_p, \qquad f \in L^p(\Omega,\mu), \tag{4.8}$$

where A_p is a constant depending on p, A_j and p_j for $j \in \{1,2\}$. Moreover, we have the bound

$$\|\mathcal{T}\|_p \le A_p \le \left(\frac{pA_1^{p_1}}{\alpha_1^{p_1}(p-p_1)} + \frac{pA_2^{p_2}}{\alpha_2^{p_2}(p_2-p)}\right)^{1/p} \tag{4.9}$$

for the p-norm of the operator $\mathcal{T}$, where α_j are any positive numbers with $\alpha_1 + \alpha_2 = 1$.

Proof. Let $f \in L^p(\Omega,\mu)$ and α_2, α_2 be such that $\alpha_1 + \alpha_2 = 1$. For fixed $t \in \,]0,\infty[$ we split $f = f_1 + f_2$, where

$$f_1(z) = \begin{cases} f(z) & \text{if } f(z) > t \\ 0 & \text{if } f(z) \le t \end{cases} \quad \text{and} \quad f_2(z) = \begin{cases} 0 & \text{if } f(z) > t \\ f(z) & \text{if } f(z) \le t \end{cases}.$$

Now

$$\int_\Omega |f_1|^{p_1} d\mu = \int_{\{z:|f(z)|>t\}} |f|^{p_1} d\mu = t^{p_1} \int_{\{z:\frac{|f(z)|}{t}>1\}} |\frac{f}{t}|^{p_1} d\mu$$

$$\leq t^{p_1} \int_{\{z:\frac{|f(z)|}{t}>1\}} |\frac{f}{t}|^{p} d\mu \leq \frac{t^{p_1}}{t^p} \int_\Omega |f|^p d\mu < \infty.$$

By similar calculation, $\int_\Omega |f_2|^{p_2} d\mu < \infty$. Therefore $f_1 \in L^{p_1}(\Omega,\mu)$ and $f_2 \in L^{p_2}(\Omega,\mu)$.

Let λ be the distribution function of $\mathcal{T}f$. By subadditivity of $\mathcal{T}$ and the assumption (4.7) we get

$$\lambda(t) = \mu(\{z \in \Omega : |\mathcal{T}f(z)| > t\}) = \mu(\{z \in \Omega : |\mathcal{T}(f_1+f_2)(z)| > (\alpha_1+\alpha_2)t\})$$

$$\leq \mu(\{z \in \Omega : |\mathcal{T}f_1(z)| + |\mathcal{T}f_2(z)| > \alpha_1 t + \alpha_2 t\})$$

$$\leq \mu(\{z \in \Omega : |\mathcal{T}f_1(z)| > \alpha_1 t\} \cup \{z \in \Omega : |\mathcal{T}f_2(z)| > \alpha_2 t\})$$

$$\leq \mu(\{z \in \Omega : |\mathcal{T}f_1(z)| > \alpha_1 t\}) + \mu(\{z \in \Omega : |\mathcal{T}f_2(z)| > \alpha_2 t\})$$

$$\leq \frac{A_1^{p_1}}{\alpha_1^{p_1} t^{p_1}} \int_\Omega |f_1|^{p_1} d\mu + \frac{A_2^{p_2}}{\alpha_2^{p_2} t^{p_2}} \int_\Omega |f_2|^{p_2} d\mu$$

$$= \frac{A_1^{p_1}}{\alpha_1^{p_1} t^{p_1}} \int_{\{z:|f(z)|>t\}} |f|^{p_1} d\mu + \frac{A_2^{p_2}}{\alpha_2^{p_2} t^{p_2}} \int_{\{z:|f(z)|<t\}} |f|^{p_2} d\mu.$$

Using this estimate, by Lemma 4.4 and Fubini's theorem we get

$$\|\mathcal{T}f\|_p^p = \int_\Omega |\mathcal{T}f|^p d\mu = p \int_0^\infty t^{p-1} \lambda(t) dt$$

$$\leq \frac{pA_1^{p_1}}{\alpha_1^{p_1}} \int_0^\infty t^{p-1-p_1} \left(\int_{\{z:|f(z)|>t\}} |f|^{p_1} d\mu \right) dt + \frac{pA_2^{p_2}}{\alpha_2^{p_2}} \int_0^\infty t^{p-1-p_2} \left(\int_{\{z:|f(z)|<t\}} |f|^{p_2} d\mu \right) dt$$

$$= \frac{pA_1^{p_1}}{\alpha_1^{p_1}} \int_0^\infty t^{p-1-p_1} \left(\int_\Omega |f|^{p_1} \chi_{\{z:|f(z)|>t\}} d\mu \right) dt + \frac{pA_2^{p_2}}{\alpha_2^{p_2}} \int_0^\infty t^{p-1-p_2} \left(\int_\Omega |f|^{p_2} \chi_{\{z:|f(z)|<t\}} d\mu \right) dt$$

$$= \frac{pA_1^{p_1}}{\alpha_1^{p_1}} \int_\Omega |f|^{p_1} \left(\int_0^\infty t^{p-1-p_1} \chi_{\{z:|f(z)|>t\}} dt \right) d\mu + \frac{pA_2^{p_2}}{\alpha_2^{p_2}} \int_\Omega |f|^{p_2} \left(\int_0^\infty t^{p-1-p_2} \chi_{\{z:|f(z)|<t\}} dt \right) d\mu$$

$$= \frac{pA_1^{p_1}}{\alpha_1^{p_1}} \int_\Omega |f|^{p_1} \left(\int_0^{|f|} t^{p-1-p_1} dt \right) d\mu + \frac{pA_2^{p_2}}{\alpha_2^{p_2}} \int_\Omega |f|^{p_2} \left(\int_{|f|}^\infty t^{p-1-p_2} dt \right) d\mu$$

$$= \frac{pA_1^{p_1}}{\alpha_1^{p_1}} \int_\Omega |f|^{p_1} \left(\frac{|f|^{p-p_1}}{p-p_1} \right) d\mu + \frac{pA_2^{p_2}}{\alpha_2^{p_2}} \int_\Omega |f|^{p_2} \left(\lim_{a\to\infty} \left(\frac{a^{p-p_2}}{p-p_2} - \frac{|f|^{p-p_2}}{p-p_2} \right) \right) d\mu$$

$$= \frac{pA_1^{p_1}}{\alpha_1^{p_1}(p-p_1)} \int_\Omega |f|^p d\mu + \frac{pA_2^{p_2}}{\alpha_2^{p_2}(p_2-p)} \int_\Omega |f|^p d\mu$$
$$= \left(\frac{pA_1^{p_1}}{\alpha_1^{p_1}(p-p_1)} + \frac{pA_2^{p_2}}{\alpha_2^{p_2}(p_2-p)} \right) \|f\|_p^p.$$

Hence we have
$$\|\mathcal{T} f\|_p \leq \left(\frac{pA_1^{p_1}}{\alpha_1^{p_1}(p-p_1)} + \frac{pA_2^{p_2}}{\alpha_2^{p_2}(p_2-p)} \right)^{1/p} \|f\|_p,$$
which proves the claim. □

The assumptions of the Marcinkiewicz theorem are slightly weaker than the assumptions of the Riesz-Thorin theorem. This can be seen as follows: If $\mathcal{T}$ is a linear operator and $x \in \Omega$, we have

$$|\mathcal{T}(f+g)(x)| = |Tf(x) + \mathcal{T}g(x)| \leq |\mathcal{T}f(x)| + |\mathcal{T}g(x)|$$

by triangle inequality. Thus a linear operator is always subadditive. Clearly, if $1 \leq p_1 < p_2 < \infty$, then $L^{p_j}(\Omega, \mu) \subset L^{p_1}(\Omega, \mu) + L^{p_2}(\Omega, \mu)$ for $j \in \{1, 2\}$, since we can choose either L^{p_1}- or L^{p_2}-part of the function $f \in L^{p_j}(\Omega, \mu)$ to be zero. Moreover, Chebyshev's inequality shows us that for every $t > 0$

$$\mu(\{z \in \Omega : |\mathcal{T}f(z)| > t\}) \leq \left(\frac{\|\mathcal{T}f\|_{p_j}}{t} \right)^{p_j} \leq \left(\frac{\|\mathcal{T}\|_{p_j}}{t} \|f\|_{p_j} \right)^{p_j}$$

as $j \in \{1, 2\}$. Thus, the estimate (4.7) holds for $f \in L^{p_1}(\Omega, \mu) \cap L^{p_2}(\Omega, \mu)$. Hence the assumptions of the Marcinkiewicz theorem hold.

The Marcinkiewicz theorem guarantees us boundedness of an operator by slightly weak assumptions. However, this boundedness only holds in an open interval $]p_1, p_2[$, contrary to by applying the Riesz-Thorin theorem. All in all, the power of the Marcinkiewicz theorem is in its weak assumptions, compared to the Riesz-Thorin theorem. Both of the theorems have proved to be essential in analysis research.

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