A coarse-grained Schrödinger cat

Johannes KOFLER and Časlav BRUKNER
Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, Boltzmanngasse 3, 1090 Wien, Austria;
Fakultät für Physik, Universität Wien, Boltzmanngasse 5, 1090 Wien, Austria

Abstract. We show that under coarse-grained measurements there is no observational difference between a quantum superposition of macroscopically distinct states ("Scrödinger-cat states") and a classical mixture of these states. Since normally our observations in every-day life are of limited accuracy, no quantum features can be observed. Remarkably, the information gain in such classical coarse-grained measurements is only half of the maximal information gain in sharp quantum measurements. This suggests a novel approach to macroscopic realism and classical physics within quantum theory.

Keywords. Schrödinger cat, coarse-grained measurements, information, randomness

Introduction

Quantum physics is in conflict with a classical world view both conceptually and mathematically. The assumptions of a genuine classical world — local realism and macroscopic realism — are at variance with quantum mechanical predictions as characterized by the violation of the Bell and Leggett–Garg inequality, respectively [1,2]. Does this mean that the classical world is substantially different from the quantum world? When and how do physical systems stop to behave quantumly and begin to behave classically? Questions like this date back to Schrödinger's "burlesque" *Gedankenexperiment* of a cat in a "hell machine" which becomes entangled with the microscopic state of a radioactive atom [3]. If, as was Schrödinger's point, quantum mechanics also applies for all macroscopic pieces of the apparatus together with the unfortunate cat, the superposition also includes the cat's states of "dead" and "alive".

In order to explain the fact that we do not see such macroscopic superpositions in our every-day experience, the opinions in the physics community still differ dramatically. Various views range from the mere experimental difficulty of sufficiently isolating any system from its environment (decoherence) [4] to the principal impossibility of superpositions of macroscopically distinct states due to the breakdown of quantum physical laws at some quantum-classical border (collapse models) [5].

Assuming that quantum physics is universally valid, could the quantum features of a Schrödinger cat state be lost, even if the cat is arbitrarily well isolated from its surrounding such that decoherence effects are negligible? Inspired by the thoughts of Peres [6], we give an affirmative answer, putting the stress on the limited observability of quantum phenomena, which is due to the restricted accuracy of the apparatuses used to measure the cat. To show quantum features of larger and larger objects better and better measurement accuracy is needed. But due to the finiteness of resources in any laboratory the accuracy of measurement devices is limited. The larger the objects the harder it is to distinguish between macroscopic superpositions and classical mixtures. If the required measurement precision is not met, *the classical world emerges as a coarse-grained view onto a fully quantum world* [7].

In this work we will illustrate our approach using the example of a Schrödinger kitten — a coherent superposition state of a spin of length $j = 10$.

1. Coarse-grained measurements

Any (pure or mixed) spin-j density matrix can be written in the diagonal representation [8]

$$\hat{\rho} = \iint_{\Omega} P(\vartheta, \varphi)\, |\vartheta, \varphi\rangle\langle\vartheta, \varphi|\, \mathrm{d}^2\Omega\,, \tag{1}$$

with $\mathrm{d}^2\Omega \equiv \sin\vartheta\, \mathrm{d}\vartheta\, \mathrm{d}\varphi$ the infinitesimal solid angle element and P a *not necessarily positive* real function (normalization $\iint_{\Omega} P(\vartheta, \varphi)\, \mathrm{d}^2\Omega = 1$). The spin-$j$ coherent states

$$|\vartheta, \varphi\rangle = \sum_{m=-j}^{j} \binom{2j}{j+m}^{1/2} \cos^{j+m}\tfrac{\vartheta}{2}\, \sin^{j-m}\tfrac{\vartheta}{2}\, \mathrm{e}^{-\mathrm{i}m\varphi}\, |m\rangle \tag{2}$$

are the eigenstates of the spin operator pointing in the (ϑ, φ)-direction, whereas $|m\rangle$ denotes the eigenstates of the spin operator's z-component. Choosing $P = \frac{1}{4\pi}$, e.g., would result in the totally mixed state.

Without loss of generality we make a measurement of the spin's z-component. The probability for an outcome m in a $\hat{J}_z$ measurement in the state (1) is denoted by $p(m)$. At the coarse-grained level of classical physics only the probability for a "slot" outcome $\bar{m}$ (containing many neighboring m) can be measured, i.e., $\bar{p}(\bar{m}) \equiv \sum_{m \in \{\bar{m}\}} p(m)$ with $\{\bar{m}\}$ the set of all m belonging to $\bar{m}$. This can be well approximated by [7]

$$\bar{p}(\bar{m}) = \int_0^{2\pi} \int_{\vartheta_1(\bar{m})}^{\vartheta_2(\bar{m})} P(\vartheta, \varphi) \sin\vartheta\, \mathrm{d}\vartheta\, \mathrm{d}\varphi\,, \tag{3}$$

where $\vartheta_1(\bar{m})$ and $\vartheta_2(\bar{m})$ are the borders of the polar angle region corresponding to a projection onto $\bar{m}$. We will show that $\bar{p}(\bar{m})$ can be obtained from a positive probability distribution of classical spin vectors which emerges from the P-function. Consider the Q-function

$$Q(\vartheta, \varphi) \equiv \frac{2j+1}{4\,\pi} \iint_{\Omega'} P(\vartheta', \varphi')\, \cos^{4j}\tfrac{\Theta}{2}\, \mathrm{d}^2\Omega' \tag{4}$$

with $\mathrm{d}^2\Omega' \equiv \sin\vartheta'\, \mathrm{d}\vartheta'\, \mathrm{d}\varphi'$ and $\Theta = 2\arccos\{\frac{1}{2}\,[1 + \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\varphi - \varphi')]\}^{1/2}$ the angle between the directions (ϑ, φ) and (ϑ', φ'). The Q-function is *positive* because it is, up to a normalization factor, the expectation value $\mathrm{Tr}[\hat{\rho}\, |\vartheta, \varphi\rangle\langle\vartheta, \varphi|]$ of the state $|\vartheta, \varphi\rangle$ (see Ref. [9]). In the case of large spins the factor $\cos^{4j}\frac{\Theta}{2}$ in the integrand is sharply peaked around vanishing relative angle Θ and significant contributions arise only from regions where $\Theta \lesssim 1/\sqrt{j}$. The normalization factor $\frac{2j+1}{4\pi}$ in eq. (4) is the inverse size of the solid angle element for which the integrand contributes significantly and makes Q normalized: $\iint_\Omega Q(\vartheta, \varphi)\, \mathrm{d}^2\Omega = 1$.

The probability for having an outcome $\bar{m}$ can now be expressed only in terms of the classical distribution Q [7]:

$$\bar{p}(\bar{m}) = \int_0^{2\pi} \int_{\vartheta_1(\bar{m})}^{\vartheta_2(\bar{m})} Q(\vartheta, \varphi) \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi \,. \tag{5}$$

The equivalence of eqs. (3) and (5) is shown by substituting eq. (4) into (5) under the constraint that the angular spread $\Delta\Theta \sim \vartheta_2(\bar{m}) - \vartheta_1(\bar{m})$ of the slot $\bar{m}$ is much larger than $1/\sqrt{j}$. Hence, for coarse-grained measurements with angular inaccuracy $\Delta\Theta \gg 1/\sqrt{j}$, a *full description* of the (quantum) situation is given by an *ensemble of classical spins with the positive and normalized probability distribution* Q [7].

2. A Schrödinger spin kitten

Let us consider a superposition of two spin-j coherent states,

$$|\psi\rangle_{\text{sup}} = c\left(|\vartheta_1, \varphi_1\rangle + \mathrm{e}^{\mathrm{i}\alpha}|\vartheta_2, \varphi_2\rangle\right). \tag{6}$$

As the spin coherent states are overcomplete, the normalization constant reads $c = 1/\sqrt{2\,(1 + \mathrm{Re}(\mathrm{e}^{\mathrm{i}\alpha}\langle\vartheta_1, \varphi_1|\vartheta_2, \varphi_2\rangle)}$. For comparison reasons we also consider the statistical mixture of the two coherent states

$$\hat{\rho}_{\text{mix}} = \frac{|\vartheta_1, \varphi_1\rangle\langle\vartheta_1, \varphi_1| + |\vartheta_2, \varphi_2\rangle\langle\vartheta_2, \varphi_2|}{2}. \tag{7}$$

Using eq. (2), the density matrices $\hat{\rho}_{\text{sup}} = |\psi\rangle_{\text{sup}}\langle\psi|$ and $\hat{\rho}_{\text{mix}}$ can now be easily written in the form

$$\hat{\rho} = \sum_{m=-j}^{j} \sum_{m'=-j}^{j} c_{m,m'}\, |m\rangle\langle m'|\,, \tag{8}$$

where the coefficients $c_{m,m'}$ depend on the given angles $(\vartheta_1, \varphi_1; \vartheta_2, \varphi_2)$ and differ of course for the states (6) and (7). The P-function reads [9]

$$P(\vartheta, \varphi) = \sum_{k=0}^{2j} \sum_{q=-k}^{k} \rho_{kq}\, Y_{kq}(\vartheta, \varphi)\, \frac{(-1)^{k-q}\sqrt{(2j-k)!\,(2j+k+1)!}}{\sqrt{4\pi}\,(2j)!}\,. \tag{9}$$

Here, Y_{kq} are the spherical harmonics and

$$\rho_{kq} = \sqrt{2k+1} \sum_{m=-j}^{j} (-1)^{j-m}\, c_{m,m-q} \begin{pmatrix} j & k & j \\ -m+q & -q & m \end{pmatrix}, \tag{10}$$

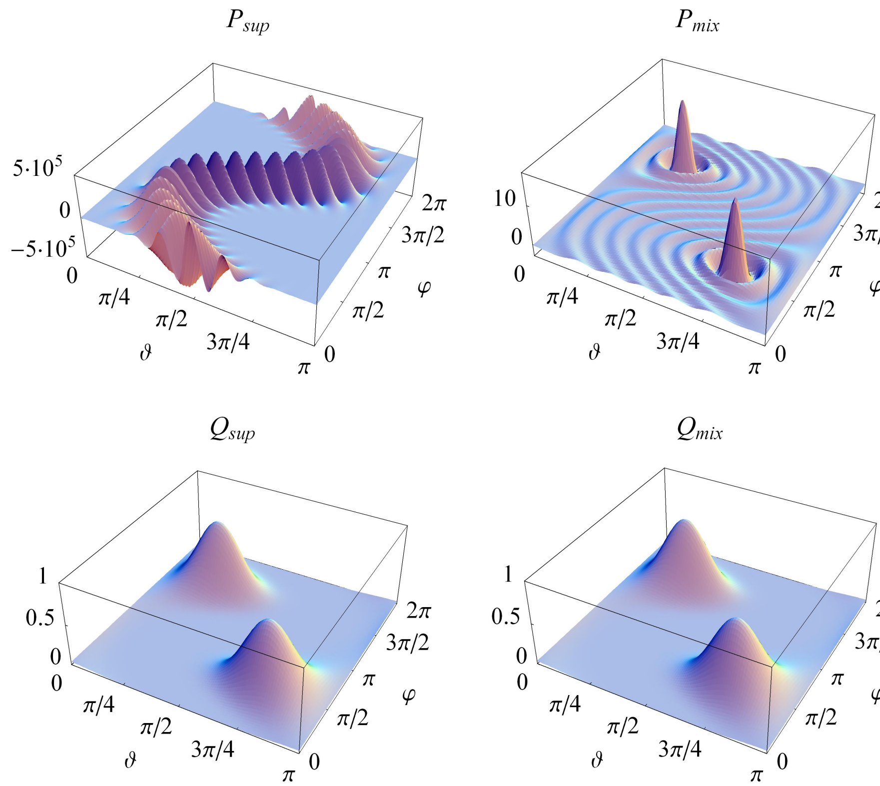

Figure 1. (Top left) The P-function P_{sup} of the superposition state (6) for $j = 10$, $\alpha = 0$, $\vartheta_1 = \frac{\pi}{4}$, $\varphi_1 = \frac{3\pi}{2}$, $\vartheta_2 = \frac{3\pi}{4}$, $\varphi_2 = \frac{\pi}{2}$. Is is wildly oscillating with very large positive and negative regions. (Top right) The P-function P_{mix} of the corresponding statistical mixture (7). (Bottom left and right) If the angular measurement resolution of our apparatuses, $\Delta\Theta$, is much weaker than $1/\sqrt{j}$, we cannot distinguish anymore between the states (6) and (7), as both lead to the same (positive) Q-functions $Q_{\text{sup}} = Q_{\text{mix}}$. Under such coarse-grained measurements both states can be seen as an ensemble of classical spin vectors in which half of the spins are pointing into the direction (ϑ_1, φ_1) and the other half into (ϑ_2, φ_2).

where the last bracket denotes the Wigner $3j$ symbol. The Q-function can be either found through integration of P, i.e. eq. (4), or via the representation [9]

$$Q(\vartheta,\varphi) = \frac{2j+1}{4\pi} \sum_{k=0}^{2j} \sum_{q=-k}^{k} \rho_{kq}\, Y_{kq}(\vartheta,\varphi)\, \frac{(-1)^{k-q}\sqrt{4\pi}\,(2j)!}{\sqrt{(2j-k)!\,(2j+k+1)!}}\,. \tag{11}$$

Let us choose the size of our Schrödinger kitten (6), i.e. the spin length, as $j = 10$ (the numerical computation of P and Q for much larger values of j is extremely time consuming). Furthermore, we set $\alpha = 0$ and choose the angles $\vartheta_1 = \frac{\pi}{4}$, $\varphi_1 = \frac{3\pi}{2}$, $\vartheta_2 = \frac{3\pi}{4}$, $\varphi_2 = \frac{\pi}{2}$ such that the two coherent states $|\vartheta_1, \varphi_1\rangle$ and $|\vartheta_2, \varphi_2\rangle$ point into opposite directions. The P-function of this superposition, P_{sup}, is shown in Fig. 1 at the top left. Is is wildly oscillating with very large positive and negative regions (note the scale in the plot). We show at the top right the P-function P_{mix} of the corresponding statistical mixture (7) which shows two pronounced peaks at (ϑ_1, φ_1) and (ϑ_2, φ_2) and slightly negative regions.

If the angular measurement resolution of our apparatuses, $\Delta\Theta$, is much weaker than $1/\sqrt{j}$, i.e. $\Delta\Theta \gg 1/\sqrt{j}$, we cannot distinguish anymore between the states (6) and (7), i.e. between a superposition and a statistical mixture, as both lead to the same Q-functions, denoted as Q_{sup} and Q_{mix}, respectively, which are shown in Fig. 1 at the bottom. Under such coarse-grained measurements both states can be seen as an ensemble of classical spins with the unique (positive) distribution $Q_{\text{sup}} = Q_{\text{mix}}$. The latter shows just two peaks, centered at the directions (ϑ_1, φ_1) and (ϑ_2, φ_2), corresponding to a classical mixture in which half of the spins are pointing into the direction (ϑ_1, φ_1) and the other half into (ϑ_2, φ_2). Interestingly, it is the heavily oscillating regions of the P_{sup}-function which vanish in the coarse-graining procedure. Its two small peaks along (ϑ_1, φ_1) and (ϑ_2, φ_2) cannot be seen on this scale.

Going to larger and larger values of j, i.e. from kittens to cats, makes it more and more difficult to observe the quantum nature of superposition states like eq. (6). The angular resolution which is necessary to distinguish a superposition from the corresponding classical mixture is of the order of $1/\sqrt{j}$.

3. Information and randomness

If we consider measurements of the spin operator's z-component in the state (6), coarse-grained measurements correspond to the fact that we cannot resolve individual eigenvalues m but only whole bunches of size Δm. The "classicality condition" $\Delta\Theta \gg 1/\sqrt{j}$ corresponds to $\Delta m \gg \sqrt{j}$. In a measurement with perfect resolution an individual out of the $2j + 1 \approx 2j$ possible results carries $\log_2(2j) = 1 + \log_2 j$ bits of information. Under coarse-grained measurements the finding that an outcome lies in a certain bunch of size $\Delta m = c\sqrt{j}$ with $c \gg 1$ carries only $\log_2(\frac{2j}{c\sqrt{j}}) = 1 - \log_2 c + \frac{1}{2}\log_2 j$ bits of information. For large j, i.e. $j \gg c \gg 1$, the information gain in a sharp quantum measurement is approximately $\log_2 j$ bits, whereas in the classical case it is only half of that, namely $\frac{1}{2}\log_2 j$ bits.

Finally, we note that — given coarse-grained measurements — it is objectively random which of the two directions, (ϑ_1, φ_1) or (ϑ_2, φ_2), one will find in a spin measurement in the Schrödinger cat state (6). Classical physics emerges out of the quantum world but the randomness in the classical mixture is still irreducible. Which possibility becomes factual is objectively random and does not have a causal reason.

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