Critical values of L-functions of residual representations of GL_4

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Abstract

In this paper we prove rationality results of critical values for L-functions attached to representations in the residual spectrum of $\mathrm{GL}_4(\mathbb{A})$. We use the Jacquet-Langlands correspondence to describe their partial L-functions via cuspidal automorphic representations of the group $\mathrm{GL}_2'(\mathbb{A})$ over a quaternion algebra. Using ideas inspired by results of Grobner and Raghuram we are then able to compute the critical values as a Shalika period up to a rational multiple.

1 Introduction

Let $\mathbb{D}$ be a division algebra over a totally real number field $\mathbb{F}$, which is non-split at every place at infinity. Denote by $\Sigma_{\mathbb{D}}$ the set of places where $\mathbb{D}$ splits and by $\mathbb{A} = \mathbb{A}_{\mathbb{F}}$ the adeles. We let $M_{n,n}$ be the algebraic variety of $n \times n$ matrices over $\mathbb{F}$ and GL_n be the general linear group over $\mathbb{F}$. Similarly, let $M'_{n,n}$ be the variety of $n \times n$ matrices with coefficients in $\mathbb{D}$ and let $\mathrm{GL}'_{n,\mathbb{D}} = \mathrm{GL}'_n$ be the group of invertible matrices in $M'_{n,n}$, where we see both varieties as algebraic groups over $\mathbb{F}$. In [15] the authors proved certain rationality results of critical values of the L-function of cohomological cuspidal irreducible automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A})$, which admit a Shalika model. The goal of this paper is to extend these results to non-cuspidal discrete series representations of $\mathrm{GL}_4(\mathbb{A})$ by lifting them from cuspidal irreducible representations of $\mathrm{GL}'_2(\mathbb{A})$ by use of the Jacquet-Langlands correspondence $|\mathrm{JL}|$, see [3].

To be more precise, let

$$\mathcal{S} := \Delta \mathrm{GL}'_n \rtimes U'_{(n,n)} = \left\{ \begin{pmatrix} h & X \\ 0 & h \end{pmatrix} : h \in \mathrm{GL}'_n,\, X \in M'_{n,n} \right\}$$

be the Shalika subgroup of GL'_{2n}. As in the split case we say that an irreducible cuspidal automorphic representation π' of $\mathrm{GL}'_{2n}(\mathbb{A})$ with central character ω admits a Shalika model with respect to a character η, if $\eta^n = \omega$ and if the Shalika period

$$\mathcal{S}^{\eta}_{\psi}(\phi)(g) := \int_{Z'_{2n}(\mathbb{A})\mathcal{S}(\mathbb{F})\backslash \mathcal{S}(\mathbb{A})} \phi(sg)\, \psi\left(\mathrm{Tr}'(X)\right)^{-1} \eta\left(\det{}'(h)\right)^{-1} \mathrm{d}s \neq 0$$

does not vanish for some $\phi \in \pi'$ and $g \in \mathrm{GL}'_{2n}(\mathbb{A})$. Here we write $s = \begin{pmatrix} h & X \\ 0 & h \end{pmatrix}$ and denote by Tr' and $\det'$ the trace and determinant maps of $M'_{n,n}$.

In the split case, *i.e.* $\mathbb{D} = \mathbb{F}$, it is well know that π' admits a Shalika model with respect to η if and only if the twisted partial exterior square L-function $L^S\left(s, \pi', \wedge^2 \otimes \eta^{-1}\right)$ has a pole at $s = 1$.

In the non-split case there is currently no analogous theorem known, however, in the special case $n = 1$ and $\mathbb{D}$ a quaternion division algebra the following was proved in [10].

Theorem 1 ([10, Theorem 1.3]). *Assume $\mathbb{D}$ is a quaternion division algebra and π' a cuspidal irreducible automorphic representation of $\mathrm{GL}_2'(\mathbb{A})$. If $|\mathrm{JL}|(\pi')$ is cuspidal and irreducible, the following assertions are equivalent.*

1. *π' admits a Shalika model with respect to η.*
2. *The twisted partial exterior square L-function $L^S\left(s, \pi', \wedge^2 \otimes \eta^{-1}\right)$ has a pole at $s = 1$ and for all $v \in \Sigma_{\mathbb{D}}$, π_v' is not isomorphic to a parabolically induced representation*

$$\mathrm{Ind}_{P_{(1,1)}}^{\mathrm{GL}_2(\mathbb{F}_v)}\left(|\det'|^{\frac{1}{2}}\tau_1 \otimes |\det'|^{-\frac{1}{2}}\tau_2\right),$$

where τ_i' are representations of $\mathrm{GL}_1'(\mathbb{A})$ with central character η_v.

If $|\mathrm{JL}|(\pi')$ is not cuspidal, $|\mathrm{JL}|(\pi')$ is a quotient of a representation $|\det|^{\frac{1}{2}}\tau \times |\det|^{-\frac{1}{2}}\tau$ for some cuspidal irreducible representation τ of $\mathrm{GL}_2(\mathbb{A})$. Then the following assertions are equivalent.

1. *π' admits a Shalika model with respect to η.*
2. *The central character ω_τ of τ equals η.*
3. *The twisted partial exterior square L-function $L^S\left(s, \pi', \wedge^2 \otimes \eta^{-1}\right)$ has a pole at $s = 2$.*

For the rest of the introduction assume $\mathbb{D}$ is a quaternion algebra, we fix

$$\mathrm{GL}_{2,\infty}' := \prod_{v \in V_\infty} \mathrm{GL}_2'(\mathbb{F}_v),\ K_\infty' = \prod_{v \in V_\infty} \mathrm{Sp}(2)\,\mathbb{R}_{\geq 0}$$

and let π' be an irreducible cuspidal cohomological automorphic representation of $\mathrm{GL}_2'(\mathbb{A})$ with respect to a coefficient system $E_\mu^\vee$, *i.e.* $E_\mu^\vee$ is a highest weight representation of $\mathrm{GL}_{2,\infty}'$ such that

$$\dim_{\mathbb{C}} H^*\left(\mathfrak{g}_\infty', K_\infty', \pi_\infty' \otimes E_\mu^\vee\right) \neq 0.$$

Note that for a cuspidal π' and $\sigma \in \mathrm{Aut}(\mathbb{C})$ one can define the σ-twist ${}^\sigma\pi_f'$ of the finite part of π_f'. Following [14] we extended this to a σ-twist ${}^\sigma\pi'$ of π', which is a discrete series representation of $\mathrm{GL}_2'(\mathbb{A})$. In [14] it was shown that if moreover $|\mathrm{JL}|(\pi')$ is cuspidal, ${}^\sigma\pi'$ is again cuspidal. We prove that the assumption of $|\mathrm{JL}|(\pi')$ being cuspidal is not necessary and extend their argument using the Mœglin-Waldspurger classification to the case when $|\mathrm{JL}|(\pi')$ is residual. Using the above criterion for admitting a Shalika model, we see that if π' admits a Shalika model then so does ${}^\sigma\pi'$. Let $\mathbb{Q}\left(\pi_f'\right)$ be the field fixed by the automorphisms fixing π_f'. In [14] it was shown that $\mathbb{Q}\left(\pi_f'\right)$ is a number field and that $\mathbb{Q}\left(\pi_f'\right) = \mathbb{Q}\left(|\mathrm{JL}|(\pi')_f\right)$. Following [17], [15] we define a finite extension $\mathbb{Q}(\pi', \eta)$ of $\mathbb{Q}\left(\pi_f'\right)$ and a $\mathbb{Q}(\pi', \eta)$-structure on the Shalika model $\mathcal{S}_{\psi_f}^{\eta_f}\left(\pi_f'\right)$ of π_f'.

As in [15] we will make use of a *numerical coincidence*, which is together with Theorem 1, Theorem 6.2.2 and Theorem 4.7.1 the reason why we must limit ourselves to the case $\mathbb{D}$ being quaternion and $n = 1$. Let q_0 be the lowest degree in which the $(\mathfrak{g}', K'_\infty)$-cohomology of $\pi'_\infty \otimes E^\vee_\mu$ does not vanish. Then $q_0 = \dim_{\mathbb{Q}} \mathbb{F}$ and

$$\dim_{\mathbb{C}} H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \pi'_\infty \otimes E^\vee_\mu\right) = 1.$$

Fixing a basis vector of this one-dimensional space, we define an isomorphism

$$\Theta_\pi \colon \mathcal{S}^{\eta_f}_{\psi_f}\left(\pi'_f\right) \to H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \pi' \otimes E^\vee_\mu\right),$$

where the right hand side inherts a $\mathbb{Q}(\pi', \eta)$-structure from its geometric realization as automorphic cohomology. Thus we can normalize the above isomorphism by a factor $\omega\left(\pi'_f\right)$, the so called Shalika period, such that it respects the $\mathbb{Q}(\pi', \eta)$-structures of both sides. Analogously to [15] we compute how $\omega\left(\pi'_f\right)$ behaves under twisting with a Hecke character χ of $\mathrm{GL}_1(\mathbb{A})$ lifted to $\mathrm{GL}'_2(\mathbb{A})$ via the determinant map. Let $\mathcal{G}\left(\chi_f\right)$ be the Gauss sum of χ_f. Then

$$\sigma\left(\frac{\omega\left(\pi_f \otimes \chi_f\right)}{\mathcal{G}\left(\chi_f\right)^4 \omega\left(\pi_f\right)}\right) = \frac{\omega\left({}^\sigma\pi_f \otimes {}^\sigma\chi_f\right)}{\mathcal{G}\left({}^\sigma\chi_f\right)^4 \omega\left({}^\sigma\pi_f\right)}$$

for $\sigma \in \mathrm{Aut}(\mathbb{C})$.

The next ingredient is the Shalika zeta-integral of [9] extended to $\mathrm{GL}'_2(\mathbb{A})$,

$$\zeta(s, \phi) := \int_{\mathrm{GL}'_1(\mathbb{A})} S^\eta_\psi(\phi)\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix}\right) |\det(g_1)|^{s-\frac{1}{2}}\, \mathrm{d}g_1$$

and its local analogs. As in [15] we fix a special vector $\xi^0_{\pi_f} \in \mathcal{S}^{\eta_f}_{\psi_f}\left(\pi_f\right)$ such that

$$\zeta_v\left(\frac{1}{2}, \xi^0_{\pi_f}\right) = L\left(\frac{1}{2}, \pi_v\right)$$

if v is a finite place at which ψ and π' are unramified. By [9] the period integral over $H'_1 = \mathrm{GL}'_1 \times \mathrm{GL}'_1$ of a cusp form is precisely the Shalika zeta integral. To show the invariance of this period integral under the action of a Galois group, we first interpret it as an instance of Poincaré duality of the top cohomology group of the space

$$\mathbf{S}^{H'_1}_{K_f} = H'_1(\mathbb{F}) \backslash H'_1(\mathbb{A}) / \left(K'_\infty \cap H'_{1\infty}\right) \iota^{-1}\left(K_f\right),$$

where $\iota \colon H'_1 \hookrightarrow \mathrm{GL}'_2$ is the block-diagonal embedding and K_f a small enough open compact subgroup of $\mathrm{GL}'_2\left(\mathbb{A}_f\right)$. To make the whole story work it is crucial that $\dim_{\mathbb{R}} \mathbf{S}^{H'_1}_{K_f} = q_0$, which only works if we restrict ourselves to the case $n = 1$ and $\mathbb{D}$ being a quaternion algebra, the aforementioned numerical coincidence. Since we assume that $|\mathrm{JL}|(\pi')$ is residual, we then compute that the critical values of $L(s, \pi')$ are all half-integers $s = \frac{1}{2} + m$, $m \in \mathbb{Z}$. We must moreover assume that for the weight μ there exists an integer p such that for all infinite places v of $\mathbb{F}$ the v-th component $\mu_v = (\mu_{v,1}, \ldots, \mu_{v,4})$ of μ satisfies $-\mu_{v,2} \leq p \leq -\mu_{v,3}$. In this case we call μ *admissible* and we say $\frac{1}{2}$ is compatible with μ if we can choose $p = 0$.

Since we assume $\mathbb{F}$ to be totally real, we show as in [15] that a certain representation $E_{(0,-w)}$ of H_1' appears in the coefficient system $E_\mu^\vee$ of π_∞' if $\frac{1}{2}$ is compatible with μ, which in turn lets us map the fixed special vector $\xi_{\pi_f}^0$ first to

$$H^{q_0}\left(\mathfrak{g}_\infty', K_\infty', \mathcal{S}_\psi^\eta(\pi)\otimes E_\mu^\vee\right)$$

and then interpret it as an element of

$$H_c^{q_0}\left(\mathbf{S}_{K_f}^{H_1'}, \mathcal{E}_\mu^\vee\right),$$

which we than map to

$$H_c^{q_0}\left(\mathbf{S}_{K_f}^{H_1'}, \mathcal{E}_{(0,-w)}\right)$$

using the map from above, where $\mathcal{E}_\mu^\vee$ and $\mathcal{E}_{(0,-w)}$ are the sheaves on $\mathbf{S}_{K_f}^{H_1'}$ associated to $E_\mu^\vee$ and $E_{(0,-w)}$. Finally, applying Poincaré duality to this last space, we show that the resulting number is essentially the value of the L-function $L(s,\pi')$ at $s=\frac{1}{2}$. Now the final result of [15] for critical values of the L-function follows analogously in our case, namely if $s=\frac{1}{2}+m$ and μ is admissible, there exist periods $\omega\left(\pi_f'\right)$ and $\omega\left(\pi_\infty', m\right)$ such that

$$\sigma\left(\frac{L\left(\frac{1}{2}+m,\pi_f'\otimes\chi_f\right)}{\omega\left(\pi_f'\right)\mathcal{G}\left(\chi_f\right)^4\omega\left(\pi_\infty',m\right)}\right)=\frac{L\left(\frac{1}{2}+m,{}^\sigma\pi_f'\otimes{}^\sigma\chi_f\right)}{\omega\left({}^\sigma\pi_f'\right)\mathcal{G}\left({}^\sigma\chi_f\right)^4\omega\left(\pi_\infty',m\right)}$$

for all $\sigma\in\mathrm{Aut}\left(\mathbb{C}/\mathbb{Q}\left(\pi',\eta\right)\right)$. Let $\mathbb{Q}\left(\pi',\eta,\chi\right)$ be the compositum of $\mathbb{Q}\left(\pi',\eta\right)$ and $\mathbb{Q}\left(\chi\right)$. This implies that

$$\frac{L\left(\frac{1}{2}+m,\pi_f'\otimes\chi_f\right)}{\omega\left(\pi_f'\right)\mathcal{G}\left(\chi_f\right)^4\omega\left(\pi_\infty',m\right)}\in\mathbb{Q}\left(\pi',\eta,\chi\right)$$

and hence, proves the main result.

Theorem 2. *Let π be a non-cuspidal discrete series representation of $\mathrm{GL}_4(\mathbb{A})$ written as $\pi\cong\mathrm{MW}(\tau|\det'|^{\frac{1}{2}}\times\tau|\det'|^{-\frac{1}{2}})$ via the Mœglin-Waldspurger classification, where τ is a cuspidal irreducible representation of $\mathrm{GL}_2(\mathbb{A})$. Assume moreover that there exists an irreducible cuspidal cohomological representation π' of $\mathrm{GL}_2'(\mathbb{A})$ with $|\mathrm{JL}|(\pi')=\pi$ which is cohomological with respect to coefficient system $E_\mu^\vee$ and μ is admissible. Let χ be a finite order Hecke-character of $\mathrm{GL}_1(\mathbb{A})$ and $s=\frac{1}{2}+m$ a critical value of $L(s,\pi')$. Then*

$$\frac{L\left(\frac{1}{2}+m,\pi_f\otimes\chi_f\right)}{\omega\left(\pi_f'\right)\mathcal{G}\left(\chi_f\right)^4\omega\left(\pi_\infty',m\right)}\in\mathbb{Q}\left(\pi',\omega_\tau,\chi\right).$$

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2 Preliminaries

2.1 Let $\mathbb{F}$ be a totally real number field of dimension $\dim_{\mathbb{Q}} \mathbb{F} = r$ with ring of integers $\mathbb{O}$ and let $\mathbb{D}$ be a central division algebra of dimension $\dim_{\mathbb{F}} \mathbb{D} = d^2$ over $\mathbb{F}$ which is non-split at all infinite places. To be more precise, let $V = V_\infty \cup V_f$ be the set of nontrivial places of $\mathbb{F}$, let V_∞ be the infinite ones and let V_f be the finite ones. For each place v of $\mathbb{F}$ we denote by $\mathbb{F}_v$ the corresponding completion of $\mathbb{F}$ and set $\mathbb{D}_v := \mathbb{D} \otimes \mathbb{F}_v$. Then $\mathbb{D}_v \cong M_{r_v}(A_v)$, where A_v is a central division algebra of dimension d_v^2 over $\mathbb{F}_v$ and $d_v r_v = d$. If $A_v = \mathbb{F}_v$ we call v a split place of $\mathbb{D}$. We let $\Sigma_{\mathbb{D}}$ be the set of non-split places of $\mathbb{D}$ and assume that $V_\infty \subseteq \Sigma_{\mathbb{D}}$. Note this implies that d is even and for $v \in V_\infty$ $\mathbb{D}_v = M_{\frac{d}{2}}(\mathbb{H})$, where $\mathbb{H}$ denotes the Hamilton quaternions. As usual, denote by $\mathbb{A} := \mathbb{A}_{\mathbb{F}}$ the adeles and by $\mathbb{A}_f := \mathbb{A}_{\mathbb{F},f}$ the finite adeles. The cardinality of the residue field of $\mathbb{F}_v$ is denoted by q_v.

2.2 Let $M_{n,n}$ be the variety whose $\mathbb{F}$ points are the $n \times n$ matrices with entries in $\mathbb{F}$ and let $M'_{n,n}$ be the variety whose $\mathbb{F}$ points are the $n \times n$ matrices with entries in $\mathbb{D}$. For $A \in M'_n(\mathbb{F})$ define $\det'(A)$ as follows. Choose an isomorphism $\phi \colon M'_n(\mathbb{F}) \otimes \mathbb{C} \to M_{nd}(\mathbb{C})$ and set $\det'(A) := \det(\phi(A \otimes 1))$. This map is independent of the chosen isomorphism and defined over $\mathbb{F}$. Thus, we can extend it to a map $\det' \colon M'_{n,n} \to M_{1,1}$. Similarly we define a trace map $\mathrm{Tr}'(A) := \mathrm{Tr}(\phi(A \otimes 1))$ and extended it to a map $\mathrm{Tr}' \colon M'_{n,n} \to M_{1,1}$. For $v \in V_f$ we extended the valuation v of $\mathbb{F}_v$ to A_v by

$$v'(x) := \frac{1}{d_v} v\left(\mathrm{Nrm}_{A_v/\mathbb{F}_v}(x)\right).$$

and define the ring of integers of A_v as

$$\mathbb{O}'_v := \{x \in A_v : v'(x) \geq 0\}.$$

Let $\mathfrak{D}$ be the absolute different of $\mathbb{F}$, *i.e.* $\mathfrak{D}^{-1} := \{x \in \mathbb{F} : \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}(x\mathbb{O}) \subseteq \mathbb{Z}\}$. Having fixed all of these conventions, let us define the reductive $\mathbb{F}$-group

$$\mathrm{GL}'_n := \{A \in M'_{n,n} : \det'(A) \neq 0\}.$$

We also define

$$\mathrm{GL}'_{n,\infty} := \prod_{v \in V_\infty} \mathrm{GL}'_n(\mathbb{F}_v) \cong \prod_{v \in V_\infty} \mathrm{GL}_{\frac{nd}{2}}(\mathbb{H}), \; H'_n := \mathrm{GL}'_n \times \mathrm{GL}'_n$$

and denote the center of GL'_n by $Z'_n \cong \mathrm{GL}_1$. Let us fix the maximal split torus of GL_n respectively GL'_n given by diagonal matrices respectively their image in GL'_n. For both groups, we fix the Borel subgroup B_n respectively minimal parabolic subgroup B'_n defined over $\mathbb{F}$ of upper triangular matrices. For the rest of the text a parabolic subgroup of GL_n respectively GL'_n will mean a standard parabolic subgroup defined over $\mathbb{F}$ containing B_n respectively B'_n. Recall that in both cases these parabolic subgroups are parameterized by ordered partitions of n. In other words, to $\alpha = (\alpha_1, \ldots, \alpha_k)$ a partition of n we associate the parabolic subgroup P_α respectively P'_α of GL_n respectively GL'_n containing the upper triangular matrices and having as a Levi-component $M_\alpha = \mathrm{GL}_{\alpha_1} \times \ldots \times \mathrm{GL}_{\alpha_k}$ and unipotent component U_α respectively Levi-component $M'_\alpha = \mathrm{GL}'_{\alpha_1} \times \ldots \times \mathrm{GL}'_{\alpha_k}$ and unipotent component U'_α.

We will also fix a non-trivial additive character

$$\psi : \mathbb{F}\backslash\mathbb{A} \to \mathbb{C}^*$$

as follows. First define

$$\psi_{\mathbb{Q}} := \left(\psi_{\mathbb{R}} \otimes \bigotimes_{p \text{ prime}} \psi_p\right) : \mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}} \to \mathbb{C}^*$$

by

$$\psi_{\mathbb{R}}(t) := e^{-2\pi i t},\ \psi_p(t) := e^{2\pi i \lambda_p(t)},\ \lambda\left(\sum_{k\in\mathbb{Z}} a_k p^k\right) = \sum_{k<0} a_k p^k.$$

By composing $\psi_{\mathbb{Q}}$ with the trace map $\mathrm{Tr} : \mathbb{A}_{\mathbb{F}} \to \mathbb{A}_{\mathbb{Q}}$, we obtain the desired character $\psi : \mathbb{F}\backslash\mathbb{A} \to \mathbb{C}^*$. In later sections, we need ψ to be compatible with $\mathfrak{D}$ in the following sense. If we write $\mathfrak{D} = \prod_{v\in V_f} \mathfrak{p}_v^{r_v}$, where $\mathfrak{p}_v$ is the prime ideal of $\mathbb{O}$ corresponding to v, then the conductor of ψ_v is precisely $\mathfrak{p}_v^{-r_v}$. Thus, the finite places where ψ ramifies correspond precisely to the prime ideals $\mathfrak{p}$ not dividing $\mathfrak{D}$.

2.3 For $v \in V_\infty$, let $Z'_{n,v}$ be the center of $\mathrm{GL}'_n(\mathbb{F}_v)$ and let K'_v be the product of the maximal compact subgroup of $\mathrm{GL}'_n(\mathbb{F}_v)$ and connected component of $Z'_{n,v}$, *i.e.*

$$K'_v := \mathrm{Sp}\left(\frac{nd}{2}\right)\mathbb{R}_{>0},\ K'_\infty := \prod_{v\in V_\infty} K'_v.$$

Similarly, we fix for $v \in V_\infty$

$$K_v := \mathrm{SO}(n)\mathbb{R}_{>0},\ K_\infty := \prod_{v\in V_\infty} K_v.$$

Moreover, we fix also compact subgroups K_v of $\mathrm{GL}'_n(\mathbb{F}_v)$ for $v \in V_f$ as follows. Note that $\mathrm{GL}'_n(\mathbb{F}_v)$ consists of invertible $nd_v \times nd_v$ matrices with entries in A_v. We then let K_v be those matrices in $\mathrm{GL}'_n(\mathbb{F}_v) = \mathrm{GL}_{nd_v}(A_v)$ which have entries in $\mathbb{O}'_v$. Denote for $v \in V_\infty$ by $\mathfrak{g}'_v$ the Lie algebra of $\mathrm{GL}'_n(\mathbb{F}_v)$ and by $\mathfrak{g}'_\infty$, the Lie algebra of $\mathrm{GL}'_{n,\infty}$.

To ensure that the periods we will consider in later sections are well defined, we will also have to fix a Haar measure on $H'_n(\mathbb{F}_v)$ for all $v \in V_f$. We do this by setting the volumes of the two copies of K_v in $H'_n(\mathbb{F}_v)$ with respect to the measures to 1. Taking the product of those measures over all $v \in V_f$, we obtain a Haar measure $\mathrm{d}_f g_1 \times \mathrm{d}_f g_2$ on $H'_n(\mathbb{A}_f)$. This in turn determines the volume

$$c := \mathrm{vol}\left(Z'_{2n}(\mathbb{F})\backslash Z'_{2n}(\mathbb{A})/\mathbb{R}^r_{>0}\right) = 2^r \cdot \mathrm{vol}\left(\overbrace{\mathbb{F}^*\backslash\mathbb{A}_f^* \times \ldots \times \mathbb{F}^*\backslash\mathbb{A}_f^*}^{r}\right).$$

Now for $v \in V_\infty$ let $\mathrm{d}_v g_1$ and $\mathrm{d}_v g_2$ be the Haar measures on the two copies of $\mathrm{GL}'_n(\mathbb{F}_v)$ such that $\mathrm{Sp}\left(\frac{nd}{2}\right) \subseteq \mathrm{GL}_{\frac{nd}{2}}(\mathbb{H})$ has volume 1. Set

$$\mathrm{d}_\infty g_1 := c \prod_{v\in V_\infty} \mathrm{d}_v g_1,\ \mathrm{d}_\infty g_2 := \prod_{v\in V_\infty} \mathrm{d}_v g_2,$$

which then gives a Haar measure $dg_1 \times dg_2$ on $H_n'(\mathbb{A})$ where $dg_i := \mathrm{d}_f g_i \, \mathrm{d}_\infty g_i, i = 1, 2$.

2.4 Let us now fix our notations regarding automorphic representation and automorphic forms. We call an irreducible $(\mathfrak{g}_\infty', K_\infty', \mathrm{GL}_n'(\mathbb{A}))$-subquotient π' of the space of automorphic forms $\mathcal{A}(\mathrm{GL}_n'(\mathbb{F}) \backslash \mathrm{GL}_n'(\mathbb{A}))$ on $\mathrm{GL}_n'(\mathbb{A})$ an (irreducible) automorphic representation of $\mathrm{GL}_n'(\mathbb{A})$. We call π' *cuspidal* if it is generated by a cusp form ϕ, *i.e.* an automorphic form ϕ such that

$$\int_{U(\mathbb{F})\backslash U(\mathbb{A})} \phi(ng)\, dn = 0,$$

for all for all $g \in \mathrm{GL}_n'(\mathbb{A})$ and all non-trivial parabolic subgroups P of GL_n' with Levi-decomposition $P = MU$.

Let

$$\omega \colon Z_n'(\mathbb{F}) \backslash Z_n'(\mathbb{A}) \to \mathbb{C}$$

be a continous, unitary character. We let $L^2(\mathrm{GL}_n'(\mathbb{F}) \backslash \mathrm{GL}_n'(\mathbb{A}), \omega)$ be the Hilbert-space with respect to the L^2-norm of the space of all measureable functions

$$f \colon \mathrm{GL}_n'(\mathbb{F}) \backslash \mathrm{GL}_n'(\mathbb{A}) \to \mathbb{C}$$

such that $f(zg) = \omega(z) f(g)$ for $z \in Z_n'(\mathbb{A}), g \in \mathrm{GL}_n'(\mathbb{A})$ and f is square-integrable as a function on $Z_n'(\mathbb{A})\mathrm{GL}_n'(\mathbb{F}) \backslash \mathrm{GL}_n'(\mathbb{A})$. This is a representation of $\mathrm{GL}_n'(\mathbb{A})$ via the right regular action. If $\tilde{\pi}$ is an irreducible subrepresentation of $L^2(\mathrm{GL}_n'(\mathbb{F}) \backslash \mathrm{GL}_n'(\mathbb{A}), \omega)$, we will denote by $\tilde{\pi}^\infty$ the smooth vectors in $\tilde{\pi}$, *cf.* [13, Chapter 11]. Moreover, the subspace of smooth, K_∞'-finite vectors in $\tilde{\pi}$ carries the structure of a $(\mathfrak{g}_\infty', K_\infty', \mathrm{GL}_n'(\mathbb{A}))$-module. The automorphic representations which can be obtained in this way will be called *discrete series representations* and every cuspidal representation is a discrete series representation. If it is clear from context, we will implicitly use the representation π' if we talk about $(\mathfrak{g}_\infty', K_\infty', \mathrm{GL}_n'(\mathbb{A}))$-modules and the corresponding representation $\tilde{\pi}$ if we talk about $\mathrm{GL}_n'(\mathbb{A})$-representations. From now on we mean by a *representation* π_v' of $\mathrm{GL}_n(\mathbb{F}_v)$ or $\mathrm{GL}_n'(\mathbb{F}_v)$ a representation which is smooth and admissible.

Coming with those two ways of looking at a discrete series representation π', we have two ways of writing it as a restricted tensor product, *cf.* [13, Chapter 14]. We again denote by $\tilde{\pi}$ the corresponding subrepresentation of $L^2(\mathrm{GL}_n'(\mathbb{F}) \backslash \mathrm{GL}_n'(\mathbb{A}), \omega)$. Then the smooth vectors $\tilde{\pi}^\infty$ admit a decomposition

$$\tilde{\pi}^\infty \cong \overline{\bigotimes_{v \in V_\infty}}_{\mathrm{pr}} \tilde{\pi}_v^\infty \otimes_{\mathrm{in}} {\bigotimes_{v \in V_f}}' \tilde{\pi}_v^\infty,$$

where $\overline{\bigotimes}_{\mathrm{pr}}$ denotes taking the completed projective tensor product, $\otimes_{\mathrm{in}}$ denotes the inductive tensor product and $\tilde{\pi}_v^\infty$ are $\mathrm{GL}_n'(\mathbb{F}_v)$-representations. For $v \in V_\infty$, taking K_v'-finite vectors gives a $(\mathfrak{g}_v', K_v')$-module π_v'. This gives us a second decomposition $\pi' \cong \bigotimes_{v \in V}' \pi_v'$, which now is a restricted tensor product of $(\mathfrak{g}_\infty', K_\infty')$- respectively $\mathrm{GL}_n'(\mathbb{F}_v)$-modules. Throughout the paper we will therefore mean $(\tilde{\pi}_v)^\infty$ if we treat π_v' as a $\mathrm{GL}_n'(\mathbb{F}_v)$-representation. We denote by

$S_{\pi'} \subseteq V_f$ the finite set of places where π' ramifies. The central character of π' will be denoted by $\omega = \omega_{\pi'}$.

2.5 We denote by $\delta_{P'_\alpha}$ the modulus character of P'_α and for $v \in V$ and (ρ, W), $\rho = \rho_1 \otimes \ldots \otimes \rho_k$ an irreducible representation of $M'_\alpha(\mathbb{F}_v)$ we set

$$\operatorname{ind}_{P'_\alpha(\mathbb{F}_v)}^{\mathrm{GL}'_n(\mathbb{F}_v)}(\rho) :=$$

$$= \{f \in C^\infty(G, W) : f(mng) = \rho(m) f(g), m \in M'_\alpha(\mathbb{F}_v), n \in U'_\alpha(\mathbb{F}_v), g \in \mathrm{GL}'_n(\mathbb{F}_v)\}, \qquad (1)$$

on which $\mathrm{GL}'_n(\mathbb{F}_v)$ acts by right translation. We call this space the parabolically induced representation of ρ. If $v \in V_\infty$ we equip $\operatorname{ind}_{P'_\alpha(\mathbb{F}_v)}^{\mathrm{GL}'_n(\mathbb{F}_v)}$ with the subspace topology induced from the Fréchet space $C^\infty(\mathrm{GL}'_n(\mathbb{F}_v), W)$. The space

$$\operatorname{Ind}_{P'_\alpha(\mathbb{F}_v)}^{\mathrm{GL}'_n(\mathbb{F}_v)}(\rho) = \rho_1 \times \ldots \times \rho_k := \operatorname{ind}_{P'_\alpha(\mathbb{F}_v)}^{\mathrm{GL}'_n(\mathbb{F}_v)}\left(\rho \otimes \delta_{P'_\alpha}^{\frac{1}{2}}\right)$$

is called the normalized parabolically induced representation.

If (ρ, W) is a discrete series representation of $M'_\alpha(\mathbb{A})$ with corresponding $M'_\alpha(\mathbb{A})$-representation $(\widetilde{\rho}, \widetilde{W})$ and μ a character of $P'_\alpha(\mathbb{A})$ we define $\operatorname{Ind}_{P'_\alpha(\mathbb{A})}^{\mathrm{GL}'_n(\mathbb{A})}(\rho \otimes \mu)$ to be the space of smooth functions $f\colon \mathrm{GL}'_n(\mathbb{A}) \to \widetilde{W}^\infty$ satisfying the normalized global analogon of the equivariance condition (1). The so obtained space admits a natural topology with which the $\mathrm{GL}'_n(\mathbb{A})$-action by right translations is continuous. It admits a decomposition

$$\operatorname{Ind}_{P'_\alpha(\mathbb{A})}^{\mathrm{GL}'_n(\mathbb{A})}(\rho \otimes \mu) \cong \overline{\bigotimes_{v \in V_\infty}}_{\mathrm{pr}} \operatorname{Ind}_{P'_\alpha(\mathbb{F}_v)}^{\mathrm{GL}'_n(\mathbb{F}_v)}((\tilde{\rho_v} \otimes \mu_v)^\infty) \otimes_{\mathrm{in}} {\bigotimes_{v \in V_f}}' \operatorname{Ind}_{P'_\alpha(\mathbb{F}_v)}^{\mathrm{GL}'_n(\mathbb{F}_v)}(\rho_v \otimes \mu_v).$$

Similarly, we define for GL_n parabolic and normalized parabolic induction.

2.6 We will now quickly recall the basic notions of the Jacquet-Langlands correspondence. For a complete discussion see [2], [3]. Let $v \in V$ be a place and recall that to each irreducible unitary representation of π_v of $\mathrm{GL}_{dn}(\mathbb{F}_v)$ respectively π'_v of $\mathrm{GL}'_n(\mathbb{F}_v)$ we can associate a trace character χ_{π_v} respectively $\chi_{\pi'_v}$. Moreover, for $g \in \mathrm{GL}_{dn}(\mathbb{F}_v)$ and $g' \in \mathrm{GL}'_n(\mathbb{F}_v)$ semisimple, we write $g \leftrightarrow g'$ if they have the same characteristic polynomial. Following [8] we call π_v d_v-compatible if there exists a unitary irreducible representation π'_v of $\mathrm{GL}'_n(\mathbb{F}_v)$ such that

$$\chi_{\pi_v}(g) = \epsilon(\pi_v)\chi_{\pi'_v}(g) \text{ if } g \leftrightarrow g',$$

where $\epsilon(\pi_v) \in \{-1, 1\}$.

Let $U'_{cp}(\mathrm{GL}_{dn}(\mathbb{F}_v))$ be the set of unitary d_v-compatible irreducible representations of $\mathrm{GL}_{dn}(\mathbb{F}_v)$ and let $U'(\mathrm{GL}'_n(\mathbb{F}_v))$ be the set of unitary irreducible representations of $\mathrm{GL}'_n(\mathbb{F}_v)$. Moreover, let $U_{cp}(\mathrm{GL}_{dn}(\mathbb{F}_v))$ respectively $U(\mathrm{GL}'_n(\mathbb{F}_v))$ be the set of representations of the form $\pi \otimes |\det|^s$ respectively $\pi' \otimes |\det'|^s$ for $\pi \in U'_{cp}(\mathrm{GL}_{dn}(\mathbb{F}_v))$ respectively $\pi' \in U'(\mathrm{GL}'_n(\mathbb{F}_v))$. Then there exists a map called the local Jacquet-Langlands correspondence

$$|\mathrm{LJ}|_v\colon U_{cp}(\mathrm{GL}_{dn}(\mathbb{F}_v)) \to U\left(\mathrm{GL}'_n(\mathbb{F}_v)\right)$$

with the following properties, see [8]:

1. If $\pi_v = \widetilde{\pi}_v \otimes |\det'|^s$ with $\widetilde{\pi}_v$ a unitary d_v-compatible irreducible representations of $\mathrm{GL}_{dn}(\mathbb{F}_v)$,

$$|\mathrm{LJ}|_v(\pi_v) = |\mathrm{LJ}|_v(\widetilde{\pi}_v) \otimes |\det'|^s.$$

2. If v is a split place of $\mathbb{D}$, $|\mathrm{LJ}|_v$ is the identity.
3. $|\mathrm{LJ}|_v$ restricted to square integrable representations is a bijection onto the square integrable representations of $\mathrm{GL}'_n(\mathbb{F}_v)$.
4. $|\mathrm{LJ}|_v$ commutes with parabolic induction.

2.7 Similarly, there is a global correspondence going from the unitary discrete series representations of $\mathrm{GL}'_n(\mathbb{A})$ into the set of unitary discrete series representations of $\mathrm{GL}_{nd}(\mathbb{A})$ which is denoted by $|\mathrm{JL}|$ and called the global Jacquet-Langlands correspondence. It satisfies the following properties:

1. $|\mathrm{LJ}|_v((|\mathrm{JL}|(\pi'))_v) \cong \pi'_v$ for all $v \in V$.
2. $|\mathrm{JL}|$ is injective.
3. If $|\mathrm{JL}|(\pi')$ is cuspidal, then π' is cuspidal.

Crucially, if π' is cuspidal $|\mathrm{JL}|(\pi')$ does not have to be cuspidal.

2.8 We also recall the following well-known description of discrete series representations called the Mœglin-Waldspurger classification.

Theorem 2.8.1 ([2],[20]). *Let $k, l \in \mathbb{Z}_{\geq 1}$, $n = lk$ and τ' be a cuspidal unitary automorphic representation of $\mathrm{GL}'_l(\mathbb{A})$. Then the parabolically induced $\mathrm{GL}'_n(\mathbb{A})$-representation*

$$\tau'|\det'|^{\frac{k-1}{2}} \times \ldots \times \tau'|\det'|^{\frac{1-k}{2}}$$

has a unique irreducible quotient, denoted by $\mathrm{MW}(\tau', k)$. It is a discrete series representation of $\mathrm{GL}'_n(\mathbb{A})$ and moreover for every discrete series representation π' of $\mathrm{GL}'_n(\mathbb{A})$, there exists l, k and τ' as above such that $\pi' \cong \mathrm{MW}(\tau', k)$. The analogous statement for GL_n instead of GL'_n holds also true.

3 Cohomological automorphic representation

3.1 Let T_∞ be the maximal split torus of $\mathrm{GL}_{n,\infty}$ consisting of diagonal matrices and $X(T_\infty)$ its characters. The choice of the Borel subgroup of upper triangular matrices fixes a set of simple roots. Let $X^+(T_\infty)$ be the set of dominant algebraic characters of T_∞, which parameterize the algebraic finite-dimensional representations of $\mathrm{GL}_{n,\infty}$ via the highest weight correspondence. For a weight $\mu = (\mu_v)_{v \in V_\infty} \in X^+(T_\infty)$ we write for each place $v \in V_\infty$ $\mu_v = (\mu_{v,1}, \ldots, \mu_{v,n})$ in the coordinates obtained coming from our fixed maximal diagonal torus, where $\mu_{v,1} \geq \ldots \geq \mu_{v,n}$. The representation corresponding to a highest weight μ will be denoted by $E_\mu = \bigotimes_{v \in V_\infty} E_{\mu_v}$. We call E_μ essentially self-dual if it is essentially self-dual at every place *i.e.* $E_{\mu_v} \cong E_{\mu_v}^\vee \otimes |\det'|^{w_v}$ for some w_v and self dual if all $w_v = 0$. Since $\mathbb{F}$ is totally real, E_{μ_v} is essentially self-dual if and only if

$$\mu_{v,l} + \mu_{v,n-l+1} = w_v$$

for all $l \in \{1, \dots, n\}$. Similarly, we define the above notations for $\mathrm{GL}'_{n,\infty}$ by passing to the split form $\mathrm{GL}_{dn,\infty}$. The representation E_μ is called regular if μ lies in the interior of the dominant Weyl chamber of $\mathrm{GL}'_{n,\infty}$, *i.e.* if for all $v \in V_\infty$ $\mu_{v,1} > \dots > \mu_{v,dn}$.

3.2 Let us now fix our notations regarding relative Lie algebra cohomology and cohomological automorphic representation. For each irreducible $(\mathfrak{g}'_\infty, K'_\infty)$-module

$$\pi'_\infty \cong \bigotimes_{v \in V_\infty} \pi'_v$$

we denote by $H^q(\mathfrak{g}'_\infty, K'_\infty, \pi'_\infty)$ the $(\mathfrak{g}'_\infty, K'_\infty)$-cohomology of degree q of π'_∞. By the Künneth formula

$$H^q(\mathfrak{g}'_\infty, K'_\infty, \pi'_\infty) \cong \bigoplus_{\sum_{v \in V_\infty} q_v = q} \bigotimes_{v \in V_\infty} H^{q_v}(\mathfrak{g}'_v, K'_v, \pi'_v)$$

A $(\mathfrak{g}', K'_\infty)$-module π'_∞ is called cohomological if there exists a highest weight representation E_μ such that $H^q(\mathfrak{g}'_\infty, K'_\infty, \pi'_\infty \otimes E_\mu)$ is nonzero for some q. We call an automorphic representation $\pi' \cong \pi'_\infty \otimes \pi'_f$ of $\mathrm{GL}_n(\mathbb{A})$ or $\mathrm{GL}'_n(\mathbb{A})$ cohomological if its archimedian component π'_∞ is cohomological. The analogous definition can be made for GL_n.

3.3 Next, we recall the classification of the cohomological irreducible unitary dual of $\mathrm{GL}_n(\mathbb{H})$ due to [27] and explicitly described in [14]. Let $\mathfrak{g}'$ be the Lie algebra of $\mathrm{GL}_n(\mathbb{H})$ and let $\mathfrak{k}'$ be the Lie algebra of $\mathrm{Sp}(n)$, which determines a Cartan involution $\theta'(X) = -\bar{X}^T$ of $\mathfrak{g}'$. Moreover, let $\mathfrak{h}'$ be a maximal compact, θ'-stable Cartan-algebra $\mathfrak{h}' = \mathfrak{a}' \oplus \mathfrak{t}'$, with

$$\mathfrak{t}' = \left\{ \begin{pmatrix} ix_1 & & 0 \\ & \ddots & \\ 0 & & ix_n \end{pmatrix} : x_j \in \mathbb{R} \right\} \text{ and } \mathfrak{a}' = \left\{ \begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix} : y_j \in \mathbb{R} \right\}.$$

Furthermore, let E_λ be a highest weight representation of $\mathrm{GL}_n(\mathbb{H})$, where λ is a highest weight with respect to the subalgebra $\mathfrak{h}'_\mathbb{C}$. To each partition $n = \sum_{i=0}^r n_i$ written as

$$\underline{n} = [n_0, \dots, n_r]$$

with $n_0 \geq 0$ and $n_i > 0$ we can associate a θ'-stable, parabolic subalgebra $\mathfrak{q}'_{\underline{n}}$ of $\mathfrak{g}'_\mathbb{C}$ whose Levi-decomposition we will denote as $\mathfrak{q}'_{\underline{n}} = \mathfrak{l}'_{\underline{n}} + \mathfrak{u}'_{\underline{n}}$, *cf.* [14, Section 4] for more details. We further assume that $\lambda|_{\mathfrak{a}'} = 0$ and that λ can be extended to an admissible character of $\mathfrak{l}'_{\underline{n}} \supseteq \mathfrak{h}'_\mathbb{C}$.

Theorem 3.3.1 ([14, Theorem 4.9]). *Let E_λ be a self-dual highest weight representation of $\mathrm{GL}_n(\mathbb{H})$.*

1. *To each ordered partition $\underline{n} = [n_0, \dots, n_r]$ of n with $n_0 \geq 0, n_i > 0$ one can assign an irreducible unitary representation $A_{\underline{n}}(\lambda)$ of $\mathrm{GL}_n(\mathbb{H})$.*
2. *All such representations are cohomological with respect to E_λ and every cohomological representation is of this form.*

3. The Poincaré polynomial of $H^*\left(\mathfrak{g}', \mathrm{Sp}(n)\mathbb{R}_{\geq 0}, E_\lambda \otimes A_{\underline{n}}(\lambda)\right)$ *is*

$$P(\underline{n}, X) = \frac{X^{\dim_{\mathbb{C}}\left(\mathfrak{g}_{\mathbb{C}}^- \cap \mathfrak{u}'_{\underline{n}}\right)}}{1+X} \prod_{i=1}^{r} \prod_{j=1}^{n_i} \left(1 + X^{2j-1}\right) \prod_{j=1}^{n_0} \left(1 + X^{4j-3}\right).$$

Here $\mathfrak{g}_{\mathbb{C}}^-$ *is the* -1*-eigenspace of* θ' *acting on* $\mathfrak{g}'_{\mathbb{C}}$*.*

For later use, we compute the following.

Lemma 3.3.2. *Let* $\underline{n} = [n_0, n_1, \ldots, n_r]$ *be a partition of* n*. Then*

$$\dim_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}}^- \cap \mathfrak{u}'_{\underline{n}}) = \sum_{i=1}^{r} \binom{n_i}{2} + 2 \sum_{0 \leq i < j \leq r} n_i n_j.$$

Proof. We first recall the definition of $\mathfrak{u}'_{\underline{n}}$, *cf.* [14, §4.2]. Let

$$x = \mathrm{diag}(\underbrace{0, \ldots, 0}_{n_0}, \underbrace{1, \ldots, 1}_{n_1}, \ldots, \underbrace{r \ldots, r}_{n_r}) \in i\mathfrak{t}'$$

and let $\Delta(\mathfrak{g}'_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$ respectively $\Delta(\mathfrak{g}^-_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$ be the set of roots coming from $\mathfrak{t}'_{\mathbb{C}}$. We have the explicit description

$$\Delta(\mathfrak{g}^-_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}}) = \{\pm e_i \pm e_j, 1 \leq i < j \leq n\},$$

where $e_j(H) = ix_j$ for $H = \mathrm{diag}(ix_1 + y_1, \ldots, ix_n + y_n)) \in \mathfrak{h}'$. Moreover,

$$\mathfrak{u}'_{\underline{n}} = \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{g}'_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}}) \\ \alpha(x) > 0}} (\mathfrak{g}'_{\mathbb{C}})_\alpha$$

and therefore

$$\mathfrak{u}'_{\underline{n}} \cap \mathfrak{g}^-_{\mathbb{C}} = \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{g}^-_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}}) \\ \alpha(x) > 0}} (\mathfrak{g}'_{\mathbb{C}})_\alpha.$$

Hence

$$\dim_{\mathbb{C}}(\mathfrak{g}^-_{\mathbb{C}} \cap \mathfrak{u}'_{\underline{n}}) = \#\{\alpha \in \Delta(\mathfrak{g}^-_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}}),\ \alpha(x) > 0\},$$

which is easily seen to be equal to the above explicit formula. □

Our next step is to showing that if τ is a cuspidal irreducible representation of $\mathrm{GL}'_n(\mathbb{A})$ and $k \in \mathbb{N}$, then τ is cohomological if $\mathrm{MW}(\tau, k)$ is. The author would like to thank Harald Grobner for pointing out the argument presented here. Before we start, we need to recall the following theorem.

Theorem 3.3.3 ([24, Theorem 1.8])**.** *Let* G *be a connected, semisimple real Lie group with finite center and Lie algebra* $\mathfrak{g}$*. Fix a maximal connected subgroup* K *of* G *with Lie algebra* $\mathfrak{k}$ *and moreover, let* π *be an irreducible unitary smooth representation of* G *with central character* χ_π*. Finally, let* U *be a finite-dimensional* $(\mathfrak{g}, K)$*-module admitting an infinitesimal character* $\chi_U = \chi_{\pi^\vee}$*. Then*

$$H^*(\mathfrak{g}, \mathfrak{k}, \pi \otimes U) \neq 0.$$

We denote for a real Lie group G by Z_G its center and by Z_G^0 the connected component of the latter.

Lemma 3.3.4. *Let $\underline{G}$ be a connected reductive group over $\mathbb{F}$, $v \in V_\infty$ and $G := \underline{G}(\mathbb{F}_v)$. Let π be an irreducible unitary representation of G and E_λ a finite dimensional highest weight representation of G over $\mathbb{C}$ such that Z_G^0, acts trivially on $E_\lambda \otimes \pi$ and $\chi_{E_\lambda} = \chi_{\pi^\vee}$. Then*

$$H^*(\mathfrak{g}, (Z_G \cdot K)^0, \pi \otimes E_\lambda) \neq 0.$$

Proof. Note that E_λ always admits a central character. Recall that

$$H^*(\mathfrak{g}, (Z_G \cdot K)^0, \pi \otimes E_\lambda) = H^*(\mathfrak{g}, \mathfrak{z}_G \oplus \mathfrak{k}, \pi \otimes E_\lambda),$$

where $\mathfrak{z}_G$ is the Lie algebra of Z_G and we use that K has finite center. Since $\mathfrak{z}_G$ acts trivially on $\pi \otimes E_\lambda$, the Künneth formula gives a decomposition

$$H^*(\mathfrak{g}, \mathfrak{z}_G \oplus \mathfrak{k}, \pi \otimes E_\lambda) \cong \bigotimes_{a+b=*} H^a(\mathfrak{z}_G, \mathfrak{z}_G, \mathbb{C}) \otimes H^b(\mathfrak{g}/\mathfrak{z}_G, \mathfrak{k}, \pi \otimes E_\lambda) = H^*(\mathfrak{g}/\mathfrak{z}_G, \mathfrak{k}, \pi \otimes E_\lambda).$$

The latter does not vanish by Theorem 3.3.3, since both π and E_λ admit the right central characters and the image of $\mathfrak{g}/\mathfrak{z}_G$ under the exponential map generates the connected, semisimple real Lie group G/Z_G . □

Let $\underline{G}$ be either GL_n or GL_n', $v \in V_\infty$, $G = \underline{G}(\mathbb{F}_v)$ and $K = K_v$ or K_v'. Moreover, let $\underline{P}$ be a standard parabolic subgroup of $\underline{G}$ and set $P = \underline{P}(\mathbb{F}_v) = L \rtimes U$. Write $L = M \times A^0$, where $A^0 = Z_L^0$. Next, let (π, V) be an irreducible, unitary representation of L. Denote now by $\mathfrak{b}_\mathbb{C}$ the complexified Cartan subalgebra of the Lie algebra of M coming from our fixed choice of Cartan subalgebra of L, *i.e.* the diagonal matrices if $\underline{G} = \mathrm{GL}_n$ or $\mathfrak{h}'$ if $\underline{G} = \mathrm{GL}_n'$. Let $\mathfrak{a}_{P,\mathbb{C}}^\vee$ be the complexified dual of the Lie-algebra of A^0 and fix $\mu \in \mathfrak{a}_{P,\mathbb{C}}^\vee$. We let $\mathfrak{p}_\mathbb{C}$ be the complexified Lie algebra of P and let ρ be the half-sum of all positive roots of $\mathfrak{p}_\mathbb{C}$ with respect to our fixed Cartan subalgebra. Denote by Δ_M the simple roots of M, W the Weyl-group of G and

$$W^P = \{w \in W : w^{-1}(\alpha) > 0 \text{ for all } \alpha \in \Delta_M\}.$$

We write

$$\mathrm{Ind}_P^G(\pi, \mu) = \{f\colon G \to V \text{ smooth} : f(maug) = a^{\rho+\mu}\pi(m)f(g),\ a \in A^0,\ m \in M,\ u \in U,\ g \in G\}$$

and use the standard parametrization of infinitesimal characters, i.e. for a highest weight representation E_λ, $\chi_{\lambda+\rho} = \chi_{E_\lambda}$.

Proposition 3.3.5. *If τ is a non-zero $(\mathfrak{g}, (Z_G \cdot K)^0)$-module, which appears as a quotient of $\mathrm{Ind}_P^G(\pi, \mu)$ and is cohomolocigal with respect to some highest weight representation $E_\lambda^\vee$, then π is cohomological as a $(\mathfrak{l}, (Z_L \cdot (L \cap K)^0)$-module with respect to $E_{w(\lambda+\rho)-\rho}^\vee$, where w is some element of W^P.*

Proof. We notice that without loss of generality A^0 acts trivially on π. Moreover, if χ_π denotes the infinitesimal character of π, $\mathrm{Ind}_P^G(\pi,\mu)$ and hence also τ have infinitesimal character $\chi_{\pi+\mu}$. On the other hand, τ is by assumption cohomological with respect to $E_\lambda^\vee$ and hence it has to have infinitesimal character $\chi_{\lambda+\rho}$ by [5, Theorem I.5.3]. Therefore the infinitesimal character of $\pi|_M$ is equal to

$$\chi_{\lambda+\rho-\mu|_{\mathfrak{b}_\mathbb{C}}}$$

and hence $\pi|_M$ has non-vanishing cohomology with respect to $E^\vee_{w(\lambda+\rho)-\rho|_{\mathfrak{b}_\mathbb{C}}}$ by Lemma 3.3.4. Now for any Konstant-representative $w \in W^P$

$$\chi_{\lambda+\rho-\mu|_{\mathfrak{b}_\mathbb{C}}} = \chi_{w(\lambda+\rho)|_{\mathfrak{b}_\mathbb{C}}}.$$

Note now that $w(\lambda+\rho)-\rho|_{\mathfrak{b}_\mathbb{C}}$ is a dominant weight, see [5, III.3.2], and hence the last character is equal to

$$\chi_{E^\vee_{w(\lambda+\rho)-\rho|_{\mathfrak{b}_\mathbb{C}}}}.$$

We can choose now w as in [5, III. Theorem 3.3] such that $Z_L^0 = A^0$ acts trivially on

$$\pi \otimes E^\vee_{w(\lambda+\rho)-\rho}.$$

Hence π is by Lemma 3.3.4 cohomological with respect to $E^\vee_{w(\lambda+\rho)-\rho}$. ☐

Corollary 3.3.5.1. *Let τ be a cuspidal irreducible representation of $\mathrm{GL}_n(\mathbb{A})$ or $\mathrm{GL}'_n(\mathbb{A})$ and $k \in \mathbb{N}$. Then τ is cohomological if $\mathrm{MW}(\tau,k)$ is cohomological.*

Proof. Since $\mathrm{MW}(\tau,k)_v^\infty$ is the quotient of

$$(\tau^\infty|\det'|^{\frac{k-1}{2}} \times \ldots \times \tau^\infty|\det'|^{\frac{1-k}{2}})_v$$

for all $v \in V_\infty$, the claim follows from Proposition 3.3.5, because

$$K'_v = \mathrm{Sp}\left(\frac{nd}{2}\right)\mathbb{R}_{>0} = (\mathrm{Sp}\left(\frac{nd}{2}\right)\mathbb{R})^0,\ K_v = \mathrm{SO}(n)\mathbb{R}_{>0} = (\mathrm{O}(n)\mathbb{R})^0.$$

☐

Lemma 3.3.6. *Assume π' is a cuspidal irreducible cohomological representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ such that $|\mathrm{JL}|(\pi')$ is not a cuspidal representation of $\mathrm{GL}_{2dn}(\mathbb{A})$. Let $2dn = kl$ and let τ be a unitary cuspidal irreducible representation of $\mathrm{GL}_l(\mathbb{A})$ such that $|\mathrm{JL}|(\pi') = \mathrm{MW}(\tau,k)$.*

Then l is even and τ_v is cohomological with respect to some highest weight representation E_{λ_v}, $\lambda_v = \left(\lambda_{v,1},\ldots,\lambda_{v,\frac{l}{2}}\right)$. For each $v \in V_\infty$, π'_v is of the form $\pi'_v = A_{\underline{nd}}(\lambda'_v)$ for

$$\underline{nd} = [0, \overbrace{k,\ldots,k}^{\frac{l}{2}}]$$

and $\lambda_{v,\frac{l}{2}} = \lambda'_{v,nd}$. *In particular, the lowest respectively highest degree in which the cohomology group*

$$H^q\left(\mathfrak{g}'_v, \mathrm{Sp}\,(nd)\,\mathbb{R}_{\geq 0}, \pi'_v \otimes E_{\lambda'_v}\right)$$

does not vanish is

$$q = nd\,(nd-1) - \frac{nd}{2}\,(k-1) \ \ \textit{respectively}\ q = nd\,(nd-1) + \frac{nd}{2}\,(k+1) - 1.$$

Proof. Fix an infinite place $v \in V_\infty$. By [14, Theorem 5.2] $\mathrm{MW}\,(\tau,k)$ is cohomological and thus by Corollary 3.3.5.1 so is τ. By [2, Theorem 18.2], $k|d$ and hence, l has to be even. Since the archimedean component of a cohomological cuspidal irreducible unitary representation of $\mathrm{GL}_l\,(\mathbb{A})$ must be tempered we may write

$$\tau_v = A_{[0,2,\ldots,2]}\,(\lambda_v)$$

and let

$$\pi'_v = A_{\underline{nd}}\left(\lambda'_v\right)$$

for suitable $\underline{nd}$ and λ_v, λ'_v, with $\underline{nd} = [n_0, n_1, \ldots, n_{l'}]$, see [14, Section 5.5]. Furthermore, τ_v is fully induced from representations of $\mathrm{GL}_2(\mathbb{R})$. To proceed with the proof, we are quickly going to recap the construction of $A_{\underline{nd}}\,(\lambda'_v)$ in the proof of [14, Theorem 5.2]. Let

$$\rho_{\mathfrak{gl}_m(\mathbb{H})} = \left(\frac{2m-1}{2}, \ldots, -\frac{2m-1}{2}\right)$$

respectively

$$\rho_{\mathfrak{gl}_m(\mathbb{C})} = \left(\left(\frac{m-1}{2}, \ldots, -\frac{m-1}{2}\right), \left(\frac{m-1}{2}, \ldots, -\frac{m-1}{2}\right)\right)$$

the smallest algebraically integral element in the interior of the dominant Weyl chamber of $\mathrm{GL}_m\,(\mathbb{H})$ respectively $\mathrm{GL}_m\,(\mathbb{C})$. Define now

$$\mu := \left(\rho_{\mathfrak{gl}_{n_0}(\mathbb{H})}, \rho_{\mathfrak{gl}_{n_1}(\mathbb{C})}, \ldots, \rho_{\mathfrak{gl}_{n_{l'}}(\mathbb{C})}\right)$$

and let P' be a certain complex parabolic subgroup of $\mathrm{GL}_{dn}\,(\mathbb{H})$, which we will specify in a moment, and having Levi-factor $\prod_{i=1}^{nd} \mathrm{GL}_1(\mathbb{H})$. For any integer $s > 0$ and $u \in \mathbb{C}$ we set

$$D\,(u,s) := D\,(s) \otimes |\det|^{-\frac{u}{2}},$$

where $D\,(s)$ is the unique irreducible discrete series representation of $\mathrm{SL}_2^{\pm}\,(\mathbb{R})$ of lowest $\mathrm{O}\,(2)$-type $s+1$. We also set

$$F\,(u,s) := F\,(s) \otimes |\det{}'|^{-\frac{u}{2}},$$

where $F\,(s)$ is the unique irreducible representation of $\mathrm{SL}_1\,(\mathbb{H})$ of dimension s. Moreover, recall the Levi decomposition $\mathfrak{q}'_{\underline{nd}} = \mathfrak{l}'_{\underline{nd}} + \mathfrak{u}'_{\underline{nd}}$ and let $\rho\,(\underline{nd}) = (\rho\,(\underline{nd})_1, \ldots, \rho\,(\underline{nd})_{nd})$ be the half-sum of all roots appearing in $\mathfrak{u}'_{\underline{nd}}$. Let $k_i = \lambda'_i + \rho\,(\underline{nd})_i$. We set

$$\sigma = \bigotimes_{i=1}^{nd-n_0} F\,(0,k_i)\,.$$

Then P' can be chosen such that (P', σ, μ) is a Langlands-datum and $A_{\underline{nd}}(\lambda'_v)$ is the unique irreducible quotient of the induced representation $\mathrm{Ind}_{P'}^{\mathrm{GL}'_{2n}(\mathbb{F}_v)}(\sigma, \mu)$. Since $\tau_v|\det'|^{\frac{k+1}{2}-j}$ is essentially tempered for every $v \in V_\infty$ and $j \in \{1, \ldots, k\}$ and τ_v is cohomological,

$$\tau_v|\det'|^{\frac{k+1}{2}-j} \cong \mathrm{Ind}_{P_j}^{\mathrm{GL}_l(\mathbb{F}_v)}(\sigma_j)$$

by [14, Section 5.5], where

$$\sigma_j = \bigotimes_{i=1}^{\frac{l}{2}} D(2j-k-1, k_{i,j})$$

for certain $k_{i,j} \in \mathbb{Z}_{>0}$ and P_j is the standard parabolic subgroup of upper triangular matrices with block size $\overbrace{(2, \ldots, 2)}^{\frac{l}{2}}$. Recall furthermore

$$A_{\underline{nd}}(\lambda'_v) = \pi'_v = |\mathrm{LJ}|_v\left(\left(|\mathrm{JL}|\left(\pi'\right)\right)_v\right) = |\mathrm{LJ}|_v(\mathrm{MW}(\tau_v, k)).$$

By [14, Theorem 5.2] and its proof the last term is equal to the Langlands quotient of

$$\mathrm{Ind}_P^{\mathrm{GL}_{nd}(\mathbb{H})}\left(\bigotimes_{j=1}^{k}\bigotimes_{i=1}^{\frac{l}{2}} F(0, k_{i,j}), \mu'\right),$$

where now P is the standard parabolic subgroup of type $\overbrace{(1, \ldots, 1)}^{nd}$ of $\mathrm{GL}_{nd}(\mathbb{H})$ and

$$\mu' = \left(\overbrace{\frac{k-1}{2}, \ldots, \frac{k-1}{2}}^{\frac{l}{2}}, \ldots, \overbrace{-\frac{k-1}{2}, \ldots, -\frac{k-1}{2}}^{\frac{l}{2}}\right).$$

Comparing μ and μ' and using the uniqueness of the Langlands quotient implies then that $n_0 = 0$ and $n_1 = \ldots = n_{l'} = k$. Moreover, by we have $\lambda_{v,\frac{l}{2}} + 1 = k_{\frac{l}{2},k} = k_{nd} = \lambda'_{v,nd} + 1$

From Lemma 3.3.2 we obtain that

$$\dim_{\mathbb{C}}\left(\mathfrak{g}_{\mathbb{C}}^- \cap \mathfrak{u}'_{\underline{nd}}\right) = 2\binom{nd}{2} - \frac{l}{2}\binom{k}{2} = nd(nd-1) - \frac{nd}{2}(k-1).$$

Therefore, by Theorem 3.3.1 the lowest degree of non-vanishing cohomology is $nd(nd-1) - \frac{nd}{2}(k-1)$ and the highest degree of non-vanishing cohomology is

$$\dim_{\mathbb{C}}\left(\mathfrak{g}_{\mathbb{C}}^- \cap \mathfrak{u}'_{\underline{nd}}\right) - 1 + \sum_{j=1}^{\frac{l}{2}}\sum_{i=1}^{k}(2i-1) = nd(nd-1) + \frac{nd}{2}(k+1) - 1.$$

□

Remark. To extend the ideas of [15] to the case $\mathrm{GL}'_{2n}(\mathbb{A})$ we need the following numerical coincidence. Namely, it will be necessary that either the lowest or highest degree in which the cohomology group

$$H^*\left(\mathfrak{g}'_v, K'_v, \pi_v \otimes E^\vee_{\lambda_v}\right)$$

does not vanish is

$$q_0 = (nd)^2 - (nd) - 1 = \dim_{\mathbb{R}}\left(H'_{\frac{nd}{2}}(\mathbb{R}) / \left(\mathrm{Sp}\left(\frac{nd}{2}\right) \times \mathrm{Sp}\left(\frac{nd}{2}\right) \times \mathbb{R}_{>0}\right)\right).$$

By Lemma 3.3.6 the only possible value for n is therefore $n = 1$, $d = 2$ and the partition $\underline{2n}$ of $2n = 2$ has to be $\underline{2n} = [0, 2]$.

3.4 Next we will recall the action of $\mathrm{Aut}(\mathbb{C})$ on representations. Let π'_f be a representation of $\mathrm{GL}'_n(\mathbb{A}_f)$ on some complex vector space W and $\sigma \in \mathrm{Aut}(\mathbb{C})$. We define the σ-twist ${}^\sigma\pi'_f$ as follows, *cf.* [28]. Let W' be a complex vector space which allows a σ-linear isomorphism $t: W' \to W$. We then set

$${}^\sigma\pi'_f := t^{-1} \circ \pi'_f \circ t.$$

An explicit example of such a space W' is the space

$$W' = W \otimes_{\mathbb{C}} {}_\sigma\mathbb{C},$$

where ${}_\sigma\mathbb{C}$ is $\mathbb{C}$ as a field but $\mathbb{C}$ acts on ${}_\sigma\mathbb{C}$ via σ^{-1}. Then W' is a $\mathbb{C}$ vector space via the right action of $\mathbb{C}$ on $\mathbb{C}$ and the map $t: W \to W \otimes_{\mathbb{C}} {}_\sigma\mathbb{C}$ is given by $w \mapsto w \otimes 1$. Similarly, we define the σ-twist ${}^\sigma\pi'_v$ of a local representation π'_v with $v \in V_f$. For a highest weight representation E_μ of $\mathrm{GL}'_{n,\infty}$, we define

$$\left({}^\sigma E_\mu\right)_v := \left(E_\mu\right)_{\sigma^{-1}\circ v},$$

where v is seen as an embedding $\mathbb{F} \hookrightarrow \mathbb{C}$ and hence, $\sigma^{-1} \circ v$ defines an infinite place of $\mathbb{F}$. For π'_f as above let

$$\mathfrak{S}\left(\pi'_f\right) := \{\sigma \in \mathrm{Aut}(\mathbb{C}) : {}^\sigma\pi'_f \cong \pi'_f\}$$

and let

$$\mathbb{Q}\left(\pi'_f\right) := \{z \in \mathbb{C} : \sigma(z) = z, \text{ for all } \sigma \in \mathfrak{S}\left(\pi'_f\right)\}$$

be the rationality field of π'_f. Analogously we define for a highest weight representation E_μ and a local representation π'_v the fields $\mathbb{Q}(E_\mu)$ and $\mathbb{Q}(\pi'_v)$. Moreover, if $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a partition of n and $\rho = \rho_1 \otimes \ldots \otimes \rho_k$ an irreducible representation of $M'_\alpha(\mathbb{F}_v)$, $v \in V_f$,

$${}^\sigma\mathrm{ind}_{P'_\alpha(\mathbb{F}_v)}^{\mathrm{GL}_n(\mathbb{F}_v)}(\rho) = \mathrm{ind}_{P'_\alpha(\mathbb{F}_v)}^{\mathrm{GL}_n(\mathbb{F}_v)}\left({}^\sigma\rho\right), \; \sigma \in \mathrm{Aut}(\mathbb{C})$$

and therefore

$${}^\sigma(\rho_1 \times \ldots \times \rho_k) = \left({}^\sigma\rho_1 \times \ldots \times {}^\sigma\rho_k\right)\epsilon_\sigma^{dn-1}, \; \epsilon_\sigma := \frac{|\det'|^{\frac{1}{2}}}{\sigma\left(|\det'|^{\frac{1}{2}}\right)} \tag{2}$$

and similarly for the split case GL_n.

Finally, we say that the representation π'_f, π'_v or E_μ with underlying vector space W is defined over some field $\mathbb{E} \subseteq \mathbb{C}$ if there exists an $\mathbb{E}$-vector space $W_{\mathbb{E}} \subseteq W$, stable under the group action of $\mathrm{GL}'_n(\mathbb{A}_f)$, $\mathrm{GL}'_n(\mathbb{F}_v)$ respectively $\prod_{v\in V_\infty}\mathrm{GL}'_n(\mathbb{F})$, such that the natural map $W_{\mathbb{E}}\otimes_{\mathbb{E}}\mathbb{C}\to W$ is an isomorphism. In this case we say W admits an $\mathbb{E}$-structure. Let E_μ be a highest weight representation and let $\mathbb{L}$ be a minimal field extension of $\mathbb{F}$ such that $\mathbb{D}$ splits over $\mathbb{L}$. Then E_μ is defined over $\mathbb{L}$, see [14, Lemma 7.1] and we set

$$\mathbb{Q}(\mu) \coloneqq \mathbb{L}\cdot\mathbb{Q}(E_\mu).$$

Lemma 3.4.1 ([7, Proposition 3.2])**.** *Let $v\in V_f$ and π'_v an irreducible representation of $\mathrm{GL}'_n(\mathbb{F}_v)$. Then π'_v admits an $\mathbb{Q}(\pi'_v)$-structure.*

Note that in the reference the lemma is only proven in the case GL_n. However, the proof carries over analogously, since the Langlands classification via multisegments used in it is also valid for GL'_n.

Theorem 3.4.2 ([14, Theorem 8.1, Proposition 8.2, Theorem 8.6])**.** *Let π' be a cuspidal irreducible representation of $\mathrm{GL}'_n(\mathbb{A})$ and let μ be a highest weight such that π' is cohomological with respect to E_μ. Then π'_f is defined over the number field*

$$\mathbb{Q}\left(\pi'\right) \coloneqq \mathbb{Q}(\mu)\,\mathbb{Q}\left(\pi'_f\right).$$

Moreover, let $S\subseteq V$ be a finite set containing all places where π'_f ramifies. Then $\mathbb{Q}\left(\pi'_f\right)$ is the compositum of the number fields $\mathbb{Q}(\pi'_v)$, $v\in V_f - S$.

We also have the following theorem by the same authors.

Theorem 3.4.3 ([14, Proposition 7.21])**.** *Let π' be a cuspidal irreducible representation of $\mathrm{GL}'_n(\mathbb{A})$ and let μ be a highest weight such that π' is cohomological with respect to E_μ. Then for all $\sigma\in\mathrm{Aut}(\mathbb{C})$ the representation ${}^\sigma\pi'_f$ is the finite part of a discrete series representation ${}^\sigma\pi'$ of $\mathrm{GL}'_n(\mathbb{A})$ which is cohomological with respect to ${}^\sigma E_\mu$. Moreover, if E_μ is regular, ${}^\sigma\pi$ is cuspidal.*

Definition. We say the $\mathrm{Aut}(\mathbb{C})$-orbit of an cuspidal irreducible representation π' of either $\mathrm{GL}'_n(\mathbb{A})$ or $\mathrm{GL}_n(\mathbb{A})$ is cuspidal cohomological if ${}^\sigma\pi'$ is cuspidal and cohomological for all $\sigma\in\mathrm{Aut}(\mathbb{C})$.

We will now show that the regularity condition on E_μ is not needed.

Proposition 3.4.4. *Let π' be a cuspidal irreducible cohomological representation of $\mathrm{GL}'_n(\mathbb{A})$. Then ${}^\sigma\pi'$ is cuspidal for all $\sigma\in\mathrm{Aut}(\mathbb{C})$. Moreover,*

$${}^\sigma|\mathrm{JL}|\left(\pi'\right) = |\mathrm{JL}|\left({}^\sigma\pi'\right)$$

for all $\sigma\in\mathrm{Aut}(\mathbb{C})$.

Proof. If $\pi \coloneqq |\mathrm{JL}|(\pi')$ is cuspidal, the two claims are proven in [14, Theorem 7.30]. More precisely, they are proven under the assumption that π is so-called *regular algebraic*, which by [7, Lemma 3.14] is equivalent to π being cohomological. Since $|\mathrm{JL}|$ sends cohomological representations to cohomological representations, this shows the first claims.

If π is not cuspidal, it is still is a discrete series and we can write $\pi = \mathrm{MW}(\rho, k)$ for some $k \geq 1$ by Theorem 2.8.1. Let $\sigma \in \mathrm{Aut}(\mathbb{C})$. We will proceed by showing that ${}^\sigma\pi'$ is cuspidal by induction on the $\mathbb{F}$-rank of GL'_n. If $n = 1$, we already know that ${}^\sigma\pi'$ is cuspidal. For $n > 1$, let τ, ρ, s and t be such that

$$\mathrm{MW}(\tau, s) = {}^\sigma\pi',\ \mathrm{MW}\left(\rho', t\right) = |\mathrm{JL}|(\tau)$$

and hence, by [3, Theorem 18.2]

$$|\mathrm{JL}|\left({}^\sigma\pi'\right) = \mathrm{MW}\left(\rho', st\right).$$

Note that

$$\begin{aligned} {}^\sigma\mathrm{MW}(\rho, k)_{V-\Sigma_{\mathbb{D}}} &= {}^\sigma|\mathrm{JL}|\left(\pi'\right)_{V-\Sigma_{\mathbb{D}}} \overset{(2.6.(2))}{=} {}^\sigma\left(\pi'_{V-\Sigma_{\mathbb{D}}}\right) \overset{(2.7)}{=} \\ = \left({}^\sigma\pi'\right)_{V-\Sigma_{\mathbb{D}}} &\overset{(2.6.(2))}{=} |\mathrm{JL}|\left({}^\sigma\pi'\right)_{V-\Sigma_{\mathbb{D}}} = \mathrm{MW}\left(\rho', st\right)_{V-\Sigma_{\mathbb{D}}}. \end{aligned} \tag{3}$$

We will need the following intermediate lemma.

Lemma 3.4.5. *We have*

$${}^\sigma\mathrm{MW}(\rho, k) = \mathrm{MW}\left({}^\sigma\rho\chi_\sigma, k\right)$$

for some quadratic character χ_σ *with* ${}^{\sigma^{-1}}\chi_\sigma = \chi_{\sigma^{-1}}^{-1}$.

Proof. Let v be a finite place where ρ, $\mathrm{MW}(\rho, k)$ and $\mathbb{D}$ are unramified. Applying [7, Lemma 3.5(ii)] both to ρ_v and $\mathrm{MW}(\rho, k)_v$ yields that the unique quotient of

$${}^\sigma\rho_v|\det|^{\frac{k-1}{2}}\chi_\sigma \times \ldots \times {}^\sigma|\det|^{\frac{1-k}{2}}\chi_\sigma$$

is

$${}^\sigma\mathrm{MW}(\rho, k)_v,$$

where χ_σ is ϵ_σ if both k and dn are odd and the trivial character otherwise. By [18, Lemma 1] $\mathrm{MW}\left({}^\sigma\rho\epsilon_\sigma, k\right)_v$ has to be the unique constituent of

$$(|\det|^{\frac{k-1}{2}}\rho\chi_\sigma \times \ldots \times |\det|^{\frac{1-k}{2}}\rho\chi_\sigma)_v$$

with a K'_v-fixed vector for almost all places $v \in V_f$. Similarly, ${}^\sigma\mathrm{MW}(\rho, k)_v$ has to be the unique constituent of

$${}^\sigma\left(|\det|^{\frac{k-1}{2}}\rho \times \ldots \times |\det|^{\frac{1-k}{2}}\rho\right)_v$$

with a K'_v-fixed vector for almost all places $v \in V_f$. Thus, ${}^\sigma\mathrm{MW}(\rho, k)$ and $\mathrm{MW}\left({}^\sigma\rho\chi_\sigma, k\right)$ have to agree at almost all places and the claim follows then from Strong Multiplicity One, *cf.* [2, §4.4]. ☐

Hence, it follows from (3) that $st = k$ and ${}^{\sigma}\rho\chi_\sigma = \rho'$ by Strong Multiplicity One. Assume now that $s > 1$. By Corollary 3.3.5.1, we know that τ is cohomological and since $s > 1$ the induction hypothesis implies that ${}^{\sigma^{-1}}\tau$ is cuspidal. We thus can consider the discrete series representation $\mathrm{MW}\left({}^{\sigma^{-1}}\tau, t\right)$. Finally,

$$|\mathrm{JL}|\left({}^{\sigma^{-1}}\tau\right)_{V-\Sigma_{\mathbb{D}}} = {}^{\sigma^{-1}}|\mathrm{JL}|(\tau)_{V-\Sigma_{\mathbb{D}}} = {}^{\sigma^{-1}}\mathrm{MW}\left({}^{\sigma}\rho\chi_\sigma, t\right)_{V-\Sigma_{\mathbb{D}}} \overset{(3.4.5)}{=}$$

$$= \mathrm{MW}\left(\rho^{\sigma^{-1}}\chi_\sigma\chi_{\sigma^{-1}}, t\right)_{V-\Sigma_{\mathbb{D}}} = \mathrm{MW}\left(\rho, t\right)_{V-\Sigma_{\mathbb{D}}}.$$

Therefore,

$$|\mathrm{JL}|\left(\mathrm{MW}\left({}^{\sigma^{-1}}\tau, s\right)\right)_{V-\Sigma_{\mathbb{D}}} = \mathrm{MW}\left(\rho, k\right)_{V-\Sigma_{\mathbb{D}}} = |JL|\left(\pi'\right)_{V-\Sigma_{\mathbb{D}}},$$

which implies $\mathrm{MW}\left({}^{\sigma^{-1}}\tau, s\right) = \pi'$ by Strong Multiplicity One and the injectivity of $|\mathrm{JL}|$, a contradiction by Theorem 2.8.1. Thus, $s = 1$ and hence, ${}^{\sigma}\pi'$ is cuspidal. Moreover,

$${}^{\sigma}|\mathrm{JL}|\left(\pi'\right)_{V-\Sigma_{\mathbb{D}}} = {}^{\sigma}\mathrm{MW}\left(\rho, k\right)_{V-\Sigma_{\mathbb{D}}} = \mathrm{MW}\left({}^{\sigma}\rho\epsilon_\sigma, k\right)_{V-\Sigma_{\mathbb{D}}} = |\mathrm{JL}|\left({}^{\sigma}\pi\right)_{V-\Sigma_{\mathbb{D}}}$$

and the second claim follows again from Strong Multiplicity One. □

4 Shalika models

4.1 Let $U'_{(n,n)}$ and $\mathcal{S}$ be the following two subgroups of GL'_{2n}. We define

$$U'_{(n,n)} := \left\{ \begin{pmatrix} 1_n & X \\ 0 & 1_n \end{pmatrix} : X \in M'_{n,n} \right\}$$

and the Shalika subgroup

$$\mathcal{S} := \Delta\mathrm{GL}'_n \rtimes U'_{(n,n)} = \left\{ \begin{pmatrix} h & X \\ 0 & h \end{pmatrix} : h \in \mathrm{GL}'_n, X \in M'_n \right\}.$$

Let ψ be the additive character fixed in Section 2.2. We extend this character to $\mathcal{S}(\mathbb{A})$ by setting

$$\psi(s) := \psi\left(\mathrm{Tr}'(X)\right), \; \eta(s) := \eta\left(\det{}'(h)\right),$$

for $s = \begin{pmatrix} h & X \\ 0 & h \end{pmatrix}$. Let π' be a cuspidal irreducible representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ and we assume there exists a Hecke character η of $\mathrm{GL}_1(\mathbb{A})$ such that for all $a \in \mathrm{GL}_1(\mathbb{A})$

$$\eta \circ \det{}' \overbrace{\begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix}}^{n} = \omega \overbrace{\begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix}}^{2n}.$$

Let S_η be the set of places where η ramifies. For $\phi \in \pi'$ a cusp form and $g \in \mathrm{GL}'_{2n}(\mathbb{A})$ we define the Shalika period integral by

$$\mathcal{S}_\psi^\eta(\phi)(g) := \int_{Z'_{2n}(\mathbb{A})\mathcal{S}(\mathbb{F})\backslash\mathcal{S}(\mathbb{A})} \phi(sg)\,\psi(s)^{-1}\,\eta(s)^{-1}\,\mathrm{d}s.$$

Note that this is well defined since

$$Z'_{2n}(\mathbb{A})\,\Delta\mathrm{GL}'_n(\mathbb{F})\backslash\Delta\mathrm{GL}'_n(\mathbb{A})$$

has finite measure and $U'_{(n,n)}(\mathbb{F})\backslash U'_{(n,n)}(\mathbb{A})$ is compact. If there exists a ϕ such that $\mathcal{S}_\psi^\eta(\phi)$ does not vanish for some $g \in \mathrm{GL}'_{2n}(\mathbb{A})$, this gives a nonzero intertwining operator of $\mathrm{GL}'_{2n}(\mathbb{A})$-representations

$$\mathcal{S}_\psi^\eta \colon \pi' \to \mathrm{Ind}_{\mathcal{S}(\mathbb{A})}^{\mathrm{GL}'_{2n}(\mathbb{A})}(\eta\otimes\psi),$$

where the second space is the vector-space consisting of smooth functions

$$\mathrm{Ind}_{\mathcal{S}(\mathbb{A})}^{\mathrm{GL}'_{2n}(\mathbb{A})}(\eta\otimes\psi) := \{f \in C^\infty\left(\mathrm{GL}'_{2n}(\mathbb{A})\right),\ f(sg) = \eta(s)\,\psi(s)\,f(g) \text{ for all } s \in \mathcal{S}(\mathbb{A})\}$$

and $\mathrm{GL}'_{2n}(\mathbb{A})$ acts by right-translation. In this case we say that π' admits a Shalika model with respect to η. For $v \in V$ we define local Shalika models of $\pi' \cong \otimes'_{v\in V}\pi'_v$ as follows. Set

$$\mathrm{Ind}_{\mathcal{S}(\mathbb{F}_v)}^{\mathrm{GL}'_{2n}(\mathbb{F}_v)}(\eta_v\otimes\psi_v) := \{f \in C^\infty\left(\mathrm{GL}'_{2n}(\mathbb{F}_v)\right),\ f(sg) = \eta_v(s)\,\psi_v(s)\,f(g) \text{ for all } s \in \mathcal{S}(\mathbb{F}_v)\}.$$

If v is a finite place in V, we say π'_v admits a local Shalika model if there exists a non-zero intertwiner

$$\pi'_v \to \mathrm{Ind}_{\mathcal{S}(\mathbb{F}_v)}^{\mathrm{GL}'_{2n}(\mathbb{F}_v)}(\eta_v\otimes\psi_v)$$

of $\mathrm{GL}'_{2n}(\mathbb{F}_v)$-representations. For $v \in V_\infty$ a priori, π'_v is by our conventions not an honest $\mathrm{GL}'_{2n}(\mathbb{F}_v)$-representation and therefore we have to consider $(\pi'_v)^\infty$, the sub-space of smooth vectors in π'_v. Then $\mathrm{Ind}_{\mathcal{S}(\mathbb{F}_v)}^{\mathrm{GL}'_{2n}(\mathbb{F}_v)}(\eta_v\otimes\psi_v)$ and $(\pi'_v)^\infty$ are both Fréchet spaces and admit a natural, smooth $\mathrm{GL}'_{2n}(\mathbb{F}_v)$-action. We say π'_v admits a local Shalika model with respect to η_v if there exists a non-zero, continuous intertwining operator of $\mathrm{GL}'_{2n}(\mathbb{F}_v)$-representations

$$\left(\pi'_v\right)^\infty \to \mathrm{Ind}_{\mathcal{S}(\mathbb{F}_v)}^{\mathrm{GL}'_n(\mathbb{F}_v)}(\eta_v\otimes\psi_v).$$

If π' admits a global Shalika model with respect to η, then so does π'_v with respect to η_v for all $v \in V$. Note that the reverse direction, *i.e.* the existence of a local Shalika model for each π'_v, $v \in V$ implying the existence of a Shalika model for π', is not true in general, see [10, Theorem 1.4].

4.2 Before we come to the precise criteria, when a cuspidal representation admits a Shalika model, we need to fix our notations regarding standard L-functions, *cf.* [12, Theorem 3.3, Theorem 13.8]. Recall that one can associate to each local representation π'_v of $\mathrm{GL}'_n(\mathbb{F}_v)$, $v \in V$ its standard local L-factor $L(s,\pi'_v)$. For $v \in V_f$ it is of the form

$$L(s,\pi'_v) = \frac{1}{P_0(s,\pi'_v)},\ P_0(s,\pi'_v) \in \mathbb{C}[q_v^{s+\frac{n-1}{2}}]$$

and $P_0(0,\pi'_v) = 1$, where we denoted by q_v the cardinality of the residue field of $\mathbb{F}_v$. If π'_v unramified, $P_0(s,\pi'_v)$ is determined by its Satake-parameters. For π' a discrete series representation of $\mathrm{GL}'_n(\mathbb{A})$, we then define

$$L(s,\pi') \coloneqq \prod_{v\in V} L(s,\pi'_v),$$

which is well-defined for $\mathfrak{Re}(s) >> 0$ and admits an analytic continuation to a meromorphic function. For S a finite subset of V, we write $L^S(s,\pi') \coloneqq \prod_{v\notin S} L(s,\pi'_v)$ respectively for its analytic continuation. In particular, we set $L(s,\pi'_f) \coloneqq L^{V_\infty}(s,\pi')$.

4.3 In the case of GL_n we have the following characterization of Shalika models.

Theorem 4.3.1 ([16, Theorem 1])**.** *Let π be a cuspidal irreducible representation of $\mathrm{GL}_{2n}(\mathbb{A})$. Then the following assertions are equivalent.*

1. *There exists $\phi \in \pi$ and $g \in \mathrm{GL}_{2n}(\mathbb{A})$ such that $\mathcal{S}^\eta_\psi(\phi)(g) \neq 0$.*
2. *Let $S \subset V$ be a finite subset of places containing V_∞ and the finite places where π and η ramify. Then the twisted partial exterior square L-function $L^S\left(s,\pi,\wedge^2\otimes\eta^{-1}\right)$ has a pole at $s = 1$.*

If $\mathbb{D}$ does not split over $\mathbb{F}$ there is no longer such a nice criterion. In the case $n = 2$ and $\mathbb{D}$ a quaternion algebra we have the following criterion by [10], which is a consequence of the global theta correspondence.

Theorem 4.3.2 ([10, Theorem 1.3])**.** *Assume $\mathbb{D}$ is a quaternion algebra, π' a cuspidal irreducible representation of $\mathrm{GL}'_2(\mathbb{A})$ and η the Hecke character we fixed above. Let $S \subset V$ be a finite subset of places containing V_∞ and the finite places where π and η ramify. If $|\mathrm{JL}|(\pi')$ is cuspidal the following assertions are equivalent.*

1. *π' admits a Shalika model with respect to η.*
2. *The twisted partial exterior square L-function $L^S\left(s,\pi',\wedge^2\otimes\eta^{-1}\right)$ has a pole at $s = 1$ and for all $v \in \Sigma_{\mathbb{D}}$ the representation π'_v is not of the form $\mathrm{Ind}_{P_{(1,1)}}^{\mathrm{GL}_2(\mathbb{F}_v)}\left(|-|^{\frac{1}{2}}\tau_1 \otimes |-|^{-\frac{1}{2}}\tau_2\right)$, where τ_1 and τ_2 are representations of $\mathrm{GL}'_1(\mathbb{F}_v)$ with central character η_v.*

If $|\mathrm{JL}|(\pi')$ is not cuspidal it is of the form $\mathrm{MW}(\tau,2)$ for some cuspidal irreducible representation τ of $\mathrm{GL}_2(\mathbb{A})$. Then the following assertions are equivalent.

1. *π' admits a Shalika model with respect to η.*
2. *The central character ω_τ of τ equals η.*
3. *The twisted partial exterior square L-function $L^S\left(s,\pi',\wedge^2\otimes\eta^{-1}\right)$ has a pole at $s = 2$.*

Thus, the situation is much more delicate in the case where $|\mathrm{JL}|(\pi')$ is cuspidal because of the second, local condition. On the other hand, if $|\mathrm{JL}|(\pi')$ is not cuspidal we have a priori $\eta^2 = \omega_\pi = \omega_\tau^2$, hence, η and ω_τ only differ by a quadratic character at most.

4.4 The connection between L-functions and Shalika models can first be seen from the next two theorems, which are extensions of [9, Proposition 2.3, Proposition 3.1, Proposition 3.3].

Theorem 4.4.1. *Let π' be a cuspidal irreducible representation of $\mathrm{GL}'_{2n}(\mathbb{A})$. Assume π' admits a Shalika model with respect to η and let $\phi \in \pi'$ be a cusp form. Consider the integrals*

$$\Psi(s,\phi) := \int_{Z'_{2n}(\mathbb{A})H'_n(\mathbb{F})\backslash H'_n(\mathbb{A})} \phi\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}\right)\left|\frac{\det'(h_1)}{\det'(h_2)}\right|^{s-\frac{1}{2}} \eta(h_2)^{-1}\, \mathrm{d}h_1\, \mathrm{d}h_2,$$

$$\zeta(s,\phi) := \int_{\mathrm{GL}'_n(\mathbb{A})} \mathcal{S}^{\eta}_{\psi}(\phi)\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix}\right)|\det'(g_1)|^{s-\frac{1}{2}} dg_1.$$

Then $\Psi(s,\phi)$ converges absolutely for all s and $\zeta(s,\phi)$ converges absolutely if $\mathfrak{Re}(s) >> 0$. Moreover, if $\zeta(s,\phi)$ converges absolutely, $\Psi(s,\phi) = \zeta(s,\phi)$.

In [9] this statement was proven for $\mathbb{D} = \mathbb{F}$, and we will show in Section 7 that their proof extends with some small adjustments to the case of $\mathbb{D}$ being a division algebra. Let $\xi_\phi \in \mathcal{S}^{\eta}_{\psi}(\pi')$ and choose an isomorphism $\mathcal{S}^{\eta}_{\psi}(\pi') \xrightarrow{\cong} \otimes_{v\in V} \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)$. Assume the image of ξ_ϕ can be written as a pure tensor

$$\xi_\phi \mapsto \bigotimes_{v\in V} \xi_{\phi,v} \in \bigotimes_{v\in V} \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v).$$

We can now consider the local version of the above integral

$$\zeta_v(s,\xi_{\phi,v}) := \int_{\mathrm{GL}'_n(\mathbb{F}_v)} \xi_{\phi,v}\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix}\right)|\det'_v(g_1)|^{s-\frac{1}{2}}\, \mathrm{d}_v g_1,$$

where $\xi_{\phi,v} \in \mathcal{S}^{\eta}_{\psi_v}(\pi'_v)$. The local Shalika integrals are then connected to the local L-factors by the following theorem.

Theorem 4.4.2. *Let π' be a cuspidal irreducible representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ and assume π' admits a Shalika model with respect to η. Then for each place $v \in V$ and $\xi_v \in \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)$ there exists an entire function $P(s,\xi_v)$, with $P(s,\xi_v) \in \mathbb{C}[q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s}]$ if $v \in V_f$, such that*

$$\zeta_v(s,\xi_v) = P(s,\xi_v)\, L(s,\pi_v)$$

and hence, $\zeta_v(s,\xi_v)$ can be analytically continued to $\mathbb{C}$. Moreover, for each place v there exists a vector ξ_v such that $P(s,\xi_v) = 1$. If v is a place where neither π' nor ψ ramify this vector can be taken as the spherical vector $\xi_{\pi'_v}$ normalized by $\xi_{\pi'_v}(\mathrm{id}) = 1$.

In the case $\mathbb{F} = \mathbb{D}$ this the existence of such a holomorphic P was proven in [9] and in [19, Corollary 5.2] it was shown that P is actually a polynomial in $\mathbb{C}[q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s}]$. Theorem 4.4.1 and Theorem 4.4.2 imply for $\xi_{\phi,f} \cong \otimes'_{v\in V_f} \xi_{\phi,f}$ and $\mathfrak{Re}(s) >> 0$

$$\zeta_f(s,\xi_{\phi,f}) := \int_{\mathrm{GL}'_n(\mathbb{A}_f)} \xi_{\phi,f}\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix}\right)|\det'(g_1)|^{s-\frac{1}{2}}\, \mathrm{d}_f g_1 = \prod_{v\in V_f} P(s,\xi_{\phi,v})\, L(s,\pi_v).$$

4.5 Let π' be a cuspidal irreducible representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ and assume π' admits a Shalika model with respect to η. It is natural to ask whether ${}^{\sigma}\pi'$ admits a Shalika model with respect to ${}^{\sigma}\eta$ assuming that ${}^{\sigma}\pi'$ is cuspidal. In the split case it was proven in the appendix of [15] that if π' admits a Shalika model with respect to η, then ${}^{\sigma}\pi'$ admits one with respect to ${}^{\sigma}\eta$.

Definition. We say the $\mathrm{Aut}(\mathbb{C})$-orbit of π' admits a Shalika model with respect to η if ${}^\sigma\pi'$ is cuspidal and admits a Shalika model with respect to ${}^\sigma\eta$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$.

In the case of $n = 1$ and $\mathbb{D}$ a quaternion algebra Theorem 4.3.2 allows us to prove the following.

Lemma 4.5.1. *Let $\mathbb{D}$ be a quaternion algebra and π' a cuspidal irreducible cohomological representation of $\mathrm{GL}_2'(\mathbb{A})$. If π' admits a Shalika model with respect to η then ${}^\sigma\pi'$ admits one with respect to ${}^\sigma\eta$.*

Proof. Note first that by Proposition 3.4.4 ${}^\sigma\pi'$ is cuspidal. Assume first that $|\mathrm{JL}|(\pi')$ is not cuspidal, *i.e.* $|\mathrm{JL}|(\pi') = \mathrm{MW}(\tau, 2)$ for some cuspidal irreducible representation of $\mathrm{GL}_2(\mathbb{A})$. From Corollary 3.3.5.1, Proposition 3.4.4 and Lemma 3.4.5 it follows that

$$|\mathrm{JL}|\left({}^\sigma\pi'\right) = {}^\sigma|\mathrm{JL}|\left(\pi'\right) = \mathrm{MW}\left({}^\sigma\tau, 2\right).$$

Since the central character ω_τ of τ equals by assumption η, the central character of ${}^\sigma\tau$ equals ${}^\sigma\eta$. Thus we are done by Theorem 4.3.2.

Next assume $|\mathrm{JL}|(\pi')$ is cuspidal and hence, $|\mathrm{JL}|(\pi')$ admits a Shalika model with respect to η by Theorem 4.3.1 and Theorem 4.3.2. Thus, ${}^\sigma|\mathrm{JL}|(\pi') = |\mathrm{JL}|({}^\sigma\pi')$ admits also a Shalika model with respect to ${}^\sigma\eta$ by [15, Theorem 3.6.2] and hence,

$$L^S\left(s, {}^\sigma\pi, \wedge^2 \otimes {}^\sigma\eta^{-1}\right)$$

has a pole at $s = 1$. Moreover, if v is a non-split place of $\mathbb{D}$ and ${}^\sigma\pi_v$ were of the form ${}^\sigma\pi_v \cong \mathrm{Ind}_{P_{(1,1)}}^{\mathrm{GL}_2(\mathbb{F}_v)}\left(|-|^{\frac{1}{2}}\tau_1 \otimes |-|^{-\frac{1}{2}}\tau_2\right)$, where τ_i are representations of $\mathrm{GL}_1'(\mathbb{F}_v)$ with central character ${}^\sigma\eta$. This would lead to the contradiction, since by (2)

$$\pi_v = {}^{\sigma^{-1}}\mathrm{Ind}_{P_{(1,1)}}^{\mathrm{GL}_2(\mathbb{F}_v)}\left(|-|^{\frac{1}{2}}\tau_1 \otimes |-|^{-\frac{1}{2}}\tau_2\right) = \mathrm{Ind}_{P_{(1,1)}}^{\mathrm{GL}_2(\mathbb{F}_v)}\left(|-|^{\frac{1}{2}}\left({}^{\sigma^{-1}}\tau_1\right) \otimes |-|^{-\frac{1}{2}}\left({}^{\sigma^{-1}}\tau_2\right)\right).$$

□

Remark. We have currently no proof in the general case $n > 2$ and unfortunately the methods of [10] do not generalize well beyond the quaternion case. Hence, we can only conjecture the following.

Conjecture 4.5.1. *Let π' be a cuspidal irreducible cohomological representation of $\mathrm{GL}_n'(\mathbb{A})$ such that $|\mathrm{JL}|(\pi')$ is residual. If π' admits a Shalika model with respect to η, then so does the $\mathrm{Aut}(\mathbb{C})$-orbit of π'.*

4.6 In [15] the authors define an action of $\mathrm{Aut}(\mathbb{C})$ on a given Shalika model and we will generalize this now to our setting. Let ψ_f be the finite part of the additive character ψ, which takes values in $\mu_\infty \subseteq \mathbb{C}^*$, the subgroup of all roots of unity of $\mathbb{Q}^*$. We will associate to an element $\sigma \in \mathrm{Aut}(\mathbb{C})$ an element $t_\sigma \in \mathbb{A}^*$ such that for all $x \in \mathbb{A}$

$$\sigma(\psi(x)) = \psi(t_\sigma x).$$

More explicitly, we construct t_σ by first restricting σ to $\mathbb{Q}(\mu_\infty)$ and sending it to $\prod_p \mathbb{Z}_p^*$ via the global symbol map of Artin reciprocity

$$\mathrm{Aut}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \xrightarrow{\cong} \widehat{\mathbb{Z}}^* = \prod_{p \text{ prime}} \mathbb{Z}_p^*,$$

then embed the so obtained element into $\mathbb{A}$ via the diagonal embedding $\mathbb{Z}_p \hookrightarrow \prod_{v|p} \mathbb{O}_v$. Next we define the action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ on the finite part $\mathcal{S}_{\psi_f}^{\eta_f}(\pi'_f)$ by sending ξ_f to

$$g_f \mapsto {}^\sigma\xi_f(g_f) \coloneqq \sigma\left(\xi_f\left(\mathbf{t}_\sigma^{-1} g_f\right)\right), \, g_f \in \mathrm{GL}'_{2n}(\mathbb{A}_f),$$

where

$$\mathbf{t}_\sigma = \mathrm{diag}\left(\overbrace{t_\sigma, \ldots, t_\sigma}^{n}, \overbrace{1, \ldots, 1}^{n}\right).$$

This gives a σ-linear intertwining operator

$$\sigma^*\colon \mathrm{Ind}_{\mathcal{S}(\mathbb{A}_f)}^{\mathrm{GL}'_{2n}(\mathbb{A}_f)}(\eta_f \otimes \psi_f) \to \mathrm{Ind}_{\mathcal{S}(\mathbb{A}_f)}^{\mathrm{GL}'_{2n}(\mathbb{A}_f)}({}^\sigma\eta_f \otimes \psi_f), \, \xi_f \mapsto {}^\sigma\xi_f. \tag{4}$$

Completely analogously we define a σ-linear intertwining operator

$$\sigma^*\colon \mathrm{Ind}_{\mathcal{S}(\mathbb{F}_v)}^{\mathrm{GL}'_{2n}(\mathbb{F}_v)}(\eta_v \otimes \psi_v) \to \mathrm{Ind}_{\mathcal{S}(\mathbb{F}_v)}^{\mathrm{GL}'_{2n}(\mathbb{F}_v)}({}^\sigma\eta_v \otimes \psi_v)$$

for every finite place v, where we use $t_{\sigma,v}$ and $\mathbf{t}_{\sigma,v}$ instead of t_σ and $\mathbf{t}_\sigma$.

4.7 The next two lemmas are the generalizations of [15, Lemma 3.8.1, Lemma 3.9]. Let π' be a cuspidal irreducible cohomological representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ which admits a Shalika model with respect to η. Let $v \in V_f$ be a finite place. In order to proceed we need the local uniqueness of the Shalika model, *i.e.* for every irreducible representation π'_v of $\mathrm{GL}'_{2n}(\mathbb{F}_v)$ the claim that

$$\dim_\mathbb{C} \mathrm{Hom}_{\mathrm{GL}'_{2n}(\mathbb{F}_v)}\left(\pi'_v, \mathrm{Ind}_{\mathcal{S}(\mathbb{F}_v)}^{\mathrm{GL}'_{2n}(\mathbb{F}_v)}(\eta_v \otimes \psi_v)\right) \leq 1.$$

By Frobenius reciprocity every such map corresponds uniquely to a Shalika functional

$$\lambda \in \mathrm{Hom}_{\mathcal{S}(\mathbb{F}_v)}\left(\pi'_v, \eta_v \otimes \psi_v\right),$$

i.e. a map

$$\lambda\colon \pi'_v \to \mathbb{C}$$

such that $\lambda(\pi'_v(s)\phi) = \eta_v(s)\psi_v(s)\lambda(\phi)$ for $s \in \mathcal{S}(\mathbb{F}_v)$ and $\phi \in \pi'_v$ and vice versa.

Definition. We say that the $\mathrm{Aut}(\mathbb{C})$-orbit of π' has a unique local Shalika model if ${}^\sigma\pi'_v$ has a unique Shalika model for all $v \in V_f$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$.

In the split case or when $\mathbb{D}$ is a quaternion algebra the following was proven in [21].

Theorem 4.7.1 ([21, Theorem 3.4])**.** *Let $\mathbb{D}$ be a field or a quaternion algebra. Then*

$$\dim_\mathbb{C} \mathrm{Hom}_{\mathrm{GL}'_{2n}(\mathbb{F}_v)}\left(\pi'_v, \mathrm{Ind}_{\mathcal{S}(\mathbb{F}_v)}^{\mathrm{GL}'_{2n}(\mathbb{F}_v)}(\eta_v \otimes \psi_v)\right) \leq 1.$$

This is yet another reason why we will have to restrict ourselves to the case $\mathbb{D}$ being quaternion in the end. Combining Proposition 3.4.4, Lemma 4.5.1, and Theorem 4.7.1 we have proved the following.

Theorem 4.7.2. *Let π' be a cuspidal irreducible cohomological representation of $\mathrm{GL}_2'(\mathbb{A})$ which admits a Shalika model with respect to η and assume that $\mathbb{D}$ is a quaternion algebra. Then the $\mathrm{Aut}(\mathbb{C})$-orbit of π' is cuspidal cohomological, admits a Shalika model with respect to η and has a unique local Shalika model.*

4.8 For the rest of the paper let us collect the following decorations of a cuspidal irreducible representation π' of $\mathrm{GL}_{2n}'(\mathbb{A})$:

1. π' is cuspidal irreducible cohomological representation of $\mathrm{GL}_{2n}'(\mathbb{A})$.
2. The $\mathrm{Aut}(\mathbb{C})$-orbit of π' is cuspidal cohomological and admits a Shalika model with respect to η.
3. The $\mathrm{Aut}(\mathbb{C})$-orbit of π' has a local unique Shalika model.

Moreover, we also fix a splitting ${}^\sigma\pi' \cong {}^\sigma\pi'_\infty \otimes {}^\sigma\pi'_f$, ${}^\sigma\pi'_f \xrightarrow{\cong} \otimes'_{v\in V_f} {}^\sigma\pi'_v$ and a Shalika model of ${}^\sigma\pi_v$

$$\mathcal{S}_{\psi_v}^{{}^\sigma\eta_v} \colon {}^\sigma\pi'_v \to \mathrm{Ind}_{\mathcal{S}(\mathbb{F}_v)}^{\mathrm{GL}_{2n}'(\mathbb{F}_v)}\left({}^\sigma\eta_v \otimes \psi_v\right)$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, $v \in V_f$.

Lemma 4.8.1. *For π' as in 4.8, $v \in V_f$ and the action of (4) we have*

$$\sigma^*\left(\mathcal{S}_{\psi_v}^{\eta_v}\left(\pi'_v\right)\right) = \mathcal{S}_{\psi_v}^{{}^\sigma\eta_v}\left({}^\sigma\pi'_v\right)$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. For any finite extension $\mathbb{K}$ of $\mathbb{Q}(\pi'_v, \eta_v)$ we have a $\mathbb{K}$-structure

$$\mathcal{S}_{\psi_v}^{\eta_v}\left(\pi'_v\right)_{\mathbb{K}} := \mathcal{S}_{\psi_v}^{\eta_v}\left(\pi'_v\right)^{\mathrm{Aut}(\mathbb{C}/\mathbb{K})}$$

on $\mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v)$.

Proof. For the first assertion, note that the representation ${}^\sigma\pi'_v$ has on the on hand the unique Shalika model $\mathcal{S}_{\psi_v}^{{}^\sigma\eta_v}({}^\sigma\pi'_v)$ with respect to ${}^\sigma\eta_v$, but on the other hand, the σ-linear map

$$\pi'_v \xrightarrow{\cong} \mathcal{S}_{\psi_v}^{\eta_v}\left(\pi'_v\right) \overset{\sigma^*}{\hookrightarrow} \mathrm{Ind}_{\mathcal{S}(\mathbb{F}_v)}^{\mathrm{GL}_{2n}'(\mathbb{F}_v)}\left({}^\sigma\eta_v \otimes \psi_v\right)$$

gives rise to a linear map

$${}^\sigma\pi'_v \hookrightarrow \mathrm{Ind}_{\mathcal{S}(\mathbb{F}_v)}^{\mathrm{GL}_{2n}'(\mathbb{F}_v)}\left({}^\sigma\eta_v \otimes \psi_v\right).$$

Therefore, the assumed local uniqueness of the Shalika model implies that up to a scalar those two maps have to agree and hence, their image is identical.

For the second assertion, we follow the line of reasoning of [17, Theorem 3.1]. We fix the following Shalika functional

$$\phi_v \mapsto \lambda(\phi_v) := \mathcal{S}_{\psi_v}^{\eta_v}(\phi_v)(1).$$

First we show that there exists a $\mathbb{Q}(\pi'_v)$-rational structure $\pi'_{v,\mathbb{Q}(\pi'_v)}$ of π'_v such that $\lambda\left(\pi'_{v,\mathbb{Q}(\pi'_v)}\right) \subseteq \overline{\mathbb{Q}}$. Let $\widetilde{\pi'_v}$ be a $\mathbb{Q}(\pi'_v)$-rational structure of π'_v, see Lemma 3.4.1. Therefore, $\widetilde{\pi'_v} \otimes_{\mathbb{Q}(\pi'_v)} \overline{\mathbb{Q}}$ is a $\overline{\mathbb{Q}}$-rational structure of π'_v. Since both η_v and ψ_v take values in $\overline{\mathbb{Q}}^*$,

$$\left(\widetilde{\pi'_v} \otimes_{\mathbb{Q}(\pi'_v)} \overline{\mathbb{Q}}\right) \otimes \left(\eta_v^{-1} \otimes \psi_v^{-1}\right)_{\mathcal{S}(\mathbb{F}_v)} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} = \left(\pi'_v \otimes \eta_v^{-1} \otimes \psi_v^{-1}\right)_{\mathcal{S}(\mathbb{F}_v)},$$

where the subscripts indicate that we take the $\mathcal{S}(\mathbb{F}_v)$-invariant vectors. Frobenius reciprocity ensures that the right-hand side is one dimensional, hence, so is the left-hand side. Thus,

$$\left(\widetilde{\pi'_v} \otimes_{\mathbb{Q}(\pi'_v)} \overline{\mathbb{Q}}\right) \otimes \left(\eta_v^{-1} \otimes \psi_v^{-1}\right)_{\mathcal{S}(\mathbb{F}_v)}$$

is non-zero and we can apply Frobenius reciprocity again to obtain a Shalika functional λ' on π'_v which by the local uniqueness differs by some constant c from λ, *i.e.* $\lambda' = c\lambda$. Setting

$$\pi'_{v,\mathbb{Q}(\pi'_v)} := c\widetilde{\pi'_v}$$

we obtain the desired rational structure. We define an action of $\mathrm{Aut}(\mathbb{C})$ on the space of η_v-Shalika functionals of π'_v by setting

$${}^{\sigma}\lambda(u) = \sigma\left(\lambda\left(\beta_\sigma^{-1}\left(\pi'_v(\mathbf{t}_{\sigma,v}^{-1})u\right)\right)\right),$$

where $\beta_\sigma\colon \pi'_v \to \pi'_v$ is defined by

$$\beta_\sigma(zu) = \sigma(z)\,u,\ u \in \pi'_{v,\mathbb{Q}(\pi'_v)},\ z \in \mathbb{C}.$$

It is easy to see that for σ fixing $\mathbb{Q}(\pi'_v, \eta_v)$, ${}^{\sigma}\lambda$ is again a η_v-Shalika functional of π'_v.

Since $\lambda \neq 0$, we can fix a vector $u_0 \in \pi'_{v,\mathbb{Q}(\pi'_v)}$ such that $\lambda(u_0) \neq 0$. Then there exists an open subgroup $\Gamma \subseteq \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi'_v, \eta_v))$, in the usual profinite topology on $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi'_v, \eta_v))$, such that

$$\mathbf{t}_\sigma u_0 = u_0,\ \sigma(\lambda(u_0)) = \lambda(v_0)$$

for all $\sigma \in \Gamma$, since λ takes values in $\overline{\mathbb{Q}}$. By the uniqueness of the local Shalika model we have constants c_σ such that ${}^{\sigma}\lambda = c_\sigma\lambda$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi'_v, \eta_v))$. Then one easily computes

$$c_\sigma\lambda(u_0) = {}^{\sigma}\lambda(u_0) = \sigma\left(\lambda\left(\beta_\sigma^{-1}\left(\pi'_v(\mathbf{t}_{\sigma^{-1},v}^{-1})u_0\right)\right)\right) = \lambda(u_0)$$

and therefore $c_\sigma = 1$, which shows that Γ fixes λ. Recall the action of $\mathrm{Aut}(\mathbb{C})$ on $\mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v)$ given by ${}^{\sigma}\xi(u) = \sigma\left(\xi\left(\pi'_v(\mathbf{t}_{\sigma^{-1},v}^{-1})u\right)\right)$. Using the embedding of

$$\mathcal{S}_{\psi_v}^{\eta_v}\colon \pi'_v \hookrightarrow \mathrm{Ind}_{\mathcal{S}(\mathbb{F}_v)}^{\mathrm{GL}'_{2n}(\mathbb{F}_v)}(\eta_v \otimes \psi_v),$$

we can pull this action back to π'_v and denote it by $u \mapsto \sigma(u)$. It is clear from how λ was defined that for $u \in \pi'_v$, $g \in \mathrm{GL}'_{2n}(\mathbb{F}_v)$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$

$$\lambda\left(\pi'_v(g)\sigma(u)\right) = \sigma\left(\lambda\left(\pi'_v\left(\mathbf{t}_{\sigma^{-1},v}g\right)u\right)\right).$$

Thus, for $\sigma \in \Gamma$, $u \in \pi'_{v,\mathbb{Q}(\pi'_v)}$ and $g \in \mathrm{GL}'_{2n}(\mathbb{F}_v)$ we deduce

$$\lambda\left(\pi'_v(g)\left(\sigma\left(u\right)\right)\right) = \lambda\left(\pi'_v(g)u\right).$$

But the irreducibly of π'_v now implies that $\sigma(u) - u = 0$, since otherwise π'_v would be spanned by $\mathrm{GL}'_{2n}(\mathbb{F}_v) \cdot (\sigma(u) - u)$ and therefore we arrive at the contradiction $\lambda = 0$. Hence, $\sigma(u) = u$ for all $u \in \pi'_{v,\mathbb{Q}(\pi'_v)}$ and $\sigma \in \Gamma$. This implies that $\pi'_{v,\mathbb{Q}(\pi'_v)}$ is contained in $\pi_v'^{\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})}$ and therefore

$$\pi_v'^{\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})} = \pi'_{v,\mathbb{Q}(\pi'_v)} \otimes_{\mathbb{Q}(\pi'_v,\eta_v)} \overline{\mathbb{Q}}.$$

But now every vector in $\pi'_{v,\mathbb{Q}(\pi'_v)} \otimes \overline{\mathbb{Q}}$ is fixed by an open subgroup of $\mathrm{Aut}\left(\overline{\mathbb{Q}}/\mathbb{Q}(\pi'_v,\eta_v)\right)$ and therefore by [25, Proposition 11.1.6]

$$\pi_v'^{\mathrm{Aut}\left(\overline{\mathbb{Q}}/\mathbb{Q}(\pi'_v,\eta_v)\right)} = \pi_v'^{\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})^{\mathrm{Aut}\left(\overline{\mathbb{Q}}/\mathbb{Q}(\pi'_v,\eta_v)\right)}}$$

is a $\mathbb{Q}(\pi'_v,\eta_v)$-rational structure of $\pi_v'^{\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})}$ and hence, of π'_v. We set

$$\mathcal{S}^{\eta_v}_{\psi_v}\left(\pi'_v\right)_{\mathbb{Q}(\pi'_v,\eta_v)} := \left(\mathcal{S}^{\eta_v}_{\psi_v}\left(\pi'_v\right)\right)^{\mathrm{Aut}\left(\overline{\mathbb{Q}}/\mathbb{Q}(\pi'_v,\eta_v)\right)},$$

which is the image of $\pi_v'^{\mathrm{Aut}\left(\overline{\mathbb{Q}}/\mathbb{Q}(\pi'_v,\eta_v)\right)}$ under the map

$$\mathcal{S}^{\eta_v}_{\psi_v}\colon \pi'_v \xrightarrow{\cong} \mathcal{S}^{\eta_v}_{\psi_v}\left(\pi'_v\right).$$

Since this map respects the actions of $\mathrm{Aut}(\mathbb{C})$ and $\pi_v'^{\mathrm{Aut}\left(\overline{\mathbb{Q}}/\mathbb{Q}(\pi'_v,\eta_v)\right)}$ is defined over $\mathbb{Q}(\pi'_v,\eta_v)$, so is $\mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)_{\mathbb{Q}(\pi'_v,\eta_v)}$. Moreover, it follows that the canonical map

$$\mathcal{S}^{\eta_v}_{\psi_v}\left(\pi'_v\right)_{\mathbb{Q}(\pi'_v,\eta_v)} \otimes_{\mathbb{Q}(\pi'_v,\eta_v)} \mathbb{C} \to \mathcal{S}^{\eta_v}_{\psi_v}\left(\pi'_v\right)$$

is an isomorphism and hence, the claim is proven. □

We introduce the following notation. Let $v \in V_f$, $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $f \in \mathbb{C}(q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s})$. We denote by f^σ the rational function obtained by applying σ to all coefficients of f for some $\sigma \in \mathrm{Aut}(\mathbb{C})$, which is the same as applying σ to the coefficients of f considered as a Laurent-series. Moreover,

$$\sigma\left(f\left(\frac{1}{2}\right)\right) = f^\sigma\left(\frac{1}{2}\right).$$

Lemma 4.8.2. *Let π' be a cuspidal irreducible automorphic representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ with local representations π_v of $\mathrm{GL}'_{2n}(\mathbb{F}_v)$ for $v \in V$. Then for every finite place v*

$$L^\sigma\left(s,\pi'_v\right) = L\left(s,{}^\sigma\pi'_v\right),$$

and hence, if $L(s,\pi'_v)$ has no pole at $s = \frac{1}{2}$, $L\left(\frac{1}{2},\pi'_v\right) \in \mathbb{Q}(\pi'_v)$.

Proof. For the first claim, note that by [2, Theorem 6.18], we have an explicit description of the local L-factors and that for π'_v a representation of $\mathrm{GL}'_m(\mathbb{F}_v)$, $L(s+\frac{md-1}{2},\pi'_v) \in \mathbb{C}(q_v^s, q_v^s)$. We denote then for $f \in \mathbb{C}(q_v^s, q_v^s)$ by ${}^\sigma f$ the coefficient-wise application of σ. Note that for m even we thus have that ${}^\sigma L(s+\frac{md-1}{2},\pi'_v) = L^\sigma(s+\frac{md}{2},\pi'_v)$. One can then carry over the proof of [7, Lemma 4.6] *muta mutandis* from the case GL_m to GL'_m to obtain that

$$ {}^\sigma L\left(s+\frac{md-1}{2},\pi'_v\right) = L\left(s+\frac{md-1}{2},{}^\sigma\pi'_v\right). $$

Thus, for $m = 2n$,

$$ L^\sigma(s+nd,\pi'_v) = L\left(s+nd,{}^\sigma\pi'_v\right) $$

and since $q_v^{nd} \in \mathbb{Q}$, the first claim follows. For the second claim, it is enough to observe that in this case

$$ \sigma\left(L\left(\frac{1}{2},\pi'_v\right)\right) = L^\sigma\left(\frac{1}{2},\pi'_v\right) = L\left(\frac{1}{2},{}^\sigma\pi'_v\right) = L\left(\frac{1}{2},\pi'_v\right) $$

for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi'_v))$. □

Lemma 4.8.3. *For π' as in 4.8 and $v \in V_f$ there exists a vector $\xi^0_{\pi',v} \in \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)_{\mathbb{Q}(\pi',\eta)}$ such that*

$$ \zeta_v\left(s,\xi^0_{\pi',v}\right) = L\left(s,\pi'_v\right) $$

if $v \notin S_{\pi'_f,\psi}$ and

$$ P^\sigma(s,\xi^0_{\pi',v}) = P(s,{}^\sigma\xi^0_{\pi',v}) $$

for all $\sigma \in \mathrm{Aut}\left(\mathbb{C}/\mathbb{Q}(\pi',\eta_v)\right)$ if $v \in S_{\pi'_f,\psi}$.

Proof. We again follow the proof of [17, Theorem 3.1]. For $v \notin S_{\pi'_f,\psi}$ we choose $\xi^0_{\pi',v}$ to be the normalized spherical vector of Theorem 4.4.2. Note that $\xi^0_{\pi',v} \in \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)_{\mathbb{Q}(\pi',\eta_v)}$, since for $v \notin S_{\pi'_f,\psi}$ the normalization $\xi^0_{\pi',v}(1) = 1$ implies that ${}^\sigma\xi^0_{\pi',v}(1) = 1$ and hence, because $\sigma^*(\mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)) = \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)$, ${}^\sigma\xi^0_{\pi',v} = \xi^0_{\pi'_v}$. Thus, $P(s,\xi^0_{\pi'_f,v}) = 1$ for all $v \notin S_{\pi'_f,\psi}$ and therefore $\zeta_v\left(s,\xi^0_{\pi',v}\right) = L\left(s,\pi'_v\right)$.

For $v \in S_{\pi'_f,\psi}$, pick any non-zero $\xi^0_{\pi',v} \in \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)_{\mathbb{Q}(\pi',\eta)}$ and recall that $P\left(s,\xi^0_{\pi',v}\right) \in \mathbb{C}[q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s}]$, see Theorem 4.4.2, and the L-function $L\left(s,\pi'_v\right)$ does not vanish at $s = \frac{1}{2}$, as it is the reciprocal of a polynomial. Since

$$ \frac{\zeta_v\left(s,\xi^0_{\pi',v}\right)}{L\left(s,\pi'_v\right)} = P\left(s,\xi^0_{\pi',v}\right), \frac{1}{L\left(s,\pi'_v\right)} \in \mathbb{C}[q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s}], $$

we have

$$ \zeta_v\left(s,\xi^0_{\pi',v}\right) \in \mathbb{C}\left(q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s}\right). $$

From the definition of $\zeta_v\left(s, \xi^0_{\pi',v}\right)$ it follows that the k-th coefficient of $q_v^{s-\frac{1}{2}}$ in $\zeta_v\left(s, \xi^0_{\pi',v}\right)$ is

$$c_k\left(\xi^0_{\pi',v}\right) := \int_{\substack{\mathrm{GL}'_n(\mathbb{F}_v), \\ |\det'(g_1)|=q^{-k}}} \xi^0_{\pi',v}\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix}\right) \mathrm{d}_v g_1,$$

which vanishes for $k << 0$ and is a finite sum, see the proof of Theorem 4.4.2. Hence, by a change of variables,

$$c_k\left({}^\sigma\xi^0_{\pi',v}\right) = \sigma\left(c_k\left(\xi^0_{\pi',v}\right)\right)$$

for all $\sigma \in \mathrm{Aut}\left(\mathbb{C}/\mathbb{Q}\left(\eta_v\right)\right)$. It follows that for all $s \in \mathbb{C}, \sigma \in \mathrm{Aut}\left(\mathbb{C}/\mathbb{Q}\left(\eta_v\right)\right)$ $\zeta_v^\sigma\left(s, \xi^0_{\pi',v}\right) = \zeta_v\left(s, {}^\sigma\xi^0_{\pi',v}\right)$ by analytic continuation. Lemma 4.8.2 shows then that

$$P^\sigma\left(s, \xi^0_{\pi',v}\right) L^\sigma\left(s, \pi'_v\right) = \zeta_v^\sigma\left(s, \xi^0_{\pi',v}\right) = \zeta_v\left(s, {}^\sigma\xi^0_{\pi',v}\right) =$$

$$= P\left(s, {}^\sigma\xi^0_{\pi',v}\right) L\left(s, {}^\sigma\pi'_v\right) = P\left(s, {}^\sigma\xi^0_{\pi',v}\right) L^\sigma\left(s, \pi'_v\right)$$

for all $\sigma \in \mathrm{Aut}\left(\mathbb{C}/\mathbb{Q}(\pi', \eta_v)\right)$ and hence,

$$P^\sigma(s, \xi^0_{\pi',v}) = P(s, {}^\sigma\xi^0_{\pi',v}).$$

□

We let $\xi_{\pi'_f,0} \in \mathcal{S}^{\eta_f}_{\psi_f}\left(\pi'_f\right)$ be the image of $\otimes_{v \in V_f} \xi^0_{\pi',v}$ under the fixed isomorphisms of Section 4.8.

5 Periods

5.1 In this section we will closely follow the strategy of [15]. Throughout the rest of the section let π' be an automorphic representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ as in 4.8. Let μ be the highest weight such that π' is cohomological with respect to $E^\vee_\mu$ and assume that $|\mathrm{JL}|(\pi') = \mathrm{MW}(\tau, k)$ for some $k > 1$ and τ a cuspidal irreducible representation of $\mathrm{GL}_l(\mathbb{A})$ with $lk = 2nd$. Note that we also fixed a splitting isomorphism

$$\pi' \xrightarrow{\cong} \pi'_\infty \otimes \pi'_f \xrightarrow{\cong} \bigotimes_{v \in V_\infty} \pi'_v \otimes \pi'_f.$$

and $|\mathrm{JL}|({}^\sigma\pi') = {}^\sigma\mathrm{MW}(\tau, k) = \mathrm{MW}({}^\sigma\tau, k)$ by Lemma 3.4.5 and Proposition 3.4.4. We also have that

$$\left({}^\sigma\pi'\right)_\infty \xrightarrow{\cong} \bigotimes_{v \in V_\infty} \pi'_{\sigma^{-1}\circ v}.$$

Indeed, by Theorem 3.4.3 ${}^\sigma\pi'$ is cohomological with respect to ${}^\sigma E^\vee_\mu$ and therefore Theorem 3.3.1 and Lemma 3.3.6 show that for $v \in V_\infty$ ${}^\sigma\pi'_v \cong A_{\underline{n'}}(\lambda_v)$, where λ_v is determined by $\mu_{\sigma^{-1}\circ v}$ and $\underline{n'}$ is determined by k and l. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ we thus can fix a splitting isomorphism

$$\left({}^\sigma\pi'\right)_\infty \otimes {}^\sigma\pi'_f \xrightarrow{\cong} \bigotimes_{v \in V_\infty} \pi'_{\sigma^{-1}\circ v} \otimes {}^\sigma\pi'_f.$$

Let us give an example that satisfies all of those properties.

Example. Let for a moment $\mathbb{F} = \mathbb{Q}$. Then in [14, § 6.11] the following representation was constructed. Set π_∞ to the Langlands quotient of $F(1, s+2) \times F(-1, s+2)$ for s a positive integer. This representation is cohomological with the coefficient system given by the weight vector $\left(\frac{s}{2}, \frac{s}{2}, -\frac{s}{2}, -\frac{s}{2}\right)$. Moreover, π_∞ can be extended to a cuspidal irreducible automorphic representation of $\mathrm{GL}'_2(\mathbb{A})$ with $\mathbb{D}$ a quaternion algebra and is regular algebraic if k is even.

5.2 Let K_f be a compact open subgroup of $\mathrm{GL}'_{2n}(\mathbb{A}_f)$ and denote the block diagonal embedding by $\iota\colon H'_n \hookrightarrow \mathrm{GL}'_{2n}$. We set

$$\mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}} := \mathrm{GL}'_{2n}(\mathbb{F}) \backslash \mathrm{GL}'_{2n}(\mathbb{A}) / K'_\infty K_f$$

and

$$\mathbf{S}_{K_f}^{H'_n} := H'_n(\mathbb{F}) \backslash H'_n(\mathbb{A}) / \left(K'_\infty \cap H'_{n,\infty}\right) \iota^{-1}\left(K_f\right).$$

Let $r = \dim_{\mathbb{Q}} \mathbb{F}$ and note that if we consider $\mathbf{S}_{K_f}^{H'_n}$ as an orbifold, its real dimension is

$$\dim_{\mathbb{R}} \mathbf{S}_{K_f}^{H'_n} = r\left((nd)^2 - nd - 1\right).$$

Lemma 5.2.1. *The embedding ι induces a proper map*

$$\iota\colon \boldsymbol{S}_{K_f}^{H'_n} \to \boldsymbol{S}_{K_f}^{\mathrm{GL}'_{2n}}.$$

Proof. It follows from [1, Lemma 2.7] that

$$H'_n(\mathbb{F}) \backslash H'_n(\mathbb{A}) / \iota^{-1}\left(K_f\right) \to \mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}}$$

is proper. But this map factors as

$$H'_n(\mathbb{F}) \backslash H'_n(\mathbb{A}) / \iota^{-1}\left(K_f\right) \to \mathbf{S}_{K_f}^{H'_n} \to \mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}}.$$

Since the first map is surjective and the composition is proper, the second map is proper. □

5.3 Next let $E_\mu^\vee$ be a highest weight representation of $\mathrm{GL}'_{2n,\infty}$ and consider the locally constant sheaf $\mathcal{E}_\mu^\vee$ on $\mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}}$, whose espace étalé is

$$\mathrm{GL}'_{2n}(\mathbb{A})/K'_\infty K_f \times_{\mathrm{GL}'_{2n}(\mathbb{F})} E_\mu^\vee,$$

We consider its cohomology groups of compact support $H_c^*\left(\mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)$ and define analogously $H_c^*\left(\mathbf{S}_{K_f}^{H'_n}, \mathcal{E}_\mu^\vee\right)$. Both carry a natural structure of a module of the Hecke algebra

$$\mathcal{H}_{K_f}^{\mathrm{GL}'_{2n}} = C_c^\infty\left(K_f \backslash \mathrm{GL}'_{2n}\left(\mathbb{A}_f\right) / K_f\right)$$

respectively

$$\mathcal{H}_{K_f}^{H'_n} = C_c^\infty\left(\iota^{-1}(K_f) \backslash H'_n\left(\mathbb{A}_f\right) / \iota^{-1}(K_f)\right),$$

where the product is as usual given by convolution. Now since ι is proper, it defines a map between compactly supported cohomology groups

$$\iota^*: H_c^*(\mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee) \to H_c^*(\mathbf{S}_{K_f}^{H'_n}, \mathcal{E}_\mu^\vee).$$

Recall that for each $\sigma \in \mathrm{Aut}(\mathbb{C})$ there exists a σ-linear isomorphism

$$\sigma: E_\mu^\vee \to {}^\sigma E_\mu^\vee$$

of $\mathrm{GL}'_{2n}(\mathbb{F})$-representations. Thus, there exist natural σ-linear isomorphisms of Hecke algebra-modules

$$\sigma^*_{\mathrm{GL}'_{2n}}: H_c^*\left(\mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right) \to H_c^*\left(\mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}}, {}^\sigma\mathcal{E}_\mu^\vee\right)$$

and

$$\sigma^*_{H'_n}: H_c^*\left(\mathbf{S}_{K_f}^{H'_n}, \mathcal{E}_\mu^\vee\right) \to H_c^*\left(\mathbf{S}_{K_f}^{H'_n}, {}^\sigma\mathcal{E}_\mu^\vee\right),$$

as well as a morphism

$${}^\sigma\iota^*: H_c^*\left(\mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}}, {}^\sigma\mathcal{E}_\mu^\vee\right) \to H_c^*\left(\mathbf{S}_{K_f}^{H'_n}, {}^\sigma\mathcal{E}_\mu^\vee\right).$$

Then the following diagram commutes.

$$\begin{array}{ccc} H_c^*\left(\mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}}, {}^\sigma\mathcal{E}_\mu^\vee\right) & \xrightarrow{\iota^*} & H_c^*\left(\mathbf{S}_{K_f}^{H'_n}, \mathcal{E}_\mu^\vee\right) \\ \downarrow \sigma^*_{\mathrm{GL}'_{2n}} & & \downarrow \sigma^*_{H'_n} \\ H_c^*\left(\mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}}, {}^\sigma\mathcal{E}_\mu^\vee\right) & \xrightarrow{{}^\sigma\iota^*} & H_c^*\left(\mathbf{S}_{K_f}^{H'_n}, {}^\sigma\mathcal{E}_\mu^\vee\right) \end{array} \tag{5}$$

Lemma 5.3.1 ([14, Lemma 7.3])**.** *The $\mathcal{H}_{K_f}^{\mathrm{GL}'_{2n}}$-module $H_c^*\left(\boldsymbol{S}_{K_f}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)$ and the $\mathcal{H}_{K_f}^{H'_n}$-module $H_c^*\left(\boldsymbol{S}_{K_f}^{H'_n}, \mathcal{E}_\mu^\vee\right)$ are defined over $\mathbb{Q}(\mu)$ by taking $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\mu))$-invariant vectors under the action given by above $\sigma^*_{\mathrm{GL}'_{2n}}$ respectively $\sigma^*_{H'_n}$.*

If $K_f \subseteq K'_f$ consider the canonical map $\mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}} \to \mathbf{S}_{K'_f}^{\mathrm{GL}'_{2n}}$. This allows us to define the space

$$\mathbf{S}^{\mathrm{GL}'_{2n}} := \varprojlim_{K_f} \mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}}$$

as a projective limit. Note that $\mathcal{E}_\mu^\vee$ naturally extends to $\mathbf{S}^{\mathrm{GL}'_{2n}}$ and hence, the cohomology $H_c^*\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)$ is a $\mathrm{GL}'_{2n}(\mathbb{A}_f)$-module.

Proposition 5.3.2 ([14, Proposition 7.16, Theorem 7.23])**.** *There exists an inclusion of the space*

$$H^q_{cusp}\left(\mathrm{GL}'_{2n}, E_\mu^\vee\right) := \bigoplus_{\pi' \ cuspidal} H^q\left(\mathfrak{g}'_\infty, K'_\infty, \pi'_\infty \otimes E_\mu^\vee\right) \otimes \pi'_f$$

into $H_c^*\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)$ respecting the $\mathrm{GL}'_{2n}\left(\mathbb{A}_f\right)$-action. We denote by $H_c^*\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)\left(\pi'_f\right)$ the image of

$$H^q\left(\mathfrak{g}'_\infty, K'_\infty, \pi'_\infty \otimes E_\mu^\vee\right) \otimes \pi'_f$$

under this inclusion.

If K_f fixes π'_f, we obtain an inclusion $H_c^*\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)(\pi'_f) \hookrightarrow H_c^*\left(\boldsymbol{S}_{K_f}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)$ and we denote its image again by $H_c^*\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)(\pi'_f)$. Moreover, the map $\sigma^*_{\mathrm{GL}'_{2n}}$ respects the decomposition, i.e. if π' and ${}^\sigma\pi'$ are both cuspidal then

$$\sigma^*_{\mathrm{GL}'_{2n}}(H_c^*\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)(\pi'_f)) = H_c^*\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)({}^\sigma\pi'_f)$$

for $\sigma \in \mathrm{Aut}(\mathbb{C})$ and thus if the $\mathrm{Aut}(\mathbb{C})$-orbit of π' is cuspidal, $H_c^*\left(\boldsymbol{S}_{K_f}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)(\pi'_f)$ is defined over $\mathbb{Q}(\pi')$.

5.4 Let q_0 be the lowest degree in which the cohomology

$$H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \pi'_\infty \otimes E_\mu^\vee\right)$$

does not vanish. Thus, by Theorem 3.3.1

$$\mathbb{C} \cong H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}^{\eta_\infty}_{\psi_\infty}(\pi'_\infty) \otimes E_\mu^\vee\right) = \left(\bigwedge^{q_0}(\mathfrak{g}'_\infty, /\mathfrak{k}'_\infty)^* \otimes \mathcal{S}^{\eta_\infty}_{\psi_\infty}(\pi'_\infty) \otimes E_\mu^\vee\right)^{K'_\infty}.$$

We fix once and for all a generator of

$$H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}^{\eta_\infty}_{\psi_\infty}(\pi'_\infty) \otimes E_\mu^\vee\right)$$

as follows. First fix an Künneth-isomorphism

$$\mathfrak{K}\colon H^*\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}^{\eta_\infty}_{\psi_\infty}(\pi'_\infty) \otimes E_\mu^\vee\right) \xrightarrow{\cong} \bigotimes_{v \in V_\infty} H^*\left(\mathfrak{g}'_v, K'_v, \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v) \otimes E_{\mu_v}^\vee\right),$$

which is determined by the already fixed isomorphism

$$\mathcal{S}^{\eta_\infty}_{\psi_\infty}(\pi'_\infty) \cong \bigotimes_{v \in V_\infty} \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v),$$

and let $q_{0,v}$ be the lowest degree in which the cohomology

$$H^{q_{0,v}}\left(\mathfrak{g}'_v, K'_v, \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v) \otimes E_{\mu_v}^\vee\right) = \left(\bigwedge^{q_{0,v}}(\mathfrak{g}'_v/\mathfrak{k}'_v)^* \otimes \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v) \otimes E_{\mu_v}^\vee\right)^{K'_v}$$

does not vanish and similarly we fix Künneth-isomorphisms $\mathfrak{K}_\sigma$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. For $v \in V_\infty$ we then choose a generator of this space of the form

$$[\pi'_v] := \sum_{\underline{i}=\left(i_1,\ldots,i_{q_{0,v}}\right)} \sum_{\alpha=1}^{\dim E_{\mu_v}^\vee} X_{\underline{i}}^* \otimes \xi_{v,\alpha,\underline{i}} \otimes e_\alpha^\vee, \tag{6}$$

where

1. Pick a $\mathbb{L}$-basis $\{Y_i\}$ of $\mathfrak{h}'_v/(\mathfrak{h}'_v \cap \mathfrak{k}'_v)$.

2. Extend $\{Y_i\}$ to a $\mathbb{L}$-basis $\{X_i\}$ of $\mathfrak{g}'_v/\mathfrak{k}'_v$, set $\{X_i^*\}$ to the corresponding dual basis of $(\mathfrak{g}'_v/\mathfrak{k}'_v)^*$ and $X_{\underline{i}}^* := \bigwedge_{i\in\underline{i}} X_i^*$.
3. A $\mathbb{Q}(\mu)$-basis $e_\alpha^\vee$ of $E_{\mu_v}^\vee$.
4. For each α and $\underline{i}$ a vector $\xi_{v,\alpha,\underline{i}} \in \mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v)$.

We then set $[\pi'_\infty] := \mathfrak{K}^{-1}(\otimes_{v\in V_\infty}[\pi'_v])$. We further assume that the X_i's are a extension of a basis of $\mathfrak{h}'_\infty/(\mathfrak{h}'_\infty \cap \mathfrak{k}'_\infty)$, where $\mathfrak{h}'_\infty$ is the Lie algebra at infinity of $H'_n(\mathbb{A})$. Finally for $\sigma \in \mathrm{Aut}(\mathbb{C})$ we set

$$\sigma([\pi'_\infty]) := [({}^\sigma\pi')_\infty] := \mathfrak{K}_\sigma^{-1}\left(\bigotimes_{v\in V_\infty}[\pi'_{\sigma^{-1}\circ v}]\right).$$

Let K_f be an open compact subgroup of $\mathrm{GL}'_{2n}(\mathbb{A}_f)$ which fixes π'. A choice of such a generator $[\pi'_\infty]$ fixes an isomorphism of $\mathcal{H}_{K_f}^{\mathrm{GL}'_{2n}}$-module

$$\Theta_{\pi'}: \mathcal{S}_{\psi_f}^{\eta_f}(\pi'_f) \xrightarrow{\cong} H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)(\pi'_f)$$

defined by

$$\mathcal{S}_{\psi_f}^{\eta_f}(\pi') \xrightarrow{\cong} \mathcal{S}_{\psi_f}^{\eta_f}(\pi') \otimes H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}_{\psi_\infty}^{\eta_\infty}(\pi'_\infty) \otimes E_\mu^\vee\right) \xrightarrow{\cong}$$
$$\xrightarrow{\cong} H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}_{\psi}^{\eta}(\pi') \otimes E_\mu^\vee\right) \xrightarrow{\cong} H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \pi' \otimes E_\mu^\vee\right) \xrightarrow{\cong} H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)(\pi'_f),$$

where the first isomorphism is the one induced by $[\pi_\infty]$ and the third isomorphism is the one induced by the inverse of $\pi' \xrightarrow{\cong} \mathcal{S}_\psi^\eta(\pi')$.

Theorem 5.4.1. *For each $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{L})$ there exists a complex number*

$$\omega({}^\sigma\pi'_f) = \omega({}^\sigma\pi'_f, [{}^\sigma\pi'_\infty]) \in \mathbb{C}^*$$

such that

$$\Theta_{{}^\sigma\pi',0} := \omega({}^\sigma\pi'_f)^{-1}\,\Theta_{{}^\sigma\pi'}$$

is $\mathrm{Aut}(\mathbb{C})$ *invariant,* i.e.

$$\begin{array}{ccc}
\mathcal{S}_{\psi_f}^{\eta_f}(\pi'_f) & \xrightarrow{\Theta_{\pi',0}} & H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)(\pi'_f) \\
\Big\downarrow \sigma^* & & \Big\downarrow \sigma^*_{\mathrm{GL}'_{2n}} \\
\mathcal{S}_{\psi_f}^{{}^\sigma\eta_f}({}^\sigma\pi'_f) & \xrightarrow{\Theta_{{}^\sigma\pi',0}} & H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, {}^\sigma\mathcal{E}_\mu^\vee\right)({}^\sigma\pi'_f)
\end{array}$$

commutes. Hence, $\Theta_{\pi',0}$ maps the $\mathbb{Q}(\pi',\eta)$-structure of $\mathcal{S}_{\psi_f}^{\eta_f}(\pi')$ to the $\mathbb{Q}(\pi',\eta)$-structure of $H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)(\pi'_f)$ and $\omega(\pi'_f)$ is well defined up to multiplication by an element of $\mathbb{Q}(\pi',\eta)$.

Proof. Since

$$\Theta_{\pi'}: \mathcal{S}_{\psi_f}^{\eta_f}(\pi'_f) \xrightarrow{\cong} H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)(\pi'_f)$$

is a morphism of irreducible $\mathrm{GL}'_{2n}\left(\mathbb{A}_f\right)$-modules it follows from Schur's Lemma that there exists a complex number $\omega\left(\pi'_f\right)$ such that $\Theta_{\pi',0} = \omega\left(\pi_f\right)^{-1}\Theta_{\pi'}$ maps one $\mathbb{Q}\left(\pi',\eta\right)$-structure onto the other, since rational structures are unique up to homothethies, see [28, Proposition 3.1]. Consider now the vector $\xi^0_{\pi'_f}$ from Lemma 4.8.3 which generates $\mathcal{S}^{\sigma\eta_f}_{\psi_f}\left(\pi'_f\right)$ as a $\mathrm{GL}'_{2n}\left(\mathbb{A}_f\right)$-module. After rescaling $\omega\left({}^\sigma\pi'_f\right)$ by an element of $\mathbb{Q}\left(\pi',\eta\right)$ we have the equality

$$\sigma^*_{\mathrm{GL}'_{2n}}\left(\Theta_{\pi',0}\left(\xi^0_{\pi'_f}\right)\right) = \Theta_{{}^\sigma\pi',0}\left(\xi^0_{{}^\sigma\pi'_f}\right),$$

since both sides of the equation lie in the same $\mathbb{Q}(\pi',\eta)$-structure. Thus we proved the assertion. □

5.5 As in [15] we discuss now how the above introduced periods behave under twisting with an algebraic character $\chi = (\widetilde{\chi}\circ\det')\cdot|\det'|^b$, where $\widetilde{\chi}$ is a Hecke character of $\mathrm{GL}_1(\mathbb{A})$ of finite order and $b\in\mathbb{Z}$. In particular, for $v\in V_\infty$ the character $\widetilde{\chi}(\det')_v$ is the trivial one. For the rest of this section fix such a character χ. The following is an easy consequence of the respective definitions.

Lemma 5.5.1. *The representation $\pi'\otimes\chi$ is cohomological with respect to $(E_{\mu+b})^\vee = E^\vee_\mu\otimes\bigotimes_{v\in S_\infty}|\det'_v|^{-b}$. If π' admits a Shalika model with respect to η then $\pi\otimes\chi$ admits one with respect to $\chi^2\eta$ and hence, $\omega\left(\pi'_f\otimes\chi_f\right)$ is well defined up to a multiple of $\mathbb{Q}\left(\pi',\chi,\eta\right)^*$.*

We fix a splitting isomorphism

$$\chi \xrightarrow{\cong} \chi_\infty\otimes\chi_f \xrightarrow{\cong} \bigotimes_{v\in V_\infty}|\det'_v|^b\otimes\chi_f,$$

which extends to a splitting isomorphism

$$\pi'\otimes\chi \xrightarrow{\cong} \pi'_\infty\otimes\chi_\infty\otimes\pi'_f\otimes\chi_f \xrightarrow{\cong} \bigotimes_{v\in V_\infty}\pi'_v\otimes|\det'_v|^b\otimes\pi'_f\otimes\chi_f.$$

Having already fixed the generator $[\pi'_v]$ we set

$$[\pi'_v\otimes\chi_v] := \sum_{\underline{i}=\left(i_1,\dots,i_{q_{0,v}}\right)}\sum_{\alpha=1}^{\dim E^\vee_{\mu_v}} X^*_{\underline{i}}\otimes|\det'_v|^{-b}\xi_{v,\alpha,\underline{i}}\otimes e^\vee_\alpha$$

and

$$[\pi'_\infty\otimes\chi_\infty] := \mathfrak{K}^{-1}_\chi\left(\bigotimes_{v\in V_\infty}[\pi'_v\otimes\chi_v]\right),$$

where $\mathfrak{K}_\chi$ is defined as the map

$$\mathfrak{K}_\chi\colon H^{q_0}\left(\mathfrak{g}'_\infty,K'_\infty,\mathcal{S}^{\chi^2_\infty\eta_\infty}_{\psi_\infty}\left(\pi'_\infty\otimes\chi_\infty\right)\otimes\left(E_{\mu+b}\right)^\vee\right)\to$$

$$\to\bigotimes_{v\in V_\infty}H^{q_0}(\mathfrak{g}'_\infty,K'_\infty,\mathcal{S}^{\chi^2_v\eta_v}_{\psi_v}\left(\pi'_v\otimes\chi_v\right)\otimes\left(E_{\mu_v+b}\right)^\vee),$$

corresponding to the splitting isomorphism

$$\pi'_\infty \otimes \chi_\infty \xrightarrow{\cong} \bigotimes_{v \in V_\infty} \pi'_v \otimes |\det'_v|^{-b}.$$

Note that for $[\pi'_v]$ as in (6) we have then

$$[\pi'_v \otimes \chi_v] := \sum_{\underline{i}=(i_1,\ldots,i_{q_{0,v}})} \sum_{\alpha=1}^{\dim E^\vee_{\mu_v}} X^*_{\underline{i}} \otimes \xi_{v,\alpha,\underline{i}} |\det'_v|^b \otimes e^\vee_\alpha.$$

5.6 We quickly recall the definition of the Gauss sum of χ_f, see [29, VII, §7]. For $v \in V_f$ let $\mathfrak{c}_v$ be the conductor of χ_v and choose $c \in \mathrm{GL}_1(\mathbb{A}_f)$ such that for all $v \in V_f$

$$\mathrm{ord}_v(c_v) = -\mathrm{ord}_v(\mathfrak{c}) - \mathrm{ord}_v(\mathfrak{D}).$$

Set

$$\mathcal{G}(\chi_v, \psi_v, c_v) := \int_{\mathbb{O}_v^*} \chi_v(u_v)^{-1} \psi(c_v^{-1} u_v)\, \mathrm{d}_v,$$

where d_v is a Haar measure on $\mathbb{O}_v^*$ normalized such that $\mathbb{O}_v^*$ has volume 1. Then $\mathcal{G}(\chi_v, \psi_v, c_v)$ is nonzero for all finite places and 1 at the places where χ and ψ is unramified, see [11, Equation 1.22]. Hence, the global Gauss sum

$$\mathcal{G}(\chi_f, c) := \prod_{v \in V_f} \mathcal{G}(\chi_v, \psi_v, c_v)$$

is well-defined. From now on we fix one such c and write $\mathcal{G}(\chi_f) := \mathcal{G}(\chi_f, c)$. If $\chi = (\widetilde{\chi} \circ \det') \cdot |\det'|^b$ is a character of GL'_{2n} as above, we set

$$\mathcal{G}(\chi_f) := \mathcal{G}(\widetilde{\chi}_f |\cdot|_f^b).$$

The periods we defined in Theorem 5.4.1 behave under twisting with such a character as follows.

Theorem 5.6.1. *Let π' be a cuspidal irreducible representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ as in Section 5.1. Let $\chi = (\widetilde{\chi} \circ \det') \cdot |\det'|^b$ with $\widetilde{\chi}$ a Hecke character of $\mathrm{GL}_1(\mathbb{A})$ of finite order and $b \in \mathbb{Z}$. For each $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\mu))$ we have*

$$\sigma\left(\frac{\omega(\pi'_f \otimes \chi_f)}{\mathcal{G}(\chi_f)^{nd}\, \omega(\pi'_f)}\right) = \frac{\omega({}^\sigma\pi'_f \otimes {}^\sigma\chi_f)}{\mathcal{G}({}^\sigma\chi_f)^{nd}\, \omega({}^\sigma\pi'_f)}.$$

In order to prove this we need three lemmata about the following maps. The first map is

$$\begin{aligned} S_\chi \colon \mathcal{S}^\eta_\psi(\pi') &\to \mathcal{S}^{\eta\chi^2}_\psi(\pi' \otimes \chi) \\ \xi &\longmapsto \left(g \mapsto \chi(\det'(g))\, \xi(g)\right), \end{aligned}$$

which splits under our fixed splitting isomorphisms into the two maps

$$\begin{aligned} S_{\chi_f}\colon \mathcal{S}_{\psi_f}^{\eta_f}\left(\pi'_f\right) &\to \mathcal{S}_{\psi_f}^{\eta_f\chi_f^2}\left(\pi'_f\otimes\chi_f\right)\\ \xi_f &\longmapsto \left(g_f\mapsto\chi_f\left(\det{}'\left(g_f\right)\right)\xi_f\left(g_f\right)\right)\end{aligned}$$

and

$$\begin{aligned} S_{\chi_\infty}\colon \mathcal{S}_{\psi_\infty}^{\eta_\infty}\left(\pi'_\infty\right) &\to \mathcal{S}_{\psi_\infty}^{\eta_\infty\chi_\infty^2}\left(\pi'_\infty\otimes\chi_f\right)\\ \xi_\infty &\longmapsto \left(g_\infty\mapsto\chi_\infty\left(\det{}'\left(g_\infty\right)\right)\xi_\infty\left(g_\infty\right)\right).\end{aligned}$$

Moreover, we define

$$\begin{aligned} A_\chi\colon \pi' &\to \pi'\otimes\chi\\ \phi &\longmapsto \left(g\mapsto\chi\left(\det{}'(g)\right)\phi(g)\right),\end{aligned}$$

where we consider $\phi\in\pi'$ as a cusp form.

Lemma 5.6.2. *With S_{χ_f} as above,*

$$\sigma^*\circ S_{\chi_f}=\sigma\left(\left(\chi_f\left(t_\sigma\right)\right)^{-nd}\right)S_{{}^\sigma\chi_f}\circ\sigma^*=\left(\frac{\sigma\left(\mathcal{G}\left(\chi_f\right)\right)}{\mathcal{G}\left({}^\sigma\chi_f\right)}\right)^{-nd}S_{{}^\sigma\chi_f}\circ\sigma^*.$$

Let $1_{E_\mu^\vee}$ be the identity map on $E_\mu^\vee$ and let $\left(A_\chi\otimes 1_{E_\mu^\vee}\right)^*$ be the induced map on cohomology

$$\left(A_\chi\otimes 1_{E_\mu^\vee}\right)^*: H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}'_{2n}},\mathcal{E}_\mu^\vee\right)\left(\pi'_f\right)\to H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}'_{2n}},\mathcal{E}_\mu^\vee\right)\left(\pi'_f\otimes\chi_f\right).$$

Lemma 5.6.3. *With A_χ as above,*

$$\left(A_\chi\otimes 1_{E_\mu^\vee}\right)^*\circ\Theta_{\pi'}=\Theta_{\pi'\otimes\chi}\circ\mathcal{S}_{\chi_f}.$$

Lemma 5.6.4. *For any $\sigma\in\mathrm{Aut}\left(\mathbb{C}/\mathbb{Q}(\mu)\right)$ we have*

$$\sigma^*_{\mathrm{GL}'_{2n}}\circ\left(A_\chi\otimes 1_{E_\mu^\vee}\right)^*=\left(A_{{}^\sigma\chi}\otimes 1_{{}^\sigma E_\mu^\vee}\right)^*\circ\sigma^*_{\mathrm{GL}'_{2n}}$$

We will first show how Theorem 5.6.1 follows from the above lemmata.

Proof of Theorem 5.6.1. We compute

$$\left(A_{{}^\sigma\chi}\otimes 1_{{}^\sigma E_\mu^\vee}\right)^*\circ\sigma^*_{\mathrm{GL}'_{2n}}\circ\Theta_\pi$$

in two different ways. On the one hand

$$\left(A_{{}^\sigma\chi}\otimes 1_{{}^\sigma E_\mu^\vee}\right)^*\circ\sigma^*_{\mathrm{GL}'_{2n}}\circ\Theta_{\pi'}\overset{(5.4.1)}{=}\left(\frac{\sigma\left(\omega\left(\pi'_f\right)\right)}{\omega\left({}^\sigma\pi'_f\right)}\right)\left(A_{{}^\sigma\chi}\otimes 1_{{}^\sigma E_\mu^\vee}\right)^*\circ\Theta_{{}^\sigma\pi'}\circ\sigma^*\overset{(5.6.3)}{=}$$

$$= \left(\frac{\sigma\left(\omega\left(\pi'_f\right)\right)}{\omega\left({}^{\sigma}\pi'_f\right)}\right)\Theta_{{}^{\sigma}\pi'\otimes{}^{\sigma}\chi}\circ\mathcal{S}_{{}^{\sigma}\chi_f}\circ\sigma^{*}.$$

But on the other hand, we see

$$\left(A_{{}^{\sigma}\chi}\otimes 1_{{}^{\sigma}E_{\mu}^{\vee}}\right)^{*}\circ\sigma^{*}_{\mathrm{GL}'_{2n}}\circ\Theta_{\pi'} \overset{(5.6.4)}{=} \sigma^{*}_{\mathrm{GL}'_{2n}}\circ\left(A_{\chi}\otimes 1_{E_{\mu}^{\vee}}\right)^{*}\circ\Theta_{\pi'} \overset{(5.6.3)}{=} \sigma^{*}_{\mathrm{GL}'_{2n}}\circ\Theta_{\pi'\otimes\chi}\circ S_{\chi_f} \overset{(5.4.1)}{=}$$

$$= \left(\frac{\sigma\left(\omega\left(\pi'_f\otimes\chi_f\right)\right)}{\omega\left({}^{\sigma}\pi'_f\otimes{}^{\sigma}\chi_f\right)}\right)\Theta_{{}^{\sigma}\pi'\otimes{}^{\sigma}\chi}\circ\sigma^{*}\circ S_{\chi_f} \overset{(5.6.2)}{=}$$

$$= \left(\frac{\sigma\left(\omega\left(\pi'_f\otimes\chi_f\right)\right)}{\omega\left({}^{\sigma}\pi'_f\otimes{}^{\sigma}\chi_f\right)}\right)\left(\frac{\sigma\left(\mathcal{G}\left(\chi_f\right)\right)}{\mathcal{G}\left({}^{\sigma}\chi_f\right)}\right)^{-nd}\Theta_{{}^{\sigma}\pi'\otimes{}^{\sigma}\chi}\circ S_{{}^{\sigma}\chi_f}\circ\sigma^{*}.$$

Hence, the desired equality follows. □

Proof of Lemma 5.6.2. The first equality follows by inserting the definition of σ^* and noticing that the determinant of $\det'(\mathbf{t}_{\sigma}) = t_{\sigma}^{nd}$. The second equality follows from [6, Theorem 2.4.3], which states

$$\sigma\left(\mathcal{G}\left(\chi_f\right)\right) = \sigma\left(\chi_f\left(t_{\sigma}\right)\right)\mathcal{G}\left({}^{\sigma}\chi_f\right).$$

□

Proof of Lemma 5.6.3. We proceed as in the proof of [23, Proposition 2.3.7]. It is enough to show that the following three diagrams commute. The first one is

$$\begin{array}{ccc}
\mathcal{S}_{\psi_f}^{\eta_f}\left(\pi'_f\right) & \longrightarrow & \mathcal{S}_{\psi_f}^{\eta_f}\left(\pi'_f\right)\otimes H^{q_0}\left(\mathfrak{g}'_{\infty},K'_{\infty},\mathcal{S}_{\psi_{\infty}}^{\eta_{\infty}}\left(\pi'_{\infty}\right)\otimes E_{\mu}^{\vee}\right) \\
\downarrow & & \downarrow \\
\mathcal{S}_{\psi_f}^{\eta_f\chi_f^2}\left(\pi'_f\right) & \longrightarrow & \mathcal{S}_{\psi_f}^{\eta_f\chi_f^2}\left(\pi'_f\otimes\chi_f\right)\otimes H^{q_0}\left(\mathfrak{g}'_{\infty},K'_{\infty},\mathcal{S}_{\psi_{\infty}}^{\eta_{\infty}\chi_{\infty}^2}\left(\pi'_{\infty}\otimes\chi_{\infty}\right)\otimes\left(\left(E_{\mu+b}\right)^{\vee}\right)\right)
\end{array}$$

Here the map on the right is

$$S_{\chi_f}\otimes\left(S_{\chi_{\infty}}\otimes 1_{E_{\mu}^{\vee}}\right)^{*},$$

where $\left(S_{\chi_{\infty}}\otimes 1_{E_{\mu}^{\vee}}\right)^{*}$ is the map on cohomology induced by the isomorphism

$$S_{\chi_{\infty}}\otimes 1_{E_{\mu}^{\vee}}\colon\mathcal{S}_{\psi_{\infty}}^{\eta_{\infty}}\left(\pi'_{\infty}\right)\otimes E_{\mu}^{\vee}\to\mathcal{S}_{\psi_{\infty}}^{\eta_{\infty}\chi_{\infty}^2}\left(\pi'_{\infty}\otimes\chi_{\infty}\right)\otimes\left(E_{\mu+b}\right)^{\vee}.$$

From this, it follows that by our choice of generators, $[\pi_{\infty}]$ is mapped to $[\pi_{\infty}\otimes\chi_{\infty}]$, and hence, this diagram commutes. The second diagram is

$$\begin{array}{ccc}
\mathcal{S}_{\psi_f}^{\eta_f}\left(\pi'_f\right)\otimes H^{q_0}\left(\mathfrak{g}'_{\infty},K'_{\infty},\mathcal{S}_{\psi_{\infty}}^{\eta_{\infty}}\left(\pi'_{\infty}\right)\otimes E_{\mu}^{\vee}\right) & \longrightarrow & H^{q_0}\left(\mathfrak{g}'_{\infty},K'_{\infty},\mathcal{S}_{\psi}^{\eta}\left(\pi'\right)\otimes E_{\mu}^{\vee}\right) \\
\downarrow & & \downarrow \\
\mathcal{S}_{\psi_f}^{\eta_f\chi_f^2}\left(\pi'_f\otimes\chi_f\right)\otimes H^{q_0}\left(\mathfrak{g}'_{\infty},K'_{\infty},\mathcal{S}_{\psi_{\infty}}^{\eta_{\infty}\chi_{\infty}^2}\left(\pi'_{\infty}\otimes\chi_{\infty}\right)\otimes E_{\mu}^{\vee}\right) & \longrightarrow & H^{q_0}\left(\mathfrak{g}'_{\infty},K'_{\infty},\mathcal{S}_{\psi}^{\eta\chi^2}\left(\pi'\otimes\chi\right)\otimes\left(E_{\mu+b}\right)^{\vee}\right)
\end{array}$$

It is induced from the following commutative diagram, and hence, commutative itself.

$$\begin{array}{ccc} \mathcal{S}_{\pi_f}^{\eta_f}(\pi_f)\otimes\mathcal{S}_{\pi_\infty}^{\eta_\infty}(\pi_\infty)\otimes E_\mu^\vee & \longrightarrow & \mathcal{S}_\pi^\eta(\pi)\otimes E_\mu^\vee \\ \downarrow & & \downarrow \\ \mathcal{S}_{\pi_f}^{\eta_f\chi_f^2}(\pi_f\otimes\chi_f)\otimes\mathcal{S}_{\pi_\infty}^{\eta_\infty\chi_\infty^2}(\pi_\infty\otimes\chi_\infty)\otimes E_\mu^\vee & \longrightarrow & \mathcal{S}_\pi^{\eta\chi^2}(\pi\otimes\chi)\otimes(E_{\mu+b})^\vee \end{array}$$

Here the left respectively right map going from top to bottom is

$$S_{\chi_f}\otimes S_{\chi_\infty}\otimes 1_{E_\mu^\vee} \text{ respectively } S_\chi\otimes 1_{E_\mu^\vee}.$$

And the third diagram is

$$\begin{array}{ccc} H^{q_0}\left(\mathfrak{g}'_\infty,K'_\infty,\mathcal{S}_\psi^\eta(\pi')\otimes E_\mu^\vee\right) & \longrightarrow & H_c^{q_0}\left(\mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}},\mathcal{E}_\mu^\vee\right)(\pi'_f) \\ \downarrow & & \downarrow \\ H^{q_0}\left(\mathfrak{g}'_\infty,K'_\infty,\mathcal{S}_\psi^{\eta\chi^2}(\pi'\otimes\chi)\otimes\left(E_{\mu+b}\right)^\vee\right) & \longrightarrow & H_c^{q_0}\left(\mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}},\mathcal{E}_{\mu+b}^\vee\right)(\pi'_f\otimes\chi_f), \end{array}$$

where the left respectively right map going from top to bottom is

$$\left(S_\chi\otimes 1_{E_\mu^\vee}\right)^* \text{ respectively } \left(A_\chi\otimes 1_{E_\mu^\vee}\right)^*,$$

which therefore clearly commutes. Thus, all three diagrams commute, and the lemma is proven. □

Proof of Lemma 5.6.4. We will proceed as in the proof [23, Proposition 2.3.6]. Recall that the cuspidal cohomology $H_c^*(\mathbf{S}^{\mathrm{GL}'_{2n}},\mathcal{E}_\mu^\vee)(\pi'_f)$ embeds injectively into the cohomology

$$H_c^q\left(\mathbf{S}_{K_f}^{\mathrm{GL}'_{2n}},\mathcal{E}_\mu^\vee\right)$$

for a suitable small open compact subgroup K_f. If moreover K_f fixes χ, we can extend the map A_χ to

$$A_{\chi,K_\chi}\colon H_c^q\left(\mathbf{S}_{K_\chi}^{\mathrm{GL}'_{2n}},\mathcal{E}_\mu^\vee\right)\to H_c^q\left(\mathbf{S}_{K_\chi}^{\mathrm{GL}'_{2n}},\mathcal{E}_{\mu+b}^\vee\right),$$

coming from the map $\mathcal{E}_\mu^\vee\to\mathcal{E}_{\mu+b}^\vee$ given simply by twisting with χ_∞. After passing to the projective limit, we thus obtain a map

$$A'_\chi\colon H_c^q\left(\mathbf{S}^{\mathrm{GL}'_{2n}},\mathcal{E}_\mu^\vee\right)\to H_c^q\left(\mathbf{S}^{\mathrm{GL}'_{2n}},\mathcal{E}_{\mu+b}^\vee\right).$$

Recall for $\sigma\in\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\mu))$ the σ-linear morphism

$$\sigma^*_{\mathrm{GL}'_{2n}}\colon H_c^q\left(\mathbf{S}^{\mathrm{GL}'_{2n}},\mathcal{E}_\mu^\vee\right)\to H_c^q\left(\mathbf{S}^{\mathrm{GL}'_{2n}},\mathcal{E}_{\sigma\mu}^\vee\right)$$

which is induced by the σ-linear morphism $l_\sigma\colon E_\mu^\vee\to{}^\sigma E_\mu^\vee$ which permutes the places according to σ. We now identify $H_c^q\left(\mathbf{S}^{\mathrm{GL}'_{2n}},\mathcal{E}_\mu^\vee\right)$ with the de Rahm cohomology $H_{dR}^q\left(\mathbf{S}^{\mathrm{GL}'_{2n}},\mathcal{E}_\mu^\vee\right)$. The maps

A_χ and $\sigma^*_{\mathrm{GL}'_{2n}}$ have then analogous incarnations $A_{\chi,dR}$ and $\sigma^*_{\mathrm{GL}'_{2n},dR}$ for the de Rahm cohomology. It therefore suffices to prove that $A_{\chi,dR}$ is $\sigma^*_{\mathrm{GL}'_{2n},dR}$ invariant. Now for $\omega \in H^*_{dR}\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}^\vee_\mu\right)$ represented by a section one has

$$\sigma^*_{\mathrm{GL}'_{2n},dR}(A_{\chi,dR}(\omega))(g) = l_\sigma \circ A_{\chi,dR}(\sigma^*_{\mathrm{GL}'_{2n},dR}(\omega))(g),\ g \in \mathbf{S}^{\mathrm{GL}'_{2n}}.$$

By fixing again a small enough compact open subgroup K_f as above, we get the explicit description

$$l_\sigma \circ A_{\chi,K_f,dR}(\omega)(g) = \sigma(\chi(g)) \cdot l_\sigma \circ \omega(g) = A_{\sigma(\chi),dR,K_f}(\sigma^*_{\mathrm{GL}'_{2n},dR}(\omega))(g),\ g \in \mathbf{S}^{\mathrm{GL}'_{2n}}_{K_f}.$$

Passing again to the inductive limit, we therefore obtain that

$$\sigma^*_{\mathrm{GL}'_{2n},dR}(A_{\chi,dR}(\omega)) = A_{\sigma(\chi),dR}(\omega).$$

Finally,

$$\sigma(\chi) = {}^\sigma\chi \tag{7}$$

since both ${}^\sigma(\widetilde{\chi} \circ \mathrm{Nrd})$ and $\widetilde{\chi} \circ \mathrm{Nrd}$ are trivial at infinity by Theorem 3.3.1 and

$$|\det'|^b = {}^\sigma|\det'|^b = \sigma(|\det'|^b)$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. □

6 Critical values of of L-functions and their cohomological interpretation

Let π' be a cuspidal irreducible cohomological representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ as in Section 5.1 and consider the standard global L-function $L(s,\pi')$ of π'. Recall that a critical point of $L(s,\pi')$ is in our case a point $s_0 \in \frac{1}{2} + \mathbb{Z}$ such that both $L(s,\pi'_\infty)$ and $L(1-s,\pi'^\vee_\infty)$ are holomorphic at s_0. Write for $v \in V_\infty$

$$\mu_v = \left(\mu_{v,1}, \ldots, \mu_{v,2nd}\right).$$

We further assume that $|\mathrm{JL}|(\pi')$ is residual, *i.e.* $|\mathrm{JL}|(\pi') = \mathrm{MW}(\tau,k)$ for some cuspidal irreducible representation τ of GL_l with $lk = 2nd$, $k > 1$. Recall that l is even by Lemma 3.3.6. To calculate the critical values of the L-function it suffices to consider the meromorphic contribution of the local L-factors from the infinite places. Let μ' be the highest weight such that τ is cohomological with respect to $E^\vee_{\mu'}$. Therefore, the L-factor of $L(s,\pi'_\infty)$ is by Lemma 3.3.6, [14, Theorem 5.2] and [3, Theorem 19.1.(b)] up to a non-vanishing holomorphic function of the form

$$\prod_{v\in V_\infty} \prod_{i=1}^{k} \prod_{j=1}^{\frac{l}{2}} \Gamma\left(s + \mu'_{v,j} + \frac{l}{2} - j + \frac{k}{2} - i + 1\right)$$

and the one of $L(1-s,\pi'^\vee_\infty)$ is of the form

$$\prod_{v\in V_\infty} \prod_{i=1}^{k} \prod_{j=1}^{\frac{l}{2}} \Gamma\left(-s - \mu'_{v,\frac{l}{2}-j+1} + \frac{l}{2} - j + \frac{k}{2} - i + 2\right).$$

Recall that the poles of the Gamma function lie precisely on the non-positive integers and it is non-vanishing everywhere else. We say $s = \frac{1}{2}$ is *compatible with* μ if for all $v \in V_\infty$

$$-\mu'_{v,\frac{l}{2}} \leq 0 \leq -\mu'_{v,\frac{l}{2}+1}$$

and we say μ is admissible if there exists $p \in \mathbb{Z}$ such that

$$-\mu'_{v,\frac{l}{2}} \leq p \leq -\mu'_{v,\frac{l}{2}+1}$$

for all $v \in V_\infty$. If k is even, all half integers are critical values

$$\mathrm{Crit}\left(\pi'\right) := \{s = \frac{1}{2} + m : m \in \mathbb{Z}\}.$$

If k is odd, a critical value $s_0 = \frac{1}{2} + m$ of $L\left(s, \pi'\right)$ has to satisfy

$$-\mu'_{v,j} - \frac{l}{2} + j + i - \frac{k+1}{2} \leq m \leq -\mu'_{v,l-j+1} + \frac{l}{2} - j - i + \frac{k+1}{2}$$

for all $v \in V_\infty$, $j \in \{1, \ldots, \frac{l}{2}\}$ and $i \in \{1, \ldots, k\}$. Since $\mu'_{v,1} \geq \ldots \geq \mu'_{v,l}$, the critical values of π' are in this case

$$\mathrm{Crit}\left(\pi'\right) := \{s = \frac{1}{2} + m : -\mu'_{v,\frac{l}{2}} + \frac{k-1}{2} \leq m \leq -\mu'_{v,\frac{l}{2}+1} + \frac{1-k}{2},\ m \in \mathbb{Z},\ v \in V_\infty\}.$$

Note that if k is odd and there exists a critical value, μ is admissible and that by Lemma 3.3.6, $\mu'_{v,\frac{l}{2}} = \mu_{v,nd}$ and $\mu'_{v,\frac{l}{2}} = \mu_{v,nd+1}$.

6.1 Following [15] we define a map $\mathcal{T}^*$.

Proposition 6.1.1 ([15, Proposition 6.3.1])**.** *Let π' be an irreducible representation as in 5.1 and assume moreover that $n = 1$. Assume $\frac{1}{2}$ is a critical value of $L(s, \pi')$ and compatible with μ. Denote by w_v the weight such that $E_{\mu_v} \cong E^\vee_{\mu_v} \otimes \det'^{w_v}$ for each $v \in S_\infty$. Let $E_{(0,-w_v)}$ be the representation $\mathbf{1} \otimes \det^{-w_v}$ of $H'(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H'(\mathbb{C})}\left(E^\vee_{\mu_v}, E_{(0,-w_v)}\right) = 1$$

for all $v \in S_\infty$.

We then let $E_{0,-w} := \bigotimes_{v \in V_\infty} E_{(0,-w_v)}$ and write $\mathcal{E}_{(0,-w)}$ for the corresponding locally constant sheaf of $\mathbf{S}^{H'_n}_{K_f}$. Since $\frac{1}{2}$ is assumed to be compatible with μ and since $\mathbb{Q}(\mu)$ contains the splitting field of $\mathbb{D}$, [17, Lemma 4.8] shows that there exists a map $\mathcal{T} = \bigotimes_{v \in S_\infty} \mathcal{T}_{\circ v}$ in above space which is defined over $\mathbb{Q}(\mu)$ and we fix a choice of such a map. Lifting this map to cohomology we obtain a morphism

$$\mathcal{T}^* : H^*_c\left(\mathbf{S}^{H'_n}_{K_f}, \mathcal{E}^\vee_\mu\right) \to H^*_c\left(\mathbf{S}^{H'_n}_{K_f}, \mathcal{E}_{(0,-w)}\right).$$

For $\sigma \in \mathrm{Aut}(\mathbb{C})$ we define the twist of $\mathcal{T}$ as

$${}^\sigma\mathcal{T} = \bigotimes_{v \in S_\infty} \mathcal{T}_{\sigma^{-1} \circ v}$$

and denote the corresponding morphism on the cohomology by ${}^{\sigma}\mathcal{T}^*$. Since $\mathcal{T}$ is defined over $\mathbb{Q}(\mu)$, we therefore obtain for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\mu))$ that ${}^{\sigma}\mathcal{T}^* = \mathcal{T}^*$. We then have the following commutative diagram for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\mu))$.

$$\begin{array}{ccc} H_c^*\left(\mathbf{S}_{K_f}^{H_n'}, \mathcal{E}_\mu^\vee\right) & \xrightarrow{\mathcal{T}^*} & H_c^*\left(\mathbf{S}_{K_f}^{H_n'}, \mathcal{E}_{(0,-w)}\right) \\ \big\downarrow \sigma_{H_n'}^* & & \big\downarrow \sigma_{H_n'}^* \\ H_c^*\left(\mathbf{S}_{K_f}^{H_n'}, {}^{\sigma}\mathcal{E}_\mu^\vee\right) & \xrightarrow{{}^{\sigma}\mathcal{T}^*} & H_c^*\left(\mathbf{S}_{K_f}^{H_n'}, {}^{\sigma}\mathcal{E}_{(0,-w)}\right) \end{array} \tag{8}$$

The next step consists of translating the computation of a critical value of $L(s, \pi')$ into an instance of Poincaré duality. But in order to apply Poincaré duality, the highest or lowest degree in which $H^*\left(\mathfrak{g}_\infty', K_\infty', \pi' \otimes E_\mu^\vee\right)$ vanishes has to equal the dimension $\dim_{\mathbb{R}} \mathbf{S}_{K_f}^{H_n'}$, which implies $n = 1, d = 2$ and $k = 2$. Indeed, Lemma 3.3.6 implies that

$$r\left((nd)^2 - nd - 1\right) = r\left((nd)^2 - nd - \frac{nd}{2}(k-1)\right) \text{ or}$$

$$r\left((nd)^2 - nd - 1\right) = r\left((nd)^2 - nd + \frac{nd}{2}(k-1) + 1\right)$$

and only the first equation can be satisfied, which leads to the above restriction.

6.2 We therefore let π' be a representation as in 5.1 and assume moreover that $\mathbb{D}$ is quaternion and $n = 1$ and μ is admissible. By Theorem 4.7.2 the respective conditions on the $\mathrm{Aut}(\mathbb{C})$-orbit on π' hold unconditionally in this case and we set $q_0 := r$, which is the real dimension of $\mathbf{S}_{K_f}^{H_1'}$ and the lowest degree in which $H^*(\mathfrak{g}_\infty', K_\infty', \pi' \otimes E_\mu^\vee)$ does not vanish.

If π' is as above and $\frac{1}{2}$ is a critical value of $L(s, \pi')$ which is compatible with μ, we choose K_f small enough so that $\iota^{-1}\left(K_f\right)$ can be written as a product $K_{f,1} \times K_{f,2}$, η_f is trivial on $\det'\left(K_{f,2}\right)$ and K_f fixes π_f'. Let $\mathcal{C}$ be the set of connected components of $\mathbf{S}_{K_f}^{H_1'}$. Using [4, Theorem 5.1] we see that $\mathcal{C}$ is finite and each $C \in \mathcal{C}$ is a quotient of

$$H_{1,\infty}' / \left(K_\infty' \cap H_{1,\infty}'\right)$$

by a discrete subgroup of $H_1'(\mathbb{F})$. Recall that the Y_i's from in Section 5.4 give a basis of $\mathfrak{h}_\infty' / (\mathfrak{h}_\infty' \cap \mathfrak{k}_\infty')$. Thus, they give an orientation on $H_{1,\infty}' / \left(K_\infty' \cap H_{1,\infty}'\right)$, since $\mathfrak{h}_\infty' / (\mathfrak{h}_\infty' \cap \mathfrak{k}_\infty')$ is parallelizable and therefore on each $C \in \mathcal{C}$ we can now consider $\mathbf{1} \times \eta^{-1}$ as a global section of $\mathcal{E}_{(0,-w)}$ and denote the corresponding cohomology class as

$$[\eta] \in H_c^{q_0}\left(\mathbf{S}_{K_f}^{H_1'}, \mathcal{E}_{(0,-w)}\right).$$

Poincaré duality on each connected component of $\mathbf{S}_{K_f}^{H_1'}$ gives rise to a surjective map

$$H_c^{q_0}\left(\mathbf{S}_{K_f}^{H_1'}, \mathcal{E}_{(0,-w)}\right) \to \mathbb{C}$$

defined by

$$\theta \mapsto \int_{\mathbf{S}^{H_1'}_{K_f}} \theta \wedge [\eta] := \sum_{C \in \mathcal{C}} \int_C \theta \wedge [\eta]$$

Lemma 6.2.1. *This map commutes with twisting by an automorphism* $\sigma \in \mathrm{Aut}(\mathbb{C})$*, i.e.*

$$\sigma\left(\int_{\mathbf{S}^{H_1'}_{K_f}} \theta \wedge [\eta]\right) = \int_{\mathbf{S}^{H_1'}_{K_f}} \sigma^*_{H_1'}(\theta) \wedge [{}^\sigma\eta].$$

Proof. Indeed,

$$\sigma\left(\int_{\mathbf{S}^{H_1'}_{K_f}} \theta \wedge [\eta]\right) = \int_{\mathbf{S}^{H_1'}_{K_f}} \sigma^*_{H_1'}(\theta \wedge [\eta]) = \int_{\mathbf{S}^{H_1'}_{K_f}} \sigma^*_{H_1'}(\theta) \wedge \sigma^*_{H_1'}([\eta]).$$

Finally, by (7) we have that $\sigma^*_{H_1'}([\eta]) = [{}^\sigma\eta]$. □

The goal now is to compute the composition of maps

$$\begin{array}{ccccc} H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}_2'}_{K_f}, \mathcal{E}^\vee_\mu\right) & \xrightarrow{\iota^*} & H_c^{q_0}\left(\mathbf{S}^{H_1'}_{K_f}, \mathcal{E}^\vee_\mu\right) & \xrightarrow{\mathcal{T}^*} & H_c^{q}\left(\mathbf{S}^{H_1'}_{K_f}, \mathcal{E}_{(0,-w)}\right) \\ \Theta_{\pi',0}\big\uparrow & & & & \big\downarrow \int_{\mathbf{S}^{H_1'}_{K_f}} \\ \mathcal{S}^{\eta_f}_{\psi_f}\left(\pi'_f\right) & & & & \mathbb{C}, \end{array}$$

i.e.

$$\xi_{\phi_f} \mapsto \int_{\mathbf{S}^{H_1'}_{K_f}} \mathcal{T}^* \iota^* \Theta_{\pi,0}\left(\xi_{\phi_f}\right) \wedge [\eta]$$

and relate this expression to the critical values of the L-function. To proceed we need the following non-vanishing result. Recall for $v \in V_\infty$

$$[\pi_v'] = \sum_{\underline{i}=\left(i_1,\dots,i_{q_{0,v}}\right)} \sum_{\alpha=1}^{\dim E^\vee_{\mu_v}} X^*_{\underline{i}} \otimes \xi_{v,\alpha,\underline{i}} \otimes e^\vee_\alpha.$$

For each $\underline{i}$ write

$$\iota^*\left(X_{\underline{i}}\right) = s\left(\underline{i}\right) Y_1^* \wedge \ldots \wedge Y^*_{q_0},$$

where $s\left(\underline{i}\right)$ is some complex number. If $\frac{1}{2} \in \mathrm{Crit}(\pi')$ is compatible with μ, we know that $\mathcal{T}$ exists and $\zeta_v(s, \cdot) = P(s, \cdot)L(s, \pi_v')$ for $v \in V_\infty$ by Theorem 4.4.2. Since $\frac{1}{2}$ is critical, we know that $\zeta_v(\frac{1}{2}, \cdot)$ is well defined for all vectors in the Shalika model. We thus can set

$$c\left(\pi_v'\right) := \sum_{\underline{i}} \sum_{\alpha=1}^{\dim E_{\mu_v}} s\left(\underline{i}\right) \mathcal{T}\left(e^\vee_\alpha\right) \zeta_v\left(\frac{1}{2}, \xi_{v,\alpha,\underline{i}}\right)$$

and $c\left(\pi'_\infty\right) := \prod_{v \in V_\infty} c\left(\pi_v'\right)$.

For $s = \frac{1}{2} + m \in \mathrm{Crit}(\pi')$, the assumption on μ being admissible gives an integer p such that $\pi' \otimes |\det'|^p$ has critical value $\frac{1}{2}$ and $\frac{1}{2}$ is compatible with $\mu - p$. We fix a smallest such p, which exists, since $-\mu_{v,2} \leq p$ for all $v \in V_\infty$, and set

$$\pi'(m) = \pi' \otimes |\det|^p \tag{9}$$

and

$$c\left(\pi'_\infty, m\right) \coloneqq c\left(\pi'(m)_\infty\right).$$

Theorem 6.2.2 ([26, Theorem A.3])**.** *For π' as in 6.2 and $s = \frac{1}{2} + m \in \mathrm{Crit}(\pi')$, the expression $c(\pi'_\infty, m)$ does not vanish.*

Proof. We will assume without loss of generality that $s = \frac{1}{2}$ is critical. Since $c(\pi'_\infty) = \prod_{v \in V_\infty} c(\pi'_v)$ we fix a place $v \in V_\infty$. We set $H \coloneqq \mathrm{GL}_1(\mathbb{H}) \times \mathrm{GL}_1(\mathbb{H})$ and $G \coloneqq \mathrm{GL}_2(\mathbb{H})$ with maximal compact subgroup K'_H respectively K'. Since $\frac{1}{2}$ is critical, it follows from Theorem 4.4.1 that the local zeta integral at v gives a functional

$$\zeta_v\left(\frac{1}{2}, \cdot\right) \in \mathrm{Hom}_H\left(\pi'_v, \chi\right),$$

where $\chi = 1_{\mathrm{GL}_1(\mathbb{H})} \otimes \det'^{w_v}$. It is nonzero, since $\zeta_v(s, \cdot) = P(s, \cdot)L(s, \pi'_v)$ and there exists by Theorem 4.4.2 $\xi_{\pi',v}$ such that $P(s, \xi_{\pi',v}) = 1$. Since the L-factors at infinity are products of Gamma-functions and non-vanishing holomorphic functions, $\zeta_v(s, \xi_{\pi',v})$ also never vanishes. Thus $\zeta_v(s, \cdot)$ never vanishes and hence $\zeta_v(\frac{1}{2}, \cdot)$ is non-zero. Let j_2 be the inclusion

$$j_2\colon \mathfrak{h}'/\mathfrak{k}'_H \hookrightarrow \mathfrak{g}'/\mathfrak{k}'$$

and consider now the map

$$\begin{aligned} \mathrm{Hom}\left(\mathfrak{g}'/\mathfrak{k}', \pi'_v \otimes E^\vee_{\mu_v}\right) &\to \mathrm{Hom}\left(\mathfrak{h}'/\mathfrak{k}'_H, \chi \otimes E_{(0,-w_v)}\right) \\ f &\longmapsto \left(\zeta_v\left(\frac{1}{2}, \cdot\right) \otimes \mathcal{T}_v\right) \circ f \circ j_2 \end{aligned}$$

By [26, Theorem A.3] the induced map

$$c\colon H^1\left(\mathfrak{g}'_\infty, K', \pi'_v \otimes E^\vee_{\mu_v}\right) \to H^1\left(\mathfrak{h}', K'_H, \chi \otimes E_{(0,-w_v)}\right)$$

does not vanish on the one dimensional space $H^1\left(\mathfrak{g}'_\infty, K', \pi'_v \otimes E^\vee_{\mu_v}\right)$. Since it is generated by $[\pi'_v]$ we conclude that $c(\pi'_v) \neq 0$. □

We set

$$c(\pi_\infty, m)^{-1} \coloneqq \omega(\pi_\infty, m).$$

Remark. The proof of [26, Theorem A.3] relies crucially on the numerical coincidence, *i.e.* that either the lowest or highest nonvanishing degree of the $(\mathfrak{g}'_\infty, K'_\infty)$-cohomology

$$H^q\left(\mathfrak{g}'_\infty, K'_\infty, \pi'_\infty \otimes E^\vee_\mu\right)$$

is $\dim_{\mathbb{R}} \mathbf{S}^{H'_1}_{K_f}$.

Theorem 6.2.3. *Let π' be a cuspidal irreducible representation of $\mathrm{GL}_2'(\mathbb{A})$ as in 6.2. Assume $s = \frac{1}{2} \in \mathrm{Crit}(\pi')$ is compatible with μ and let $\xi^0_{\pi'_f}$ be the vector of Lemma 4.8.3. Then*

$$\int_{\mathbf{S}^{H_1'}_{K_f}} \mathcal{T}^* \iota^* \Theta_{\pi',0}\left(\xi^0_{\pi'_f}\right) \wedge [\eta] = \frac{L\left(\frac{1}{2}, \pi'_f\right) \prod_{v \in S_{\pi'_f,\psi}} P\left(\frac{1}{2}, \xi^0_{\pi',v}\right)}{\omega\left(\pi'_f\right) \omega\left(\pi'_\infty\right) \mathrm{vol}\left(\iota^{-1}\left(K_f\right)\right)}$$

for every small enough open compact subgroup K_f of $\mathrm{GL}_2'\left(\mathbb{A}_f\right)$.

Proof. The proof of this theorem can be carried out in the same way as the proof of [15, Theorem 6.7.1]. We only include it for completeness. Recall from Section 2.3 that

$$c = \mathrm{vol}\left(\mathbb{F}^* \backslash \mathbb{A}^* / \mathbb{R}^r_{>0}\right).$$

We choose K_f such that it fixes $\xi^0_{\pi'_f}$.

Plugging $\xi^0_{\pi'_f}$ in the definition of the terms of the integral and using the K_f-invariance of $\xi^0_{\pi'_f}$ we obtain the following identity.

$$\int_{\mathbf{S}^{H_1'}_{K_f}} \mathcal{T}^* \iota^* \Theta_{\pi',0}\left(\xi^0_{\pi'_f}\right) \wedge [\eta] =$$

$$= \mathrm{vol}\left(\iota^{-1}\left(K_f\right)\right)^{-1} c^{-1} \omega\left(\pi'_f\right)^{-1} \sum_{\underline{i},\alpha} s\left(\underline{i}\right) \mathcal{T}\left(e_\alpha\right) \int_{H_1'(\mathbb{F}) \backslash H_1'(\mathbb{A}) / \mathbb{R}^d_+} \eta \, \phi^0_{\underline{i},\alpha}\big|_{H_1'(\mathbb{A})} \,\mathrm{d}h,$$

where

$$\phi^0_{\underline{i},\alpha} := \left(\mathcal{S}^\eta_\psi\right)^{-1} \left(\bigotimes_{v \in V_\infty} \xi_{v,\underline{i},\alpha} \otimes \xi^0_{\pi'_f}\right).$$

We compute now the latter integral over $H_1'(\mathbb{F}) \backslash H_1'(\mathbb{A}) / \mathbb{R}^d_+$ for fixed $\underline{i}$ and α. Again plugging in the definitions yields

$$\int_{H_1'(\mathbb{F}) \backslash H_1'(\mathbb{A}) / \mathbb{R}^d_+} [\eta] \, \phi^0_{\underline{i},\alpha}\big|_{H_1'(\mathbb{A})} \,\mathrm{d}h =$$

$$= \int_{Z'(\mathbb{A}) H_1'(\mathbb{F}) \backslash H_1'(\mathbb{A})} \int_{Z'(\mathbb{F}) \backslash Z'(\mathbb{A}) / \mathbb{R}^d_+} \left(\phi^0_{\underline{i},\alpha}\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} z\right) \eta^{-1}\left(\det'\left(h_2 z\right)\right) dz\right) \mathrm{d}h_1 \,\mathrm{d}h_2.$$

We can now pull the $z = \mathrm{diag}\,(a,a)$-contribution out of $\phi^0_{\underline{i},\alpha}$ and $\eta^{-1}(\det')$, which yields a factor of $\omega(z)\, \eta\left(\det'(a)\right)^{-1} = 1$ and hence, the integral simplifies to

$$c \int_{Z'(\mathbb{A}) H_1'(\mathbb{F}) \backslash H_1'(\mathbb{A})} \phi^0_{\underline{i},\alpha}\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} z\right) \eta^{-1}\left(\det'(h_2)\right) \mathrm{d}h_1 \,\mathrm{d}h_2.$$

Recall the equality of Theorem 4.4.1 and the properties of the special vector $\xi^0_{\pi'_f}$

$$\int_{Z'(\mathbb{A}) H_1'(\mathbb{F}) \backslash H_1'(\mathbb{A})} \phi^0_{\underline{i},\alpha}\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} z\right) \left|\frac{\det'(h_1)}{\det'(h_2)}\right|^{s-\frac{1}{2}} \eta^{-1}\left(\det'(h_2)\right) \mathrm{d}h_1 \,\mathrm{d}h_2 =$$

$$= \zeta_\infty\left(s, \xi^0_{\infty,\underline{i},\alpha}\right)\zeta_f\left(s, \xi^0_{\pi'_f}\right) = \zeta_\infty\left(s, \xi^0_{\infty,\underline{i},\alpha}\right) L(s, \pi'_f) \prod_{v \in S_{\pi'_f,\psi}} P\left(\frac{1}{2}, \xi^0_{\pi',v}\right)$$

for $\mathfrak{Re}(s) >> 0$. But the integral converges absolutely for all s hence, we obtain the equality for all s. Recall that $L(s,\pi)$ is an entire function and hence, $L\left(\frac{1}{2}, \pi'_f\right) \in \mathbb{C}$ since $s = \frac{1}{2}$ is critical. Therefore,

$$\int_{Z'(\mathbb{A})H'_1(\mathbb{F})\backslash H'_1(\mathbb{A})} \phi^0_{\underline{i},\alpha}\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} z\right)\eta^{-1}\left(\det{}'(h_2)\right) \mathrm{d}h_1\,\mathrm{d}h_2 =$$

$$= \zeta_\infty\left(\frac{1}{2}, \xi^0_{\infty,\underline{i},\alpha}\right) L(\frac{1}{2}, \pi'_f) \prod_{v \in S_{\pi'_f,\psi}} P\left(\frac{1}{2}, \xi^0_{\pi',v}\right).$$

Plugging this in the above sum over $\underline{i}$ and α, we obtain the desired identity. □

We are now ready to prove our analog of [15, Theorem 7.1.2].

Theorem 6.2.4. *Let $\mathbb{D}$ be a quaternion algebra and let π' be a cuspidal irreducible cohomological representation of $\mathrm{GL}'_2(\mathbb{A})$ which admits a Shalika model with respect to η. Let μ be the highest weight such that π' is cohomological with respect to $E^\vee_\mu$ and assume μ is admissible. We further assume that $|\mathrm{JL}|(\pi')$ is residual,* i.e. *$|\mathrm{JL}|(\pi') = \mathrm{MW}(\tau, 2)$ for τ a cuspidal irreducible cohomological representation of $\mathrm{GL}_2(\mathbb{A})$. Let moreover $\chi = \widetilde{\chi} \circ \det'$, where $\widetilde{\chi}$ is a Hecke-character of $\mathrm{GL}_1(\mathbb{A})$ of finite order. Then, for $\frac{1}{2} + m$ a critical value of π',*

$$\frac{L\left(\frac{1}{2} + m, \pi'_f \otimes \chi_f\right)}{\omega\left(\pi'_f\right)\mathcal{G}\left(\chi_f\right)^4 \omega\left(\pi'_\infty, m\right)} \in \mathbb{Q}\left(\pi', \chi, \eta\right).$$

Proof. Again the proof can be adapted from [15] to the situation at hand. To show the claim it is enough to show that the ratio stays invariant under all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi', \chi, \eta))$. First assume that $m = 0$, $\frac{1}{2}$ is compatible with μ and $\chi = 1$. We are going to compute

$$\sigma\left(\int_{\mathbf{S}^{H'_1}_{K_f}} \mathcal{T}^* \iota^* \Theta_{\pi',0}\left(\xi^0_{\pi'_f}\right) \wedge [\eta]\right) \tag{10}$$

for some $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi', \chi, \eta))$ in two different ways, where K_f is a sufficiently small open compact subgroup of $\mathrm{GL}'_{2n}(\mathbb{A}_f)$. On the one hand, (10) equals by Theorem 6.2.3 and Lemma 4.8.3 to

$$\sigma\left(\frac{L\left(\frac{1}{2}, \pi'_f\right)\prod_{v \in S_{\pi'_f,\psi}} P\left(\frac{1}{2}, \xi^0_{\pi',v}\right)}{\omega\left(\pi'_f\right)\omega\left(\pi'_\infty\right)\mathrm{vol}\left(\iota^{-1}\left(K_f\right)\right)}\right) =$$

$$= \sigma\left(\frac{L\left(\frac{1}{2}, \pi'_f\right)}{\omega\left(\pi'_f\right)\omega\left(\pi'_\infty\right)}\right) \cdot \frac{\prod_{v \in S_{\pi'_f,\psi}} P\left(\frac{1}{2}, \xi^0_{\pi',v}\right)}{\mathrm{vol}\left(\iota^{-1}\left(K_f\right)\right)},$$

where we used that $\mathrm{vol}(\iota^{-1}(K_f)) \in \mathbb{Q}^*$. On the other hand , by pulling σ into the integral (10), we compute

$$\sigma\left(\int_{\mathbf{S}^{H_1'}_{K_f}} \mathcal{T}^*\iota^*\Theta_{\pi',0}\left(\xi^0_{\pi'_f}\right)\wedge[\eta]\right) \overset{(6.2.1)}{=} \int_{\mathbf{S}^{H_1'}_{K_f}} \sigma^*_{H_1'}(\mathcal{T}^*\iota^*\Theta_{\pi',0}\left(\xi^0_{\pi'_f}\right))\wedge[{}^\sigma\eta] \overset{(8)}{=}$$

$$= \int_{\mathbf{S}^{H_1'}_{K_f}} {}^\sigma\mathcal{T}^*\sigma^*_{H_1'}(\iota^*\Theta_{\pi',0}\left(\xi^0_{\pi'_f}\right))\wedge[{}^\sigma\eta] \overset{(5)}{=} \int_{\mathbf{S}^{H_1'}_{K_f}} {}^\sigma\mathcal{T}^*\iota^*\sigma^*_{\mathbf{S}^{\mathrm{GL}_2'}_{K_f}}(\Theta_{\pi',0}\left(\xi^0_{\pi'_f}\right))\wedge[{}^\sigma\eta] \overset{(5.4.1)}{=}$$

$$= \int_{\mathbf{S}^{H_1'}_{K_f}} {}^\sigma\mathcal{T}^*\iota^*\Theta_{{}^\sigma\pi',0}\left(\xi^0_{{}^\sigma\pi'_f}\right)\wedge[{}^\sigma\eta].$$

By Theorem 4.7.2 ${}^\sigma\pi'$ admits a Shalika model with respect to ${}^\sigma\eta$ and is cohomological and therefore the last integral equals by Theorem 6.2.3

$$\frac{L\left(\frac{1}{2},{}^\sigma\pi'_f\right)}{\omega\left({}^\sigma\pi'_f\right)\omega\left({}^\sigma\pi'_\infty\right)}\cdot\frac{\prod_{v\in S_{\pi'_f,\psi}} P\left(\frac{1}{2},\xi^0_{\pi',v}\right)}{\mathrm{vol}\left(\iota^{-1}\left(K_f\right)\right)},$$

which proves the assertion.

If $\frac{1}{2}+m$ is an arbitrary critical value and $\chi=1$, consider $\pi'(m)=\pi'\otimes|\det'|^p$, where p is as in (9) and hence, $\frac{1}{2}$ is compatible with $\mu-p$. Recall also that $\mathcal{G}\left(|\det'|^n_f\right)=1$, thus Theorem 5.6.1 proves the claim. Finally, to obtain the result for $\frac{1}{2}+m$ is an arbitrary critical value and $\chi\neq 1$, we apply Theorem 5.6.1 again and note that $\pi'_\infty=\pi'_\infty\otimes\chi_\infty$, since χ is of finite order. □

Recall that since $|\mathrm{JL}|(\pi')=\mathrm{MW}(\tau,2)$, the partial L-functions of π' and $\mathrm{MW}(\tau,2)$ coincide. We therefore obtain a new result on critical values for residual representations of GL_4. Note that for any place $v\in V_f$, $L\left(\frac{1}{2}+m,\pi'_v\otimes\chi_v\right)\in\mathbb{Q}(\pi',\chi)$ by Lemma 4.8.2 and by Theorem 4.3.2 π' admits a Shalika model with respect to ω_τ. The following is therefore an easy consequence of Theorem 6.2.4.

Theorem 6.2.5. *Let $\pi=\mathrm{MW}(\tau,2)$ be a discrete series representation of $\mathrm{GL}_4(\mathbb{A})$ such that there exists a cuspidal irreducible representation π' of $\mathrm{GL}_2'(\mathbb{A})$ with $|\mathrm{JL}|(\pi')=\pi$, which is cohomological with respect to the coefficient system $E^\vee_\mu$ and μ admissible. Let χ be a finite order Hecke-character of $\mathrm{GL}_1(\mathbb{A})$ and let $s=\frac{1}{2}+m$ be a critical value of π'. Then*

$$\frac{L\left(\frac{1}{2}+m,\pi_f\otimes\chi_f\right)}{\omega\left(\pi'_f\right)\mathcal{G}\left(\chi_f\right)^4\omega\left(\pi'_\infty,m\right)}\in\mathbb{Q}\left(\pi',\omega_\tau,\chi\right).$$

7 Proof of Theorem 4.4.1 and Theorem 4.4.2

We will now show how to adapt the proof given in [9] to the situation at hand. Almost all of the arguments remain unchanged and we only include them for completness. Throughout this section π' will be a cuspidal irreducible representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ which admits a Shalika model with respect to η and $\phi \in \pi$ will be a cusp form.

7.1 If H is an algebraic subgroup of GL'_{2n} containing Z'_{2n} we denote by

$$H^0(\mathbb{A}) = \{h \in H(\mathbb{A}) : |\det'(h)| = 1\}.$$

Given a Haar-measure $\mathrm{d}h$ on $H(\mathbb{A})$, there exists a Haar measure $\mathrm{d}z$ on $Z'_{2n}(\mathbb{F})\backslash Z'_{2n}(\mathbb{A})$ such that for all $s \in \mathbb{C}$ and f a smooth function on $H(\mathbb{A})$

$$\int_{Z_{2n}(\mathbb{A})H(\mathbb{F})\backslash H(\mathbb{A})} f(h)|\det'(h)|^s \,\mathrm{d}h = \int_{Z'_{2n}(\mathbb{F})\backslash Z'_{2n}(\mathbb{A})} |\det'(z)|^s \int_{H(\mathbb{F})\backslash H^0(\mathbb{A})} f(hz)\,\mathrm{d}h\,\mathrm{d}z, \qquad (11)$$

assuming the first integral converges. Indeed, this follows from the fact that $H^0(\mathbb{A})\backslash H(\mathbb{A})$ can be identified with $Z'^0_{2n}(\mathbb{A})\backslash Z'_{2n}(\mathbb{A}) \times Z'^0_{2n}(\mathbb{A})\backslash Z'_{2n}(\mathbb{A})$ and that the integral of $|\det'|^s$ over $Z'^0_{2n}(\mathbb{A})\backslash Z'_{2n}(\mathbb{A})$ is the same as the integral of $|\det'|^s$ over $Z'_{2n}(\mathbb{F})\backslash Z_{2n}(\mathbb{A})$. We will denote by 1_n the n-dimensional identity matrix. Let $q, p \in \mathbb{Z}^+$ be such that $p + q = 2n$, let $U'_{(q,p)} \subseteq P'_{(q,p)}$ be the corresponding unipotent subgroup of GL'_{2n} and let A be the group of diagonal matrices of GL_{2n} embedded into GL'_{2n}. We identify $U'_{(q,p)}$ from time to time with the linear space of $p \times q$ matrices $M'_{q,p}$. To each $\beta \in M'_{q,p}(\mathbb{F})$ we associate the character θ_β of $U'_{(q,p)}(\mathbb{A})$

$$u = \begin{pmatrix} 1_p & v \\ 0 & 1_q \end{pmatrix} \mapsto \psi\left(\mathrm{Tr}'(v\beta)\right).$$

Moreover, let $H = \mathrm{GL}'_p \times \mathrm{GL}'_q$ be the Levi-component of $P_{(q,p)}$. Then for

$$\gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \in H(\mathbb{F})$$

it is straightforward to see that

$$\theta_\beta\left(\gamma^{-1}u\gamma\right) = \theta_{\gamma_2^{-1}\beta\gamma_1}(u).$$

The additive group $M'_{q,p}(\mathbb{A})$ is isomorphic to $M_{dq,dp}(\mathbb{A})$. It is well known that the additive characters of the latter group are parametrized by the space of linear functionals $\mathrm{Hom}_{\mathbb{F}}\left(M_{dq,dp}(\mathbb{F}), \mathbb{F}\right)$ by identifying $X \in \mathrm{Hom}_{\mathbb{F}}\left(M_{dq,dp}(\mathbb{F}), \mathbb{F}\right)$ with the additive character

$$v \mapsto \psi(X(v))).$$

Identifying $M_{dq,dp}(\mathbb{F})$ with $M'_{q,p}(\mathbb{F})$ again, we obtain that all characters of $U'_{(q,p)}(\mathbb{A})$ are of the form θ_β and $\theta_\beta = \theta_{\beta'}$ if and only if β'. This allows us to consider for a cuspform $\phi \in \pi'$ its Fourier expansion

$$\phi(g) = \sum_{M'_{q,p}(\mathbb{F})} \phi_\beta(g),$$

where

$$\phi_\beta(g) := \int_{U'_{(q,p)}(\mathbb{F})\backslash U'_{(q,p)}(\mathbb{A})} \phi(gu)\,\theta_\beta(u)\,du.$$

It is again easy to see that

$$\phi_\beta(\gamma g) = \phi_{\gamma_2^{-1}\beta\gamma_1}(g)$$

and $\phi_0 = 0$, since ϕ is cuspidal.

Lemma 7.1.1.

$$\int_{H(\mathbb{F})\backslash H^0(\mathbb{A})} \sum_{\beta \in M'_{q,p}(\mathbb{F})} |\phi_\beta (h)| \, \mathrm{d}h < \infty$$

Proof. It suffices to show that the integral is finite over a standard Siegel set of H^0, *i.e.* let H_{2n} be the Cartan subgroup of GL'_{2n} consisting of the diagonal matrices with entries in a fixed maximal subfield $\mathbb{E} \subseteq \mathbb{D}$, let Ω be a compact subset of $\mathrm{GL}'_{2n}(\mathbb{A})$, let C be a positive constant and let $S(C)$ be the connected component of 1_{2n} of the diagonal matrices

$$a = \mathrm{diag}(a_1, \ldots, a_p, a_{p+1}, \ldots, a_{2n})$$

with $a \in H_{2n}$ satisfying $\left|\frac{a_i}{a_{i+1}}\right| \geq C$ for $i \neq p, 2n$ and $\prod_{i=1}^{p} a_i = \prod_{i=p+1}^{2n} a_i = 1$, see [22, Theorem 4.8]. Hence, we have to show that there exists a constant D such that

$$\sum_{\beta \in M'_{q,p}} |\phi_\beta (a\omega)| < D$$

for all $a \in S(C)$ and $\omega \in \Omega$. We consider the function $u \mapsto \phi(ua\omega)$, $u \in U'_{(q,p)}(\mathbb{A})$ as a smooth, periodic function in u for fixed a and ω. Then its Fourier series is also smooth and converges absolutely. To prove that this convergence is uniform, *i.e.* independent of a and ω, it suffices to show like in the proof of [9, Lemma 2.1] that firstly, there exists a compact open subgroup $U_f \subseteq U'_{(q,p)}(\mathbb{A}_f)$ such that

$$\phi\left(uu'a\omega\right) = \phi(ua\omega)$$

for all $u' \in U_f$ and secondly, there exists a constant D' independent of a and ω such that for any X of the enveloping universal algebra of $\mathfrak{u}_{(q,p),\infty}$, the Lie algebra of $\prod_{v \in V_\infty} U_{(q,p)}(\mathbb{F}_v)$,

$$|\lambda(X)\phi(ua\omega)| < D'.$$

Here we denote by λ the left action of U_∞ and ρ its right action. The existence of U_f as above follows immediately from the smoothness of ϕ, since Ω is compact and $S(C)$ normalizes $U'_{(q,p)}(\mathbb{A}_f)$.

To prove the second claim, we fix $v \in V_\infty$, a root α of H_{2n} in $U'_{(q,p)}$ and a root vector X_α of α in the Lie algebra of $\mathfrak{u}_{(q,p),\infty}$. Recall that since H_{2n} is a Cartan subgroup, such root vectors span $\mathfrak{u}_{(q,p),\infty}$. Then

$$\lambda(-X_\alpha)\phi(ua\omega) = \alpha(a_v)^{-1} \rho\left(\mathrm{ad}(\omega^{-1})X_\alpha\right)\phi(ua\omega).$$

Now $\mathrm{ad}(\omega^{-1})X_\alpha$ is a linear combination of basis elements of $\mathfrak{g}'_v$, whose coefficients are bounded, because Ω is compact. Since $a \in S(C)$, $\alpha(a_v)^{-1}$ is bounded by a constant multiple of $|a_p|^{-M}|a_{2n}|^M$ for some $M \geq 0$.

Therefore, $\lambda(-X_\alpha)\phi(ua\omega)$ is bounded above by

$$\sum_j |a_p|^{-M_j}|a_{2n}|^{M_j}|\phi_j(ua\omega)|,$$

for $\phi_j \in \pi$. The following lemma will be useful in this and the following proof.

Lemma 7.1.2 ([9, Lemma 2.2]). *Let ϕ be a cusp form of a reductive group G, which is invariant under the split component of the center of G. Let R be a maximal proper parabolic subgroup of G, let δ_R be the module of the group $R(\mathbb{A})$ and let Ω be a compact subset of $G(\mathbb{A})$. Then for every $M \geq 0$ there exists a constant D such that*

$$\delta_R(r)^M |\phi(r\omega)| \leq D$$

for all $r \in R(\mathbb{A})$ and $\omega \in \Omega$.

Since ua is contained in $P'_{(q-1,p+1)}(\mathbb{A})$ and $P'_{(q+1,p-1)}$ with respective modules

$$\delta_{P'_{(p-1,q+1)}}(ua) = |a_p|^{-2nd}, \delta_{P'_{(p+1,q-1)}}(ua) = |a_{p+1}|^{2nd},$$

we deduce that $|a_p|^{-M_j}|a_{2n}|^{M_j}|\phi_j(ua\omega)|$ is bounded above. This finishes the proof. □

7.2 The next step is to observe that even though we are dealing with matrices over a division algebra, Gauss elimination still holds true in $M'_{q,p}(\mathbb{F})$. Therefore, the $H(\mathbb{F}) = \mathrm{GL}'_p(\mathbb{F}) \times \mathrm{GL}'_q(\mathbb{F})$-orbits on $M'_{q,p}(\mathbb{F})$ under the action $\gamma \cdot \beta = \gamma_2^{-1}\beta\gamma_1$ are precisely given by the possible ranks of the matrices. To be more precise, we say a matrix β has rank r if it is in the orbit of

$$\beta_r := \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}.$$

The stabilizer $H_{1_r}(\mathbb{F})$ of the matrix β_r is the subgroup of matrices of the form

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \text{ with } g_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}, g_2 = \begin{pmatrix} a_2 & 0 \\ c_1 & d_1 \end{pmatrix}, \tag{12}$$

where d_1 is a square matrix of dimension r, a_1 is a square matrix of dimension $q-r$ and a_2 is a square matrix of dimension $p-r$. Now we can write

$$\phi(g) = \sum_{r=1}^{\min(q,p)} \sum_{\gamma \in H_{1_r}(\mathbb{F}) \backslash H(\mathbb{F})} \phi_{\beta_r}(\gamma g).$$

Next comes the proof of the generalization of [9, Proposition 2.1].

Proposition 7.2.1. *Let $q > p$. Then for any cusp form $\phi \in \pi'$,*

$$\int_{G'_q(\mathbb{F})\backslash G'^0_q(\mathbb{A})} \phi\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1_p \end{pmatrix}\right) \mathrm{d}g_1 = 0.$$

Proof. By Lemma 7.1.1 we can exchange the integral and sum in the Fourier series of ϕ, *i.e.*

$$\int_{G'_q(\mathbb{F})\backslash G'^0_q(\mathbb{A})} \phi\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1_p \end{pmatrix}\right) \mathrm{d}g_1 = \int_{G'_q(\mathbb{F})\backslash G'^0_q(\mathbb{A})} \sum_{r=1}^{p} \sum_{\gamma \in H_{1_r}(\mathbb{F})\backslash H(\mathbb{F})} \phi_{\beta_r}\left(\gamma \begin{pmatrix} g_1 & 0 \\ 0 & 1_p \end{pmatrix}\right) \mathrm{d}g_1 =$$

$$= \sum_{r=1}^{q} \sum_{\gamma \in H_{1_r}(\mathbb{F}) \backslash H(\mathbb{F})} \int_{G'_q(\mathbb{F}) \backslash G'^{0}_q(\mathbb{A})} \phi_{\beta_r} \left(\gamma \begin{pmatrix} g_1 & 0 \\ 0 & 1_p \end{pmatrix} \right) \mathrm{d}g_1.$$

This equals by (12)

$$\sum_{r=1}^{q} \sum_{\gamma_2} \int_{P_{(q-r,r)}(\mathbb{F}) \backslash G^0_q{}'(\mathbb{A})} \phi_{\beta_r} \left(\begin{pmatrix} g_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \right) \mathrm{d}g_1,$$

where γ_2 is a matrix of $G'_q(\mathbb{F})$ modulo matrices of the form $\begin{pmatrix} a_2 & 0 \\ c_1 & 1_r \end{pmatrix}$. Taking the integral first over

$$U'_{(q-r,r)}(\mathbb{F}) \backslash U'_{(q-r,r)}(\mathbb{A})$$

and then over

$$P_{(q-r,r)}(\mathbb{F})\, U'_{(q-r,r)}(\mathbb{A}) \backslash R(\mathbb{A})$$

shows that the claim follows if

$$\int_{U'_{(q-r,r)}(\mathbb{F}) \backslash U'_{(q-r,r)}(\mathbb{A})} \phi_{\beta_r}(vg)\, \mathrm{d}v \tag{13}$$

vanishes for all $g \in \mathrm{GL}'_{2n}(\mathbb{A})$. To see this, observe first that the character θ_{β_r} can be extended to $U'_{(q,p)}(\mathbb{A}) U'_{(q-r,r)}(\mathbb{A})$, trivially on $U'_{(q-r,r)}$, since $U'_{(q-r,r)}$ normalizes $U'_{(q,p)}$. Thus, the integral (13) can be written as

$$\int_{U'_{(q,p)} U'_{(q-r,r)}(\mathbb{F}) \backslash U'_{(q,p)} U'_{(q-r,r)}(\mathbb{A})} \phi(u'g)\, \theta_{\beta_r}(u')\, du'.$$

This integral however factors through an integral over $U'_{(p-r,p-r)}(\mathbb{F}) \backslash U'_{(p-r,q-r)}(\mathbb{A})$, on which θ_{β_r} is trivial. Since ϕ is cuspidal, the assertion follows. □

7.3 We now come to the proof Theorem 4.4.1. Before we begin the proof, let us remark the following.

Remark. If π' admits a Shalika model with respect to η, π'_v admits a Shalika model with respect to η_v, *i.e.* a continous intertwining map

$$(\pi'_v)^\infty \to \mathrm{Ind}_{\mathcal{S}(\mathbb{F}_v)}^{\mathrm{GL}'_{2n}(\mathbb{F}_v)} (\psi_v \otimes \eta_v)$$

if $v \in V_\infty$ and a morphism of $\mathrm{GL}'_{2n}(\mathbb{F}_v)$-representations

$$\pi'_v \to \mathrm{Ind}_{\mathcal{S}(\mathbb{F}_v)}^{\mathrm{GL}'_{2n}(\mathbb{F}_v)} (\psi_v \otimes \eta_v)$$

if $v \in V_f$.

In both cases Frobenius reciprocity gives us a continuous morphism

$$\lambda_v \in \mathrm{Hom}_{\mathcal{S}(\mathbb{F}_v)} \left((\pi'_v)^\infty, \psi_v \otimes \eta_v \right) \text{ respectively } \lambda_v \in \mathrm{Hom}_{\mathcal{S}(\mathbb{F}_v)} \left(\pi'_v, \psi_v \otimes \eta_v \right).$$

If $v \in V_\infty$ the so obtained map is a priori just an intertwiner of group actions, but not necessarily continuous. However, the space of smooth vectors satisfies the Heine-Borel property, *i.e.* a subset of $(\pi'_v)^\infty$ is compact if and only if it is bounded on bounded sets and closed. Since a linear map of Fréchet spaces is continuous if and only if it is bounded, the claim follows. If $v \in V_f$ we obtain λ_v without any extra steps.

For a cuspform $\phi = \otimes_{v\in V} \phi_v \in \pi'$ we have that $|\phi(g)| \leq \beta(\phi)$, where β is a semi-norm on $(\pi'_\infty)^\infty \otimes \pi_f^{K_f}$ for any open compact subgroup K_f, since cusp forms are of rapid decay. Letting λ be the Shalika functional associated via Frobenius reciprocity to our Shalika model, we obtain that $g \mapsto \lambda(\pi'(g)\phi)$ is bounded. Thus, if restrict λ to smooth vectors obtain so the local Shalika functionals λ_v, $v \in V$, we also have that $g_v \mapsto \lambda_v(\pi'_v(g_v)\phi_v)$ is bounded for all $v \in V$.

Theorem 7.3.1. *Let π' be a cuspidal irreducible representation of $\mathrm{GL}'_{2n}(\mathbb{A})$. Assume π' admits a Shalika model with respect to η and let $\phi \in \pi'$ be a cusp form. Consider the integrals*

$$\Psi(s,\phi) := \int_{Z'_{2n}(\mathbb{A})H'_n(\mathbb{F})\backslash H'_n(\mathbb{A})} \phi\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2\end{pmatrix}\right)\left|\frac{\det'(h_1)}{\det'(h_2)}\right|^{s-\frac{1}{2}} \eta(h_2)^{-1}\, \mathrm{d}h_1\,\mathrm{d}h_2,$$

$$\zeta(s,\phi) := \int_{\mathrm{GL}'_n(\mathbb{A})} \mathcal{S}_\psi^\eta(\phi)\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1\end{pmatrix}\right)|\det'(g_1)|^{s-\frac{1}{2}}\,\mathrm{d}g_1.$$

Then $\Psi(s,\phi)$ converges absolutely for all s and $\zeta(s,\phi)$ converges absolutely if $\mathfrak{Re}(s) >> 0$. Moreover, if $\zeta(s,\phi)$ converges absolutely, $\Psi(s,\phi) = \zeta(s,\phi)$.

Proof. We apply Lemma 7.1.2 to the case $R' = P'_{(n,n)} \subseteq \mathrm{GL}'_{2n}$ to see that

$$\phi\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2\end{pmatrix}\right)\left|\frac{\det'(h_1)}{\det'(h_2)}\right|^{M}$$

is bounded above for any M, hence, $\Psi(s,\phi)$ converges absolutely. Indeed, recall that

$$Z'_n(\mathbb{A})\mathrm{GL}'_n(\mathbb{F})\backslash\mathrm{GL}'_n(\mathbb{A})$$

has finite volume and hence

$$Z'_{2n}(\mathbb{A})H'_n(\mathbb{F})\backslash H'_n(\mathbb{A}) = (1_n \times Z'_n(\mathbb{A}))\Omega,$$

where Ω has finite volume. Since above M can be chosen arbitrarily small, the claim follows. For a suitable measure $\mathrm{d}z$ on $Z'_{2n}(\mathbb{F})\backslash Z'_{2n}(\mathbb{A})$ we have by (11)

$$\Psi(s,\phi) = \int_{Z'_{2n}(\mathbb{F})\backslash Z'_{2n}(\mathbb{A})} |\det'(z)|^{s-\frac{1}{2}}$$

$$\int_{\mathrm{GL}'_n(\mathbb{F})\backslash\mathrm{GL}'^0_n(\mathbb{A})}\int_{\mathrm{GL}'_n(\mathbb{F})\backslash\mathrm{GL}'^0_n(\mathbb{A})} \phi\left(\begin{pmatrix} h_1 z & 0 \\ 0 & h_2\end{pmatrix}\right)\eta(h_2)\,\mathrm{d}h_1\,\mathrm{d}h_2\,\mathrm{d}z.$$

Inserting the Fourier series we see that the contribution of the matrices with rank $r < n$ is 0 by Proposition 7.2.1 and hence,

$$\int_{\mathrm{GL}'_n(\mathbb{F})\backslash\mathrm{GL}'^0_n(\mathbb{A})}\int_{\mathrm{GL}'_n(\mathbb{F})\backslash\mathrm{GL}'^0_n(\mathbb{A})} \phi\left(\begin{pmatrix} h_1 z & 0 \\ 0 & h_2\end{pmatrix}\right)\eta(h_2)\,\mathrm{d}h_1\,\mathrm{d}h_2 =$$

$$= \int_{\mathrm{GL}'_n(\mathbb{F})\backslash \mathrm{GL}'^0_n(\mathbb{A}) \times \mathrm{GL}'_n(\mathbb{F})\backslash \mathrm{GL}'^0_n(\mathbb{A})} \sum_{\gamma_1 \in \mathrm{GL}'_n(\mathbb{F})} \phi_{\beta_n}\left(\begin{pmatrix} \gamma_1 h_1 z & 0 \\ 0 & h_2 \end{pmatrix}\right) \eta(h_2)\, \mathrm{d}h_1\, \mathrm{d}h_2. \tag{14}$$

Contracting the sum and the integral and performing a change of variables, it follows that (14) is equal to

$$\int_{\mathrm{GL}'^0_n(\mathbb{A})} \mathcal{S}^\eta_\psi(\phi)\left(\begin{pmatrix} gz & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}\right) \eta(g)\, \mathrm{d}g.$$

Thus,

$$\Psi(s,\phi) = \int_{Z'_{2n}(\mathbb{F})\backslash Z'_{2n}(\mathbb{A})} |\det{}'(z)|^{s-\frac{1}{2}} \int_{\mathrm{GL}'^0_n(\mathbb{A})} \mathcal{S}^\eta_\psi(\phi)\left(\begin{pmatrix} gx & 0 \\ 0 & 1_n \end{pmatrix}\right) \eta(g)\, \mathrm{d}g\, \mathrm{d}z = \zeta(s,\phi), \tag{15}$$

where the last equation is valid only once we show that $\zeta(s,\phi)$ converges absolutely for $\mathfrak{Re}(s) >> 0$. To show the convergence we use the Dixmier-Malliavin theorem.

Theorem 7.3.2 (Dixmier-Malliavin theorem)**.** *Suppose G to be a Lie group and π a continuous representation of G on a Fréchet space V. Then every smooth vector $v \in V^\infty$ can be represented as a finite sum*

$$v = \sum_k \pi(f_k)\, v_k,$$

with $v_k \in V$, f_k a smooth, compactly supported function on G and

$$\pi(f)\, w := \int_G f(x)\, \pi(x)\, w\, \mathrm{d}x$$

for some fixed Haar measure on G.

Remark. Note that if G is a reductive group over $\mathbb{F}$ and (π'_f, V_f) is a smooth representation of $G(\mathbb{A}_f)$, we can write $v = \pi(f)v := \int_{G(\mathbb{A}_f)} f(x)\, \pi(x)\, v\, \mathrm{d}x$ for some smooth, *i.e.* locally constant, function f as every vector in V_f is fixed by some open compact subgroup.

We consider the action of $\mathrm{GL}'_{2n}(\mathbb{A})$ on $\mathcal{S}^\eta_\psi(\pi')$. The cusp form ϕ is a smooth vector in π', where we consider π' as a proper $\mathrm{GL}'_{2n}(\mathbb{A})$-subrepresentation of the corresponding L^2-space. Applied to our case this yields that $\mathcal{S}^\eta_\psi(\phi)(g)$ can be written as a finite sum

$$\sum_k \int_{U'_{(n,n)}(\mathbb{A})} \xi_k\left(g \begin{pmatrix} 1_n & u \\ 0 & 1_n \end{pmatrix}\right) \phi_k(u)\, \mathrm{d}u,$$

where the ϕ_k are compactly supported, smooth functions on $U'_{(n,n)}(\mathbb{A})$ and $\xi_k \in \mathcal{S}^\eta_\psi(\pi)$. Moreover, all ξ_k satisfy the equivariance property under η and ψ and are therefore bounded by the remark of Section 7.3. Applying this, we deduce

$$\mathcal{S}^\eta_\psi\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1_n \end{pmatrix}\right) = \sum_k \xi_k\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1_n \end{pmatrix}\right) \widehat{\phi_k}(g_1),$$

where $\widehat{\phi_k}$ is the Fourier transform of ϕ_k. Recalling the definition of $\zeta(s,\phi)$, we obtain

$$\zeta(s,\phi) = \int_{\mathrm{GL}_n'(\mathbb{A})} \sum_k \xi_k \left(\begin{pmatrix} g_1 & 0 \\ 0 & 1_n \end{pmatrix} \right) \widehat{\phi_k}(g_1) \left|\det{}'(g_1)\right|^{s-\frac{1}{2}} \mathrm{d}g_1$$

Since the ξ_k are bounded, $\zeta(s,\phi)$ is thus bounded by a multiple of

$$\sum_k \int_{\mathrm{GL}_n'(\mathbb{A})} \widehat{\phi_k}(g_1) \left|\det g_1\right|^{s-\frac{1}{2}} dg_1,$$

which converges absolutely for s with real part sufficiently large by [12, Theorem 13.8] and thus $\zeta(s,\phi)$ converges for $\mathfrak{Re}(s) >> 0$. □

7.4 We now come to the proof of Theorem 4.4.2.

Theorem 7.4.1. *Let π' be a cuspidal irreducible representation of $\mathrm{GL}_{2n}'(\mathbb{A})$ and assume π' admits a Shalika model with respect to η. Then for each place $v \in V$ and $\xi_v \in \mathcal{S}_{\psi_v}^{\eta_v}(\pi_v')$ there exists an entire function $P(s,\xi_v)$, with $P(s,\xi_v) \in \mathbb{C}[q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s}]$ if $v \in V_f$, such that*

$$\zeta_v(s,\xi_v) = P(s,\xi_v)\, L(s,\pi_v')$$

and hence, $\zeta_v(s,\xi_v)$ can be analytically continued to $\mathbb{C}$. Moreover, for each place v there exists a vector ξ_v such that $P(s,\xi_v) = 1$. If v is a place where neither π' nor ψ ramify this vector can be taken as the spherical vector $\xi_{\pi_v'}$ normalized by $\xi_{\pi_v'}(\mathrm{id}) = 1$.

We follow closely the proofs of [9, Proposition 3.1, Proposition 3.2]. We denote by $S(M_{s,t})$ respectively $S(M_{s,t}')$ the space of Schwartz-functions on $M_{s,t}(\mathbb{F}_v)$ respectively $M_{s,t}'(\mathbb{F}_v)$.

Proof. The first step is to prove the following lemma

Lemma 7.4.2. *There exists, depending on ξ_v, a positive Schwartz-function $\Theta \in S(M_{n,n})$, such that*

$$\left| \xi_v \left(\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix} g \right) \right) \right| \leq \Theta\left(b^{-1} g_1 a\right)$$

for the Iwasawa decomposition

$$g = u \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} k,\ u \in U_{(n,n)}'(\mathbb{F}_v),\ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in H_n'(\mathbb{F}_v),\ k \in K_v'.$$

Proof. We first assume that we are in the archimedean case. Using the Dixmier-Malliavin theorem, it is enough to prove the claim in the case ξ_v being of the form

$$\int_{\mathrm{GL}_{2n}'(\mathbb{F}_v)} \xi_{v,1}(gh)\, \Psi(h)\, \mathrm{d}_v h,$$

where Ψ is smooth function of $\mathrm{GL}_{2n}'(\mathbb{F}_v)$ with compact support. Write g as

$$h = \begin{pmatrix} 1_n & u_1 \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} k_1,\ g = \begin{pmatrix} 1_n & u_2 \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix} k_2.$$

We compute

$$\xi_v\left(\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1_n \end{pmatrix} g\right)\right) = \psi\left(\mathrm{Tr}'\left(g_1 u_2\right)\right) \int_{\mathrm{GL}'_n(\mathbb{F}_v)\times\mathrm{GL}'_n(\mathbb{F}_v)\times K'_v} \xi_{v,1}\left(\begin{pmatrix} g_1 a_2 a_1 & 0 \\ 0 & b_2 b_1 \end{pmatrix} k_1\right)$$

$$\Xi\left(b_2^{-1} g_1 a_2; k, k_2, a_1, b_1\right) \left|\det'_v\left(a_1 b_1^{-1}\right)\right|^{-nd} \mathrm{d}_v a_1 \, \mathrm{d}_v b_1 \, \mathrm{d}_v k_1,$$

where $\Xi(v; k_1, k_2, a_1, b_2)$ is the Fourier transform of

$$u_1 \mapsto \Psi\left(k_2^{-1}\begin{pmatrix} 1_n & u_1 \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} k_2\right)$$

This function and its derivatives have compact support, which is independent of the parameters k_2, a_1, b_1, k_1. Thus, the respective Fourier transform are contained in a bounded set in the space of Schwartz-functions on $U'_{(n,n)}(\mathbb{F}_v)$. Hence, there exists a positive Schwartz-function Θ_1 and a function Θ_2 with compact support such that

$$\left|\Xi(v; k_1, k_2, a_1, b_1)\right| \leq \Theta_1(v)\,\Theta_2(a_1, b_1).$$

This is enough to show the majorization, since $\xi_{v,1}$ is bounded by the remark of Section 7.3.

In the case where $\mathbb{F}_v$ is non-archimedean we do not need the Dixmier-Malliavin lemma, since we automatically can write ξ_v in integral form by the remark after Theorem 7.3.2. □

In the next step we let $v \in V$ be a place and consider integrals of the form

$$Z(\xi_v, \Psi, s) := \int_{\mathrm{GL}'_{2n}(\mathbb{F}_v)} \xi_v(g)\,\Psi(g)\left|\det'_v(g)\right|^{s-\frac{2nd-1}{2}} \mathrm{d}_v g$$

for $\xi_v \in \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)$ and $\Psi \in S\left(M'_{2n,2n}\right)$. Since ξ_v is bounded, this integral converges for $\mathfrak{Re}(s) >> 0$, see for example the proof of [12, Theorem, 3.3].

Lemma 7.4.3. *The function*

$$\frac{Z(\xi_v, \Psi, s)}{L(s, \pi'_v)}$$

is meromorphic and if $v \in V_f$

$$\frac{Z(\xi_v, \Psi, s)}{L(s, \pi'_v)} \in \mathbb{C}[q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s}].$$

Moreover, there exists $\xi_{v,j} \in \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)$, $\Psi_j \in S\left(M'_{2n,2n}\right)$ *such that we can write the local L-factor as a finite sum of the form*

$$L\left(s, \pi'_v\right) = \sum_j Z\left(\xi_{v,j}, \Psi_j, s\right).$$

Proof. We first assume that $\mathbb{F}_v$ is non-archimedean. Let $I(\pi'_v)$ be the $\mathbb{C}$ vector-space spanned by the integrals of the form

$$Z(f, \Psi, s) := \int_{\mathrm{GL}'_{2n}(\mathbb{F}_v)} \Psi(g)\,f(g)\left|\det'_v(g)\right|^{s-\frac{2nd-1}{2}} \mathrm{d}_v g,$$

where f is a smooth matrix coefficient of π'_v and $\Psi \in S\left(M'_{2n,2n}\right)$. To be more precise, the integrals converge for $\mathfrak{Re}(s) >> 0$ and admit a meromorphic continuation. By [12, Theorem 3.3] $I(\pi'_v)$ is a $\mathbb{C}[q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s}]$-ideal in $\mathbb{C}\left(q_v^{s-\frac{1}{2}}\right)$ generated by $L(s,\pi'_v)$.

We will now show that the $\mathbb{C}$-vector space spanned by the $Z(\xi_v,\Psi,s)$ is $I(\pi'_v)$, which consequently will show the claims of the lemma. To do so we introduce the space $\mathscr{U}$ consisting of smooth matrix coefficients of the form

$$g \mapsto \int_{K'_v} \xi_v\left(k^{-1}g\right) e\left(x\right) \mathrm{d}_v k,\ g \in \mathrm{GL}'_{2n}(\mathbb{F}_v),\ \xi_v \in \mathcal{S}^{\eta_v}_{\psi_v}\left(\pi'_v\right),$$

where e is an idempotent under the usual convolution product on the functions supported on K'_v.

Given ξ_v and Ψ, we define

$$g \mapsto f\left(g\right) := \int_{K'_v} \xi_v\left(x^{-1}g\right) e\left(k\right) \mathrm{d}_v k,\ g \in \mathrm{GL}'_{2n}(\mathbb{F}_v),$$

which is a smooth matrix coefficient of π'_v and hence, for such f

$$Z\left(\xi_v,\Psi,s\right) = \int_{\mathrm{GL}'_{2n}(\mathbb{F}_v)} f\left(g\right)\Psi\left(g\right)\left|\det{}'_v\left(g\right)\right|^{s-\frac{2nd-1}{2}} \mathrm{d}_v g \in \mathscr{U}.$$

On the other hand, for every $f \in \mathscr{U}$ and Schwartz-function Ψ there exists $\xi_v \in \mathcal{S}^{\eta_v}_{\psi_v}\left(\pi'_v\right), \Psi' \in S\left(M'_{2n,2n}\right)$ such that $Z\left(f,\Psi,s\right) = Z\left(\xi_v,\Psi',s\right)$. Indeed,

$$Z(f,\Psi,s) = \int_{\mathrm{GL}'_{2n}(\mathbb{F}_v)} \int_{K'_v} \xi_v\left(k^{-1}g\right) e\left(k\right)\Psi\left(g\right)\left|\det{}'_v\left(g\right)\right|^{s-\frac{2nd-1}{2}} \mathrm{d}_v k\, \mathrm{d}_v g =$$

$$= \int_{\mathrm{GL}'_{2n}(\mathbb{F}_v)} \xi_v\left(g\right) \int_{K'_v} \left(e\left(k\right)\Psi\left(kg\right)\right| \det{}'_v\left(g\right)|^{s-\frac{2nd-1}{2}} \mathrm{d}_v k\, \mathrm{d}_v g = Z(\xi_v,\Psi',s),$$

where $\Psi'(g) := \int_{K'_v} e\left(k\right)\Psi\left(kg\right) \mathrm{d}_v k$. This shows that the space spanned by the $Z(\xi_v,\Psi,s)$ is the space spanned by $\mathscr{U}$. It therefore suffices to show that the span of $\mathscr{U}$ is $I(\pi'_v)$. But since $\mathscr{U}$ is closed under right translations under $\mathrm{GL}'_{2n}(\mathbb{F}_v)$ and π'_v is irreducible, any smooth matrix coefficient f of π'_v can be written as a finite sum

$$f\left(g\right) = \sum_i f_i\left(g_i g\right)$$

for some suitable $g_i \in \mathrm{GL}'_{2n}(\mathbb{F}_v)$ and $f_i \in \mathscr{U}$. Therefore, the final claim follows because then

$$Z(f,\Psi,s) = \int_{\mathrm{GL}'_{2n}(\mathbb{F}_v)} \sum_i f_i\left(g_i g\right)\Psi\left(g\right)\left|\det g\right|^{s-\frac{2nd-1}{2}} \mathrm{d}_v g =$$

$$= \int_{\mathrm{GL}'_{2n}(\mathbb{F}_v)} \sum_i f_i\left(g g_i\right)\Psi\left(g_i^{-1} g g_i\right)\left|\det g\right|^{s-\frac{2nd-1}{2}} \mathrm{d}_v g,$$

where the last expression is of the desired form. In the case where $\mathbb{F}_v$ is archimedean, we argue as above, replacing the action of $\mathrm{GL}'_{2n}\left(\mathbb{F}_v\right)$ by the action of the Lie algebra and K'_v and using the Dixmier-Malliavin lemma. □

We return now to the proof of Theorem 7.4.1 and assume from now on that v is archimedean, since the non-archimedean case can be dealt with analogously. We start with the second assertion. We will only prove the archimedean case, since the non-archimedean case follows analogously. Let $\mathrm{SL}'_{2n} := \{g \in \mathrm{GL}'_{2n} : \det'_v(g) = 1\}$ and $K_0 := \mathrm{SL}'_{2n}(\mathbb{F}_v) \cap K'_v$. Using the Iwasawa decomposition we can write

$$Z(\xi_v, \Psi, s) = \int_{H'_n(\mathbb{F}_v)\times U'_{(n,n)}(\mathbb{F}_v)\times K_0} \xi_v\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right) \Psi\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right)$$

$$|\det'_v(a)|^{s-\frac{1}{2}} |\det'_v(b)|^{s-\frac{nd-1}{2}} \,\mathrm{d}_v a \,\mathrm{d}_v b \,\mathrm{d}_v x \,\mathrm{d}_v k.$$

We introduce the function

$$\Xi(u, t, w; k) := \int_{H'_n(\mathbb{F}_v)} \Psi\left(\begin{pmatrix} x & y \\ 0 & w \end{pmatrix} k\right) \left(\mathrm{Tr}'(yt) - \mathrm{Tr}'(xu)\right) \mathrm{d}_v x \,\mathrm{d}_v y. \tag{16}$$

If we put issues of convergence aside for a moment, the Fourier inversion formula and a change of variables imply that

$$Z(\xi_v, f, s) = \int_{\mathrm{SL}'_{2n}(\mathbb{F}_v)\times \mathrm{GL}'_n(\mathbb{F}_v)} \xi_v\left(\begin{pmatrix} a & 0 \\ 0 & 1_n \end{pmatrix} x\right) |\det'_v(a)|^{s-\frac{1}{2}} \,\mathrm{d}\mu_\Psi(x) \,\mathrm{d}_v a, \tag{17}$$

where we define the measure μ_Ψ on $\mathrm{SL}'_{2n}(\mathbb{F}_v)$ by

$$\int_{\mathrm{SL}'_{2n}(\mathbb{F}_v)} \Psi(x) \,\mathrm{d}\mu_\Psi(x) :=$$

$$= \int_{\mathrm{GL}'_n(\mathbb{F}_v)\times U'_{(n,n)}(\mathbb{F}_v)\times K'_v} f\left(\begin{pmatrix} b^{-1} & 0 \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} 1_n & u \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} 1_n & 0 \\ 0 & b \end{pmatrix} k\right)$$

$$\Xi\left(u, b, b^{-1}; k\right) |\det'_v(b)|^{nd} \,\mathrm{d}_v b \,\mathrm{d}_v u \,\mathrm{d}_v k.$$

Let us now argue how to put the issues of convergence to rest in the integral of (17). Following [9] we consider the unimodular subgroup Q of GL'_{2n} consisting of matrices of the form

$$Q = \left\{\begin{pmatrix} b^{-1} & u \\ 0 & b \end{pmatrix} : b \in \mathrm{GL}'_n,\ u \in M'_{n,n}\right\}.$$

Thus, $\mathrm{d}\mu_\Psi = \mu_\Psi(q,k)\,\mathrm{d}_v q\,\mathrm{d}_v k$, where μ_Ψ is a smooth function on $Q(\mathbb{F}_v) \times K'_v$ and it and its derivatives are rapidly decreasing, *i.e.*

$$||q||^N |\mu_\Psi(q,k)|$$

is bounded for all natural numbers N, where $||q||$ denotes the usual height of q. Recall the majorization α of the beginning of the proof and the remark before Theorem 7.3.1 and that we obtained from Lemma 7.4.2 that

$$\int_{\mathrm{SL}'_{2n}(\mathbb{F}_v)\times \mathrm{GL}'_n(\mathbb{F}_v)} \left|\xi_v\left(\begin{pmatrix} a & 0 \\ 0 & 1_n \end{pmatrix} x\right) |\det'_v(a)|^{s-\frac{1}{2}}\right| \mathrm{d}\mu_\Psi(x) \,\mathrm{d}_v a \le$$

$$\leq \int_{\mathrm{GL}'_n(\mathbb{F}_v)\times Q(\mathbb{F}_v)} \Theta(b^{-1}ab^{-1})|\det'_v(a)|^{\mathfrak{Re}(s)-\frac{1}{2}}|\mu_\Psi(q,k)|\,\mathrm{d}_vq\,\mathrm{d}_vk\,\mathrm{d}_va$$

for a suitable Schwartz-function Θ. After changing $a\mapsto bab$, we can bound this integral for $\mathfrak{Re}(s)>>0$ by a multiple of

$$\int_{Q(\mathbb{F}_v)}|\det'_v(b)|^{2\mathfrak{Re}(s)-1}|\mu_\Psi(q,k)|\,\mathrm{d}_vq\,\mathrm{d}_vk\,\mathrm{d}_va,$$

which converges since μ_Ψ is rapidly decreasing. Thus, we justified the rewriting of the integral (17) and showed that the operator

$$\int_{Q\times K'_v}\pi'_v(qk)\,\mu_\varphi(q,k)\,\mathrm{d}_vq\,\mathrm{d}_vk \tag{18}$$

preserves the local Shalika model, since a priori the operator does not preserve smoothness.

By collecting the results so far, we can prove the following. By Lemma 7.4.3 we find $\xi_{v,j}\in\mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)$, $\Psi_j\in S\left(M'_{2n,2n}\right)$ such that

$$L\left(s,\pi'_v\right)\overset{(7.4.3)}{=}\sum_j\int_{\mathrm{GL}'_{2n}(\mathbb{F}_v)}\xi_{v,j}(g)\,\Psi_j(g)\,|\det'_v(g)|^{s-nd-\frac{1}{2}}\,\mathrm{d}_vg\overset{(17)}{=}$$

$$=\sum_j\int_{\mathrm{GL}'_n(\mathbb{F}_v)}\xi'_{v,j}\left(\begin{pmatrix}g_1&0\\0&1_n\end{pmatrix}\right)|\det'_v(g)|^{s-\frac{1}{2}}\,\mathrm{d}_vg_1,$$

where

$$\xi'_{v,j}(g)=\int_{\mathrm{SL}'_{2n}(\mathbb{F}_v)}\xi_{v,j}(gx)\,\mu_{\Psi_j}(x)\,\mathrm{d}_vx.$$

Since we showed that (18) preserves $\mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)$, we have $\xi'_{v,j}\in\mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)$ and therefore we proved the second claim of Theorem 7.4.1.

Next, we show the first claim of Theorem 7.4.1. We apply the Dixmier-Malliavin lemma to $Q\times K'_v$ and write

$$\xi_v(g)=\sum_j\int_{\mathrm{GL}'_n(\mathbb{F}_v)\times U'_{(n,n)}(\mathbb{F}_v)\times K'_v}\xi_{v,j}\left(g\begin{pmatrix}b^{-1}&0\\0&1_n\end{pmatrix}\begin{pmatrix}1_n&u\\0&1_n\end{pmatrix}\begin{pmatrix}1_n&0\\0&b\end{pmatrix}k\right)$$

$$\Gamma_j(u,b,k)\,|\det'_v(b)|^{nd}\,\mathrm{d}_vb\;\mathrm{d}_vu\;\mathrm{d}_vk,$$

where Γ_j are smooth functions with compact support on $\mathrm{GL}'_n(\mathbb{F}_v)\times U'_{(n,n)}(\mathbb{F}_v)\times K'_v$. Let Λ_j be the projection of the support of Γ_j to $U'_{(n,n)}(\mathbb{F}_v)$ and let $\Psi\in S\left(M'_{n,n}\right)$ be such that $\Psi\left(b^{-1}\right)=1$ for $b\in\bigcup_j\Lambda_j$, where we identify $U'_{(n,n)}$ with $M'_{n,n}$. Define

$$\Gamma'_j\left(\begin{pmatrix}a&x\\0&b\end{pmatrix};k\right):=\int_{U'_{(n,n)}(\mathbb{F}_v)\times U'_{(n,n)}(\mathbb{F}_v)}\Gamma_j(u,b,k)\,\Psi(v)\,\psi(\mathrm{Tr}(xu-yv))\,\mathrm{d}_vu\,\mathrm{d}_vv.$$

Then $\zeta_v(s,\xi_v)$ can be written as

$$\zeta_v(s,\xi_v) =$$
$$= \sum_j \int_{H'_n(\mathbb{F}_v)\times U'_{(n,n)}(\mathbb{F}_v)\times K'_v} \xi_{v,j}\left(\begin{pmatrix} a & x \\ 0 & b\end{pmatrix} k\right)\Gamma'_j\left(\begin{pmatrix} a & x \\ 0 & b\end{pmatrix}; k\right) \tag{19}$$
$$|\det'_v(a)|^{s+nd-\frac{1}{2}}||\det'_v(b)|^{s+nd-\frac{1}{2}}\,\mathrm{d}_v a\,\mathrm{d}_v b\,\mathrm{d}_v x\,\mathrm{d}_v k.$$

Let

$$\Omega_1 := \{(a,b)\in M'_{n,2n} : (a,b) : \mathbb{D}^{2n}\to\mathbb{D}^n \text{ surjective}\}.$$

The group $\mathrm{GL}'_n(\mathbb{F}_v)$ acts from the left and the group K'_v from the right on $\Omega_1(\mathbb{F}_v)$. The resulting action of $\mathrm{GL}'_n(\mathbb{F}_v)\times K'_v$ is transitive. The stabilizer of $(0,1_n)$ is the group $\left(k_2^{-1},k\right)$, where

$$k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2\end{pmatrix} \in K'_v \cap P'_{(n,n)}(\mathbb{F}_v)$$

for some k_1. Let

$$\Omega := \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in M'_{2n,2n} : (c,d)\in\Omega_1\right\}.$$

Let $\mathscr{S}(\Omega)$ be the space of smooth functions $\varphi\colon\Omega(\mathbb{F}_v)\to\mathbb{C}$ such that

1. $|a|^{2n}|b|^{2n}\varphi(g)$, $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ is bounded for all $n\in\mathbb{ZZ}$,
2. The projection of the support of ϕ to $\Omega_1(\mathbb{F}_v)$ is compact,
3. If D is a differential operator which commutes with additive changes in (a,b) then $D\varphi\in\mathscr{S}(\Omega)$.

Analogously we define the space

$$\Omega_0 := \left\{\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} \in M'_{2n,2n} : \det'_v(d)\neq 0\right\}$$

and $\mathscr{S}(\Omega_0\times K'_v)$. The natural map

$$r\colon\Omega_0(\mathbb{F}_v)\times K'_v\to\Omega(\mathbb{F}_v),\ (p,k)\mapsto pk$$

is surjective, proper, and a submersion and the inverse image of pk is

$$r^{-1}(pk) = \{\left(pk'^{-1},k'k\right) : k'\in K'_v\cap P'_{(n,n)}(\mathbb{F}_v)\}.$$

Lemma 7.4.4. *Let $\varphi\in\mathscr{S}(\Omega_0\times K')$. Then*

$$\varphi_*(pk) = \int_{K'_v\cap P_{(n,n)}(\mathbb{F}_v)} \varphi\left(pk'^{-1},k'k\right)\mathrm{d}_v k'$$

belongs to $\mathscr{S}(\Omega)$.

We postpone the proof of this lemma to see how it finishes the proof of the first claim of Theorem 7.4.1. Recall

$$\zeta\left(s,\pi_v'\right) \overset{(19)}{=} \sum_j \int_{H_n'(\mathbb{F}_v)\times U_{(n,n)}'(\mathbb{F}_v)\times K_v'} \xi_{v,j}\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right)$$

$$\Gamma_j'\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}; k\right) |\mathrm{det}_v'(a)|^{s+nd-\frac{1}{2}} |\mathrm{det}_v'(b)|^{s+nd-\frac{1}{2}} \,\mathrm{d}_v a \,\mathrm{d}_v b \,\mathrm{d}_v x \,\mathrm{d}_v k.$$

Observe that changing Γ_j' by

$$(p,k) \mapsto \Gamma_j'\left(pk'^{-1}, k'k\right)$$

for some fixed $k' \in K_v' \cap P_{(n,n)}'(\mathbb{F}_v)$ does not change the integral. Thus,

$$\int_{H_n'(\mathbb{F}_v)\times U_{(n,n)}'(\mathbb{F}_v)\times K_v'} \xi_{v,j}\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right)$$

$$\Gamma_j'\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}; k\right) |\mathrm{det}_v'(a)|^{s-\frac{1}{2}} |\mathrm{det}_v'(b)|^{s+nd-\frac{1}{2}} \,\mathrm{d}_v a \,\mathrm{d}_v b \,\mathrm{d}_v x \,\mathrm{d}_v k =$$

$$= \int_{\Omega_0(\mathbb{F}_v)\times K_v'} \xi_{v,j}(r(p,k)) |\mathrm{det}_v'(p)|^{s-nd-\frac{1}{2}} \int_{K_v'\cap P_{(n,n)}'(\mathbb{F}_v)} \Gamma_j'(pk'^{-1}; k'k) \,\mathrm{d}k' \,\mathrm{d}p \,\mathrm{d}k,$$

where we extend $\xi_{v,j}$ trivially to $\Omega(\mathbb{F}_v)$ and $\mathrm{d}p$ is a right $P_{(n,n)}'(\mathbb{F}_v)$-invariant measure on $\Omega_0(\mathbb{F}_v)$. Thus,

$$\zeta_v\left(s,\pi_v'\right) = \sum_j \int_{\Omega(\mathbb{F}_v)} \xi_{v,j}(g)\, \Gamma_{j\,*}'(g)\, |\mathrm{det}_v'(g)|^{s-nd-\frac{1}{2}} \,\mathrm{d}_v g =$$

$$= \sum_j \int_{\mathrm{GL}_{2n}'(\mathbb{F}_v)} \xi_{v,j}(g)\, \Gamma_{j\,*}'(g)\, |\mathrm{det}_v'(g)|^{s-nd-\frac{1}{2}} \,\mathrm{d}_v g,$$

for a suitable $\mathrm{GL}_{2n}'(\mathbb{F}_v)$-invariant measure on $\Omega(\mathbb{F}_v)$. Extending $\Gamma_{j\,*}'$ to a Schwartz-function of $M_{2n}'(\mathbb{F}_v)$ proves the first claim of Theorem 7.4.1 by Lemma 7.4.3.

Proof of Lemma 7.4.4. It is easy to check that φ_* is well defined and that the first two properties are satisfied, so it remains to check the third. Let D be a differential operator of order 1 on Ω, which is independent of (a,b). Since r is submersive, there exists a pullback differential operator D^* on $\Omega_0(\mathbb{F}_v)\times K_v'$ such that $(D^*\phi)_* = D\phi_*$, hence, it is enough to show that D^* leaves $\mathscr{S}(\Omega_0 \times K')$ invariant. Assume that D is an operator in (a,b), hence, without loss of generality it acts on a function φ' by

$$\frac{d}{dt}\varphi'\left(\begin{pmatrix} a+tX & b+tX \\ c & d \end{pmatrix}\right)\Big|_{t=0}$$

at a matrix X. Then we can choose D^* such that it acts on a function φ by

$$\frac{d}{dt}\varphi\left(\begin{pmatrix} x+tXk^{-1} & y+tYk^{-1} \\ 0 & m \end{pmatrix}; k\right)\Big|_{t=0},$$

which is a differential operator in the variables x, y, whose coefficients depend only on k. Therefore, the obtained function stays in $\mathscr{S}\left(\Omega_0 \times K'_v\right)$.

The second possibility is that D is a differential operator on Ω_1. Since any such operator is the linear combination of operators defined by invariant vector fields on $\mathrm{GL}'_n(\mathbb{F}_v)$ and K'_v. First assume that D acts on φ' by

$$\frac{d}{dt}\varphi'\begin{pmatrix} a & b \\ \exp\left(tX\right)c & \exp\left(tX\right)d \end{pmatrix}\Bigg|_{t=0}$$

where X is an element of the Lie algebra of $\mathrm{GL}'_n(\mathbb{F}_v)$. Then we can choose again D^* such that it acts on φ by

$$\frac{d}{dt}\varphi\left(\begin{pmatrix} x & y \\ 0 & \exp\left(tX\right)m \end{pmatrix}; k\right)\Bigg|_{t=0},$$

which clearly leaves $\mathscr{S}\left(\Omega_0 \times K'\right)$ invariant.

Finally, for an element $Y \in \mathfrak{k}'_v$, the value of D on φ' is the difference, of the two operators D_1 and D_2 applied to φ', given by

$$\frac{d}{dt}\varphi'\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\exp\left(tY\right)\right)\Bigg|_{t=0}$$

and

$$\frac{d}{dt}\varphi'\begin{pmatrix} a\exp\left(tY\right) & b\exp\left(tY\right) \\ c & d \end{pmatrix}\Bigg|_{t=0}.$$

We can then choose D_1^* to act as

$$\frac{d}{dt}\phi\left(p, k\exp\left(tY\right)\right)\big|_{t=0},$$

which preserves $\mathscr{S}\left(\Omega_0 \times K'\right)$. By the first case we considered, D_2* does so too, since it is a differential operator in (a, b) with polynomial coefficients. □

The last step in the proof of Theorem 7.4.1 concerns the special case of π'_v and ψ_v being unramified. Thus, assume π'_v is unramified and let $\xi_{v,0}$ be the corresponding vector in the Shalika model, *i.e.* the one fixed by $\mathrm{GL}_{2d_v n}\left(\mathbb{O}'_v\right)$. Then, using [12, Lemma 6.10], we know that

$$L\left(s, \pi'_v\right) = \int_{\mathrm{GL}_{2d_v n}\left(\mathbb{O}'_v\right)} f\left(g\right)\left|\det'_v\left(g\right)\right|^{s-nd-\frac{1}{2}}\,\mathrm{d}_v g,$$

where f is a spherical function attached to π'_v, *i.e.* the matrix coefficient of π'_v is of the form $g \mapsto v_0^\vee(\pi'_v(g)v_0)$, where v_0 and $v_0^\vee$ are non-zero vectors of π'_v and $\pi'^\vee_v$ fixed by by the maximal open compact subgroup $\mathrm{GL}_{2d_v n}\left(\mathbb{O}'_v\right)$. Let Ψ_v be the characteristic function of $\mathrm{GL}_{2d_v n}\left(\mathbb{O}'_v\right)$. Then following the proof of Lemma 7.4.3 shows that

$$L\left(s, \pi'_v\right) = \int_{\mathrm{GL}'_{2n}(\mathbb{F}_v)} \xi_{v,0}\left(g\right)\Psi_v\left(g\right)\left|\det'_v\left(g\right)\right|^{s-nd-\frac{1}{2}}\,\mathrm{d}_v g$$

Recall, that $\zeta(s,\xi_{v,0})$ can by (17) also be written as

$$\zeta_v(s,\xi_{v,0}) = \int_{H'_n(\mathbb{F}_v)\times U'_{(n,n)}(\mathbb{F}_v)\times K'_v} \xi_{v,0}\left(\begin{pmatrix} a & 0 \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} b^{-1} & 0 \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} 1_n & u \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} 1_n & 0 \\ 0 & b \end{pmatrix} k\right)$$

$$\Xi\left(u,b,b^{-1};k\right)|\det'_v(a)|^{s-\frac{1}{2}}|\det'_v(b)|^{nd}\,\mathrm{d}_v a\,\mathrm{d}_v b\,\mathrm{d}_v u\,\mathrm{d}_v k, \tag{20}$$

where we plugged in the definition of μ_{Ψ_v} and Ξ is defined by (16). It is easy to see that Ξ vanishes unless u, b and b^{-1} have entries in $\mathbb{O}'_v$, since the conductor of ψ_v is $\mathbb{O}_v$. Therefore, (20) equals to

$$\int_{\mathrm{GL}'_{2n}(\mathbb{F}_v)} \xi_{v,0}(g)\,\Psi_v(g)\,|\det'_v(g)|^{s-nd-\frac{1}{2}}\,\mathrm{d}_v g,$$

which proves the claim. □

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