On modular representations of inner forms of GL_n over a local non-archimedean field

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Abstract

Let F be a local non-archimedean field of residue characteristic p and $\overline{\mathbb{F}}_\ell$ an algebraic closure of a finite field of characteristic $\ell \neq p$. We extend the results of [21] concerning $\square$-irreducible representations of inner forms of $\mathrm{GL}_n(\mathrm{F})$ to representations over $\overline{\mathbb{F}}_\ell$. As applications, we compute the Godement-Jacquet L-factor for any smooth irreducible representation over $\overline{\mathbb{F}}_\ell$ and show that the local factors of a representation agree with the ones of its C-parameter defined in [18]. Moreover, we reprove that the classification of irreducible representations via multisegments due to Vignéras and Mínguez-Sécherre is indeed exhaustive without using the results of [2]. Finally, we characterize the irreducible constituents of certain parabolically induced representations, as was already done by Zelevinsky over $\mathbb{C}$.

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Contents

1 Introduction

Let F be a local non-archimedean field of residual characteristic p and G a reductive group over F. The classification of irreducible, smooth admissible representations of $G(\mathrm{F})$ over an algebraically closed field R is of great importance in the Langlands program. This has been achieved by Bernstein and Zelevinsky in [38], [6], [7] for the case $G(\mathrm{F}) = \mathrm{GL}_n(\mathrm{F})$ and $R = \mathbb{C}$ and was later extended by Tadić in [34] to the case $G(\mathrm{F}) = \mathrm{GL}_n(\mathrm{D})$, where D is a central division algebra over F of dimension d^2 and char $\mathrm{F} = 0$. For char $\mathrm{F} \neq 0$, the classification was completed in [5].

The main focus of this paper will lie on the case $G(F) = \mathrm{GL}_n(\mathrm{D})$ and R of characteristic $\ell := \mathrm{char}\, R$ possibly non-zero. Moreover, we restrict ourselves to the case $\ell \neq p$ to ensure the existence of Haar measures on reductive groups over F. The theory of such representations has been explored and developed in [35], [36]. Passing from the $\ell = 0$ to the $\ell \neq 0$ case, many results carry over, however also several new phenomena appear. For example the notions of *cuspidal* and *supercuspidal* no longer coincide. Vignéras [36] and Mínguez-Sécherre [31] generalized the classification of Bernstein and Zelevinsky and Tadic to the case of $\ell \neq p$. The irreducible representations are classified by purely combinatorial objects called *multisegments*, formal finite sums of segments, which we will define in a moment. To each multisegment $\mathfrak{m}$ an irreducible representation $\mathrm{Z}(\mathfrak{m})$ of $\mathrm{GL}_n(\mathrm{D})$, for a suitable n, can be assigned and every irreducible representation of $\mathrm{GL}_n(\mathrm{D})$ is isomorphic to a representation coming from such a multisegment. However, in the case $\ell \neq 0$ it might happen that two multisegments give rise to isomorphic representations. Because of this one has to restrict the map $\mathfrak{m} \mapsto \mathrm{Z}(\mathfrak{m})$ to so called *aperiodic* multisegments to obtain a bijection.

Before we come to the main results of the paper, we need to introduce some notation. For the rest of the paper a representation will be a smooth representation of finite length of $G_n := \mathrm{GL}_n(\mathrm{D})$ over some fixed algebraically closed field $R \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell\}$. We denote the image of a representation π of G_n in the Grothendieck group of representations of G_n as $[\pi]$. If π_1, π_2 are representations of G_{n_1} and G_{n_2}, we let $P_{(n_1,n_2)}$ be the parabolic subgroup of $\mathrm{GL}_{n_1+n_2}(\mathrm{D})$ defined over F containing the upper triangular matrices with Levi-component the block diagonal matrices $G_{n_1} \times G_{n_2}$. We then denote by $\pi_1 \times \pi_2$ the normalized parabolically induced representation of $\pi_1 \otimes \pi_2$ and by $r_{(n_1,n_2)}$ normalized parabolic reduction with respect to $P_{(n_1,n_2)}$. For a character χ of F^*, we will write by abuse of notation also χ for the character $g \mapsto \chi(\det'(g))$ of G_n, where $\det'$ denotes the determinant map. Write $\mathfrak{Irr}_n$ for the irreducible representations of G_n and $\mathfrak{Irr} := \bigcup_{n \geq 0} \mathfrak{Irr}_n$.

Let ρ be a cuspidal irreducible representation of G_m. Then there exists an unramified character v_ρ such that the representation $\rho \times \rho\chi$ is reducible if and only if $\chi \cong v_\rho^{\pm 1}$ and we set

$$o(\rho) := \#\{[\rho v_\rho^k] : k \in \mathbb{Z}\}.$$

In [32, Section 4] the authors proved that $o(\rho)$ is the order of $q(\rho)$ in R^*, where $q(\rho)$ is an explicit power of q depending on ρ. Set

$$e(\rho) := \begin{cases} o(\rho) & \text{if } o(\rho) > 1, \\ \ell & \text{if } o(\rho) = 1. \end{cases}$$

For $a \leq b \in \mathbb{Z}$ and ρ a cuspidal representation of G_m, we define a segment as a sequence

$$[a,b]_\rho := (\rho v_\rho^a, \dots, \rho v_\rho^b).$$

Let the cuspidal support of $[a,b]_\rho$ be $[\rho v_\rho^a] + \dots + [\rho v_\rho^b]$, its length $l([a,b]_\rho) := b - a + 1$ and its degree $\deg([a,b]_\rho) := m \cdot l([a,b])$. To a segment $[a,b]_\rho$ of length n we can also associate an irreducible subrepresentation

$$\mathrm{Z}([a,b]_\rho)$$

of

$$\rho v_\rho^a \times \dots \times \rho v_\rho^b,$$

which corresponds to the trivial character of the affine Hecke-algebra $\mathscr{H}_R(n, q(\rho))$ under the map of [32, §4.4]. Two segments $[a,b]_\rho$ and $[a',b']_{\rho'}$ are called equivalent if they have the same length and $[\rho v_\rho^{a+i}] = [\rho' v_{\rho'}^{a'+i}]$ for all $i \in \mathbb{Z}$ and equivalent segments give isomorphic representations. We let for ρ a cuspidal representation $\mathcal{S}(\rho)$ be the set of segments of the form $[a,b]_\rho$ and we call it the set of ρ-segments. A multisegment $\mathfrak{m} = \Delta_1 + \dots + \Delta_k$ is a formal sum of equivalence classes of segments and we extend the notion of cuspidal support, length, degree and equivalence linearly to multisegments. We call a multisegment *aperiodic* if it does not contain a sub-multisegment equivalent to

$$[a,b]_\rho + \dots + [a + e(\rho) - 1, b + e(\rho) - 1]_\rho.$$

Denote by $\mathcal{MS}_{ap}$ the set of equivalence classes of aperiodic segments. For some fixed cuspidal representation ρ we denote by $\mathcal{MS}(\rho)$ the set of all multisegments of the form

$$\mathfrak{m} = [a_1, b_1]_\rho + \dots + [a_k, b_k]_\rho$$

and $\mathcal{MS}(\rho)_{ap}$ the aperiodic multisegments contained in $\mathcal{MS}(\rho)$. In [31, Section 9.5] the authors define for a multisegment $\mathfrak{m} = \Delta_1 + \dots + \Delta_k$ a particular irreducible subquotient $\mathrm{Z}(\mathfrak{m})$ of

$$\mathrm{I}(\mathfrak{m}) := [\mathrm{Z}(\Delta_1) \times \dots . \times \mathrm{Z}(\Delta_k)]$$

such that $\mathrm{Z}(\mathfrak{m}) \cong \mathrm{Z}(\mathfrak{n})$ implies $\mathfrak{m} = \mathfrak{n}$ if both are aperiodic.

Following [21], we call an irreducible representation π of G_n $\square$- *irreducible* if $\pi \times \pi$ is irreducible. We study these representations in Section 3.2 using the theory of intertwining operators introduced in [9], which we recall in Section 3.1. The main lemma of this section is then the following.

Proposition 1 (*cf.* [21], [14], [15])**.** *Let $\pi \in \mathfrak{Irr}$. The following are equivalent.*

1. *π is $\square$-irreducible.*
2. *For all $\sigma \in \mathfrak{Irr}$, $\mathrm{soc}(\pi \times \sigma)$ is irreducible and appears with multiplicity 1 in $\pi \times \sigma$.*

3. For all $\sigma \in \mathfrak{Irr}$, $\mathrm{soc}(\sigma \times \pi)$ is irreducible and appears with multiplicity 1 in $\sigma \times \pi$.

If any of the above statements hold true, the maps

$$\sigma \mapsto \mathrm{soc}(\sigma \times \pi),\ \sigma \mapsto \mathrm{soc}(\pi \times \sigma)$$

are injective.

Note that over $R = \overline{\mathbb{Q}}_\ell$, this was already proven in [21]. We adapt their argument, which is originally due to [14], [15]. The proposition allows us to give the following definition. Let $\pi' \in \mathfrak{Irr}$. If $\pi' \cong \mathrm{soc}(\sigma \times \pi)$ respectively $\pi' \cong \mathrm{soc}(\pi \times \sigma)$, we define the right-$\pi$-derivative $\mathcal{D}_{r,\pi}(\pi') \coloneqq \sigma$ respectively the left-π-derivative $\mathcal{D}_{l,\pi}(\pi') \coloneqq \sigma$. If π' does not lie in the image of these maps, we set the derivatives to 0.

One of the main differences between the $\overline{\mathbb{F}}_\ell$-case and the complex case is that not all cuspidal representations are $\square$-irreducible. Recall that if ρ is a cuspidal representation, then ρ is $\square$-irreducible if and only if $o(\rho) > 1$. The behavior of $\square$-irreducible and $\square$-reducible cuspidal representations differs quite drastically. For example, if ρ is a $\square$-reducible cuspidal representation, its L-factor will always be trivial, and $\rho \times \rho$ it is of length 2 if $\ell > 2$ and of length 3 if $\ell = 2$. We say an irreducible representation has $\square$-irreducible cuspidal support if its cuspidal support consists of $\square$-irreducible cuspidal representations.

In Section 4 we study the specific situation where ρ is a $\square$-irreducible cuspidal representation. The notion of ρ-derivatives was first introduced independently in [25] and [13] and further developed in [19] in the case G_n and $R = \mathbb{C}$. Moreover, the theory was extended to classical groups in [3]. Here we extend the notion of derivatives to the case $\ell \neq 0$ and observe that most of the results carry over.

Corollary 2. *If $\mathcal{D}_{r,\rho}(\pi) \cong \mathcal{D}_{r,\rho}(\pi') \neq 0$ or $\mathcal{D}_{l,\rho}(\pi) \cong \mathcal{D}_{l,\rho}(\pi') \neq 0$, then $\pi \cong \pi'$. Furthermore,*

$$\mathcal{D}_{r,\rho}(\pi)^\vee \cong \mathcal{D}_{l,\rho^\vee}(\pi^\vee).$$

For ρ a $\square$-irreducible cuspidal representation, we moreover define four maps

$$\mathrm{soc}(\rho, \cdot), \mathrm{soc}(\cdot, \rho), \mathcal{D}_{r,\rho}, \mathcal{D}_{l,\rho} \colon \mathcal{MS}_{ap}(\rho) \to \mathcal{MS}_{ap}(\rho)$$

using a combinatorial description such that

$$\mathcal{D}_{l,\rho}(\mathrm{soc}(\rho, \mathfrak{m})) = \mathfrak{m} = \mathcal{D}_{r,\rho}(\mathrm{soc}(\mathfrak{m}, \rho)).$$

We then can compute the ρ-derivatives of the corresponding representation $\mathrm{Z}(\mathfrak{m})$ simply as follows.

Proposition 3. *Let $\mathfrak{m}$ be an aperiodic multisegment and ρ a $\square$-irreducible cuspidal representation.*

$$\mathcal{D}_{r,\rho}(\mathrm{Z}(\mathfrak{m})) = \mathrm{Z}(\mathcal{D}_{r,\rho}(\mathfrak{m})),\ \mathcal{D}_{l,\rho}(\mathrm{Z}(\mathfrak{m})) = \mathrm{Z}(\mathcal{D}_{l,\rho}(\mathfrak{m})).$$

As a corollary we obtain that $\mathrm{Z}(\mathrm{soc}(\rho, \mathfrak{m})) = \mathrm{soc}(\rho \times \mathrm{Z}(\mathfrak{m}))$. In particular, this is used to give a new proof of the following theorem.

Theorem 4. *Let π be an irreducible smooth representation of G_n whose cuspidal support is $\square$-irreducible cuspidal. Then there exists a multisegment $\mathfrak{m}$ such that $\mathrm{Z}(\mathfrak{m}) \cong \pi$.*

Remark. There are two main reasons why such a new proof could be of interest. The first is that the original proof is quite involved and relies heavily on the techniques from geometric representation theory. Secondly, it does not give much information on the inverse Z^{-1} of Z, which is something that we will need in the computations of the L-factors. Let us now expand on both of these points. We start by recalling the existing proof of the subjectivity of Z in [31]. One first reduces the claim to the situation where all representations and multisegments involved have cuspidal support $\mathfrak{s}$ contained in $\mathcal{MS}(\rho)$ for some fixed ρ. Let $\mathcal{MS}(\mathfrak{s})$ be the set of aperiodic multisegments with cuspidal support $\mathfrak{s}$ and $\mathfrak{Irr}(\mathfrak{s})$ the set of isomorphism classes of irreducible representations with cuspidal support $\mathfrak{s}$. It is then possible to define a central character $\chi_{\mathfrak{s}}$ of the affine Hecke-algebra $\mathscr{H}_R(n, q(\rho))$ with the property that there exists a bijection between $\mathfrak{Irr}(\mathfrak{s})$ and

$$\mathfrak{Irr}(\mathscr{H}_R(n, q(\rho)), \mathfrak{s}),$$

the set of isomorphism classes modules of $\mathscr{H}_R(n, q(\rho))$ with central character $\chi_{\mathfrak{s}}$. The proof of the classification of simple modules of the Hecke algebra has two very different flavors, depending on whether $o(\rho) > 1$ or $o(\rho) = 1$. If $o(\rho) = 1$, the central character of $\mathscr{H}_R(n, 1)$ is always trivial and hence the representation theory of the Hecke-algebra reduces in this case to the representation theory of the group algebra $R[\mathrm{S}_n]$ or said differently, to the modular representation theory of the symmetric group. Although some questions on modular representations of the symmetric groups remain to this day unsolved, the classification of irreducible representations falls not among them. It is a classical result that the irreducible representations are classified by ℓ-regular partitions and there is an explicit construction of these, see Section 2.7. Thus one has an explicit description of the inverse map Z^{-1} in this case.

On the other hand, if we assume that $o(\rho) > 1$, the situation is much more involved. Here one uses [2, Theorem B] to show that $\mathcal{MS}(\mathfrak{s})$ and $\mathfrak{Irr}(\mathscr{H}_R(n, q(\rho)), \mathfrak{s})$ have the same cardinality and hence obtains as a consequence that the irreducible representations are classified by aperiodic multisegments. In the proof of Theorem B of [2] the author first uses the geometric representation theory of [8] to show that in the case $\ell \neq 0$, $\mathfrak{Irr}(\mathscr{H}_R(n, q(\rho)), \mathfrak{s})$ is parametrized by simple modules with a particular central character $\tilde{\chi}_{\mathfrak{s}}$ of an affine Hecke-algebra $\mathscr{H}_{\mathbb{C}}(n, \xi)$ over $\mathbb{C}$ at a root of unity ξ. Using geometric representation theory again, the author then argues that those are classified by the aperiodic multisegments. Thus we proved that $\mathfrak{Irr}(\mathfrak{s})$ is finite and of the same cardinality as $\mathcal{MS}(\mathfrak{s})$. Since the map $\mathfrak{m} \mapsto \mathrm{Z}(\mathfrak{m})$ induces an injective map between these two sets, it has to be bijective. This gives unfortunately not the kind of information on the inverse map Z^{-1} we need later on. However, since a cuspidal representation is $\square$-irreducible if and only if $o(\rho) > 1$, this second, harder, case is covered by our new subjectivity proof, and thus gives us a more representation-theoretic description of Z^{-1}.

Let us now return to the remaining contents of this paper. We continue by discussing how the Aubert-Zelevinsky dual of an irreducible representation interacts with taking derivatives. Recall that the Aubert-Zelevinsky dual is a map

$$(\cdot)^{*}\colon \mathfrak{Irr} \to \mathfrak{Irr},\ \pi \mapsto \pi^{*}$$

preserving the cuspidal support. For $\mathfrak{m}$ an aperiodic multisegment, we write $\langle \mathfrak{m} \rangle := \mathrm{Z}(\mathfrak{m})^{*}$. Using derivatives we then compute in Section 5.3 the Godement-Jacquet local L-factors $\mod \ell$ introduced in [26].

Theorem 5. *Fix an additive character ψ of F^{*} and let $\Delta = [a,b]_{\rho}$ be a segment over R. If $\rho \cong \chi$ for an unramified character χ of* F *and $q^{d} \neq 1$, then*

$$L(\langle \Delta \rangle, T) = \frac{1}{1 - \chi(\varpi_{\mathrm{F}}) q^{-db + \frac{1-d}{2}} T},$$

where ϖ_{F} is a uniformizer of $\mathfrak{o}_{\mathrm{F}}$. Otherwise

$$L(\langle \Delta \rangle, T) = 1.$$

More generally, if $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ is an aperiodic multisegment then

$$L(\langle \mathfrak{m} \rangle, T) = \prod_{i=1}^{k} L(\langle \Delta_i \rangle, T),\ \epsilon(T, \langle \mathfrak{m} \rangle, \psi) = \prod_{i=1}^{k} \epsilon(T, \langle \Delta_i \rangle, \psi)$$

and

$$\gamma(T, \langle \mathfrak{m} \rangle, \psi) = \prod_{i=1}^{k} \gamma(T, \langle \Delta_i \rangle, \psi).$$

We recall now the definition of the map

$$\mathrm{C}\colon \mathfrak{Irr}_{n} \xrightarrow{\cong} \{\mathrm{C} - \text{parameters of length } n\},$$

cf. [18]. Let W_{F} be the Weil group of F and ν the unique unramified character of W_{F} acting on the Frobenius as $\nu(\mathrm{Frob}) = q^{-1}$. The set of C-parameters over $\overline{\mathbb{F}}_{\ell}$ is then a subset of the equivalence classes of W_{F}-semisimple Deligne $\overline{\mathbb{F}}_{\ell}$-representations, *i.e.* pairs (Φ, U), where Φ is a semi-simple W_{F}-representation over $\overline{\mathbb{F}}_{\ell}$ and $U \in \mathrm{Hom}_{W_{\mathrm{F}}}(\nu\Phi, \Phi)$. The length of such a representation refers to its dimension. For a precise definition see Section 5.3.1. In [18] to each such C-parameter (Φ, U) an L-factor $L((\Phi, U), T)$, an ϵ-factor $\epsilon(T, (\Phi, U), \psi)$ and a γ-factor $\gamma(T, (\Phi, U), \psi)$ is then associated.

Corollary 6. *The map* C *respects the local factors,* i.e. *for* $\pi \in \mathfrak{Irr}$

$$L(\pi,T) = L(\mathrm{C}(\pi),T),\ \epsilon(T,\pi,\psi) = \epsilon(T,\mathrm{C}(\pi),\psi),\ \gamma(T,\pi,\psi) = \gamma(T,\mathrm{C}(\pi),\psi).$$

In Section 6 we describe the structure of the representations of the form $\mathrm{I}(\mathfrak{m})$, which is heavily motivated by their analysis done in [38] in the case $R = \overline{\mathbb{Q}}_\ell$. Fix a cuspidal representation ρ and a cuspidal support $\mathfrak{s} \in \mathcal{MS}(\rho)$. Let $Q = A^+_{o(\rho)-1}$ be the affine Dynkin quiver with $o(\rho)$ vertices numbered $\{0,\dots,o(\rho)-1\}$ and an arrow from i to j if $j = i+1 \mod o(\rho)$. Moreover, $\mathfrak{s} = d_0[\rho] + \dots + d_{o(\rho)-1}[\rho v_\rho^{o(\rho)-1}]$ gives rise to a graded vector space

$$V(\mathfrak{s}) \coloneqq \bigoplus_{i=1}^{o(\rho)-1} \mathbb{C}^{d_i}.$$

We let $N(V(\mathfrak{s}))$ be the $\mathbb{C}$-variety of nilpotent representations of Q with underlying vector space $V(\mathfrak{s})$. Then $N(V(\mathfrak{s}))$ admits an action by

$$\mathrm{GL}_{d_0}(\mathbb{C}) \times \dots \times \mathrm{GL}_{d_{o(\rho)-1}}(\mathbb{C}),$$

whose orbits are in bijection with irreducible representations $\mathrm{Z}(\mathfrak{m})$ whose cuspidal support is $\mathfrak{s}$. We write $X_\mathfrak{m}$ for the orbit corresponding to $\mathrm{Z}(\mathfrak{m})$ and write $\mathfrak{n} \leq \mathfrak{m}$ if $X_\mathfrak{m} \subseteq \overline{X_\mathfrak{n}}$

Theorem 7. *Let* $\mathfrak{m},\mathfrak{n} \in \mathcal{MS}(\rho)$ *not necessarily aperiodic. Then* $\mathrm{Z}(\mathfrak{n})$ *appears as a subquotient of* $\mathrm{I}(\mathfrak{m})$ *if and only if the cuspidal supports of* $\mathfrak{n}$ *and* $\mathfrak{m}$ *agree and* $\mathfrak{n} \leq \mathfrak{m}$.

It is thus expected that the singularities of the space $N(V(\mathfrak{s}))$ allow one to obtain a deeper understanding of the irreducible representations with cuspidal support $\mathfrak{s}$. In this spirit, we extended [20, Conjecture 4.2] to representations over $\overline{\mathbb{F}}_\ell$.

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2 Notation and preliminaries

2.1 Let F be a local non-archimedean field, $\mathfrak{o}_\mathrm{F}$ its ring of integers with uniformizer ϖ_F and $\mathrm{k} \coloneqq \mathfrak{o}_\mathrm{F}/(\varpi_\mathrm{F})$ its residue field. Let $p \coloneqq \operatorname{char}\mathrm{k}$ and q the cardinality of k. Finally, let D be a division algebra of degree d over F. Fix an algebraically closed field $R \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell\}$ with $\ell \neq p$. For a reductive group G over F, we let $\mathfrak{Rep}_G = \mathfrak{Rep}_G(R)$ be the category of smooth, finite length representations of $G(F)$ over R. We will focus on the case $G_n := \mathrm{GL}_n(D)$, in which case we write $\mathfrak{Rep}_n = \mathfrak{Rep}_{G_n}$ and set

$$\mathfrak{Rep} = \mathfrak{Rep}(R) := \bigcup_{n \geq 0} \mathfrak{Rep}_n.$$

We let $\mathfrak{R}_n(R) = \mathfrak{R}_n$ be the Grothendieck group of representations in $\mathfrak{Rep}_n$ and set

$$\mathfrak{R} = \mathfrak{R}(R) := \bigoplus_{n \geq 0} \mathfrak{R}_n.$$

Let $\mathfrak{Irr}_n = \mathfrak{Irr}_n(R)$ be the set of isomorphism classes of irreducible smooth admissible representations of G_n and set

$$\mathfrak{Irr} := \bigcup_{n \geq 0} \mathfrak{Irr}_n.$$

From now on a representation π of G_n means a smooth, finite length representation of G_n over R and its image in $\mathfrak{R}_n$ is denoted as $[\pi]$ and $\deg(\pi) = n$. We write $\pi^\vee$ for the contragredient representation, $\tau \hookrightarrow \pi$ if τ is a subrepresentation of π and $\tau \leq \pi$ if τ is a subquotient of π, *i.e.* the quotient of a subrepresentation of π. For an element $[\mathrm{X}] \in \mathfrak{R}$, we write $[\mathrm{X}] \geq 0$ if the multiplicity of all isomorphism classes of irreducible representations in $[\mathrm{X}]$ is greater or equal than 0 and we write $[\mathrm{X}] \geq [\mathrm{Y}]$ if $[\mathrm{X}] - [\mathrm{Y}] \geq 0$. Finally, if S is a set we let $\mathbb{N}(S)$ be the commutative monoid consisting of functions $S \to \mathbb{N}$ with finite support. Finally, we fix once and for all an additive character $\widetilde{\psi}$ of F^* taking values in $\overline{\mathbb{Z}}_\ell$ and let ψ be its reduction mod ℓ.

2.2 Let $n \in \mathbb{Z}_{>0}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_t)$ a partition of n and S_n the n-th symmetric group. Its reverse partition $\overline{\alpha}$ is denoted by $\overline{\alpha} := (\alpha_t, \dots, \alpha_1)$ and we call α ordered if

$$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_t.$$

If $\alpha' = (\alpha'_1, \dots, \alpha'_{t'})$ is a second partition of n we define the intersection $\alpha \cap \alpha'$ recursively as follows.

$$\alpha \cap \alpha' := \begin{cases} (\alpha_1, (\alpha'_2, \dots, \alpha'_{t'}) \cap (\alpha_2, \dots, \alpha_t)) & \text{if } \alpha_1 = \alpha'_1, \\ (\alpha_1, (\alpha'_1 - \alpha_1, \dots, \alpha'_{t'}) \cap (\alpha_2, \dots, \alpha_t)) & \text{if } \alpha_1 < \alpha'_1, \\ (\alpha'_1, (\alpha'_2, \dots, \alpha'_{t'}) \cap (\alpha_1 - \alpha'_1, \dots, \alpha_t)) & \text{if } \alpha_1 > \alpha'_1. \end{cases}$$

If α and α' are ordered partitions of n we say $\alpha \leq \alpha'$ if

$$\sum_{i=1}^{j} \alpha_j \leq \sum_{i=1}^{j} \alpha'_j$$

for all $j \in \{1, \dots, \min(t, t')\}$. We let

$$G_\alpha := G_{\alpha_1} \times \dots \times G_{\alpha_t}$$

and every irreducible representation π of G_α is then of the form

$$\pi \cong \pi_1 \otimes \ldots \otimes \pi_t,$$

for $[\pi_i] \in \mathfrak{Irr}_{\alpha_i}$ and $i \in \{1, \ldots, t\}$. As above let $\mathfrak{Rep}_\alpha$ be the category of smooth, finite length representations of G_α. The Grothendieck group of representations in $\mathfrak{Rep}_\alpha$ of finite length is denoted by $\mathfrak{R}_\alpha$, and we write $\mathfrak{Irr}_\alpha$ for the set of isomorphism classes of irreducible representations of G_α. For $j \in \{1, \ldots, t\}$ let $(\beta_{1,j}, \beta_{2,j}, \ldots, \beta_{i_j,j})$ be a partition of α_j. We then call

$$(\beta_{1,1}, \ldots, \beta_{i_1,1}, \beta_{1,2}, \ldots, \beta_{i_2,2}, \ldots, \beta_{1,t}, \ldots, \beta_{i_t,t})$$

a subpartition of α.

Recall the exact functors of *parabolic induction* and *parabolic reduction.* In order to be well defined we fix a choice of a square root $\sqrt{q}$ in R. If $P = M \rtimes U$ is a parabolic subgroup defined over F of G_α with Levi-component M and unipotent-component U, we let δ_P be the module of P, *i.e.* the map $\delta_P \colon M \to \mathbb{C}^*$

$$\delta_P(m) = \sqrt{q}^{v_P(m)},$$

where v_P is a certain morphism from M to $\mathbb{Z}$ depending on P. We denote then by

$$\mathrm{Ind}_P^{G_\alpha} : \mathfrak{Rep}_M \to \mathfrak{Rep}_{G_\alpha} \text{ resp. } r_{G_\alpha,P} : \mathfrak{Rep}_{G_\alpha} \to \mathfrak{Rep}_M.$$

Recall that the functor $\mathrm{Ind}_P^{G_\alpha}$ is the left adjoint of $r_{G_\alpha,P}$, *i.e.*

$$\mathrm{Hom}_{G_\alpha}(-, \mathrm{Ind}_P^{G_\alpha}(-)) = \mathrm{Hom}_M(r_{G_\alpha,P}(-), -) \text{ (Frobenius reciprocity).}$$

We also fix the minimal parabolic subgroup P_0 defined over F of upper triangular matrices in G_α. Note that each parabolic subgroup P defined over F containing P_0 corresponds to a subpartition β of α in such a way that the Levi subgroup of P is G_β. If $\alpha = (n)$ we write $r_\beta := r_{G_n,P_\beta}$, $\mathrm{Ind}_\beta := \mathrm{Ind}_{P_\beta}^{G_n}$ and if

$$(\beta_{1,1}, \ldots, \beta_{i_1,1}, \beta_{1,2}, \ldots, \beta_{i_2,2}, \ldots, \beta_{1,t}, \ldots, \beta_{i_t,t})$$

is a subpartition of α and

$$\pi = \pi_{1,1} \otimes \ldots \otimes \pi_{i_1,1} \otimes \pi_{1,2} \otimes \ldots \otimes \pi_{i_2,2} \otimes \ldots \otimes \pi_{1,t} \otimes \ldots \otimes \pi_{i_t,t}$$

is an object of $\mathfrak{Rep}_\beta$, we write

$$\pi_{1,1} \times \ldots \times \pi_{i_1,1} \otimes \pi_{1,2} \times \ldots \times \pi_{i_2,2} \otimes \ldots \otimes \pi_{1,t} \times \ldots \times \pi_{i_t,t} := \mathrm{Ind}_{P_\beta}^{G_\alpha}(\pi).$$

Since both induction and reduction are exact, they induce functors on the corresponding Grothendieck groups. In particular, parabolic induction induces a product

$$\times : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}, ([\pi_1], [\pi_2]) \mapsto [\pi_1 \times \pi_2],$$

which equips $\mathfrak{R}$ with the structure of an commutative algebra, see [35, 1.16] for $\ell > 2$, [38, Theorem 1.9] for the case $R = \mathbb{C}$ and [31, Proposition 2.6] in general. If π is a representation we will write $\pi^k := \pi \times \ldots \times \pi$ for the product of k copies of π. We will also use this opportunity to state two lemmas, which were proven in [20] over $\overline{\mathbb{Q}}_\ell$, but the presented proofs work as well *muta mutandis* over $\overline{\mathbb{F}}_\ell$.

Lemma 2.2.1 ([20, Lemma 2.1]). *Let $P = M \rtimes U$ be a parabolic subgroup of $G = G_n$ and $Q_1 = N_1 \rtimes V_1$, $Q_2 = N_2 \rtimes V_2$ parabolic subgroups of G containing P. Let $\pi \in \mathfrak{Rep}(M)$, $\tau_1 \in \mathfrak{Rep}(N_1)$, $\tau_2 \in \mathfrak{Rep}(N_2)$ such that*

$$\tau_1 \hookrightarrow \mathrm{Ind}_{N_1 \cap P}^{N_1}(\pi),\ \tau_2 \hookrightarrow \mathrm{Ind}_{N_2 \cap P}^{N_2}(\pi)$$

and there exists

$$\mathrm{Ind}_{Q_1}^{G}(\tau_1) \hookrightarrow \mathrm{Ind}_{Q_2}^{G}(\tau_2)$$

such that the diagram

$$\begin{tikzcd}
\mathrm{Ind}_{Q_1}^{G}(\tau_1) \arrow[r, hook] \arrow[d, hook] & \mathrm{Ind}_{P}^{G}(\pi) \\
\mathrm{Ind}_{Q_2}^{G}(\tau_2) \arrow[ru, hook] &
\end{tikzcd}$$

commutes. Then there exists $\tau \hookrightarrow \pi$ such that

$$\tau_1 \hookrightarrow \mathrm{Ind}_{N_1 \cap P}^{N_1}(\tau),\ \mathrm{Ind}_{N_2 \cap P}^{N_2}(\tau) \hookrightarrow \tau_2.$$

Corollary 2.2.2 ([20, Corollary 2.2], [15, Lemma 3.1]). *Let $n_1, n_2, n_3 \in \mathbb{Z}_{>0}$ and*

$$\pi_i \in \mathfrak{Rep}(G_{n_i}),\ i \in \{1,2,3\}.$$

Let σ be a subrepresentation of $\pi_1 \times \pi_2$ and τ a subrepresentation of $\pi_2 \times \pi_3$ such that as subrepresentations of $\pi_1 \times \pi_2 \times \pi_3$

$$\sigma \times \pi_3 \hookrightarrow \pi_1 \times \tau.$$

Then there exists a subrepresentation ω of π_2 such that

$$\sigma \hookrightarrow \pi_1 \times \omega,\ \omega \times \pi_3 \hookrightarrow \tau.$$

In particular, if π_2 is irreducible and $\sigma \neq 0$, then $\sigma = \pi_2 \times \pi_3$

2.3 Geometric Lemma

One of the key tools we will use is the Geometric Lemma, *cf.* [35, Theorem 2.19], [7, Theorem 5.2]. Let A be the maximal torus defined over F of our minimal parabolic subgroup P_0 and $W(G, A)$ the Weyl group of G_n. The choice of P_0 corresponds to a choice of a basis S of the root system ϕ of G_n and hence of a set of positive roots ϕ^+. Let α be a partition of n corresponding to a parabolic subgroup containing P_0, which in turn corresponds to a subset $I \subseteq S$.

For $w \in W(G,A)$ we let α^w be the partition corresponding to the subset $w(I) \subseteq S$. For α_1, α_2 partitions of n, which correspond to subsets I_1, I_2 of S, we denote

$$W(\alpha_1,\alpha_2) := \{w \in W(G,A) : w(I_1) \subseteq \phi^+ \text{ and } w^{-1}(I_2) \subseteq \phi^+\}$$

and for $w \in W(\alpha_1,\alpha_2)$ we set $\alpha_2' := \alpha_2 \cap \alpha_1^w$ and $\alpha_1' = \alpha_1 \cap \alpha_2^{w^{-1}}$. For $g \in G_{\alpha_2'}$ and $w \in W(\alpha_1,\alpha_2)$ the element $w^{-1}gw \in G_{\alpha_1'}$ and hence pulling back via the group morphism $g \mapsto w^{-1}gw$ allows us to define a functor

$$w : \mathfrak{Rep}_{\alpha_1'} \to \mathfrak{Rep}_{\alpha_2'}.$$

Set

$$F(w) := \mathrm{Ind}_{P_{\alpha_2'}}^{G_{\alpha_2}} \circ w \circ r_{G_{\alpha_1},P_{\alpha_1'}} : \mathfrak{Rep}_{\alpha_1} \to \mathfrak{Rep}_{\alpha_2}.$$

Let $\leq$ the Bruhat order on $W(\alpha_1,\alpha_2)$ and $w_0 \geq \ldots \geq w_k$ the corresponding ordering of the elements in $W(\alpha_1,\alpha_2)$. Then

$$r_{\alpha_2} \circ \mathrm{Ind}_{\alpha_1} : \mathfrak{Rep}_{\alpha_1} \to \mathfrak{Rep}_{\alpha_2}$$

is glued together from the exact functors $F(w), w \in W(\alpha_1,\alpha_2)$, *i.e.* there exists a filtration

$$0 = F_{-1} \subseteq F_0 \subseteq \ldots \subseteq F_k = r_{\alpha_2} \circ \mathrm{Ind}_{\alpha_1}$$

of $r_{\alpha_2} \circ \mathrm{Ind}_{\alpha_1}$ such that

$$F_i \backslash F_{i-1} \cong F(w_i),\ i \in \{0,\ldots,k\}.$$

As a consequence

$$[r_{\alpha_2} \circ \mathrm{Ind}_{\alpha_1}] = \sum_{w \in W(\alpha_1,\alpha_2)} [F(w)].$$

This can be described in a more combinatorial way, see for example [19, Section 1.2]. Let $\alpha = (n)$, $\alpha_1 = (m_1,\ldots,m_t)$, $\alpha_2 = (n_1,\ldots,n_{t'})$, $\pi_1 \otimes \ldots \otimes \pi_t \in \mathfrak{Rep}_{\alpha_1}$ and $\mathrm{Mat}^{\alpha_1,\alpha_2}$ the set of $t \times t'$ matrices $B = (b_{i,j})$ with non-negative integer coefficients such that for all $(i,j) \in \{1,\ldots,t\} \times \{1,\ldots,t'\}$

$$\sum_{j=1}^{t'} b_{i,j} = m_i,\ \sum_{i=1}^{t} b_{i,j} = n_j.$$

For $B \in \mathrm{Mat}^{\alpha_1,\alpha_2}$ and $i \in \{1,\ldots,t\}$, $\beta_{B,i} := (b_{1,i},\ldots,b_{i,t'})$ is a partition of m_i. Let l_i be the length of $r_{\beta_{B,i}}(\pi_i)$ and write the irreducible decomposition factors of $r_{\beta_{B,i}}(\pi_i)$ as

$$\sigma_i^{(k)} = \sigma_{i,1}^{(k)} \otimes \ldots \otimes \sigma_{i,t'}^{(k)},\ k \in \{1,\ldots,l_i\}$$

with $[\sigma_{i,j}^{(k)}] \in \mathfrak{Irr}_{b_{i,j}}$. For $j \in \{1,\ldots,t'\}$ and a sequence $\underline{k} = (k_1,\ldots,k_t)$ with $k_i \in \{1,\ldots,l_i\}$ set

$$\Sigma_j^{B,\underline{k}} := \sigma_{1,j}^{(k_1)} \otimes \ldots \otimes \sigma_{t,j}^{(k_t)}.$$

Then

$$[r_{\alpha_2}(\pi_1 \times \ldots \times \pi_t)] = \sum_{B \in \mathrm{Mat}^{\alpha_1,\alpha_2},\, \underline{k}} [\Sigma_1^{B,\underline{k}} \otimes \ldots \otimes \Sigma_{t'}^{B,\underline{k}}].$$

2.4 An irreducible representation ρ of G_m is called *cuspidal* if $r_\alpha(\rho) = 0$ for all nontrivial partitions α of m and *supercuspidal* if ρ is not the subquotient of a nontrivially parabolically induced representation. We denote the corresponding subsets of $\mathfrak{Irr}$ as $\mathfrak{C}$ resp. $\mathfrak{S}$. From Frobenius reciprocity follows immediately that if ρ is supercuspidal it is cuspidal.

Let ρ, ρ' be cuspidal representations of G_m and $G_{m'}$. Then by [32, § 4.5] there exists an unramified character v_ρ of G_m such that $\rho \times \rho'$ is irreducible if and only if $\rho' \cong \rho v_\rho$ or $\rho' \cong \rho v_\rho^{-1}$. We set

$$\mathbb{Z}[\rho] := \{[\rho v_\rho^k], k \in \mathbb{Z}\}$$

and let $o(\rho)$ be the cardinality of $\mathbb{Z}[\rho]$. One can associate to ρ a finite extension $\mathrm{k}(\rho)$ of k of cardinality $q(\rho)$, see [32, Section 4], and define a number $e(\rho)$ as the smallest integer k such that

$$1 + q(\rho) + \ldots + q(\rho)^{k-1} = 0 \mod \ell.$$

Moreover, set

$$e(\rho) := \begin{cases} o(\rho) & \text{if } o(\rho) > 1, \\ \ell & \text{if } o(\rho) = 1. \end{cases}$$

Moreover in [32, Lemma 4.41] it was shown that $o(\rho)$ is the order of $q(\rho)$ in R^*.

There exist surjective maps with finite fibres

$$\mathrm{cusp} : \mathfrak{Irr} \to \mathbb{N}(\mathfrak{C}) \text{ and } \mathrm{scusp} : \mathfrak{Irr} \to \mathbb{N}(\mathfrak{S})$$

called *cuspidal support* and *supercuspidal support.* They are defined by

$$\mathrm{cusp}^{-1}([\rho_1] + \ldots + [\rho_k]) = \{[\pi] \in \mathfrak{Irr} : \pi \hookrightarrow \rho_{\sigma(1)} \times \ldots \times \rho_{\sigma(k)} \text{ for some } \sigma \in S_k\},$$

$$\mathrm{scusp}^{-1}([\rho_1] + \ldots + [\rho_k]) = \{[\pi] \in \mathfrak{Irr} : [\pi] \le [\rho_1 \times \ldots \times \rho_k]\}.$$

We call k the length of the cuspidal support $[\rho_1] + \ldots + [\rho_k]$. A cuspidal representation ρ is called $\square$-*irreducible* if $o(\rho) > 1$ and we say an irreducible representation has $\square$-irreducible cuspidal support if its cuspidal support consists of $\square$-representations.

Lemma 2.4.1. *Let π be an irreducible representation and $\mathfrak{s} = \mathrm{cusp}(\pi)$. We write*

$$\mathfrak{s} = \mathfrak{s}_1 + \ldots + \mathfrak{s}_k$$

with $\mathfrak{s}_\mathfrak{i} \in \mathbb{N}(\mathbb{Z}[\rho_i])$ and $\mathbb{Z}[\rho_i] \neq \mathbb{Z}[\rho_j]$ for $i \neq j$. Then there exist irreducible representations $\pi_1, \ldots, \pi_k$ with $\mathrm{cusp}(\pi_i) = \mathfrak{s}_i$ and

$$\pi \cong \pi_1 \times \ldots \times \pi_k$$

Proof. Since parabolic induction is exact, we can find irreducible representations $\pi_1, \dots, \pi_k$ such that $\mathrm{cusp}(\pi_i) = \mathfrak{s}_i$ and $\pi \hookrightarrow \pi_1 \times \dots \times \pi_k$. In [31, Proposition 5.9] it was proven that if $k = 2$, $\pi_1 \times \dots \times \pi_k$ is irreducible and their method extends easily to the general case $k > 2$. Thus the claim follows. □

2.5 Multisegments We will now fix our notations regarding segments, the central combinatorial objects in the classification of [38] and [31]. For ρ a cuspidal representation of G_m and integers $a \leq b$ we define a *segment* as the sequence

$$[a, b]_\rho := (\rho v_\rho^a, \rho v_\rho^{a+1}, \dots, \rho v_\rho^b).$$

The length of such a segment is

$$l([a, b]_\rho) := b - a + 1$$

and its degree is

$$\deg([a, b]_\rho) := m(b - a + 1).$$

We will also sometimes consider an empty segment [] of length and degree 0. Two segments $[a, b]_\rho$ and $[a', b']_{\rho'}$ are said to be equivalent if they have the same length and $[\rho v_\rho^{a+i}] = [\rho' v_{\rho'}^{a'+i}]$ for all $i \in \mathbb{Z}$. The set of equivalence classes of segments will be denoted by $\mathcal{S}$ and for a fixed ρ the set of segments of the form $[a, b]_\rho$ will be denoted as $\mathcal{S}(\rho)$ and we call them ρ-segments.

If $\Delta = [a, b]_\rho$ is a segment, we let $a(\Delta) = a$ and $b(\Delta) = b$. We write

$$^-\Delta := [a+1, b]_\rho,\ \Delta^- := [a, b-1]_\rho,\ \Delta^+ := [a, b+1]_\rho,\ ^+\Delta := [a-1, b]_\rho,\ [a]_\rho := [a, a]_\rho$$

and $\Delta^\vee := [-b, -a]_{\rho^\vee}$. The operation $\Delta \mapsto {}^-\Delta$ will sometimes be called *shortening the segment* Δ *by* 1 *on the left* and similarly for the others. If $a + 1 > b$, $^-\Delta$ is the empty segment and similarly for the other operations. A segment $\Delta = [a, b]_\rho$ *precedes* $\Delta' = [a', b']_{\rho'}$ if the sequence

$$(\rho v_\rho^a, \dots, \rho v_\rho^b, \rho' v_{\rho'}^{a'}, \dots, \rho' v_{\rho'}^{b'})$$

contains a subsequence which is, up to isomorphism, a segment of length greater than $l(\Delta)$ and $l(\Delta')$. The segments Δ and Δ' are called *unlinked* if Δ does not precede Δ' and Δ' does not precede Δ.

A *multisegment* $\mathfrak{m} = \Delta_1 + \dots + \Delta_k$ is a formal finite sum of equivalence classes of segments, *i.e.* an element in $\mathbb{N}[\mathcal{S}]$. We extend the notion of length and degree linearly to multisegments as well as the dual operation $(-)^\vee$. The set of multisegments is denoted by $\mathcal{MS}$ and $\mathcal{MS}(\rho)$ is the set of multisegments $\mathfrak{m} = \Delta_1 + \dots + \Delta_k$ such that the equivalence class of Δ_i is contained in $\mathcal{S}(\rho)$ for all $i \in \{1, \dots, n\}$. A multisegment $\mathfrak{m}$ contains a multisegment $\mathfrak{m}'$ if there exists a multisegment $\mathfrak{m}''$

such that $\mathfrak{m}$ is equivalent to $\mathfrak{m}' + \mathfrak{m}''$. Moreover, $\mathfrak{m}$ is called *aperiodic* if it does not contain a multisegment of the form

$$[a,b]_\rho + [a+1,b+1]_\rho + \ldots + [a+e(\rho)-1, b+e(\rho)-1]_\rho.$$

We let $\mathcal{MS}_{ap}$ be the set of aperiodic multisegments and $\mathcal{MS}(\rho)_{ap} := \mathcal{MS}_{ap} \cap \mathcal{MS}(\rho)$. Let $\mathfrak{m} = [a_1,b_1]_{\rho_1} + \ldots + [a_k,b_k]_{\rho_k}$ be a multisegment. We set

$$\mathfrak{m}^1 := [b_1]_{\rho_1} + \ldots + [b_k]_{\rho_k},$$

$$\mathfrak{m}^- := [a_1, b_1 - 1]_{\rho_1} + \ldots + [a_k, b_k - 1]_{\rho_k}$$

and define recursively $\mathfrak{m}^{-(s+1)} := (\mathfrak{m}^{-s})^-$ and $\mathfrak{m}^{s+1} := (\mathfrak{m}^{-s})^1$. Let l be the largest natural number such that $\mathfrak{m}^l \neq 0$ and define the partition $\mu_\mathfrak{m} := (\deg(\mathfrak{m}^1), \ldots, \deg(\mathfrak{m}^l))$ of $\deg(\mathfrak{m})$.

We call $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ unlinked if Δ_i and Δ_j are pairwise unlinked for all $i \neq j$, $i, j \in \{1, \ldots, k\}$. As for representations, there exists a surjective map

$$\mathrm{cusp}_{\mathcal{MS}} : \mathcal{MS}_{ap} \to \mathbb{N}(\mathfrak{C}),$$

which maps $[a,b]_\rho \mapsto [\rho v_\rho^a] + \ldots + [\rho v_\rho^b]$ and is extended linearly to multisegments.

2.6 Classification We are now going to recall the classifications proved in [31] and [35]. Let ρ be a supercuspidal representation of G_m and $n = e(\rho)\ell^r$ for some $r \in \mathbb{Z}_{\geq 0}$. Then the representation

$$\rho \times \rho v_\rho \times \ldots \times \rho v_\rho^{n-1}$$

contains a unique cuspidal subquotient, which is denoted by $\mathrm{St}(\rho, n)$ and $o(\mathrm{St}(\rho, n)) = 1$. Moreover, every cuspidal non-supercuspidal representation is of the above form and if $\mathrm{St}(\rho, n) \cong \mathrm{St}(\rho', n')$ then $n = n'$ and $\mathbb{Z}[\rho] = \mathbb{Z}[\rho']$, see [31, Section 6]. In particular, every $\square$-irreducible cuspidal representation is supercuspidal. We let $\mathfrak{C}^\square(G_m)$ be the set of isomorphism classes of $\square$-irreducible cuspidal representations of G_m and $\mathfrak{C}^\square := \bigcup_{m \geq 0} \mathfrak{C}^\square(G_m)$. We write

$$\mathrm{scusp}_{\mathcal{MS}} : \mathcal{MS} \to \mathbb{N}(\mathfrak{S})$$

for the linear map sending a segment

$$[a,b]_{\mathrm{St}(\rho,n)} \mapsto n \cdot ([\rho v_\rho^a] + \ldots + [\rho v_\rho^b]).$$

Fix a cuspidal representation ρ of G_m. To each segment $[a,b]_\rho$ we are going to associate two irreducible representations $\mathrm{Z}([a,b]_\rho)$ and $\mathrm{L}([a,b]_\rho)$. This requires us to first consider $\mathscr{H}_R(n, q(\rho))$, the R-Hecke-algebra generated by $S_1, \ldots, S_{n-1}, X_1, \ldots, X_n$ satisfying the relations

1. $(S_i + 1)(S_i - q(\rho)) = 0$, $1 \leq i \leq n-1$
2. $S_i S_j = S_j S_i$, $|i-j| \geq 2$
3. $S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$, $1 \leq i \leq n-2$
4. $X_i X_j = X_j X_i$, $1 \leq i, j \leq n$
5. $X_j S_i = S_i X_j$, $i \notin \{j, j-1\}$
6. $S_i X_i S_i = q(\rho) X_{i+1}$, $1 \leq i \leq n-1$

We let $\mathfrak{Irr}(\Omega_{\rho,n})^*$ be the set of all isomorphism classes of irreducible representations π of G_{nm} with cuspidal support contained in $\mathbb{N}(\mathbb{Z}[\rho])$. By [32, §4.4] there exists a bijection

$$\xi_{\rho,n} \colon \mathfrak{Irr}(\Omega_{\rho,n})^* \xrightarrow{\cong} \{\text{isomorphism classes of simple } \mathscr{H}(n, q(\rho))\text{-modules}\}.$$

Let $a, b \in \mathbb{Z}$ such that $b - a + 1 = n$. Then $\mathscr{H}_R(n, q(\rho))$ has two 1-dimensional modules, $\mathscr{Z}(a,b)$, defined by

$$S_i \mapsto q(\rho),\ X_j \mapsto q(\rho)^{a+j-1}$$

and $\mathcal{L}(a,b)$, defined by

$$S_i \mapsto -1,\ X_j \mapsto q(\rho)^{b-j+1}.$$

To a segment $[a,b]_\rho$ we can associate now an irreducible subrepresentation $\mathrm{Z}([a,b]_\rho)$ resp. irreducible quotient $\mathrm{L}([a,b]_\rho)$ of

$$\rho v_\rho^a \times \ldots \times \rho v_\rho^b$$

by demanding

$$\xi_{\rho,n}(\mathrm{Z}([a,b]_\rho) = \mathscr{Z}(a,b) \text{ and } \xi_{\rho,n}(\mathrm{L}([a,b]_\rho) = \mathcal{L}(a,b).$$

For the empty segment, we let $Z([])$ be the trivial representation of the trivial group. Note that equivalent segments give rise to isomorphic representations. The representations $\mathrm{Z}([a,b]_\rho)$ and $\mathrm{L}([a,b]_\rho)$ behave very well under parabolic restriction.

Lemma 2.6.1 ([31, Lemma 7.16])**.** *Let $k \in \{a, \ldots, b\}$. Then*

$$r_{((k-a)m,(b-k+1)m)}(\mathrm{Z}([a,b]_\rho)) = \mathrm{Z}([a,k-1]_\rho) \otimes \mathrm{Z}([k,b]_\rho),$$

$$r_{((b-k+1)m,(k-a)m)}(\mathrm{L}([a,b]_\rho)) = \mathrm{L}([k,b]_\rho) \otimes \mathrm{L}([a,k-1]_\rho).$$

The structure of representations induced from representations of the form $\mathrm{Z}([a,b]_\rho)$ or $\mathrm{L}([a,b]_\rho)$ is more complex and will preoccupy us throughout Section 6. A starting point is the following theorem.

Theorem 2.6.2 ([31, Theorem 7.26]). *Let* $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$. *Then the following are equivalent.*

1. $\mathrm{Z}(\Delta_1) \times \ldots \times \mathrm{Z}(\Delta_k)$ *is irreducible.*
2. $\mathrm{L}(\Delta_1) \times \ldots \times \mathrm{L}(\Delta_k)$ *is irreducible.*
3. $\mathfrak{m}$ *is unlinked.*

If some of the segments Δ_i are linked, the induced representation is no longer irreducible. The following two lemmata already hint how these induced representations might look like.

Lemma 2.6.3 ([31, Proposition 7.17]). *If* $\rho \in \mathfrak{C}^{\square}$ *then* $\mathrm{Z}([a,b]_\rho)$ *is the socle of*

$$\mathrm{Z}([a,b-1]) \times \rho v_\rho^b \text{ and } \rho v_\rho^a \times \mathrm{Z}([a+1,b])$$

and the coscole of

$$\mathrm{Z}([a+1,b]) \times \rho v_\rho^a \text{ and } \rho v_\rho^b \times \mathrm{Z}([a,b-1]).$$

Next we recall the definition of a residually non-degenerate representation. We will use an alternative definition from the one in [31], however it was proven in Theorem 9.10 of said paper that the following definition is equivalent to theirs. An irreducible representation π is called *residually non-degenerate* if there exists a multisegment $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$, $\Delta_i = [a_i, b_i]_{\rho_i}$ for $i \in \{1, \ldots, k\}$ such that

$$[\pi] = [\mathrm{L}(\Delta_1) \times \ldots \times \mathrm{L}(\Delta_k)]$$

and $l(\Delta_i) < e(\rho_i)$ for $i \in \{1, \ldots, k\}$. More generally, for $\alpha = (\alpha_1, \ldots, \alpha_t)$ a partition, we call a representation of the form $\pi = \pi_1 \otimes \ldots \otimes \pi_t$ of G_α residually non-degenerate if each of the π_i's is residually non-degenerate.

Lemma 2.6.4 ([31, Corollary 8.5]). *If* π *is residually non-degenerate and*

$$[\pi] \leq [\mathrm{Z}(\Delta_1) \times \ldots \times \mathrm{Z}(\Delta_k)]$$

then $l(\Delta_i) = 1$ *for all* $i \in \{1, \ldots, k\}$.

Let ρ be a cuspidal representation of G_m, α a partition of nm and π an irreducible representation of G_{nm}. We say π is α*-degenerate* if $r_\alpha(\pi)$ contains a residually non-degenerate representation. It follows from the definition given in [31, §8] that if α' is a permutation of α then π is α-degenerate if and only if it is α'-degenerate. For $\mathfrak{m} = \Delta_1 + \ldots + \Delta_n$ we define

$$\mathrm{I}(\mathfrak{m}) = [\mathrm{Z}(\Delta_1) \times \ldots \times \mathrm{Z}(\Delta_k)],$$

which is well defined since × is commutative in the Grothendieck group.

Theorem 2.6.5 ([31, Section 8]). *Let* $\mathfrak{m}$ *be a multisegment of the form* $\mathfrak{m} = [a_1]_{\rho_1} + \ldots + [a_k]_{\rho_k}$*. Then* $\mathrm{I}(\mathfrak{m})$ *contains a unique residually non-degenerate representation denoted* $\mathrm{Z}(\mathfrak{m})$ *and any cuspidal representation is residually non-degenerate.*

For $\mathfrak{m}$ a multisegment and l maximal such that $\mathfrak{m}^l \neq 0$ we set

$$\mathrm{St}(\mathfrak{m}) := \mathrm{Z}(\mathfrak{m}^l) \otimes \ldots \otimes \mathrm{Z}(\mathfrak{m}^1).$$

We can now define an irreducible representation $\mathrm{Z}(\mathfrak{m})$ for a general multisegment $\mathfrak{m}$.

Theorem 2.6.6 ([31, Theorem 9.19]). *Let* $\mathfrak{m}$ *be a multisegment. Then* $\mathrm{I}(\mathfrak{m})$ *contains a unique irreducible subquotient denoted by* $\mathrm{Z}(\mathfrak{m})$ *which is* $\mu_{\mathfrak{m}}$*-degenerate. The multiplicity of* $\mathrm{Z}(\mathfrak{m})$ *in* $\mathrm{I}(\mathfrak{m})$ *is* 1 *and the unique irreducible constituent of* $r_{\overline{\mu_{\mathfrak{m}}}}(\mathrm{Z}(\mathfrak{m}))$ *which is residually non-degenerate is* $\mathrm{St}(\mathfrak{m})$*.*

If $\alpha = (\alpha_1, \ldots, \alpha_k)$ *is an ordered partition of* $\deg(\mathfrak{m})$ *and* π *is an irreducible* α*-degenerate subquotient of* $\mathrm{I}(\mathfrak{m})$ *then* $\alpha \leq \mu_{\mathfrak{m}}$ *with equality if and only if* $\pi = \mathrm{Z}(\mathfrak{m})$*.*

The theorem gives rise to a map

$$\mathrm{Z} : \mathcal{MS} \to \mathfrak{Irr}.$$

Proposition 2.6.7 ([31, Proposition 9.28]). *Let* $\mathfrak{m}$ *be a multisegment and write* $\mathfrak{m} = \mathfrak{m}_1 + \ldots + \mathfrak{m}_k$ *as a sum of multisegments. Then*

$$[\mathrm{Z}(\mathfrak{m})] \leq [\mathrm{Z}(\mathfrak{m}_1) \times \ldots \times \mathrm{Z}(\mathfrak{m}_k)].$$

Theorem 2.6.8 ([31, Theorem 9.36]). *The map* $\mathrm{Z} : \mathcal{MS} \to \mathfrak{Irr}$ *restricted to aperiodic multisegments is injective and* $\mathrm{Z}(\mathfrak{m})^\vee \cong \mathrm{Z}(\mathfrak{m}^\vee)$*. Moreover, if* $\mathfrak{m}$ *is an multisegment*

$$\mathrm{scusp}_{\mathcal{MS}}(\mathfrak{m}) = \mathrm{scusp}(\mathrm{Z}(\mathfrak{m}))$$

and if $\mathfrak{m}$ *is moreover aperiodic, then*

$$\mathrm{cusp}_{\mathcal{MS}}(\mathfrak{m}) = \mathrm{cusp}(\mathrm{Z}(\mathfrak{m})).$$

Lemma 2.6.9. *Let* $\mathfrak{m}$ *be a multisegment. Then*

$$[r_{(\deg(\mathfrak{m}^-),\deg(\mathfrak{m}^1))}(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{Z}(\mathfrak{m}^-) \otimes \mathrm{Z}(\mathfrak{m}^1)].$$

Proof. By Theorem 2.6.6

$$[r_{\overline{\mu_{\mathfrak{m}}}}(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{St}(\mathfrak{m})].$$

Let $\pi \in \mathfrak{Irr}_{\deg(\mathfrak{m}^-)}$ such that

$$[r_{\overline{\mu_{\mathfrak{m}^-}}}(\pi)] \geq [\mathrm{St}(\mathfrak{m}^-)],$$

$$[r_{(\deg(\mathfrak{m}^-),\deg(\mathfrak{m}^1)}(\mathrm{I}(\mathfrak{m}))] \geq [r_{(\deg(\mathfrak{m}^-),\deg(\mathfrak{m}^1))}(\mathrm{Z}(\mathfrak{m}))] \geq [\pi \otimes \mathrm{Z}(\mathfrak{m}^1)].$$

The first inequality implies that π is $\mu_{\mathfrak{m}^-}$-degenerate. On the other hand, the second inequality implies, using Lemma 2.6.4 and the Geometric Lemma, that $[\pi] \leq \mathrm{I}(\mathfrak{m}^-)$. Theorem 2.6.6 forces therefore $\pi \cong \mathrm{Z}(\mathfrak{m}^-)$. ☐

In the same article it was proven that Z is also surjective, using by passing to the Hecke-algebra and invoking a counting argument. We will reprove in Section 5 this surjectivity for all representations with □-irreducible cuspidal support.

2.7 Symmetric group

In this section we quickly review the representation theory of S_n over R, see for example [12]. Let $\lambda = (\lambda_1, \dots, \lambda_t)$ be an ordered partition of n and denote by $\mathbf{1}_n$ the trivial representation of S_n. We set

$$\mathrm{S}_\lambda \coloneqq \mathrm{S}_{\lambda_1} \times \dots \times \mathrm{S}_{\lambda_t},$$

$$\mathbf{1}_\lambda \coloneqq \mathbf{1}_{\lambda_1} \otimes \dots \otimes \mathbf{1}_{\lambda_t}$$

and

$$M^\lambda \coloneqq \mathrm{Ind}_{\mathrm{S}_\lambda}^{\mathrm{S}_n}(\mathbf{1}_\lambda) = \mathrm{Hom}_{\mathrm{S}_\lambda}(\mathrm{S}_n, \mathbf{1}_\lambda),$$

called the *Young permutation module*. The representation M^λ contains the *Specht-module* S^λ with multiplicity 1. If $R = \overline{\mathbb{Q}}_\ell$, S^λ is irreducible for all λ and the Specht-modules parameterize all irreducible representations of S_n, however if $R = \overline{\mathbb{F}}_\ell$, S^λ might be reducible. To bypass this issue in the case $R = \overline{\mathbb{F}}_\ell$, one equips M^λ with a natural S_n-invariant, non-degenerate and symmetric bilinear pairing $\langle \cdot, \cdot \rangle$. Under this pairing, let $(S^\lambda)^\perp$ be the orthogonal complement of S^λ in M^λ and set

$$D_\lambda \coloneqq S^\lambda / (S^\lambda \cap (S^\lambda)^\perp).$$

We recall that λ is ℓ-regular if for all

$$a_i \coloneqq \#\{j \in \{1, \dots, t\} : \lambda_j = i\},$$

$a_i < \ell$. Now D_λ is irreducible if it is non-zero and it is non-zero if and only if λ is ℓ-regular. Moreover, if μ is a second ordered partition of n, D_λ appears in M^μ if and only if $\lambda \leq \mu$. Finally M^μ is contained in M^λ in the Grothendieck group of R-representations of S_n if and only if $\mu \leq \lambda$. More precisely, the multiplicity of S^λ in M^μ is the Kostka number $K_{\lambda,\mu}$.

2.8 Integral structures

Let $\overline{\mathbb{Z}}_\ell$ be the ring of integers of $\overline{\mathbb{Q}}_\ell$ with residue field $\overline{\mathbb{F}}_\ell$. We choose the square-roots in Section 2.2 in a compatible way, *i.e.* if $R = \overline{\mathbb{Q}}_\ell$, we let $\sqrt{q} \in \overline{\mathbb{Z}}_\ell$, such that its reduction modulo the maximal ideal in $\overline{\mathbb{Z}}_\ell$ gives the chosen square root in $\overline{\mathbb{F}}_\ell$.

A $\overline{\mathbb{Q}}_\ell$-representation (π, V) of G_n is called *integral* if it is admissible and admits an integral structure, *i.e.* a G_n-stable $\overline{\mathbb{Z}}_\ell$-submodule $\mathfrak{o}$ of V such that the natural map $\mathfrak{o} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell \to V$ is an isomorphism. If $P = M \rtimes U$ is a parabolic subgroup of G_n and σ a representation of M with integral structure $\mathfrak{o}$, then $\mathrm{Ind}_P^{G_n}(\mathfrak{o})$ is an integral structure of $\mathrm{Ind}_P^{G_n}(\sigma)$, see [35, II.4.14]. For a representation π with integral structure $\mathfrak{o}$ one can consider the reduction mod ℓ of π, which is defined as

$$\mathrm{r}_\ell(\pi) := \mathfrak{o} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell.$$

This is not an invariant of π, it depends on the chosen integral structure. However, its image in the Grothendieck group is. This leads to the following theorem. Let $\mathfrak{R}_n^{en}(\overline{\mathbb{Q}}_\ell)$ be the subgroup of $\mathfrak{R}(\overline{\mathbb{Q}}_\ell)$ generated by irreducible integral representations.

Theorem 2.8.1 ([35, II.4.12],[32, Theorem A], [31, Theorem 9.39], [35, II.4.14, II.5.11]). *The morphism* reduction mod ℓ *of groups*

$$\mathrm{r}_\ell : \mathfrak{R}_n^{en}(\overline{\mathbb{Q}}_\ell) \to \mathfrak{R}_n(\overline{\mathbb{F}}_\ell)$$

is well defined. Moreover, r_ℓ *satisfies the following properties.*

1. *If $\tilde{\rho}$ is a cuspidal representation over $\overline{\mathbb{Q}}_\ell$, it admits an integral structure if and only if its central character is integral.*
2. *If ρ is a supercuspidal representation over $\overline{\mathbb{F}}_\ell$ then there exists an integral cuspidal representation $\tilde{\rho}$ of G_m over $\overline{\mathbb{Q}}_\ell$ such that $\mathrm{r}_\ell(\tilde{\rho}) = \rho$. We refer to such $\tilde{\rho}$ as a lift of ρ.*
3. *If $\tilde{\rho}$, ρ are cuspidal such that $\mathrm{r}_\ell(\tilde{\rho}) = \rho$ then for integers $a \leq b$, $\mathrm{Z}([a,b]_{\tilde{\rho}})$ admits an integral structure and*

$$\mathrm{r}_\ell(\mathrm{Z}([a,b]_{\tilde{\rho}}) = \mathrm{Z}([a,b]_\rho).$$

4. r_ℓ *commutes with parabolic induction,* i.e. *there is a natural isomorphism*

$$\mathrm{Ind}_P(\mathfrak{o}) \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell \to \mathrm{Ind}_P(\mathfrak{o} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_l).$$

A lift of a segment $[a,b]_\rho$ over $\overline{\mathbb{F}}_\ell$ to $\overline{\mathbb{Q}}_\ell$ is a segment $[a',b']_{\tilde{\rho}}$ such that $\tilde{\rho}$ is a lift of ρ and $a' \leq b' \in \mathbb{Z}$ with $a - b = a' - b'$ and $a = a' \mod o(\rho)$. A multisegment $\widetilde{\mathfrak{m}} = \widetilde{\Delta}_1 + \ldots + \widetilde{\Delta}_k$ over $\overline{\mathbb{Q}}_\ell$ is a lift of a mulitsegment $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ over $\overline{\mathbb{F}}_\ell$ if $\widetilde{\Delta}_i$ is a lift of Δ_i for $i \in \{1, \ldots, k\}$. In this case we write

$$\mathrm{r}_\ell(\widetilde{\mathfrak{m}}) = \mathfrak{m}.$$

Theorem 2.8.2 ([31, Theorem 9.39]). *Let* $\mathfrak{m}$ *be a multisegment and* $\widetilde{\mathfrak{m}}$ *a lift of* $\mathfrak{m}$. *Then* $\mathrm{Z}(\widetilde{\mathfrak{m}})$ *admits an integral structure and* $\mathrm{r}_\ell(\mathrm{Z}(\widetilde{\mathfrak{m}}))$ *contains* $\mathrm{Z}(\mathfrak{m})$ *with multiplicity* 1.

2.9 Haar-measures If one tries to translate the computation of local L-factors from $\overline{\mathbb{Q}}_\ell$ to $\overline{\mathbb{F}}_\ell$, one quickly runs into the problem that, contrary to the situation in $\overline{\mathbb{Q}}_\ell$, for a compactly supported smooth function $f\colon G \to R$ the following identity does not necessarily have to hold:

$$\int_{G_n} f(g)\,\mathrm{d}g = \int_P \int_{K_n} f(pk)\,\mathrm{d}k\,\mathrm{d}_l p,$$

where P is a parabolic subgroup of G, K_n is the open compact subgroup $K_n := \mathrm{GL}_n(\mathfrak{o}_\mathrm{D})$ and $\mathrm{d}g$, $\mathrm{d}k$, $\mathrm{d}_l p$ are left Haar-measures on the respective groups, see for example in [26]. However one can save the situation via the following method, *cf.* [17, §2.2]. For H a closed subgroup of G_n let $C_c^\infty(H,R)$ be the set of smooth, compactly supported functions on H taking values in R. If $H' \subseteq H$ is a closed subgroup, we let $C_c^\infty(H'\backslash H) = C_c^\infty(H'\backslash H, \delta, R)$ be the space of smooth functions on H which transform by $\delta := \delta_H^{-1}\delta_{H'}$ under left translation by H'. We let $C_{c,en}^\infty(H'\backslash H, \delta, \overline{\mathbb{Q}}_\ell)$ be the functions with values in $\overline{\mathbb{Z}}_\ell$ and we denote reduction mod ℓ by

$$\mathrm{r}_\ell\colon C_{c,en}^\infty(H'\backslash H, \delta, \overline{\mathbb{Q}}_\ell) \to C_c^\infty(H'\backslash H, \delta, \overline{\mathbb{F}}_\ell).$$

We will denote from now on by $\mathrm{d}_l h$ a left Haar-measure on H and by $\mathrm{d}_r h$ a right Haar-measure on H, and we choose measures compatible with r_ℓ, i.e. for $f \in C_{c,en}^\infty(H'\backslash H, \delta, \overline{\mathbb{Q}}_\ell)$,

$$\int_{H'\backslash H} f(h)\,\mathrm{d}_r h \in \overline{\mathbb{Z}}_\ell$$

and

$$\mathrm{r}_\ell\left(\int_{H'\backslash H} f(h)\,\mathrm{d}_r h\right) = \int_{H'\backslash H} \mathrm{r}_\ell(f(h))\,\mathrm{d}_r h$$

and similarly for $\mathrm{d}_l h$. In [17, §2.2] the authors show that if one replaces the left Haar-measure $\mathrm{d}_l p$ with a right Haar-measure $\mathrm{d}_r p$ on P and $(K_n \cap P)\backslash K_n$, above identity holds also in $\overline{\mathbb{F}}_\ell$:

$$\int_{G_n} f(g)\,\mathrm{d}g = \int_P \int_{(K_n\cap P)\backslash K_n} f(pk)\delta_P^{-1}(p)\,\mathrm{d}k\,\mathrm{d}_r p. \tag{1}$$

Moreover for H a closed subgroup of G_n and $f \in C_c^\infty(H\backslash G_n)$ we can find $F \in C_c^\infty(G_n)$ such that

$$f(g) = \int_H F(gh)\delta^{-1}(h)\,\mathrm{d}_r h$$

and write in this situation $f = F^H$. As a consequence we have for $f \in C_c^\infty(P\backslash G_n)$ and P a parabolic subgroup of G_n

$$\int_{P\backslash G_n} F^P(g)\,\mathrm{d}g = \int_{(K_n\cap P)\backslash K_n} \int_P F(pk)\delta_P^{-1}(p)\,\mathrm{d}_r p\,\mathrm{d}k = \int_{(K_n\cap P)\backslash K_n} F^P(k)\,\mathrm{d}k. \tag{2}$$

Finally, we recall the following facts about smooth functions on G_n. Let $C^\infty(G_n)$ denote the representation of smooth R-valued functions on which $G_n \times G_n$ acts by left-right translation. Its dual, $C_c^\infty(G_n)$ is given by the compactly supported smooth functions often referred to as *Schwartz-functions* and such a Schwartz-function ϕ acts on $f \in C^\infty(G_n)$ by

$$\phi(f) := \int_{G_n} f(g)\phi(g)\,\mathrm{d}g,$$

where $\phi \in C_c^\infty(G_n)$. Finally, if P is a parabolic subgroup of G_n with opposite parabolic subgroup $\overline{P}$, let $C_c^\infty(\overline{P}P)$ be the set of Schwartz-functions on $\overline{P}P$ on which $\overline{P} \times P$ acts by left-right translation. It is the dual of $C^\infty(\overline{P}P)$, given by just smooth functions on $\overline{P}P$.

2.10 Godement-Jacquet local L-factors In this section we will recall the Godement-Jacquet L-functions for general irreducible representations, *cf.* [10] and [26]. Let us quickly review their construction. Let (π, V) be a representation over R of G_n and let for $v \in V, v^\vee \in V^\vee$

$$f := g \mapsto v^\vee(\pi(g)v)$$

be a matrix coefficient of π. In general we say that a smooth function f is a matrix coefficient of π if it is the finite sum of functions of the above form. Let $\mathscr{S}_R(M_n(\mathrm{D}))$ be the space of Schwartz-functions on $M_n(\mathrm{D})$, *i.e.* the space locally constant functions with compact support $f\colon M_n(\mathrm{D}) \to R$. For $H \subseteq G_n$, we write for $N \in \mathbb{Z}$,

$$H(N) := \{h \in H : |h| = q^{-N}\}.$$

For a matrix coefficient f we denote by

$$\widehat{f} := g \mapsto f(g^{-1})$$

and for $\phi \in \mathscr{S}_R(M_n(\mathrm{D}))$ we let $\widehat{\phi}$ be its Fourier transform with respect to our fixed additive character ψ, see [26, Section 1.3]. For f a matrix coefficient of π and $\phi \in \mathscr{S}_R(M_n(\mathrm{D}))$ one can then construct a formal Laurent series $Z(\phi, T, f) \in R((T))$, which is linear in f and ϕ. The coefficient of T^N of $Z(\phi, T, f)$ is given by the integral

$$\int_{G_n(N)} \phi(g)f(g)\,\mathrm{d}g.$$

Theorem 2.10.1 ([26, Theorem 2.3])**.** *Let (π, V) be a representation of G_n which is a subquotient of a representation induced from irreducible representations.*

1. *There exists $P_0(\pi, T) \in R[T]$ such that for each matrix coefficent f and $\phi \in \mathscr{S}_R(M_n(\mathrm{D}))$*

$$P_0(\pi, T)Z(\phi, T, f) \in R[T, T^{-1}].$$

2. There exists $\gamma(T,\psi,f) \in R(T)$ such that for each matrix coefficent f and $\phi \in \mathscr{S}_R(M_n(\mathrm{D}))$

$$Z(\widehat{\phi}, T^{-1}q^{-\frac{dn+1}{2}}, \widehat{f}) = \gamma(T,\pi,\psi)Z(\phi, Tq^{-\frac{dn-1}{2}}, f).$$

Note that the statement of this theorem in [26] is only stated for irreducible representations, however the proof they give, together with [26, Proposition 2.5] show that it is true for all above representations.

Corollary 2.10.2 ([26, Corollary 2.4])**.** *Let $\mathcal{L}(\pi)$ be the R-subspace of $R((T))$ generated by $Z(\phi, q^{-\frac{dn-1}{2}}T, f)$ where f ranges over all matrix coefficients of π and $\phi \in \mathscr{S}_R(M_n(\mathrm{D}))$ over all Schwartz-functions. Then $\mathcal{L}(\pi)$ is a fractional ideal of $R[T,T^{-1}]$ containing the constant functions. It admits a generator*

$$\frac{1}{P(\pi,T)}$$

with $P(\pi,T) \in R[T]$ and $P(\pi,0) = 1$.

One then defines

$$L(\pi,T) \coloneqq \frac{1}{P(\pi,T)}$$

and $\epsilon(T,\pi,\psi)$ by

$$\gamma(T,\pi,\psi) = \epsilon(T,\pi,\psi)\frac{L(\pi^\vee, q^{-1}T^{-1})}{L(\pi,T)}.$$

Then

$$\epsilon(T,\pi,\psi)\epsilon(q^{-1}T^{-1},\pi^\vee,\psi) = \omega(-1),$$

where ω is the central character of π and $\epsilon(T,\pi,\psi) = C_{\pi,\psi}T^{k(\pi,\psi)}$ for some constant $C_{\pi,\psi} \in R$ and $k(\pi,\psi) \in \mathbb{Z}$. If $R = \overline{\mathbb{Q}}_\ell$ and π is an entire representation then $P(\pi,T)$ has integer coefficients and hence so do the rational functions $\gamma(T,\pi,\psi)$ and $\epsilon(T,\pi,\psi)$. We write for a rational function $F \in \overline{\mathbb{Z}}_\ell[T,T^{-1}]$, $\mathrm{r}_\ell(F) \in \overline{\mathbb{F}}_\ell[T,T^{-1}]$ for the coefficient-wise reduction $\mod \ell$ of F and $\mathrm{r}_\ell\left(\frac{1}{F}\right) \coloneqq \frac{1}{\mathrm{r}_\ell(F)}$ if $\mathrm{r}_\ell(F) \neq 0$.

If $R = \mathbb{C}$, setting $T = q^{-s}$ gives the usual Godement-Jacquet L-factors of [10].

Lemma 2.10.3 ([26, Corollary 4.2])**.** *Let $\widetilde{\pi}$ be an integral admissible representation over $\overline{\mathbb{Q}}_\ell$ of G_n and $\mathfrak{o}$ an integral structure of $\widetilde{\pi}$. Then he polynomial $P(\mathfrak{o} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell, T)$ divides $\mathrm{r}_\ell(P(\widetilde{\pi},T))$ in $\overline{\mathbb{F}}_\ell[T]$ and*

$$\gamma(T, \mathfrak{o} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell, \psi) = \mathrm{r}_\ell(\gamma(T,\widetilde{\pi},\widetilde{\psi})) = \mathrm{r}_\ell(\epsilon(T,\widetilde{\pi},\widetilde{\psi}))\frac{\mathrm{r}_\ell(L(\widetilde{\pi}^\vee, q^{-1}T^{-1}))}{\mathrm{r}_\ell((L(\widetilde{\pi},T))}.$$

Another useful lemma is the following:

Lemma 2.10.4 ([26, Proposition 2.5]). *Let τ, π be both admissible representations of G_n over $\overline{\mathbb{F}}_\ell$ or $\overline{\mathbb{Q}}_\ell$ and τ a subquotient of π. Then $P(\tau, T)$ divides $P(\pi, T)$ and*

$$\gamma(T, \pi, \psi) = \gamma(T, \tau, \psi).$$

3 Intertwining operators and square-irreducible representations

We will quickly discuss the intertwining operators of [9] and square-irreducible representations, *cf.* [20].

3.1 Intertwining operators

We start with the intertwining operators in arbitrary characteristic introduced in [9, §7] and state some lemmas, which are probably already well-known to the experts. Fix for the rest of the section a parabolic subgroup $P = P_{(n_1,n_2)} = M \rtimes U$ with Levi-component $M = G_{n_1} \times G_{n_2}$ of $G = G_n$ and σ a smooth representation of M of finite length. We write $R(T)$ and denote by $\sigma_{R(T)} = \sigma \otimes_R R(T)$ the basechange of σ to $R(T)$. Moreover, we equip $R(T)$ with a valuation $v \colon R(T) \to \mathbb{R}$ which restricts to 0 on R and satisfies $v(T) < 0$. Finally, we denote by $\psi_{un} : M \to R(T)$ the generic unramified character sending

$$(m_1, m_2) \mapsto T^{-\log_q(|m_1|)}.$$

We write $P' = P_{(n_2,n_1)} = M' \rtimes U'$ with Levi-component $M' = G_{n_2} \times G_{n_1}$ and we write $\overline{\sigma}$ for the representation obtained by twisting σ with the inner automorphism given by the element

$$w := \begin{pmatrix} 0 & 1_{n_1} \\ 1_{n_2} & 0 \end{pmatrix},$$

e.g. if $\sigma = \sigma_1 \otimes \sigma_2$, we have $\overline{\sigma_1 \otimes \sigma_2} = \sigma_2 \otimes \sigma_1$. We define the admissible $R(T)$-representation

$$\sigma_{un} := \sigma_{R(T)} \otimes \psi_{un}.$$

By [9, Lemma 3.7]

$$\mathrm{Hom}_{G_n}(\mathrm{Ind}_P^G(\sigma_{un}), \mathrm{Ind}_{P'}^G(\overline{\sigma}_{un}))$$

is non-zero and contains a certain class of non-zero morphisms $J_P(\sigma)$'s in it, which are up to a choice of a Haar-measure uniquely determined by the condition that if $f \in \mathrm{Ind}_P^G(\sigma_{un})$ is compactly supported in $PwP'g$, then

$$J_P(\sigma)(f)(g) = \int_{U'} f(wu'g)\,\mathrm{d}u'.$$

For $k \in \mathbb{Z}$ we denote by $\mathfrak{l}_k := \sigma \otimes_R (T-1)^k R[T]_{(T-1)} \otimes \psi_{un}$.

Lemma 3.1.1. *Let* $\sigma \in \mathfrak{Rep}(M)$*. Then there exists a morphism*

$$J_P(\sigma)\colon \mathrm{Ind}_P^G(\sigma_{un}) \to \mathrm{Ind}_{P'}^G(\overline{\sigma}_{un}),$$

which sends $\mathrm{Ind}_P^G(\mathfrak{l}_0)$ *to* $\mathrm{Ind}_{P'}^G(\overline{\mathfrak{l}_0})$ *and defines a non-zero morphism* R_σ

$$\begin{tikzcd}
\mathrm{Ind}_P^G(\mathfrak{l}_0) \arrow[r, "J_P(\sigma)"] \arrow[d, two heads, "T\mapsto 1"] & \mathrm{Ind}_{P'}^G(\overline{\mathfrak{l}_0}) \arrow[d, two heads, "T\mapsto 1"] \\
\mathrm{Ind}_P^G(\sigma) \arrow[r, "R_\sigma"] & \mathrm{Ind}_{P'}^G(\overline{\sigma})
\end{tikzcd}$$

Such R_σ *is called an* intertwining operator*.*

Proof. Recall that the Geometric Lemma gives inclusions

$$\begin{tikzcd}
\overline{\sigma_{un}} \arrow[r, hook] & r_{P'}(\mathrm{Ind}_P^G(\sigma_{un})) \\
\overline{\mathfrak{l}_0} \arrow[r, hook] \arrow[u, hook] & r_{P'}(\mathrm{Ind}_P^G(\mathfrak{l}_0)) \arrow[u, hook]
\end{tikzcd}$$

which are determined by a suitable choice of Haar-measures, *i.e.* up to a scalar in $R[T]_{(T-1)}$. We will use the notation of [9, §2] and note that by [9, Lemma 3.7] σ_{un} is $(\overline{P}, P)$-regular. Following [9, Lemma 2.10], we can therefore define a retraction r of the inclusion

$$\begin{tikzcd}
\overline{\sigma_{un}} \arrow[r, hook] \arrow[rr, bend right, "1"'] & r_{P'}(\mathrm{Ind}_P^G(\sigma_{un})) \arrow[r, "r"] & \overline{\sigma_{un}},
\end{tikzcd}$$

Moreover, this retraction defines by Frobenius reciprocity the intertwiner $J_P(\sigma)$ up to scalar multiplication in $R(T)$.

The lattice $r_{P'}(\mathrm{Ind}_P^G(\mathfrak{l}_0))$ is an integral structure of $r_{P'}(\mathrm{Ind}_P^G(\sigma_{un}))$ in the sense of [35, I.9] by [9, Proposition 6.7]. The image of $r_{P'}(\mathrm{Ind}_P^G(\mathfrak{l}_0))$ under r is an $R[T]_{(T-1)}$-integral structure of $\overline{\sigma_{un}}$ by [35, I.9.3]. Now every such integral structure is of the form $\sigma \otimes_R \mathfrak{l} \otimes \psi_{un}$, where $\mathfrak{l}$ is a non-trivial fractional ideal of $R(T)$. To see this, note that since the integral structure has to be $R[G]$- and $R[T]_{(T-1)}$-stable, it is of the form $\sigma' \otimes_R \mathfrak{l} \otimes \psi_{un}$ for $\mathfrak{l}$ an ideal and σ' a subrepresentation of σ. Moreover,

$$\sigma'_{un} = \sigma' \otimes_R \mathfrak{l} \otimes \psi_{un} \otimes_{R[T]_{(T-1)}} R(T) = \sigma_{un}$$

and therefore $\sigma' = \sigma$. Finally, note that every fractional ideal of $R(T)$ is of the form $(T-1)^k$ for some $k \in \mathbb{Z}$. We thus obtain $k \in \mathbb{Z}$ and the following diagram.

$$\begin{tikzcd}
\overline{\mathfrak{l}_0} \arrow[r, hook] \arrow[rr, bend right, "\iota"'] & r_{P'}(\mathrm{Ind}_P^G(\mathfrak{l}_0)) \arrow[r, "r"] & \overline{\mathfrak{l}_k},
\end{tikzcd}$$

where $\iota\colon \mathfrak{l}_0 \hookrightarrow \mathfrak{l}_k$ is the natural inclusion and in particular $k \leq 0$. We now multiply r by $(T-1)^j$, $j \in \mathbb{N}$ such that the image of $r' := r(T+1)^j$ is contained in $\overline{\mathfrak{l}_0}$ but not in $\overline{\mathfrak{l}_1}$. We thus have a diagram

$$\begin{tikzcd}
\overline{\mathfrak{l}_0} \arrow[r, hook] \arrow[rr, bend right, "(T-1)^j\cdot"'] & r_{P'}(\mathrm{Ind}_P^G(\mathfrak{l}_0)) \arrow[r, "r'"] & \overline{\mathfrak{l}_0}
\end{tikzcd}$$

and we fix $J_P(\sigma)$ to the intertwiner coming from Frobenius reciprocity applied to r'. Moreover, we set $r(\sigma) := j \geq 0$ and it is easy to see that this is independent of the choices made. Next observe that the sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ind}_P^G(\mathfrak{l}_1) \arrow[r] & \mathrm{Ind}_P^G(\mathfrak{l}_0) \arrow[r, "T\mapsto 1"] & \mathrm{Ind}_P^G(\sigma) \arrow[r] & 0
\end{tikzcd}$$

is exact. The only non-trivial part is the surjectivity of the last map. It follows from passing to K-fixed vectors for a suitable, small enough open compact subgroup K of G and using [35, I.5.6] to see that

$$\mathrm{Ind}_P^G(\mathfrak{l}_0)^K = \mathrm{Ind}_P^G(\sigma)^K \otimes_R R[T]_{(T-1)}.$$

We thus have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Ind}_P^G(\sigma_{un}) \arrow[r, "J_P(\sigma)"] & \mathrm{Ind}_{P'}^G(\overline{\sigma}_{un}) \\
\mathrm{Ind}_P^G(\mathfrak{l}_0) \arrow[u, hook] \arrow[r, "J_P(\sigma)"] \arrow[d, two heads, "T\mapsto 1"] & \mathrm{Ind}_{P'}^G(\overline{\mathfrak{l}_0}) \arrow[u, hook] \arrow[d, two heads, "T\mapsto 1"] \\
\mathrm{Ind}_P^G(\sigma) \arrow[r, "R_\sigma"] & \mathrm{Ind}_{P'}^G(\overline{\sigma})
\end{tikzcd}$$

Now if R_σ vanishes, $J_P(\sigma)(\mathrm{Ind}_P^G(\mathfrak{l}_0)) \subseteq \mathrm{Ind}_{P'}^G(\overline{\mathfrak{l}_1})$, however applying Frobenius reciprocity would give a contradiction to the construction of $J_P(\sigma)$. ☐

Note that $J_P(\sigma)$ in the above lemma is unique up to a unit in $R[T]_{(T-1)}$ and R_σ is unique up to a unit in R. We call $r(\sigma)$ the order of the pole of the intertwining operator at σ. Note that $r(\sigma) = 0$ if and only if for all $f \in \mathrm{Ind}_P^G(\sigma)$ with support contained in $PwP'g$,

$$R_\sigma(f)(g) = \int_{U'} f(wu'g)\,\mathrm{d}u'.$$

If $\sigma = \pi_1 \otimes \pi_2$, we will also denote by

$$R_{\pi_1,\pi_2}\colon \pi_1 \times \pi_2 \to \pi_2 \times \pi_1$$

the so obtained morphism and r_{π_1,π_2} the order of vanishing. The next lemma is the generalization of the list of properties needed to deal with square irreducible representations and collected for $R = \mathbb{C}$ in [20, Lemma 2.3].

Lemma 3.1.2. *Let* $0 \to \sigma_1 \hookrightarrow \sigma \twoheadrightarrow \sigma_2 \to 0$ *be a short exact sequence of representations of* M *and* π_1, π_2, π_3 *representations of* G_{k_i}, $i \in \{1,2,3\}$.

1. *If* $M = G$, R_σ *is a scalar.*
2. *The order of the poles satisfies* $r(\sigma) \geq r(\sigma_1)$ *and*
$$J_P(\sigma)\big|_{\sigma_1} = (T-1)^{r(\sigma)-r(\sigma_1)} J_P(\sigma_1).$$
3. R_σ *restricts either to an intertwining operator or to* 0, i.e. *up to a scalar*
$$R_\sigma\big|_{\sigma_1} = \begin{cases} R_{\sigma_1} & \text{if } r(\sigma) = r(\sigma_1), \\ 0 & \text{otherwise.} \end{cases}$$
4. *If* $\sigma = \pi_1 \times \pi_2 \otimes \pi_3$, $r_{\pi_1 \times \pi_2, \pi_3} \leq r_{\pi_1,\pi_2} + r_{\pi_2,\pi_3}$ *and*
$(T-1)^{r_{\pi_1,\pi_2}+r_{\pi_2,\pi_3}-r_{\pi_1\times\pi_2,\pi_3}} J_P(\sigma) = (J_P(\pi_1 \otimes \pi_3) \times 1_{\pi_2}) \circ (1_{\pi_1} \times J_P(\pi_2 \otimes \pi_3))$.
Therefore,
$$(R_{\pi_1,\pi_3} \times 1_{\pi_2}) \circ (1_{\pi_1} \times R_{\pi_2,\pi_3}) = \begin{cases} R_{\pi_1\times\pi_2,\pi_3} & \text{if } r_{\pi_1\times\pi_2,\pi_3} = r_{\pi_1,\pi_2} + r_{\pi_2,\pi_3}, \\ 0 & \text{otherwise,} \end{cases}$$
up to a scalar.
5. *Moreover,* $r_{\pi_1\times\pi_2,\pi_3} = r_{\pi_1,\pi_2} + r_{\pi_2,\pi_3}$ *if at least one of the* R_{π_i,π_3}, $i \in \{1,2\}$ *is an isomorphism or if* π_3 *is irreducible.*
6. *If* R_σ *is an isomorphism, then* $R_{\overline{\sigma}} \circ R_\sigma$ *is a scalar.*

Proof. The first point is trivially true. Since the following diagram commutes up to a non-zero scalar in $R(T)$, (2) and (3) also follow immediately.

$$\begin{tikzcd}
\mathrm{Ind}_P^G(\sigma_{2un}) \arrow[r, "J_P(\sigma_2)"] & \mathrm{Ind}_{P'}^G(\overline{\sigma_2}_{un}) \\
\mathrm{Ind}_P^G(\sigma_{un}) \arrow[r, "J_P(\sigma)"] \arrow[u, two heads] & \mathrm{Ind}_{P'}^G(\overline{\sigma}_{un}) \arrow[u, two heads] \\
\mathrm{Ind}_P^G(\sigma_{1un}) \arrow[r, "J_P(\sigma_1)"] \arrow[u, hook] & \mathrm{Ind}_{P'}^G(\overline{\sigma_1}_{un}) \arrow[u, hook]
\end{tikzcd}$$

The first part of (4) follows from the following, up to multiplication by some non-zero element of $R(T)$, commutative diagram.

$$\begin{tikzcd}
\mathrm{Ind}_P^G((\pi_1 \times \pi_2 \otimes \pi_3)_{un}) \arrow[r, "J_P(\pi_1\times\pi_2\otimes\pi_3)"] \arrow[d, "1_{\pi_1}\times J_P(\pi_2\otimes\pi_3)"'] & \mathrm{Ind}_{P'}^G((\pi_3 \otimes \pi_1 \times \pi_2)_{un}) \\
\mathrm{Ind}_{P_{k_1,k_3,k_2}}^G((\pi_1 \otimes \pi_3 \otimes \pi_2)_{un}) \arrow[ur, bend right, "J_P(\pi_1\otimes\pi_3)\times 1_{\pi_2}"] &
\end{tikzcd}$$

To see that it commutes, apply [9, Proposition (ii)] to the bottom two arrows and [9, Proposition (iii)] to the top arrow. Then the commutativity follows from [9, Proposition (i)]. For (6), [9, Proposition 7.8(i)] tells us that $J_P(\sigma) \circ J_{P'}(\overline{\sigma})$ is a scalar in $R(T)$. Thus, after multiplying by the suitable normalization factors and setting $T = 1$, the claim follows. Moreover, it also is easy to check that this diagram implies the claims regarding the poles.

It remains to prove (5) and here we argue as in [20, Theorem 2.3]. If one of the R_{π_i,π_3} is an isomorphism, the composition

$$(R_{\pi_1,\pi_3} \times 1_{\pi_2}) \circ (1_{\pi_1} \times R_{\pi_2,\pi_3}) \neq 0$$

and hence $r_{\pi_1\times\pi_2,\pi_3} = r_{\pi_1,\pi_2} + r_{\pi_2,\pi_3}$. If π_3 is irreducible and $r_{\pi_1\times\pi_2,\pi_3} < r_{\pi_1,\pi_2} + r_{\pi_2,\pi_3}$, then

$$(R_{\pi_1,\pi_3} \times 1_{\pi_2}) \circ (1_{\pi_1} \times R_{\pi_2,\pi_3}) = 0$$

and hence

$$\pi_1 \times \mathrm{Im}(R_{\pi_2\times\pi_3}) \hookrightarrow \mathrm{Ker}(R_{\pi_1,\pi_3}) \times \pi_2 \hookrightarrow \pi_1 \times \pi_3 \times \pi_2.$$

Twisting with the inner automorphism given by

$$w = \begin{pmatrix} 0 & 0 & 1_{k_2} \\ 0 & 1_{k_3} & 0 \\ 1_{k_1} & 0 & 0 \end{pmatrix},$$

we obtain that

$$\mathrm{Im}(R_{\pi_2\times\pi_3})^{w_1} \times \pi_1 \hookrightarrow \pi_2 \times \mathrm{Ker}(R_{\pi_1,\pi_3})^{w_2} \hookrightarrow \pi_2 \times \pi_3 \times \pi_1,$$

where $*^{w'}$ denotes the twist by the inner automorphism$g \mapsto w'^{-1}gw'$, and

$$w_1 = \begin{pmatrix} 0 & 1_{k_2} \\ 1_{k_3} & 0 \end{pmatrix}, \; w_2 = \begin{pmatrix} 0 & 1_{k_3} \\ 1_{k_1} & 0 \end{pmatrix}.$$

By Corollary 2.2.2, the irreducibility of π_3 implies that $\mathrm{Ker}(R_{\pi_1,\pi_3}) = \pi_3 \times \pi_1$, a contradiction. ☐

3.2 Square irreducible representation Using above mod ℓ-analogous of the results of [20] , one can arrive via the same arguments to the following results. We say $\Pi \in \mathfrak{Rep}(G_n)$ is *SI* if Π is socle-irreducible, and its scole appears with multiplicity one in its composition series. Similarly, we say Π is *CSI*, if it cosocle-irreducible, and its coscole appears with multiplicity one in its composition series. The following three theorems follow *muta mutandis* as in [20], thanks to Lemma 3.1.2 and Corollary 2.2.2.

Lemma 3.2.1 ([20, Corollary 2.5], [15, Corollary 3.3]). *A representation $\pi \in \mathfrak{Irr}_n$ is called square-irreducible or $\square$-irreducible if one of the following equivalent conditions holds.*

1. $\pi \times \pi$ *is SI.*

2. $\pi \times \pi$ *is irreducible.*

3. $\dim_R \operatorname{End}_{G_{2n}}(\pi \times \pi) = 1$.

4. $R_{\pi,\pi}$ *is a scalar.*

Note that a cuspidal representation is square-irreducible if and only if it is □-irreducible cuspidal and we denote the subset of square-irreducible representations of $\mathfrak{Irr}_n$ by $\mathfrak{Irr}^\square(G_n)$. More generally, for $\Delta = [a,b]_\rho$ a segment, $\mathrm{Z}(\Delta)$ is □-irreducible if and only if $b - a + 1 < o(\rho)$.

Lemma 3.2.2 ([20, Lemma 2.8], [14, Theorem 3.1])**.** *Let* $\pi \in \mathfrak{Irr}^\square(G_n)$ *and* $\tau \in \mathfrak{Irr}_m$*. Then both* $\pi \times \tau$ *and* $\tau \times \pi$ *are SI and CSI.*

Lemma 3.2.3 ([21, Theorem 4.1.D])**.** *Let* $\pi \in \mathfrak{Irr}^\square(G_n)$*. The maps*

$$\operatorname{soc}(\pi \times \cdot\,) = \cos(\,\cdot \times \pi)\colon \mathfrak{Irr} \to \mathfrak{Irr},\ \operatorname{soc}(\,\cdot \times \pi) = \cos(\pi \times \cdot\,)\colon \mathfrak{Irr} \to \mathfrak{Irr}$$

are injective.

Lemma 3.2.4. *If* $\pi \in \mathfrak{Irr}^\square(G_n)$ *then* $\pi^k \in \mathfrak{Irr}^\square(G_{nk})$ *for all* $k \in \mathbb{Z}_{>0}$*.*

Proof. By Lemma 3.2.1 it is enough to show that R_{π^k,π^k} is a scalar. We start by showing that $R_{\pi^k,\pi}$ is a scalar for all k, which we do by induction on k. By assumption, we know that $R_{\pi,\pi}$ is a scalar. By Lemma 3.1.2(4) and (5)

$$R_{\pi^k,\pi} = (R_{\pi^{k-1},\pi} \times 1_\pi) \circ (1_{\pi^{k-1}} \times R_{\pi,\pi})$$

is a scalar by the induction hypothesis on k. By Lemma 3.1.2(6) also R_{π,π^k} is a scalar. Next we prove by induction on k that R_{π^k,π^k} is a scalar. Again by Lemma 3.1.2(4) and (5), we know that

$$R_{\pi^k,\pi^{k-1}} = (R_{\pi^{k-1},\pi^{k-1}} \times 1_\pi) \circ (1_{\pi^{k-1}} \times R_{\pi,\pi^{k-1}})$$

is a scalar. By Lemma 3.1.2(6) also R_{π^{k-1},π^k} is a scalar. Applying a third time Lemma 3.1.2(4) and (5) we see that

$$R_{\pi^k,\pi^k} = (R_{\pi^{k-1},\pi^k} \times 1_\pi) \circ (1_{\pi^{k-1}} \times R_{\pi,\pi^k}),$$

which therefore is a scalar. □

Lemma 3.2.5. *Let* ρ *be a cuspidal representation such that* $o(\rho) = 1$ *and* $\mathfrak{m} \in \mathcal{MS}(\rho)_{ap}$*. Then* $\mathrm{Z}(\mathfrak{m})$ *is not square-irreducible.*

Proof. Assume otherwise that $\mathrm{Z}(\mathfrak{m})$ is square-irreducible. Then by Lemma 3.2.4 for all $k \in \mathbb{Z}_{>0}$ we have that $\mathrm{Z}(\mathfrak{m})^k$ is square-irreducible and hence $\mathrm{Z}(\mathfrak{m})^{2k} \cong \mathrm{Z}(2k\mathfrak{m})$ by Proposition 2.6.7. But if $2k > \ell = e(\rho)$, the multisegment $2k\mathfrak{m}$ is no longer aperiodic, and hence its cuspidal support is no longer contained in $\mathbb{N}(\mathbb{Z}[\rho])$. But on the other hand, the cuspidal support of $\mathrm{Z}(\mathfrak{m})^{2k}$ must be contained in $\mathcal{MS}(\rho)$, a contradiction. □

3.3 P-regular representations Motivated by the notation of (P,Q) -regularity introduced in [9, §2], we will introduce a similar condition called P-regularity. Note that being (P,Q)-regular is a slightly more fine-tuned notion, but for our purposes the following more blunt definition is sufficient.

Let P be a parabolic subgroup of $G = G_n$ with Levi-decomposition $P = M \rtimes U$ and opposite parabolic subgroup $\overline{P}$. A representation σ of M is called *P-regular* if the natural restriction map

$$\mathrm{Hom}_{G\times G}(C_c^\infty(G), \mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \mathrm{Ind}_P^G(\sigma^\vee)) \to$$

$$\to \mathrm{Hom}_{\overline{P}\times P}(C_c^\infty(\overline{P}P), \mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \mathrm{Ind}_P^G(\sigma^\vee))$$

is an isomorphism of R-vector spaces.

Lemma 3.3.1. *Let $\sigma \in \mathfrak{Rep}(M)$.*

1. *If σ is irreducible and P-regular, then*

$$\dim_R \mathrm{Hom}_{G\times G}(C_c^\infty(G), \mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \mathrm{Ind}_P^G(\sigma^\vee)) = 1.$$

2. *If σ is irreducible and it appears in $r_{\overline{P}}(\mathrm{Ind}_{\overline{P}}^G(\sigma))$ with multiplicity 1, it is P-regular.*

Proof. For the first claim, note that by Frobenius reciprocity, we have

$$\dim_R \mathrm{Hom}_{\overline{P}\times P}(C_c^\infty(\overline{P}P), \mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \mathrm{Ind}_P^G(\sigma^\vee)) =$$

$$= \dim_R \mathrm{Hom}_M(r_{\overline{P}\times P}(C_c^\infty(\overline{P}P)), \sigma \otimes \sigma^\vee) =$$

$$= \dim_R \mathrm{Hom}_M(C_c^\infty(M), \sigma \otimes \sigma^\vee) = \dim_R \mathrm{Hom}_M(\sigma, \sigma) = 1.$$

For the second claim, note that by the first part and the existence of intertwining operators, it follows that it is enough to show that a non-zero map $f: C_c^\infty(G) \mapsto \mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \mathrm{Ind}_P^G(\sigma^\vee)$ restricts to a non-zero map on $C_c^\infty(\overline{P}P)$. We now apply Frobenius reciprocity with respect to r_P to obtain a non-zero map

$$f': r_{\{1\}\times P}(C_c^\infty(G))) \cong \mathrm{Ind}_{P\times M}^{G\times M}(C_c^\infty(M)) \to \mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \sigma^\vee$$

and Frobenius reciprocity with respect to $r_{\overline{P}}$ to obtain a non-zero map

$$f'': r_{\overline{P}\times\{1\}}(\mathrm{Ind}_{P\times M}^{G\times M}(C_c^\infty(M))) \to \sigma \otimes \sigma^\vee.$$

After applying the Geometric Lemma to the left side, we will show that f'' restricts to a non-zero map on

$$F(1)(C_c^\infty(M)) = C_c^\infty(M) \hookrightarrow r_{\overline{P}\times\{1\}}(\mathrm{Ind}_{P\times M}^{G\times M}(C_c^\infty(M))) \to \sigma \otimes \sigma^\vee.$$

Indeed, let $w_i \in G$ be a permutation matrix and assume that $Pw_i\overline{P}$ is the smallest element in $P\backslash G/\overline{P}$ such that f' does not vanish on $F(w_i)(C_c^\infty(M))$, *i.e.* we obtain a non-zero map

$$f'' \colon F(w_i)(C_c^\infty(M)) \to \sigma \otimes \sigma^\vee.$$

Using the specific form of $F(w_i)$, *cf.* [7, Geometric Lemma], and applying first Bernstein reciprocity and then Frobenius reciprocity we obtain a non-zero morphism

$$C_c^\infty(M) \to F(w_i^{-1})(\sigma) \otimes \sigma^\vee.$$

where here $F(w_i^{-1})(\sigma)$ is now the subquotient coming from the Geometric Lemma applied to $r_{\overline{P}}(\mathrm{Ind}_{\overline{P}}^G(\sigma))$. This in turn gives a non-zero morphism

$$\sigma \to F(w_i^{-1})(\sigma).$$

By assumption σ appears only with multiplicity 1 in $r_{\overline{P}}(\mathrm{Ind}_{\overline{P}}^G(\sigma))$ and $F(1)(\sigma) \cong \sigma$. Thus $w_i^{-1} \in \overline{P}$ and hence $Pw_i\overline{P} = P\overline{P}$. We therefore showed that f'' does not vanish on $F(1)(C_c^\infty(M))$. But now $F(1)(C_c^\infty(M)) = r_{\overline{P}\times P}(C_c^\infty(\overline{P}P))$ and thus by Frobenius reciprocity f does not vanish on on $C_c^\infty(\overline{P}P)$. $\square$

Assume now $P = P_{(n_1,n_2)}$, $\sigma = \sigma_1 \otimes \sigma_2$ is irreducible and either σ_1 or σ_2 is $\square$-irreducible. If moreover σ is P-regular and $\pi := \cos(\mathrm{Ind}_P^G(\sigma)) \cong \mathrm{soc}(\mathrm{Ind}_{\overline{P}}^G(\sigma))$, we call σ *strongly* P-regular. Note that by Lemma 3.2.2 π is irreducible. By Lemma 3.3.1(1), the only morphism up to a scalar in

$$\mathrm{Hom}_G(\mathrm{Ind}_P^G(\sigma), \mathrm{Ind}_{\overline{P}}^G(\sigma))$$

is given by

$$\mathrm{Ind}_P^G(\sigma) \twoheadrightarrow \pi \hookrightarrow \mathrm{Ind}_{\overline{P}}^G(\sigma),$$

since

$$\dim_R \mathrm{Hom}_G(\mathrm{Ind}_P^G(\sigma), \mathrm{Ind}_{\overline{P}}^G(\sigma)) =$$

$$= \dim_R \mathrm{Hom}_{G\times G}(C_c^\infty(G), \mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \mathrm{Ind}_P^G(\sigma^\vee)) = 1.$$

For the next corollary we identify

$$\mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \mathrm{Ind}_P^G(\sigma^\vee) \cong \mathrm{Ind}_{\overline{P}}^G(\sigma^\vee)^\vee \otimes \mathrm{Ind}_P^G(\sigma)^\vee$$

Corollary 3.3.2. *Let $\sigma \in \mathfrak{Irr}(M)$ be a strongly P-regular representation and let $\pi = \mathrm{soc}(\mathrm{Ind}_{\overline{P}}^G(\sigma))$. Let*

$$T \in \mathrm{Hom}_{\overline{P}\times P}(C_c^\infty(\overline{P}P), \mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \mathrm{Ind}_P^G(\sigma^\vee)),$$

written as

$$\phi \mapsto (f_1 \otimes f_2 \mapsto T(\phi)(f_1 \otimes f_2)),\ f_1 \otimes f_2 \in \mathrm{Ind}_{\overline{P}}^G(\sigma^\vee) \otimes \mathrm{Ind}_P^G(\sigma).$$

Then for all $f_1 \otimes f_2 \in \mathrm{Ind}_{\overline{P}}^{G}(\sigma^\vee) \otimes \mathrm{Ind}_{P}^{G}(\sigma)$ *there exists a matrix coefficient* f *of* π *such that for all* $\phi \in C_c^\infty(\overline{P}P)$

$$T(\phi)(f_1 \otimes f_2) = \int\limits_{\overline{P}P} \phi(p) f(p)\,\mathrm{d}p.$$

Proof. Since σ is irreducible, Lemma 3.3.1(1) tells us that the natural inclusion

$$\mathrm{Hom}_{G\times G}(C_c^\infty(G), \pi \otimes \pi^\vee) \hookrightarrow \mathrm{Hom}_{G\times G}(C_c^\infty(G), \mathrm{Ind}_{\overline{P}}^{G}(\sigma) \otimes \mathrm{Ind}_{P}^{G}(\sigma)^\vee)$$

is an isomorphism. Note that every morphism in the former space is given by sending a matrix coefficient f and a Schwartz-function $\phi \in C_c^\infty(G)$ to

$$\int\limits_{G} f(g)\phi(g)\,\mathrm{d}_l g.$$

The claim then follows from the definition of P-regularity. ☐

Next we see that strongly P-regular representation will interact nicely with L-factors. Recall that for π an irreducible representation,

$$P(\pi, T) = \frac{1}{L(\pi, T)}.$$

Proposition 3.3.3. *Let* $P = P_{(n_1,n_2)} = M \rtimes U$ *be a parabolic subgroup of* G_n *and* $\sigma = \sigma_1 \otimes \sigma_2$ *a strongly* P*-regular representation. Set* $\pi = \cos(\mathrm{Ind}_P^G(\sigma))$*. Then* $P(\sigma_2, T)$ *divides* $P(\pi, T)$*.*

Before we come to the proof, let us observe some parallels between this statement and a special case of it, namely [11, Theorem 2.7]. In there Jacquet uses the following fact, which only holds true over $\overline{\mathbb{Q}}_\ell$ in full generality. If $\mathfrak{m}$ is a multisegment over $\overline{\mathbb{Q}}_\ell$ and ρv_ρ^b a cuspidal representation in the cuspidal support of $\mathfrak{m}$ with b maximal, let $\Delta_1 + \ldots + \Delta_k$ be the sub-multisegment of $\mathfrak{m}$ consisting of all segments of the form $[a, b]_\rho$. Then

$$\bigtimes_{i=1}^{k} \langle \Delta_i \rangle$$

is square-irreducible and

$$\sigma = \bigtimes_{i=1}^{k} \langle \Delta_i \rangle \otimes \langle \mathfrak{m} - \sum_{i=1}^{k} \Delta_i \rangle$$

is strongly-P regular with

$$\cos(\mathrm{Ind}_P^{G_n}(\sigma)) = \langle \mathfrak{m} \rangle.$$

Thus Proposition 3.3.3 can be seen as a generalization of this fact over $\overline{\mathbb{Q}}_\ell$ and as we will see, also its proof follows essentially the same idea.

Before we start with the proof, we define the following. Let $H\colon G_n \times G_n \to R$ be a smooth function such that

$$H(u_1 m g_1, u_2 m g_2) = H(g_1, g_2)$$

for all $u_1 \in \overline{U}$, $u_2 \in U$, $m \in M, g_1, g_2 \in G_n$ and

$$m \mapsto H(g_1, m g_2)$$

is a matrix coefficient of $\sigma \otimes \delta_P^{\frac{1}{2}}$. Then for $\phi \in C_c^\infty(\overline{P}P)$, define the integral

$$T_H(\phi) \coloneqq \int\limits_{U\times M\times\overline{U}} H(1,m)\phi(\overline{u}^{-1}um)\,\mathrm{d}\overline{u}\,\mathrm{d}u\,\mathrm{d}m.$$

Lemma 3.3.4. *The integrals $T_H(\phi)$ are well defined for $\phi \in C_c^\infty(\overline{P}P)$ and define a non-zero $\overline{P} \times P$-equivariant morphism*

$$C_c^\infty(\overline{P}P) \to \mathrm{Ind}_{\overline{P}}^{G_n}(\sigma) \otimes \mathrm{Ind}_P^{G_n}(\sigma^\vee).$$

Proof. By the assumption on ϕ the integral converges. Note that above H's are precisely linear combinations of functions of the form

$$H(g_1, g_2) = f^\vee(g_1)(f(g_2))$$

for $f^\vee \in \mathrm{Ind}_{\overline{P}}^{G_n}(\sigma^\vee)$, $f \in \mathrm{Ind}_P^{G_n}(\sigma)$. ☐

Proof of Proposition 3.3.3. Having introduce above machinery, we can now mimic the proof of [11, Theorem 2.7]. We recall $\delta_P(m_1, m_2) = |m_1|^{dn_2}|m_2|^{-dn_1}$. Note that the claim of the proposition is equivalent to

$$\mathcal{L}(\sigma_2) \subseteq \mathcal{L}(\pi).$$

To show this, we need to find for each $\phi_2 \in \mathscr{S}_R(M_{n_2}(\mathrm{D}))$ and matrix coefficient f_2 of σ_2, a $\phi \in \mathscr{S}_R(M_n(\mathrm{D}))$ and a matrix coefficient f of π such that

$$Z(\phi, Tq^{-\frac{dn-1}{2}}, f) = Z(\phi_2, Tq^{-\frac{dn_2-1}{2}}, f_2).$$

Choose now a coefficient f_1 of σ_1 and $\phi_1 \in \mathscr{S}_R(M_{n_1}(\mathrm{D}))$ supported in a suitable neighbourhood of 1 such that $Z(\phi_1, Tq^{-\frac{dn_1-1}{2}}, f_1) = 1$ and choose

$$\phi_{1,2} \in C_c^\infty(\overline{U}),\ \phi_{2,1} \in C_c^\infty(U)$$

which are supported in a suitable neighbourhood of 1_n such that

$$\phi(\overline{u}um) = \phi_{1,2}(\overline{u})\phi_{2,1}(u)\phi_1(m_1)\phi_2(m_2),\ m = (m_1, m_2)$$

satisfies

$$\int\limits_{\overline{U}\times U} \phi(\overline{u}^{-1}um)\,\mathrm{d}\overline{u}\,\mathrm{d}u = \phi_1(m_1)\phi_2(m_2).$$

Note that in particular for all $N \in \mathbb{Z}$,

$$\phi\chi_{G_n(N)} \in C_c^\infty(\overline{P}P),$$

where $\chi_{G_n(N)}$ is the characteristic function of $G_n(N)$ and ϕ can be extended by 0 to a function on $\mathscr{S}_R(M_n(\mathrm{D}))$. Moreover, the coefficient of T^N in

$$Z(\phi_2, Tq^{-\frac{dn_2-1}{2}}, f_2) = Z(\phi_1, Tq^{-\frac{dn_1-1}{2}}, f_1)Z(\phi_2, Tq^{-\frac{dn_2-1}{2}}, f_2)$$

is

$$\int\limits_{M(N)} |m_1|^{\frac{dn_1-1}{2}}|m_2|^{\frac{dn_2-1}{2}} f_1(m_1)f_2(m_2) \int\limits_{\overline{U}\times U} \phi(\overline{u}^{-1}um)\,\mathrm{d}\overline{u}\,\mathrm{d}u\,\mathrm{d}(m_1, m_2) =$$

$$= \int\limits_{M(N)} |m|^{\frac{dn-1}{2}}\delta_P(m)^{\frac{1}{2}} f_1(m_1)f_2(m_2) \int\limits_{\overline{U}\times U} \phi(\overline{u}^{-1}um)\,\mathrm{d}\overline{u}\,\mathrm{d}u\,\mathrm{d}m. \qquad (3)$$

Note that here we used that $\phi_1(m_1)$ is supported in a sufficiently small neighborhood of 1 and hence in the support of ϕ_1, we have $|m_1|= 1$. We can now find H as in Lemma 3.3.4 such that $H(1, m) = f(m_1)f(m_2)\delta_P(m_1, m_2)^{\frac{1}{2}}$ and hence we find by Corollary 3.3.2 a matrix coefficient f of π, which is independent of N, such that above integral (3) equals

$$\int_{G_n} |g|^{\frac{dn-1}{2}} f(g)(\phi\chi_{G_n(N)})(g)\,\mathrm{d}ug = \int_{G_n(N)} |g|^{\frac{dn-1}{2}} f(g)\phi(g)\,\mathrm{d}g.$$

This in turn is nothing else then the T^N-th coefficient of

$$Z(\phi, Tq^{-\frac{dn-1}{2}}, f).$$

□

4 Derivatives

In this section we define derivatives with respect to a □-irreducible cuspidal representation, following the ideas of [25] and [13], who establish all results in this section in the case $R = \overline{\mathbb{Q}}_\ell$. Derivatives will also give rise to most of the P-regular representations we will encounter throughout the rest of the paper.

4.1 Basic properties of derivatives Let $\rho, \pi' \in \mathfrak{Irr}$. We call a representation $\sigma \in \mathfrak{Irr}$ a *right-ρ-derivative* respectively a *left-ρ-derivative* of π' if

$$\pi' \hookrightarrow \sigma \times \rho \text{ respectively } \pi' \hookrightarrow \rho \times \sigma.$$

As a consequence of Lemma 3.2.2 and Lemma 3.2.3 we obtain the following lemma and its corollary.

Lemma 4.1.1. *Let $\rho \in \mathfrak{Irr}^{\square}$ and $\pi' \in \mathfrak{Irr}$. The left- and right-ρ-derivative of π' are up to isomorphism unique. We denote them by $\mathcal{D}_{l,\rho}(\pi')$ and $\mathcal{D}_{r,\rho}(\pi')$ respectively and set them to 0 if no such representation exists.*

Moreover, if $\mathcal{D}_{r,\rho}(\pi') \neq 0$, then $\pi' = \operatorname{soc}(\mathcal{D}_{r,\rho}(\pi') \times \rho)$ and it appears in its decomposition series with multiplicity 1. Analogously, if $\mathcal{D}_{l,\rho}(\pi') \neq 0$, then $\pi' = \operatorname{soc}(\rho \times \mathcal{D}_{l,\rho}(\pi'))$ and it appears in its decomposition series with multiplicity 1.

Corollary 4.1.2. *Let $\pi_1, \pi_2 \in \mathfrak{Irr}$ and $\rho \in \mathfrak{Irr}^{\square}$. If $\mathcal{D}_{l,\rho}(\pi_1) \cong \mathcal{D}_{l,\rho}(\pi_2) \neq 0$ or $\mathcal{D}_{r,\rho}(\pi_1) \cong \mathcal{D}_{r,\rho}(\pi_2) \neq 0$, then $\pi_1 \cong \pi_2$.*

We now focus our attention on the case $\rho \in \mathfrak{C}^{\square}$.

Lemma 4.1.3. *Let ρ be a cuspidal representation of G_m and $\pi \in \mathfrak{Irr}_n$. Then π admits a right-derivative with respect to ρ if and only if there exists an irreducible representation $\pi' \in \mathfrak{Irr}_{n-m}$ such that*

$$[\pi' \otimes \rho] \leq [r_{(n-m,m)}(\pi)].$$

Similarly, π admits a left-derivative with respect to ρ if and only if there exists an irreducible representation $\pi' \in \mathfrak{Irr}_{n-m}$ such that

$$[\rho \otimes \pi'] \leq [r_{(m,n-m)}(\pi)].$$

Proof. We only prove the claim for right-derivatives, the one for left-derivatives follows analogously. Note that if π admits a right-derivative π', we obtain by Frobenius reciprocity $r_{(n-m,m)}(\pi) \twoheadrightarrow \pi' \otimes \rho$. On the other hand, let π' be such that

$$[\pi' \otimes \rho] \leq [r_{(n-m,m)}(\pi)].$$

We now argue inductively on n that then π admits a right derivative. Choose ρ' a cupsidal representation of $G_{m'}$ and $\tau \in \mathfrak{Irr}_{n-m'}$ such that $\pi \hookrightarrow \tau \times \rho'$. If $\rho' \cong \rho$ we are done, thus we assume otherwise. Choose k maximal such that there exists $\tau' \in \mathfrak{Irr}_{n-km-m'}$

$$\tau \hookrightarrow \tau' \times \rho^k.$$

By induction we note that this implies that there exists no $\tau'' \in \mathfrak{Irr}_{n-(k+1)m-m'}$ such that

$$[\tau'' \otimes \rho] \leq [r_{(n-(k+1)m-m',m)}(\tau')].$$

We thus have $\pi \hookrightarrow \tau' \times \sigma$, where σ is an irreducible subquotient of $\rho^k \times \rho'$. We will now show that σ admits a right-derivative with respect to ρ, which would show the claim. Firstly,

$$[\pi' \otimes \rho] \leq [r_{(n-m,m)}(\pi)] \leq [r_{(n-m,m)}(\tau' \times \sigma)].$$

The Geometric Lemma thus implies that $k > 0$ and there exists σ' such that

$$\sigma' \otimes \rho \leq [r_{((k-1)m+m',m)}(\sigma)].$$

We first note that if $\rho' \not\cong \rho v_\rho^{\pm 1}$, by Lemma 2.4.1 $\sigma \cong \rho^k \times \rho' \cong \rho' \times \rho^k$ in which the claim would follow immediately. Thus we assume $\rho' \cong \rho v_\rho^{\pm 1}$. Since we already assume that $\rho' \not\cong \rho$, we obtain that $o(\rho) > 1$. We differentiate two cases, namely $o(\rho) > 2$ and $o(\rho) = 2$.

Case 1: $o(\rho) > 2$

Then $\rho \times \rho'$ has two subquotients σ_1 and σ_2 such that $\rho^{k-1} \times \sigma_1$ and $\rho^{k-1} \times \sigma_2$ are irreducible by Theorem 2.6.2. Thus if $k > 1$, the claim follows from the commutativity of parabolic induction. If $k = 1$,

$$\sigma' \otimes \rho \leq [r_{(m,m)}(\sigma)]$$

implies thus by Lemma 2.6.1 that $\sigma \hookrightarrow \rho v_\rho^{-1} \times \rho$.

Case 1: $o(\rho) = 2$

Then $\rho \times \rho'$ has three subquotients $\sigma_1 = \mathrm{Z}([0,1]_\rho)$, $\sigma_2 = \mathrm{Z}([1,2]_\rho)$ and $\sigma_3 = St(\rho, 2)$ cuspidal. Note that $\rho^{k-1} \times \sigma_3$ is irreducible by Lemma 2.4.1 and hence if $\sigma \hookrightarrow \rho^{k-1} \times \sigma_3$, we have that $k > 1$ and the claim follows from the commutativity of parabolic induction.

If $k = 1$, it follows again from Lemma 2.6.1 that $\sigma = \sigma_2 \hookrightarrow \rho v_\rho^{-1} \times \rho$.

Thus we assume $k > 1$ and also note that if $\sigma \hookrightarrow \rho^{k-1} \times \sigma_2$, then we are done since, $\sigma_2 \hookrightarrow \rho v_\rho^{-1} \times \rho$. We are therefore in the situation that $\sigma \hookrightarrow \rho^{k-1} \times \sigma_1$. Next we show that

$$[\rho \times \sigma_1] = [\mathrm{Z}([0,2]_\rho)] + [\mathrm{Z}([0,1]_\rho + [0,0]_\rho)]$$

and moreover,

$$r_{(2m,m)}(\mathrm{Z}([0,2]_\rho)) = \mathrm{Z}([0,1]_\rho) \otimes \rho,\ r_{(2m,m)}(\mathrm{Z}([0,1]_\rho + [0,0]_\rho)) = \rho \times \rho \otimes \rho v_\rho \quad (4)$$

Indeed, note that by the Geometric Lemma

$$[r_{(2m,m)}(\mathrm{Z}([0,1]_\rho) \times \rho)] = [\mathrm{Z}([0,1]_\rho) \otimes \rho] + [\rho \times \rho \otimes \rho v_\rho]$$

and since it does not contain a cuspidal representation by Lemma 2.6.4, it is at most of length 2. On the other hand, it also contains by Lemma 2.6.3 and

Theorem 2.6.6 the two representations in question. Since by Lemma 2.6.1 the parabolic reduction of $\mathrm{Z}([0,2]_\rho)$ is of above-mentioned form, the claim follows. In particular, if $\sigma \hookrightarrow \rho^{k-2} \times Z([0,2]_\rho)$ we are done by Lemma 2.6.3 and therefore we can assume that

$$\sigma \hookrightarrow \rho^{k-2} \times \mathrm{Z}([0,1]_\rho + [0,0]_\rho).$$

By what we just proved about the parabolic reduction of $\mathrm{Z}([0,1]_\rho + [0,0]_\rho)$, we obtain that $k > 2$. It is now enough to show that $\rho \times \mathrm{Z}([0,1]_\rho + [0,0]_\rho)$ is irreducible, since then

$$\rho \times \mathrm{Z}([0,1]_\rho + [0,0]_\rho) \cong \mathrm{Z}([0,1]_\rho + [0,0]_\rho) \times \rho.$$

To see this note that

$$[\mathrm{Z}([0,1]_\rho + [0,0]_\rho + [0,0]_\rho)] \leq [\rho \times \mathrm{Z}([0,1]_\rho + [0,0]_\rho)]$$

and

$$[r_{(m,3m)}(\mathrm{Z}([0,1]_\rho + [0,0]_\rho + [0,0]_\rho))] \geq [\rho \otimes \mathrm{Z}([0,0]_\rho + [1,1]_\rho + [0,0]_\rho)] \cong$$
$$\cong [\rho \otimes \mathrm{Z}([0,0]_\rho + [1,1]_\rho) \times \rho],$$

since $\mathrm{Z}([0,0]_\rho + [1,1]_\rho)$ is cuspidal. Hence there exists π_3' such that

$$[r_{(3m,m)}(\mathrm{Z}([0,1]_\rho + [0,0]_\rho + [0,0]_\rho))] \geq [\pi_3' \otimes \rho].$$

On the other hand the Geometric Lemma and (4) imply

$$[r_{(3m,m)}(\mathrm{Z}([0,1]_\rho + [0,0]_\rho) \times \rho)] \geq [\rho^3 \otimes \rho v_\rho]$$

by (4). Since this representation is residually non-degenerate, Theorem 2.6.6 implies that the irreducible subquotient of

$$\mathrm{Z}([0,1]_\rho + [0,0]_\rho) \times \rho,$$

whose parabolic reduction contains $\pi_3' \otimes \rho$ must be

$$\mathrm{Z}([0,1]_\rho + [0,0]_\rho + [0,0]_\rho).$$

But

$$[r_{(3m,m)}(\mathrm{Z}([0,1]_\rho + [0,0]_\rho) \times \rho)] = [\rho^3 \times \rho v_\rho] + [\mathrm{Z}([0,1]_\rho + [0,0]_\rho) \otimes \rho]$$

by the Geometric Lemma and (4) and hence

$$[r_{(3m,m)}(\mathrm{Z}([0,1]_\rho + [0,0]_\rho) \times \rho)] = [r_{(3m,m)}(\mathrm{Z}([0,1]_\rho + [0,0]_\rho + [0,0]_\rho))].$$

This implies that any other possible subquotient π_4' of $\mathrm{Z}([0,1]_\rho + [0,0]_\rho) \times \rho$ must satisfy

$$r_{(3m,m)}(\pi_4) = 0.$$

But this yields a contradiction as follows. The cuspidal support of π_4 is either $3[\rho] + [\rho v_\rho]$ or $Z([0,0]_\rho + [1,1]_\rho) + 2[\rho]$. In both cases it follows that either the right-derivative of π_4' with respect to ρ or ρv_ρ is non-zero. But then Frobenius reciprocity gives the desired contradiction. □

Recall that ρ^l is irreducible if ρ is $\square$-irreducible for all $l \in \mathbb{Z}_{>0}$ by Theorem 2.6.2.

Lemma 4.1.4. *Let $\pi \in \mathfrak{Irr}_n$, $\rho \in \mathfrak{C}_m^{\square}$ and let $l > 0$ and $\pi' \in \mathfrak{Irr}_{n-lm}$ be such that*

$$\pi \hookrightarrow \pi' \times \rho^l$$

and π' admits no right derivative with respect to ρ. Then $\pi' \otimes \rho^l$ is strongly $P_{(lm,n-lm)}$-regular. Moreover, if π'' is an irreducible representation such that $\pi'' \otimes \rho^l$ appears in $r_{(n-lm,lm)}(\pi)$, then $\pi'' \cong \pi'$ and π appears with multiplicity 1 in $\pi' \times \rho^l$.

Proof. To prove the first claim, we observe that for such maximal l, $r_{(n-lm,lm)}(\pi' \times \rho^l)$ contains the representation $\pi' \otimes \rho^l$ with multiplicity one by Lemma 4.1.3 and the Geometric Lemma. Thus it is $P_{(lm,n-lm)}$-regular by Lemma 3.3.1(2), since $\overline{P_{(lm,n-lm)}}$ is conjugated to $P_{(n-lm,lm)}$. Moreover, it is also clear that if $\pi'' \otimes \rho^l$ appears in $r_{(n-lm,lm)}(\pi' \times \rho^l)$, $\pi'' \cong \pi'$. Since by Frobenius reciprocity $r_{(n-lm,lm)}(\pi)$ contains $\pi' \otimes \rho^l$, π appears with multiplicity 1 in $\pi' \times \rho^l$ and it is the only irreducible subrepresentation.

Finally, to see that $\pi' \otimes \rho^l$ is strongly $P_{(lm,n-lm)}$-regular, recall that $r_{(n-lm,lm)}(\pi)^\vee \cong r_{\overline{P_{(n-lm,lm)}}}(\pi^\vee)$ by Bernstein reciprocity. Thus

$$[\pi' \otimes \rho^l] \leq [r_{(n-lm,lm)}(\pi)]$$

if and only if

$$[\pi'^\vee \otimes (\rho^\vee)^l] \leq [r_{\overline{P_{(n-lm,lm)}}}(\pi^\vee)].$$

Since $\overline{P_{(n-lm,lm)}}$ is conjugated to $P_{(lm,n-lm)}$ this is equivalent to

$$[(\rho^\vee)^l \otimes \pi'^\vee] \leq [r_{(lm,n-lm)}(\pi^\vee)].$$

Now one can prove completely analogously to above that this is equivalent to $\pi^\vee \hookrightarrow (\rho^\vee)^l \times \pi'^\vee$ and $\pi'^\vee$ does not admit a left derivative with respect to $\rho^\vee$. But this is equivalent to $\rho^l \times \pi' \twoheadrightarrow \pi$. $\square$

Remark. Note that Lemma 4.1.4 allows one to give an alternative proof of Lemma 4.1.1 and Corollary 4.1.2 for $\square$-irreducible cuspidal representations.

We now define

$$\mathcal{D}^k_{r,\rho}(\pi) \coloneqq \overbrace{\mathcal{D}_{r,\rho} \circ \cdot \circ \mathcal{D}_{r,\rho}}^{k}(\pi),\ \mathcal{D}^k_{l,\rho}(\pi) \coloneqq \overbrace{\mathcal{D}_{l,\rho} \circ \cdot \circ \mathcal{D}_{l,\rho}}^{k}(\pi),\ \mathcal{D}^0_{r,\rho} = \mathcal{D}^0_{l,\rho} = 1,$$

$d_{r,\rho}(\pi)$ the maximal k such that $\mathcal{D}^k_{r,\rho}(\pi) \neq 0$, $\mathcal{D}_{r,\rho,max}(\pi) \coloneqq \mathcal{D}^{d_{l,\rho}(\pi)}_{l,\rho}(\pi)$, $d_{l,\rho}(\pi)$ the maximal k such that $\mathcal{D}^k_{l,\rho}(\pi) \neq 0$ and $\mathcal{D}_{l,\rho,max}(\pi) \coloneqq \mathcal{D}^{d_{l,\rho}(\pi)}_{l,\rho}(\pi)$. As a corollary of Lemma 4.1.3 we obtain the following.

Corollary 4.1.5. *For $\pi \in \mathfrak{Irr}_n$ and $\rho \in \mathfrak{C}^\square(G_m)$, $d_{r,\rho}(\pi)$ is the maximal k such that there exists $\pi' \in \mathfrak{Irr}_{n-mk}$ with*

$$[\pi' \otimes \rho^k] \leq [r_{(n-km,km)}(\pi)].$$

In this case $\pi' \cong \mathcal{D}_{r,\rho,max}(\pi)$. Analogously, $d_{l,\rho}(\pi)$ is the maximal k such that there exists $\pi' \in \mathfrak{Irr}_{n-mk}$ with

$$[\rho^k \otimes \pi'] \leq [r_{(km,n-km)}(\pi)].$$

In this case $\pi' \cong \mathcal{D}_{l,\rho,max}(\pi)$.

Finally,

$$\mathcal{D}_{r,\rho,max}(\pi) \otimes \rho^{d_{r,\rho}(\pi)} \text{ respectively } \rho^{d_{l,\rho}(\pi)} \otimes \mathcal{D}_{l,\rho,max}(\pi)$$

are strongly $P_{(n-d_{r,\rho}(\pi)m,d_{r,\rho}(\pi)m)}$- respectively $P_{(d_{l,\rho}(\pi)m,n-d_{l,\rho}(pi)m)}$-regular.

Lastly, dualizing behaves well with respect to taking derivatives.

Lemma 4.1.6. *Let $\pi \in \mathfrak{Irr}_n$ and $\rho \in \mathfrak{C}^\square(G_m)$. Then*

$$\mathcal{D}_{r,\rho}(\pi)^\vee \cong \mathcal{D}_{l,\rho^\vee}(\pi^\vee),\ d_{r,\rho}(\pi) = d_{l,\rho^\vee}(\pi^\vee),\ \mathcal{D}_{r,\rho,max}(\pi)^\vee \cong \mathcal{D}_{l,\rho^\vee,max}(\pi^\vee).$$

Proof. Note that $\overline{P_{(n-lm,lm)}}$ is conjugated to $P_{(lm,n-lm)}$ for all l. Thus, it follows from $r_{(lm,n-lm)}(\pi)^\vee \cong r_{P_{(lm,n-lm)}}(\pi^\vee)$ that

$$[\pi' \otimes \rho^l] \leq [r_{(n-lm,lm)}(\pi)]$$

if and only if

$$[(\rho^\vee)^l \otimes \pi'^\vee] \leq [r_{(lm,n-lm)}(\pi^\vee)].$$

Therefore it follows from Corollary 4.1.5 that $d_{r,\rho}(\pi) = d_{l,\rho^\vee}(\pi^\vee)$ and $\mathcal{D}_{r,\rho,max}(\pi)^\vee \cong \mathcal{D}_{l,\rho^\vee,max}(\pi^\vee)$. Hence by Lemma 4.1.1 either $\mathcal{D}_{r,\rho}(\pi)^\vee = \mathcal{D}_{l,\rho^\vee}(\pi^\vee) = 0$ or

$$\mathcal{D}_{r,\rho}(\pi)^\vee \cong \mathcal{D}_{l,\rho^\vee}(\pi^\vee).$$

□

4.2 Derivatives and multisegments

In this section we will answer the following question: Given an aperiodic multisegment $\mathfrak{m}$ and $\rho \in \mathfrak{C}^\square(G_m)$, what is $\mathcal{D}_{r,\rho}(\mathrm{Z}(\mathfrak{m}))$ and $\mathcal{D}_{l,\rho}(\mathrm{Z}(\mathfrak{m}))$?

4.2.1 Derivatives of multisegments Given an aperiodic multisegment $\mathfrak{m}$, we will define four maps

$$\mathcal{D}_{l,\rho}, \mathcal{D}_{r,\rho}, \mathrm{soc}(\,\cdot\,,\rho),\ \mathrm{soc}(\rho,\,\cdot\,)\colon \mathcal{MS}_{ap} \to \mathcal{MS}_{ap}$$

as follows, *cf.* [19]. Let $\mathfrak{m}$ be an aperiodic multisegment. We will first decompose $\mathfrak{m}$ as $\mathfrak{m} = \mathfrak{m}_{\rho,f} + \mathfrak{m}_{\rho,P}$ and define a sequence of tuples of segments $\mathrm{Pairs}_r(\mathfrak{m},\rho)$. If for all ρ-segments Δ in $\mathfrak{m}$ ending in $0 \mod o(\rho)$ there exists no ρ-segment Δ' in $\mathfrak{m}$ which ends in $-1 \mod o(\rho)$ and satisfies $l(\Delta') \geq l(\Delta)$, set

$$\mathfrak{m}_{\rho,f} \coloneqq \mathfrak{m},\ \mathfrak{m}_{\rho,P} \coloneqq 0,\ \mathrm{Pairs}_r(\mathfrak{m},\rho) \coloneqq \varnothing.$$

If there exists Δ and Δ' as above with $l(\Delta') \geq l(\Delta)$, choose the longest ρ-segment Δ_1 ending in $0 \mod o(\rho)$ such that there exist a ρ-segment Δ_2 ending in $-1 \mod o(\rho)$ and choose the shortest such Δ_2 with $l(\Delta_2) \geq l(\Delta_1)$. Set then recursively

$$\mathfrak{m}_{\rho,f} \coloneqq (\mathfrak{m} - \Delta_1 - \Delta_2)_{\rho,f},\ \mathfrak{m}_{\rho,P} \coloneqq (\mathfrak{m} - \Delta_1 - \Delta_2)_{\rho,P} + \Delta_1 + \Delta_2,$$

$$\mathrm{Pairs}_r(\mathfrak{m},\rho) \coloneqq (\mathrm{Pairs}_r(\mathfrak{m} - \Delta_1 - \Delta_2,\rho), (\Delta_1,\Delta_2))$$

and we call such (Δ_1,Δ_2) a *maximal pair* in $\mathfrak{m}$. We also decompose $\mathfrak{m}$ as $\mathfrak{m} = \mathfrak{m}^{\rho,f} + \mathfrak{m}^{\rho,P}$ by setting

$$\mathfrak{m}^{\rho,f} = ((\mathfrak{m}^\vee)_{\rho^\vee,f})^\vee,\ \mathfrak{m}^{\rho,P} = ((\mathfrak{m}^\vee)_{\rho^\vee,P})^\vee,\ \mathrm{Pairs}_r(\mathfrak{m},\rho) = (\mathrm{Pairs}_r(\mathfrak{m}^\vee,\rho^\vee))^\vee.$$

We call the ρ-segments in $\mathfrak{m}_{\rho,f}$ ending in $0 \mod o(\rho)$ the *free* ρ-segments and the ρ-segments in $\mathfrak{m}_{\rho,f}$ ending in $-1 \mod o(\rho)$ the *extendable* ρ-segments. We then set

1. $d_{r,\rho}(\mathfrak{m})$ to the number of free ρ-segments in $\mathfrak{m}$,
2. $\mathcal{D}_{r,\rho}(\mathfrak{m})$ to
$$\begin{cases} 0 & \text{if there exists no free } \rho\text{-segment in } \mathfrak{m}, \\ \mathfrak{m} - \Delta + \Delta^- & \text{where } \Delta \text{ is the shortest free } \rho\text{-segment otherwise,} \end{cases}$$
3. $\mathcal{D}_{r,\rho,max}(\mathfrak{m})$ to the multisegment obtained from $\mathfrak{m}$ by replacing all free ρ-segments Δ by Δ^-,
4. $\mathrm{soc}(\mathfrak{m},\rho)$ to
$$\begin{cases} \mathfrak{m} + [0,0]_\rho & \text{if there exists no extendable } \rho\text{-segment in } \mathfrak{m}, \\ \mathfrak{m} + \Delta^+ - \Delta & \text{with } \Delta \text{ the longest extendable } \rho\text{-segment otherwise,} \end{cases}$$
5. $d_{l,\rho}(\mathfrak{m}) \coloneqq d_{r,\rho^\vee}(\mathfrak{m}^\vee)$, $\mathcal{D}_{l,\rho}(\mathfrak{m}) \coloneqq \mathcal{D}_{r,\rho^\vee}(\mathfrak{m}^\vee)^\vee$, $\mathcal{D}_{l,\rho,max}(\mathfrak{m}) \coloneqq \mathcal{D}_{r,\rho^\vee,max}(\mathfrak{m}^\vee)^\vee$, $\mathrm{soc}(\rho,\mathfrak{m}) \coloneqq \mathrm{soc}(\mathfrak{m}^\vee,\rho^\vee)^\vee$.

We start with the following lemma.

Lemma 4.2.1. *Let* $\mathfrak{m}$ *multisegment,* $(\Delta_1, \Delta_2) \in \mathrm{Pairs}_r(\mathfrak{m}, \rho)$ *and* Δ *a free or extendable* ρ*-segment of* $\mathfrak{m}$*. Then either*

$$l(\Delta) \geq l(\Delta_2) \geq l(\Delta_1) \text{ or } l(\Delta) \leq l(\Delta_1) \leq l(\Delta_2).$$

If Δ *is extendable and* $l(\Delta) = l(\Delta_1)$*, then* $\Delta = \Delta_2$*. If* Δ *is free and* $l(\Delta) = l(\Delta_2)$*, then* $\Delta = \Delta_1$*.*

Furthermore, if in Δ *is a free* ρ*-segment and* Δ' *is an extendable* ρ*-segment of* $\mathfrak{m}$*, then* $l(\Delta') < l(\Delta)$*.*

Proof. We argue by induction on the number of segments in $\mathfrak{m}$. Let (Δ_1', Δ_2') be the maximal pair of $\mathfrak{m}$. Since Δ is a segment in $\mathfrak{m}_{\rho,f}$ ending in 0 or $-1 \mod o(\rho)$, we have the following possibilities.

1. $l(\Delta_2') < l(\Delta)$,
2. $l(\Delta) < l(\Delta_1')$,
3. $l(\Delta_2') = l(\Delta) = l(\Delta_1')$.

Thus the first claim follows for $(\Delta_1, \Delta_2) = (\Delta_1', \Delta_2')$. For any other (Δ_1, Δ_2), we know that $(\Delta_1, \Delta_2) \in \mathrm{Pairs}_r(\mathfrak{m} - \Delta_1' - \Delta_2', \rho)$ and $\Delta \in \mathfrak{m}_{\rho,f} = (\mathfrak{m} - \Delta_1' - \Delta_2')_{\rho,f}$. In this case, we use the induction-hypothesis to deduce the first claim for (Δ_1, Δ_2). For the last claim, recall that by construction $(\mathfrak{m}_{\rho,f})_{\rho,P} = 0$ and hence the claim follows. ☐

Lemma 4.2.2. *Let* $\mathfrak{m}$ *be an aperiodic multisegment. Then*

$$(\mathrm{soc}(\mathfrak{m}, \rho))_{\rho,P} = \mathfrak{m}_{\rho,P}$$

and if $\mathcal{D}_{r,\rho}(\mathfrak{m}) \neq 0$*,*

$$\mathcal{D}_{r,\rho}(\mathfrak{m})_{\rho,P} = \mathfrak{m}_{\rho,P}.$$

Proof. We will show that $(\mathrm{soc}(\mathfrak{m}, \rho))_{\rho,P} = \mathfrak{m}_{\rho,P}$ by induction on the number of segments on $\mathfrak{m}$. If $\mathrm{Pairs}_r(\mathfrak{m}, \rho)$ is empty, *i.e.* $\mathfrak{m} = \mathfrak{m}_{\rho,f}$, also $\mathrm{soc}(\mathfrak{m}, \rho) = (\mathrm{soc}(\mathfrak{m}, \rho))_{\rho,f}$. Indeed, by the maximality of the longest extendable ρ-segment, we have that every ρ-segment in $\mathrm{soc}(\mathfrak{m}, \rho)$ ending in $0 \mod o(\rho)$ is free by the last claim ofLemma 4.2.1. Thus $\mathrm{soc}(\mathfrak{m}, \rho) = (\mathrm{soc}(\mathfrak{m}, \rho))_{\rho,f}$.

We let Δ be the longest extendable ρ-segment or if it does not exist, the empty segment. In the latter case we set $\Delta^+ = [0, 0]_\rho$ for this proof. If $\mathrm{Pairs}_r(\mathfrak{m}, \rho)$ is nonempty, let (Δ_1, Δ_2) be a maximal pair in $\mathfrak{m}$. By induction on the number of segments we know that

$$\mathrm{soc}(\mathfrak{m} - \Delta_1 - \Delta_2, \rho)_{\rho,P} = (\mathrm{soc}(\mathfrak{m} - \Delta_1 - \Delta_2), \rho)_{\rho,P}.$$

It suffices now to show that

$$\mathrm{soc}((\mathfrak{m} - \Delta_1 - \Delta_2), \rho)_{\rho,P} = (\mathrm{soc}(\mathfrak{m}, \rho) - \Delta_1 - \Delta_2)_{\rho,P},$$

and that (Δ_1, Δ_2) is also a maximal pair in $\mathrm{soc}(\mathfrak{m}, \rho)$. Note that for this it actually suffices to just show that (Δ_1, Δ_2) is also a maximal pair in $\mathrm{soc}(\mathfrak{m}, \rho)$.

Assume otherwise and hence $l(\Delta^+) > l(\Delta_1)$ and there exists a ρ-segment Δ_3 in $\mathfrak{m}$ such that $(\Delta^+, \Delta_3) \in \mathrm{Pairs}_r(\mathrm{soc}(\mathfrak{m}, \rho), \rho)$ is the maximal pair. Thus by definition we have that

$$l(\Delta) < l(\Delta^+) \leq l(\Delta_3).$$

On the other hand, Δ is the longest extendable ρ-segment and hence there must exist Δ_4 such that $(\Delta_4, \Delta_3) \in \mathrm{Pairs}_r(\mathfrak{m}, \rho)$. Thus Lemma 4.2.1 implies that $l(\Delta) < l(\Delta_4)$ or equivalently $l(\Delta^+) \leq l(\Delta_4)$. Since we assumed that $l(\Delta^+) > l(\Delta_1)$, we obtain that $l(\Delta_4) > l(\Delta_1)$. But since we assumed that (Δ_1, Δ_2) is a maximal pair in $\mathfrak{m}$, we obtain a contradiction.

Next we show that $\mathcal{D}_{r,\rho}(\mathfrak{m})_{\rho,P} = \mathfrak{m}_{\rho,P}$ assuming the derivartive does not vanish. To do so we argue as above, i.e. on the number of segments in $\mathfrak{m}$. If $\mathrm{Pairs}_r(\mathfrak{m}, \rho)$ is empty, *i.e.* $\mathfrak{m} = \mathfrak{m}_{\rho,f}$, also $\mathcal{D}_{r,\rho}(\mathfrak{m}) = \mathcal{D}_{r,\rho}(\mathfrak{m})_{\rho,f}$. Indeed, by the minimality of the shortest free ρ-segment, we have that every ρ-segment in $\mathcal{D}_{r,\rho}(\mathfrak{m})$ ending in 0 mod $o(\rho)$ is free by the last claim ofLemma 4.2.1. Thus $\mathcal{D}_{r,\rho}(\mathfrak{m}) = \mathcal{D}_{r,\rho}(\mathfrak{m})_{\rho,f}$.

Le let Δ be the shortest free ρ-segment. If $\mathrm{Pairs}_r(\mathfrak{m}, \rho)$ is nonempty, let (Δ_1, Δ_2) be a maximal pair in $\mathfrak{m}$. By induction on the number of segments we know that

$$\mathcal{D}_{r,\rho}(\mathfrak{m} - \Delta_1 - \Delta_2)_{\rho,P} = \mathcal{D}_{r,\rho}(\mathfrak{m} - \Delta_1 - \Delta_2)_{\rho,P}.$$

It suffices now to show that

$$\mathcal{D}_{r,\rho}(\mathfrak{m} - \Delta_1 - \Delta_2)_{\rho,P} = (\mathcal{D}_{r,\rho}(\mathfrak{m}) - \Delta_1 - \Delta_2)_{\rho,P},$$

which again is the same as showing that (Δ_1, Δ_2) is also a maximal pair in $\mathcal{D}_{r,\rho}(\mathfrak{m})$. Assume otherwise and hence one of the following two possibilities might happen. The first case is that there exists a ρ-segment Δ_3 in $\mathfrak{m}$ with $l(\Delta_3) > l(\Delta_1)$ and $(\Delta_3, \Delta^-) \in \mathrm{Pairs}_r(\mathcal{D}_{r,\rho}(\mathfrak{m}), \rho)$ is the maximal pair. The second case that might occur is that $(\Delta_1, \Delta^-) \in \mathrm{Pairs}_r(\mathcal{D}_{r,\rho}(\mathfrak{m}), \rho)$ is the maximal pair and $l(\Delta_2) < l(\Delta^-)$.

In the first case we have $l(\Delta_3) < l(\Delta)$ and hence by the minimality of Δ, Δ_3 was not a free ρ-segment, i.e. there must exist Δ_4 in $\mathfrak{m}$ such that $(\Delta_3, \Delta_4) \in \mathrm{Pairs}_r(\mathfrak{m}, \rho)$. But since $l(\Delta_1) < l(\Delta_3)$, (Δ_1, Δ_2) was not the maximal pair of $\mathfrak{m}$, a contradiction.

In the second case, we have $l(\Delta_1) < l(\Delta)$ and hence by Lemma 4.2.1 $l(\Delta_2) < l(\Delta)$. But then (Δ_1, Δ_2) is the maximal pair of $\mathcal{D}_{r,\rho}(\mathfrak{m}, \rho)$ and not (Δ_1, Δ^-) as assumed. This finishes the argument. □

Lemma 4.2.3. *Let* $\mathfrak{m}$ *be a multisegment.*

$$\mathcal{D}_{r,\rho}(\mathrm{soc}(\mathfrak{m},\rho)) = \mathfrak{m}.$$

If moreover, $\mathcal{D}_{r,\rho}(\mathfrak{m}) \neq 0$*,*

$$\mathrm{soc}(\mathcal{D}_{r,\rho}(\mathfrak{m}),\rho) = \mathfrak{m}$$

and similarly for $\mathcal{D}_{l,\rho}$ *and* $\mathrm{soc}(\rho,\cdot)$*.*

Proof. We only prove,

$$\mathcal{D}_{r,\rho}(\mathrm{soc}(\mathfrak{m},\rho)) = \mathfrak{m},$$

the other claims follow analogously. To do so, we first note that by Lemma 4.2.2

$$\mathcal{D}_{r,\rho}(\mathrm{soc}(\mathfrak{m},\rho))_{\rho,P} = \mathfrak{m}_{\rho,P}$$

and hence we can assume that

$$\mathfrak{m} = \mathfrak{m}_{\rho,f}.$$

But again by Lemma 4.2.2, it follows that $\mathrm{soc}(\mathfrak{m},\rho)_{\rho,f} = \mathrm{soc}(\mathfrak{m},\rho)$ and the claim follows then immediately from the definitions. ☐

Finally, we show that the four maps indeed take aperiodic multisegments to aperiodic multisegments.

Lemma 4.2.4. *If* $\mathfrak{m}$ *is aperiodic,*

$$\mathcal{D}_{r,\rho}(\mathfrak{m}), \mathcal{D}_{l,\rho}(\mathfrak{m}), \mathrm{soc}(\mathfrak{m},\rho), \mathrm{soc}(\rho,\mathfrak{m})$$

are aperiodic.

Proof. We will only show the claim for $\mathcal{D}_{r,\rho}(\mathfrak{m})$ and $\mathrm{soc}(\mathfrak{m},\rho)$, the others then follow immediately from them. Let us first check that $\mathcal{D}_{r,\rho}(\mathfrak{m})$ is aperiodic, *i.e.* it contains a multisegment equivalent to

$$[a,0]_\rho + \ldots + [a+e(\rho)-1, e(\rho)-1]_\rho,$$

for a suitable a. We will now show that then $[a+e(\rho)-1, e(\rho)-1]_\rho$ appears in $\mathfrak{m}$. If this were the case, we would get a contradiction to the assumption that $\mathfrak{m}$ is aperiodic.

Let Δ be a the shortest free ρ-segment in $\mathfrak{m}_{\rho,f}$. Note that since $\mathcal{D}_{r,\rho}(\mathfrak{m}) = \mathfrak{m} + \Delta^- - \Delta$ and we assume $[a+e(\rho)-1, e(\rho)-1]_\rho$ not to appear in $\mathfrak{m}$, we have $\Delta^- = [a+e(\rho)-1, e(\rho)-1]_\rho$. Since $\mathfrak{m}$ contains then both $[a,0]_\rho$ and $[a+e(\rho)-1, e(\rho)]_\rho$ and

$$l([a,0]_\rho) < l(\Delta),$$

the minimality of Δ implies that there exists Δ' in $\mathfrak{m} - \Delta$ such that

$$([a,0]_\rho, \Delta') \in \mathrm{Pairs}_r(\mathfrak{m},\rho).$$

By Lemma 4.2.1 we obtain that

$$-a+2 = l([a+e(\rho)-1, e(\rho)]_\rho) \geq l(\Delta') \geq l([a,0]_\rho) = -a+1.$$

Since $[a+e(\rho)-1, e(\rho)]_\rho \neq [a,0]_\rho$, we have that $-a+2 > l(\Delta')$ and hence $l(\Delta') = -a+1$. This implies that $\Delta' = [a-1,-1]_\rho$. But since $[a,0]_\rho + \ldots + [a+e(\rho)-2, e(\rho)-2]_\rho$ is contained in $\mathfrak{m}$,

$$[a,0]_\rho + \ldots + [a+e(\rho)-2, e(\rho)-2]_\rho + \Delta'$$

is contained in $\mathfrak{m}$, which again contradicts the assumption that $\mathfrak{m}$ is aperiodic.

Next we show that $\operatorname{soc}(\mathfrak{m},\rho)$ is aperiodic. We will assume otherwise and show that then $\mathfrak{m}$ has to contain

$$[a,0]_\rho + \ldots + [a+e(\rho)-1, e(\rho)-1]_\rho,$$

for a suitable a. Let Δ be the longest extendable segment of $\mathfrak{m}$ and assume $\operatorname{soc}(\mathfrak{m},\rho)$ contains

$$[a,0]_\rho + \ldots + [a+e(\rho)-1, e(\rho)-1]_\rho.$$

If $\mathfrak{m}$ is aperiodic, then $\Delta = [a,-1]_\rho$. Then

$$l(\Delta) < l([a+e(\rho)-1, e(\rho)-1]_\rho).$$

Now the maximality of Δ and Lemma 4.2.1 imply that $[a+e(\rho)-1, e(\rho)-1]_\rho \in \mathfrak{m}_{\rho,P}$. Let $\Delta' \in \mathfrak{m} - \Delta$ be such that

$$(\Delta', [a+e(\rho)-1, e(\rho)-1]_\rho) \in \operatorname{Pairs}_r(\mathfrak{m},\rho).$$

Then $l(\Delta') \leq -a+1$ and $l(\Delta') \neq -a+1$, since otherwise $\mathfrak{m}$ would be not be aperiodic. Thus $l(\Delta') \leq -a = l(\Delta)$ and hence by Lemma 4.2.1 $\Delta = [a+e(\rho)-1, e(\rho)-1]_\rho$, a contradiction. ☐

Lemma 4.2.5. *Let* $k \in \{0,\ldots,d_{r,\rho}(\mathfrak{m})\}$. *Then* $\mathcal{D}^k_{r,\rho}(\mathfrak{m})$ *is the multisegment obtained from* $\mathfrak{m}$ *by shortening the shortest* k *free* ρ*-segments in* $\mathfrak{m}_{\rho,f}$ *by one on the right.*

Proof. Note that by Lemma 4.2.2 we have $\mathfrak{m}_{\rho,P} = \mathcal{D}^k_{r,\rho}(\mathfrak{m})_{\rho,P}$ and from this the claim follows straightforwardly. ☐

Lemma 4.2.6. *The tuple of pairs* $\operatorname{Pairs}_r(\mathfrak{m}^-,\rho v_\rho^{-1})$ *consists precisely of the pairs of segments* (Δ_1^-,Δ_2^-) *such that* $(\Delta_1,\Delta_2) \in \operatorname{Pairs}_r(\mathfrak{m},\rho)$ *and* $l(\Delta_1) > 1$.

Proof. We prove this claim by induction on the number of segments in $\mathfrak{m}$. If $\operatorname{Pairs}_r(\mathfrak{m},\rho)$ is empty, $\operatorname{Pairs}_r(\mathfrak{m}^-,\rho v_\rho^{-1})$ must also be empty. Indeed, if $(\Delta_1,\Delta_2) \in \operatorname{Pairs}_r(\mathfrak{m}^-,\rho v_\rho^{-1})$, then (Δ_1^+,Δ_2^+) are segments in $\mathfrak{m}$ and contradict the last claim of Lemma 4.2.1.

Thus assume $\mathrm{Pairs}_r(\mathfrak{m},\rho)$ is non-empty. Let (Δ_1',Δ_2') be a maximal pair of $\mathfrak{m}$. If $l(\Delta_1')>1$ then it is easy to see from the definitions that

$$\mathrm{Pairs}_r(\mathfrak{m}^-,\rho v_\rho^{-1})=(\mathrm{Pairs}_r(\mathfrak{m}^- - \Delta_1'^- - \Delta_2'^-,\rho v_\rho^{-1}),(\Delta_1'^-,\Delta_2'^-)).$$

The induction hypothesis gives us now the desired description of $\mathrm{Pairs}_r(\mathfrak{m}^- - \Delta_1'^- - \Delta_2'^-,\rho v_\rho^{-1})$ in terms of $\mathrm{Pairs}_r(\mathfrak{m}-\Delta_1'-\Delta_2',\rho)$. If on the other hand $l(\Delta_1')=1$, then by the maximality of Δ_1' the tuple $\mathrm{Pairs}_r(\mathfrak{m}^-,\rho v_\rho^{-1})$ has to be empty. Indeed, if (Δ_1,Δ_2) would be an element of it, (Δ_1^+,Δ_2^+) would appear in $\mathfrak{m}$ and contradict the maximality of $([0,0]_\rho,\Delta_2')$ since $l(\Delta_1^+)\geq 2$. □

As an immediate corollary, we obtain the following.

Corollary 4.2.7. *Let* $\mathfrak{m}$ *be an aperiodic multisegment and* $\rho\in\mathfrak{C}^\square$ *and assume* $\mathcal{D}_{r,\rho}(\mathfrak{m})\neq 0$. *Let* Δ *be the shortest free* ρ*-segment in* $\mathfrak{m}_{\rho,f}$. *Then*

$$\begin{cases}\mathcal{D}_{r,\rho}(\mathfrak{m})^-=\mathcal{D}_{r,\rho v_\rho^{-1}}(\mathfrak{m}^-),\ \mathcal{D}_{r,\rho}(\mathfrak{m})^1=\mathfrak{m}^1-[0,0]_\rho+[-1,-1]_\rho & \text{if } l(\Delta)>1,\\ \mathcal{D}_{r,\rho}(\mathfrak{m})^-=\mathfrak{m}^-,\ \mathcal{D}_{r,\rho}(\mathfrak{m})^1=\mathfrak{m}^1-[0,0]_\rho & \text{if } l(\Delta)=1.\end{cases}$$

4.2.2 We will now make the relation between derivatives of multisegments and derivatives of representations explicit.

Theorem 4.2.8. *For* $\mathfrak{m}$ *an aperiodic multisegment and* $\rho\in\mathfrak{C}^\square$, *we have*

$$\mathcal{D}_{r,\rho}(\mathrm{Z}(\mathfrak{m}))=\mathrm{Z}(\mathcal{D}_{r,\rho}(\mathfrak{m})),\ \mathcal{D}_{l,\rho}(\mathrm{Z}(\mathfrak{m}))=\mathrm{Z}(\mathcal{D}_{l,\rho}(\mathfrak{m})),$$

$$d_{r,\rho}(\mathrm{Z}(\mathfrak{m}))=d_{r,\rho}(\mathfrak{m}),\ d_{l,\rho}(\mathrm{Z}(\mathfrak{m}))=d_{l,\rho}(\mathfrak{m})$$

and

$$\mathcal{D}_{r,\rho,max}(\mathrm{Z}(\mathfrak{m}))=\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\mathfrak{m})),\ \mathcal{D}_{l,\rho,max}(\mathrm{Z}(\mathfrak{m}))=\mathrm{Z}(\mathcal{D}_{l,\rho,max}(\mathfrak{m})).$$

We will take great care in the proof not to use the subjectivity of Z for representations of □-irreducible cuspidal support, since we want to eventually use above theorem to give a new proof of it. We also remark that the theorem is already known over $\overline{\mathbb{Q}}_\ell$, *cf.* [27, §4]. Let $\mathfrak{m}$ be an aperiodic multisegment over $\overline{\mathbb{F}}_\ell$ and ρ a supercuspidal representation. We call a lift $\widetilde{\mathfrak{m}}$ of $\mathfrak{m}$ *ρ-(right)-derivative compatible* if for all ρ-segments in $\mathfrak{m}$ ending in 0 respectively $-1 \mod o(\rho)$, their lift in $\widetilde{\mathfrak{m}}$ ends in 0 respectively -1. Similarly, a lift $\widetilde{\mathfrak{m}}$ of $\mathfrak{m}$ *ρ-left-derivative compatible* if for all ρ-segments in $\mathfrak{m}$ starting in 0 respectively $1 \mod o(\rho)$, their lift in $\widetilde{\mathfrak{m}}$ starts in 0 respectively 1. If $\rho\in\mathfrak{C}^\square$, there exists for all $\mathfrak{m}\in\mathcal{MS}(\rho)$ such lifts since ρ is supercuspidal. The following is then easy to see.

Lemma 4.2.9. *Let* $\widetilde{\mathfrak{m}}$ *be a* ρ*-right-derivative compatible lift of a multisegment* $\mathfrak{m}$. *Then* $\mathrm{r}_\ell(\mathcal{D}_{r,\rho}(\widetilde{\mathfrak{m}}))=\mathcal{D}_{r,\rho}(\mathfrak{m})$ *and similarly for* ρ*-left-derivative compatible lifts.*

Before we start with the proof of Theorem 4.2.8, we need the following three lemmas.

Lemma 4.2.10. *Let* $\mathrm{Z}(\mathfrak{n})$ *be a residually-degenerate representation and* $\rho \in \mathfrak{C}^{\square}$. *There exists* τ' *such that* $\mathrm{Z}(\mathfrak{n}) \hookrightarrow \tau' \times \rho$ *if and only if there exists less copies of* $[-1,-1]_\rho$*'s than of* $[0,0]_\rho$ *in* $\mathfrak{n}$. *Similarly, there exists* τ' *such that* $\mathrm{Z}(\mathfrak{n}) \hookrightarrow \rho v_\rho^{-1} \times \tau'$ *if and only if there exists more* $[-1,-1]_\rho$*'s than* $[0,0]_\rho$*'s in* $\mathfrak{n}$

Proof. By Lemma 2.4.1 we can reduce to the claim where $\mathfrak{n}$ has cuspidal support in $\mathbb{N}(\mathbb{Z}[\rho])$. Thus $\mathfrak{n}$ is an aperiodic, banale multisegment as in [30] consisting of segments of length 1. Let $\widetilde{\mathfrak{n}}$ be a ρ-derivative compatible lift of $\mathfrak{n}$. Since $\mathfrak{n}$ is banale, it follows that $\mathrm{r}_\ell(\mathrm{Z}(\widetilde{\mathfrak{n}})) = \mathrm{Z}(\mathfrak{n})$ and we recall that parabolic reduction commutes with reduction mod ℓ. Hence the claim follows from Lemma 4.1.3 since we assume Theorem 4.2.8 to be true over $\overline{\mathbb{Q}}_\ell$. □

Lemma 4.2.11. *Let* π *be a non-residually degenerate representation of* G_n *such that there exists* $\rho \in \mathfrak{C}^{\square}$ *of* G_m *with*

$$[r_{(m,n-m)}(\pi)] \geq [\rho v_\rho^{-1} \otimes \mathrm{Z}(\mathfrak{n})]$$

and $\mathfrak{n}$ *consists only of segments of length* 1. *Then*

$$[r_{(n-m,m)}(\pi)] \geq [\mathrm{Z}(\mathfrak{n} + [-1,-1]_\rho - [0,0]_\rho) \otimes \rho].$$

Proof. Using Lemma 2.4.1 it is easy to see that it is enough to show the claim for π having cuspidal support contained in $\mathbb{N}(\mathbb{Z}[\rho])$. If $[\pi] \leq \mathrm{I}(\mathfrak{m})$ with $\mathfrak{m}$ a multisegment containing at least one segment not of length 1, Theorem 2.6.6 shows that above situation can happen if and only if $\mathfrak{m} = \mathfrak{n} + [-1,0]_\rho - [0,0]_\rho$ and $\pi \cong \mathrm{Z}(\mathfrak{m})$. In this case it is also easy that the only $(\deg(\mathfrak{n}), m)$-degenerate representation appearing in the parabolic reduction of $\mathrm{I}(\mathfrak{m})$ is $\mathrm{Z}(\mathfrak{n}+[-1,-1]_\rho - [0,0]_\rho) \otimes \rho$. Since the parabolic reduction of π must contain a $(\deg(\mathfrak{n}), m)$-degenerate representation, we are done in this case.

We thus assume that π does not appear as a subquotient of such a $\mathrm{I}(\mathfrak{m})$. Let $\widetilde{\mathfrak{m}}$ be a ρ-derivative compatible lift of $\mathfrak{n}' = \mathfrak{n} + [-1,-1]_\rho$ and $\pi' = \mathrm{Z}(\widetilde{\mathfrak{m}}')$ be such that π' appears in $\mathrm{I}(\widetilde{\mathfrak{m}})$ and π appears in

$$[\mathrm{r}_\ell(\pi')] \leq \mathrm{r}_\ell(\mathrm{I}(\widetilde{\mathfrak{m}}')) = \mathrm{I}(\mathrm{r}_\ell(\widetilde{\mathfrak{m}}')).$$

By assumption on π, we have that $\widetilde{\mathfrak{m}}' = \widetilde{\mathfrak{m}}$. We are now going to derive a contradiction to the assumption that π is not residually-degenerate, namely that $\pi \cong \mathrm{Z}(\mathfrak{n}')$.

We let k_1 be the number of times $[-1,-1]_{\tilde{\rho}}$ appears in $\widetilde{\mathfrak{m}}$ and k_0 be the number of times $[0,0]_{\tilde{\rho}}$ appears in $\widetilde{\mathfrak{m}}$. By Lemma 4.2.10 we have that $k_1 > k_0$ and

$$[\rho v_\rho^{-1} \otimes (\rho v_\rho^{-1})^{k_1-k_0-1} \otimes \mathrm{Z}([\mathfrak{n}' - (k_1-k_0)[-1,-1]_\rho)] \leq$$

$$\leq [r_{((m,(k_1-k_0-1)m,\deg(\mathfrak{m})-(k_1-k_0)m)}(\rho v_\rho^{-1} \otimes \mathrm{Z}(\mathfrak{n}))] \leq$$
$$\leq [r_{((m,(k_1-k_0-1)m,\deg(\mathfrak{m})-(k_1-k_0)m)}(\pi)].$$

Thus

$$[(\rho v_\rho^{-1})^{k_1-k_0} \otimes \mathrm{Z}([\mathfrak{n}' - (k_1-k_0)[-1,-1]_\rho)] \leq [r_{(k_1-k_0)m,\deg(\mathfrak{m})-(k_1-k_0)m)}(\pi)].$$

Since we already know Theorem 4.2.8 to be true over $\overline{\mathbb{Q}}_\ell$, we know by Corollary 4.1.5 that

$$(\tilde{\rho} v_{\tilde{\rho}}^{-1})^{k_1-k_0} \otimes \mathrm{Z}([\widetilde{\mathfrak{m}} - (k_1-k_0)[-1,-1]_{\tilde{\rho}})$$

appears with multiplicity 1 in $\mathrm{Z}(\widetilde{\mathfrak{m}})$. Moreover, since parabolic reduction commutes with reduction mod ℓ, Corollary 4.1.5 also shows that

$$(\rho v_\rho^{-1})^{k_1-k_0} \otimes \mathrm{Z}([\mathfrak{n}' - (k_1-k_0)[-1,-1]_\rho) \tag{5}$$

appears in $\mathrm{r}_\ell(\mathrm{Z}(\widetilde{\mathfrak{m}}))$ with multiplicity 1. But by Lemma 4.2.10 (5) appears also in the parabolic reduction of $\mathrm{Z}(\mathfrak{n}')$, thus showing that $\pi \cong \mathrm{Z}(\mathfrak{n}')$. ☐

Lemma 4.2.12. *Let $\mathfrak{m}$ be an aperiodic multisegment and define the partition $\alpha := (\mu_{\overline{\mathcal{D}_{r,\rho,max}(\mathfrak{m})}}, md_{r,\rho}(\mathfrak{m}))$. Then*

$$[r_\alpha(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{St}(\mathcal{D}_{r,\rho,max}(\mathfrak{m})) \otimes \rho^{d_{r,\rho}(\mathfrak{m})}].$$

Proof. We argue by induction on t, the maximal length of any segment in $\mathfrak{m}$. Note that we have by Lemma 2.6.9

$$[r_{(\deg(\mathfrak{m}^-),\deg(\mathfrak{m}^1))}(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{Z}(\mathfrak{m}^-) \otimes \mathrm{Z}(\mathfrak{m}^1)]$$

and hence by the induction hypothesis, we know that

$$[r_{(\alpha_{\mathfrak{m}^-,\rho v_\rho^{-1}},\deg(\mathfrak{m}^1))}(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{St}(\mathcal{D}_{r,\rho v_\rho^{-1},max}(\mathfrak{m}^-)) \otimes (\rho v_\rho^{-1})^{d_{r,\rho v_\rho^{-1}}} \otimes \mathrm{Z}(\mathfrak{m}^1)].$$

Let $\pi \in \mathfrak{Irr}_N$, $N := m \cdot d_{r,\rho v_\rho^{-1}} + \deg(\mathfrak{m}_1)$, $l := d_{r,\rho v_\rho^{-1}}$ be such that

$$[r_{(\deg \mathfrak{m}-N,N)}(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{St}(\mathcal{D}_{r,\rho v_\rho^{-1},max}(\mathfrak{m}^-)) \otimes \pi], \tag{6}$$

$$[r_{(N-\deg(\mathfrak{m}^1),\deg(\mathfrak{m}_1))}(\pi)] \geq [(\rho v_\rho^{-1})^l \otimes \mathrm{Z}(\mathfrak{m}^1)]. \tag{7}$$

From Theorem 2.6.6 it follows that $ml \leq \deg \mathfrak{m}_1$ and the maximal ordered partition α' such that π is α'-degenerate is

$$(ml, \deg(\mathfrak{m}_1)).$$

Since we know from (7) that

$$[r_{(m,\dots,m,\deg(\mathfrak{m}_1))}(\pi)] \geq [\rho v_\rho^{-1} \otimes \dots \rho v_\rho^{-1} \otimes \otimes \mathrm{Z}(\mathfrak{m}^1)]$$

we can just apply Lemma 4.2.11 l-times and obtain

$$[r_{(\deg(\mathfrak{m}_1),m,\ldots,m)}(\pi)] \geq [\mathrm{Z}(\mathfrak{m}^1 + l[-1,-1]_\rho - l[0,0]_\rho) \otimes \rho \otimes \ldots \otimes \rho].$$

Now since $o(\rho) > 1$ and ρ is supercuspidal, this implies

$$[r_{(\deg(\mathfrak{m}_1),N-\deg(\mathfrak{m}^1))}(\pi)] \geq [\mathrm{Z}(\mathfrak{m}^1 + l[-1,-1]_\rho - l[0,0]_\rho) \otimes \rho^l]. \tag{8}$$

Write $\mathfrak{n} = \mathfrak{m}^1 + l[-1,-1]_\rho - l[0,0]_\rho$. Combining (6) and (8) shows that

$$[r_{(\deg \mathfrak{m}-N,\deg(\mathfrak{m}_1),N-\deg(\mathfrak{m}^1))}(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{St}(\mathcal{D}_{r,\rho v_\rho^{-1},max}(\mathfrak{m}^-)) \otimes \mathrm{Z}(\mathfrak{n}) \otimes \rho^l]. \tag{9}$$

Write now $\mathfrak{n} = \mathfrak{n}' + k[0,0]_\rho$ such that $\mathfrak{n}'$ contained at least as many copies of $[-1,-1]_\rho$ as of $[0,0]_\rho$. Applying Lemma 4.2.10 k-times in combination with Lemma 4.1.3, we see that

$$[r_{(\deg(\mathfrak{n}'),m,\ldots,m)}(\mathrm{Z}(\mathfrak{n}))] \geq [\mathrm{Z}(\mathfrak{n}') \otimes \rho \otimes \ldots \otimes \rho]$$

or equivalently

$$[r_{(\deg(\mathfrak{n}'),km)}(\mathrm{Z}(\mathfrak{n}))] \geq [\mathrm{Z}(\mathfrak{n}') \otimes \rho^k]. \tag{10}$$

Combining (10) and (9), we obtain that

$$[r_{(\deg \mathfrak{m}-N,\deg(\mathfrak{n}'),N-\deg(\mathfrak{n}'))}(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{St}(\mathcal{D}_{r,\rho v_\rho^{-1},max}(\mathfrak{m}^-)) \otimes \mathrm{Z}(\mathfrak{n}') \otimes \rho^{l+k}].$$

Now $l + k$ is the number of free ρ-segments in $\mathfrak{m}_{\rho,f}$ since l is by Corollary 4.2.7 the number of free ρ-segments of length greater than 1 and k is the number of free ρ-segments of length precisely 1. Moreover, Lemma 4.2.6 also shows that

$$\mathcal{D}_{r,\rho,max}(\mathfrak{m})^- = \mathcal{D}_{r,\rho v_\rho^{-1},max}(\mathfrak{m}^-),\ \mathcal{D}_{r,\rho,max}(\mathfrak{m})^1 = \mathfrak{m}^1 - (k+l)[0,0]_\rho + l[1,1]_\rho.$$

Thus

$$\mathrm{St}(\mathcal{D}_{r,\rho v_\rho^{-1}(\mathfrak{m}),max}(\mathfrak{m}^-)) \otimes \mathrm{Z}(\mathfrak{n}') = \mathrm{St}(\mathcal{D}_{r,\rho,max}(\mathfrak{m})),\ \rho^{l+k} = \rho^{d_{r,\rho}(\pi)},$$

which finishes the induction step. ☐

Proof of Theorem 4.2.8. We will only proof $\mathcal{D}_{r,\rho,max}(\mathrm{Z}(\mathfrak{m})) = \mathrm{Z}(\mathcal{D}_{r,\rho,max}(\mathfrak{m}))$, the other claims follow then quickly using Corollary 4.1.5 and Lemma 4.1.6.

We will show that $\mathrm{Z}(\mathfrak{m}) = \mathrm{soc}(\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\mathfrak{m})) \times \rho^{d_{r,\rho}(\mathfrak{m})})$. To do so we first can reduce the claim via Lemma 2.4.1 to the assumption that $\mathfrak{m}$ has cuspidal support contained in $\mathbb{N}(\mathbb{Z}[\rho])$. We choose a ρ-derivative compatible lift $\widetilde{\mathfrak{m}}$ of $\mathfrak{m}$. Then $\mathrm{r}_\ell(\mathcal{D}_{r,\rho,max}(\widetilde{\mathfrak{m}})) = \mathcal{D}_{r,\rho,max}(\mathfrak{m})$ and by [27, §4] we already know that

$$\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\widetilde{\mathfrak{m}})) = \mathcal{D}_{r,\rho,max}(\mathrm{Z}(\widetilde{\mathfrak{m}})).$$

Recall also that r_ℓ commutes with parabolic reduction on the Grothendieck groups. Thus if $\pi = \mathrm{soc}(\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\mathfrak{m})) \times \rho^{d_{r,\rho}(\mathfrak{m})})$, Frobenius reciprocity implies that the parabolic reduction of π contains $\mathrm{Z}(\mathcal{D}_{r,\tilde{\rho},max}(\mathfrak{m})) \otimes \rho^{d_{r,\rho}(\mathfrak{m})}$. Let

$\alpha = (\deg(\mathcal{D}_{r,\rho,max}(\mathfrak{m})), m \cdot d_{r,\rho}(\mathfrak{m}))$. By Corollary 4.1.5 and Theorem 2.8.2 $\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\mathfrak{m})) \otimes \rho^{d_{r,\rho}(\mathfrak{m})}$ appears with multiplicity 1 in

$$r_\alpha(\mathrm{r}_\ell(\mathrm{Z}(\mathcal{D}_{r,\tilde{\rho},max}(\widetilde{\mathfrak{m}})) \times \tilde{\rho}^{d_{r,\rho}(\mathfrak{m})})).$$

Moreover, if

$$[\pi' \otimes \rho^{d_{r,\rho}(\mathfrak{m})}] \leq [r_\alpha(\mathrm{r}_\ell(\mathrm{Z}(\mathcal{D}_{r,\tilde{\rho},max}(\widetilde{\mathfrak{m}})) \times \tilde{\rho}^{d_{r,\rho}(\mathfrak{m})}))],$$

then $[\pi'] \leq [\mathrm{r}_\ell(\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\widetilde{\mathfrak{m}}))]$. Indeed, let

$$[\pi'' \otimes \tilde{\rho}^{l_1 \cdot o(\rho)} \times \ldots \tilde{\rho}^{l_{d_{r,\rho}(\mathfrak{m})} o(\rho)}] \leq [r_\alpha((\mathrm{Z}(\mathcal{D}_{r,\tilde{\rho},max}(\widetilde{\mathfrak{m}})) \times \tilde{\rho}^{d_{r,\rho}(\mathfrak{m})})]$$

be such that its reduction $\mod \ell$ contains $\pi' \otimes \rho^{d_{r,\rho}(\mathfrak{m})}$, where $l_i \in \mathbb{Z}$. But by the construction of $\widetilde{\mathfrak{m}}$ we know that

$$\mathcal{D}_{r,\tilde{\rho}^{l_{d_{r,\rho}(\mathfrak{m})} o(\rho)}} \circ \ldots \mathcal{D}_{r,\tilde{\rho}^{l_1 o(\rho)}}(\widetilde{\mathfrak{m}}) \neq 0$$

if and only if all of the $l_i = 0$. Thus $\pi' \otimes \rho^{d_{r,\rho}(\mathfrak{m})}$ is contained in the parabolic reduction of $\pi'' \otimes \tilde{\rho}^{d_{r,\rho}(\mathfrak{m})}$ and hence the claim follows from Corollary 4.1.5.

But on the other hand,

$$[\mathrm{Z}(\mathfrak{m})] \leq [\mathrm{r}_\ell(\mathrm{Z}(\widetilde{\mathfrak{m}}))]$$

and by Lemma 4.2.12 there exists π' which is $\mu_{\overline{\mathcal{D}_{r,\rho,max}(\mathfrak{m})}} = \mu_{\overline{\mathcal{D}_{r,\tilde{\rho},max}(\widetilde{\mathfrak{m}})}}$-degenerate such that

$$[r_\alpha(\mathrm{Z}(\mathfrak{m}))] \geq [\pi' \otimes \rho^{d_{r,\rho}(\mathfrak{m})}].$$

Since $[\pi'] \leq [\mathrm{r}_\ell(\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\widetilde{\mathfrak{m}})))]$, $\pi' \cong \mathrm{Z}(\mathcal{D}_{r,\rho,max}(\mathfrak{m}))$ by Theorem 2.6.6 and hence $\pi \cong \mathrm{Z}(\mathfrak{m})$. □

5 Applications of ρ-derivatives

In this section we give several applications of the above developed machinery.

5.1 The first consequence is the surjection of the map Z for representations with □-irreducible cuspidal support.

Theorem 5.1.1. *Let $\pi \in \mathfrak{Irr}_n$ with □-irreducible cuspidal support. Then there exists an aperiodic multisegment $\mathfrak{m}$ such that $\pi \cong \mathrm{Z}(\mathfrak{m})$.*

Proof. We argue by induction on n. Let π be an irreducible representation of G_n with □-irreducible cuspidal support and let $\pi' \in \mathfrak{Irr}$ and ρ be a □-irreducible cuspidal representation such $\pi \hookrightarrow \pi' \times \rho$. By the induction hypothesis we have that there exists an aperiodic $\mathfrak{m}'$ such that $\pi' \cong \mathrm{Z}(\mathfrak{m}')$. By Theorem 4.2.8 we have then that

$$\pi \hookrightarrow \mathrm{soc}(\pi' \times \rho) = \mathrm{Z}(\mathrm{soc}(\mathfrak{m}', \rho)).$$

□

5.2 We will now give an explicit description of the Aubert-Zelevinsky dual π^* using the theory of derivatives. Recall the Grothendieck group $\mathfrak{R} = \bigcup_{n\in\mathbb{N}} \mathfrak{R}_n$. In [38] the Aubert-Zelevinsky involution

$$\mathrm{D}\colon \mathfrak{R} \to \mathfrak{R},$$

which sends $[\pi] \in \mathfrak{Irr}_n$ to

$$\mathrm{D}([\pi]) \coloneqq \sum_{\alpha} (-1)^{r(\alpha)} [\mathrm{Ind}_\alpha \circ r_\alpha(\pi)]$$

was introduced, where the sum is over all partitions of n and $r(\alpha)$ is the number of elements of α. It satisfies the following properties.

Theorem 5.2.1. *Let* D *be the Aubert-Zelevinsky involution.*

1. *If π_1, π_2 are irreducible representations then*

$$\mathrm{D}([\pi_1 \times \pi_2]) = \mathrm{D}([\pi_1]) \times \mathrm{D}([\pi_2]).$$

2. *If β is a partition of n and π an irreducible representation then*

$$\mathrm{D}([r_\beta(\pi)]) = \mathrm{Ad}(w_\beta) \circ r_\beta(\mathrm{D}([\pi]))),$$

where w_β is the longest element of the Weyl group $W(\beta,(n))$, cf. *Section 2.3.*

3. *If π is irreducible, there exists a unique irreducible representation π^* with the same cuspidal support as π appearing in $\mathrm{D}([\pi])$ with sign $(-1)^{r(\pi)}$, where $r(\pi)$ is the length of the cuspidal support of π.*

Proof. For (1) and (2) see [4, Theorem 1.7] and [30, Proposition A.2]. For (3) see [29] for the case $R = \overline{\mathbb{Q}}_\ell$ and [28, Theorem 2.5] for the general case $\ell \neq p$. ☐

Theorem 5.2.2. *Let $\pi \in \mathfrak{Irr}_n$ and $\rho \in \mathfrak{C}^\square$. Then*

$$\mathcal{D}_{r,\rho}(\pi)^* \cong \mathcal{D}_{l,\rho}(\pi^*).$$

Proof. Pick ρ a cuspidal $\square$-representation of G_m such that $\mathcal{D}_{r,\rho}(\pi) \neq 0$. If $\pi = \rho^k$, the claim follows from Theorem 5.2.1(1). Thus we can assume that $\mathcal{D}_{r,\rho,max}(\pi) \neq 0$. By Frobenius reciprocity

$$[r_{(n-d_{r,\rho}(\pi)m, d_{r,\rho}(\pi)m)}(\pi)] \geq [\mathcal{D}_{r,\rho,max}(\pi) \otimes \rho^{d_{r,\rho}(\pi)}]$$

and hence by Theorem 5.2.1 (2)

$$r_{(d_{r,\rho}(\pi)m, n-d_{r,\rho}(\pi)m)}(\mathrm{D}([\pi]))]$$

contains the representation $\rho^{d_{r,\rho}(\pi)} \otimes \mathcal{D}_{r,\rho,max}(\pi)^*$ with a non-zero coefficient $(-1)^{d_{r,\rho}(\pi)}(-1)^{r(\mathcal{D}_{r,\rho,max}(\pi))}$. Recall that $(-1)^{r(\pi)}[\pi^*]$ is the unique constituent of $\mathrm{D}([\pi])$ with cuspidal support the same as π, it follows that

$$(-1)^{r(\pi)}[r_{(d_{r,\rho}(\pi)m,n-d_{r,\rho}(\pi)m)}(\pi^*)] \geq$$

$$\geq (-1)^{d_{r,\rho}(\pi)}(-1)^{r(\mathcal{D}_{r,\rho,max}(\pi))}[\rho^{d_{r,\rho}(\pi)} \otimes \mathcal{D}_{r,\rho,max}(\pi)^*] =$$

$$= (-1)^{r(\pi)}[\rho^{d_{r,\rho}(\pi)} \otimes \mathcal{D}_{l,\rho,max}(\pi^*)],$$

where the last equality follows from the induction hypothesis The claim then follows from Corollary 4.1.5. □

If $\mathfrak{m}$ is an aperiodic multisegment $\mathfrak{m}$, we set $\mathfrak{m}^*$ to the aperiodic multisegment such that $\mathrm{Z}(\mathfrak{m})^* = \mathrm{Z}(\mathfrak{m}^*)$.

For $\mathfrak{m}$ an aperiodic multisegment with □-irreducible cuspidal support we can thus offer the following algorithm to compute $\mathfrak{m}^*$, where $\mathfrak{m}^*$ is the aperiodic multisegment such that $\mathrm{Z}(\mathfrak{m})^* = \mathrm{Z}(\mathfrak{m}^*)$. It works recursively on $\deg(\mathfrak{m})$. Let $\rho \in \mathfrak{C}^\square$ such that $\mathcal{D}_{r,\rho}(\mathfrak{m}) \neq 0$. We then compute $\mathfrak{m}^*$ by first computing $\mathcal{D}_{r,\rho}(\mathfrak{m})$. From this we can already compute $\mathcal{D}_{r,\rho}(\mathfrak{m})^*$ recursively. But by Theorem 5.2.1 and Theorem 4.2.8 $\mathcal{D}_{r,\rho}(\mathfrak{m})^* = \mathcal{D}_{l,\rho}(\mathfrak{m}^*)$. By Lemma 4.1.1 and Theorem 4.2.8

$$\mathrm{soc}(\rho, \mathcal{D}_{r,\rho}(\mathfrak{m})^*) = \mathfrak{m}^*.$$

Note that we thus rediscover the algorithm of computing the Aubert-Zelevinsky dual of [22].

5.3 Godement-Jacquet local factors

In this section we will compute the local L-factors associated to an irreducible smooth representation over $\overline{\mathbb{F}}_\ell$ and show that the map

$$\mathrm{C}\colon \mathfrak{Irr}_n \overset{\cong}{\to} \{\mathrm{C}-\text{parameters of length } n\}$$

defined in [18] respects these local L-factors. We start by proving the following lemma, which in the case $R = \overline{\mathbb{Q}}_\ell$ was proved in [11, Proposition 2.3].

Lemma 5.3.1. *Let $\sigma_1, \ldots, \sigma_k$ be irreducible representations of $G_{n_1}, \ldots, G_{n_k}$ over R. Then*

$$L(\sigma_1 \times \ldots \times \sigma_k, T) = \prod_{i=1}^k L(\sigma_i, T)$$

and

$$\gamma(T, \sigma_1 \times \ldots \times \sigma_k, \psi) = \prod_{i=1}^k \gamma(T, \sigma_i, \psi).$$

Proof. We can mimic the proof of [11, Proposition 2.3] more or less *muta mutandis*. We start by showing the inclusion

$$\mathcal{L}(\sigma_1 \times \ldots \times \sigma_k) \subseteq \prod_{i=1}^{k} \mathcal{L}(\sigma_i).$$

Set $\sigma := \sigma_1 \times \ldots \times \sigma_k$, $\alpha = (n_1, \ldots, n_k)$ and $P := P_\alpha$ with Levi-component M and unipotent component U. Recall that

$$\delta_P(m) = \prod_{i=1}^{k} |m_i|^{d\delta_i},\ \delta_i := -n_1 - \ldots - n_{i-1} + n_{i+1} + \ldots + n_k.$$

Moreover, recall that as in [11, Proposition 2.3] the matrix coefficients of σ are exactly given by integrals of the form

$$f(g) = \int_{P\backslash G_n} H(g'g, g')\,\mathrm{d}g', \tag{11}$$

$$H(g_1, g_2) = h^\vee(g_1)(h(g_2)),\ h \in \sigma_1 \times \ldots \times \sigma_k,\ h^\vee \in \sigma_1^\vee \times \ldots \times \sigma_k^\vee.$$

Note that H is smooth, it satisfies for all $p \in P$, $g_1, g_2 \in G_n$

$$H(pg_1, pg_2) = \delta_P(p) H(g_1, g_2)$$

and $m \mapsto H(mg_1, g_2)$ is a matrix coefficient of $\sigma_1 \otimes \ldots \otimes \sigma_k \otimes \delta_P^{\frac{1}{2}}$. Furthermore, given such H, the function

$$f(g) = \int_{P\backslash G_n} H(g'g, g')\,\mathrm{d}g' = \int_{(K_n\cap P)\backslash K_n} H(kg, k)\,\mathrm{d}k$$

is a matrix coefficient of σ. Fix now $\phi \in \mathscr{S}_{\overline{\mathbb{F}}_\ell}(M_n(\mathrm{D}))$ and as in [11, Proposition 2.3] we see that the coefficient of T^N in $Z(\phi, Tq^{-\frac{dn-1}{2}}, f)$ is

$$\int_{(K_n\cap P)\backslash K_n} \int_{(K_n\cap P)\backslash K_n} \int_{P(N)} \phi(k^{-1}pk')\delta_P(p)^{-1}|p|^{\frac{dn-1}{2}} H(pk', k)\,\mathrm{d}_r p\,\mathrm{d}k\,\mathrm{d}k' \tag{12}$$

We write for

$$m = (m_1, \ldots m_k),\ u = \begin{pmatrix} 1_{n_1} & \ldots & u_{i,j} \\ 0 & \ddots & \vdots \\ 0 & 0 & 1_{n_k} \end{pmatrix},\ p(m,u) := \begin{pmatrix} m_1 & \ldots & u_{i,j} \\ 0 & \ddots & \vdots \\ 0 & 0 & m_k \end{pmatrix}$$

and express $\mathrm{d}_r p$ as

$$\mathrm{d}_r p = \mathrm{d}u\,\mathrm{d}m \prod_{i=1}^{k} |m_i|^{\frac{d(n-\delta_i-n_i)}{2}}.$$

Thus above integral equals to

$$\int_{(K_n\cap P)\backslash K_n}\int_{(K_n\cap P)\backslash K_n}\int_{M(N)}\int_U \phi(k^{-1}p(m,u)k')\,\mathrm{d}u$$

$$\prod_{i=1}^{k}|m_i|^{\frac{d(n_i-n-\delta_i)}{2}}|m|^{\frac{dn-1}{2}}H(mk',k)\,\mathrm{d}m\,\mathrm{d}k\,\mathrm{d}k'.$$

For $m\in M$ and $k,k'\in(K_n\cap P)\backslash K_n$ we set

$$\phi(m;k,k'):=\int_U \phi(k^{-1}p(m,u)k)\,\mathrm{d}u,\ h(m;k,k'):=H(mk',k)\delta_P^{-\frac{1}{2}}(m). \tag{13}$$

Thus the coefficient of T^N is

$$\int_{(K_n\cap P)\backslash K_n}\int_{(K_n\cap P)\backslash K_n}\int_{M(N)} \phi(m;k,k')h(m;k,k')\prod_{i=1}^{k}|m_i|^{\frac{dn_i-1}{2}}\,\mathrm{d}m\,\mathrm{d}k\,\mathrm{d}k'. \tag{14}$$

As in [11, Proposition 2.3] one sees then that this is the coefficient of T^N of a finite sum of elements of the form

$$\prod_{i=1}^{k}Z(\phi_i,Tq^{-\frac{dn_i-1}{2}},f_i)$$

with f_i a matrix coefficient of σ_i and $\phi_i\in\mathscr{S}_{\overline{\mathbb{F}}_\ell}(M_{n_i}(\mathrm{D}))$ and hence

$$\mathcal{L}(\sigma_1\times\ldots\times\sigma_k)\subseteq\prod_{i=1}^{k}\mathcal{L}(\sigma_i).$$

Next we show that

$$\prod_{i=1}^{k}\mathcal{L}(\sigma_i)\subseteq\mathcal{L}(\sigma_1\times\ldots\times\sigma_k).$$

Given $\phi_i\in\mathscr{S}_{\overline{\mathbb{F}}_\ell}(M_{n_i}(\mathrm{D}))$, we can find $\phi\in\mathscr{S}_{\overline{\mathbb{F}}_\ell}(M_n(\mathrm{D}))$ such that

$$\int_U \phi(p(m,u))\,\mathrm{d}u=\prod_{i=1}^{k}\phi_i(m_i),\ m=(m_1,\ldots,m_k)\in M.$$

Moreover, we choose two smooth functions $\xi,\xi'\in C_c^\infty((K_n\cap P)\backslash K_n)$ such that for all $p\in P$ and N

$$\int_{(K_n\cap P)\backslash K_n}\int_{(K_n\cap P)\backslash K_n}\int_P (\phi\chi_{G_n(N)})(k^{-1}pk')\xi(k)\xi'(k')\delta_P^{-1}(p)\,\mathrm{d}_rp\,\mathrm{d}k\,\mathrm{d}k'=$$

$$=\int_{P(n)}\phi(p)\delta_P^{-1}(p)\,\mathrm{d}_rp,$$

where $\chi_{G_n(N)}$ denotes the characteristic function of $G_n(N)$. For example, one could take the characteristic functions of sufficiently small open compact subgroups in G_n whose pro-order is invertible in R and rescale them appropriately. Projecting them into $C_c^\infty((K_n \cap P)\backslash K_n)$ gives then the desired functions. Then the coefficient of T^N in

$$\prod_{i=1}^{k} Z(\phi_i, Tq^{-\frac{dn_i-1}{2}}, f_i)$$

is

$$\int\limits_{M(N)} \prod_{i=1}^{k} f_i(m_i)\phi_i(m_i)|m_i|^{\frac{dn_i-1}{2}} \,\mathrm{d}m =$$

$$= \int\limits_{(K_n\cap P)\backslash K_n} \int\limits_{(K_n\cap P)\backslash K_n} \int\limits_{P(N)} |p|^{\frac{dn-1}{2}} \delta_P^{\frac{1}{2}}(m)\phi(k^{-1}pk')\delta_P^{-1}(p) \tag{15}$$

$$\prod_{i=1}^{k} f_i(m_i)\xi(k)\xi'(k')\,\mathrm{d}k\,\mathrm{d}k'\,\mathrm{d}_r p.$$

Setting

$$H_1(p,k,k') := \prod_{i=1}^{k} f_i(m_i)\xi(k)\xi'(k')$$

and choosing H as in (11) with $H(pk,k') = H_1(p,k,k')$ shows that (15) is the coefficient of T^N in $Z(\phi, Tq^{-\frac{dn-1}{2}}, f)$ as in (12).

Finally, to prove the equality of the γ-factors, we go back to (13) and define analogously for $\sigma_1^\vee \times \ldots \times \sigma_k^\vee$. $h^\vee(m;k,k')$ and $\phi^\vee(m;k,k')$. Observe that if $h^\vee$ is defined via $\widehat{f}$, the dual matrix coefficient of f, then

$$h^\vee(m;k,k') = h(m^{-1};k,k')$$

and if $\phi^\vee(m;k,k')$ is defined with the Fourier-transform of ϕ, then

$$\phi^\vee(m;k,k') = (\widehat{\phi})(m;k,k').$$

The claim follows then by writing $Z(\phi, Tq^{-\frac{dn-1}{2}}, f)$ and $Z(\widehat{\phi}, T^{-1}q^{-\frac{dn+1}{2}}, \widehat{f})$ in the form of (14). ☐

We write for $\mathfrak{m}$ an aperiodic multisegment $\langle\mathfrak{m}\rangle := \mathrm{Z}(\mathfrak{m})^*$.

Theorem 5.3.2. *Let $\Delta = [a,b]_\rho$ be a segment over R. If $\rho \cong \chi$ for an unramified character χ of F and $q^d \neq 1$, then*

$$L(\langle\Delta\rangle, T) = \frac{1}{1-\chi(\varpi_{\mathrm{F}})q^{-db+\frac{1-d}{2}}T},$$

where ϖ_{F} *is a uniformizer of* $\mathfrak{o}_{\mathrm{F}}$*. Otherwise*

$$L(\langle\Delta\rangle, T) = 1.$$

More generally, if $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ *is an aperiodic multisegment then*

$$\mathcal{L}(\langle\mathfrak{m}\rangle) = \mathcal{L}(\langle\Delta_1\rangle \times \ldots \times \langle\Delta_k\rangle)$$

and

$$L(\langle\mathfrak{m}\rangle, T) = L(\langle\Delta_1\rangle \times \ldots \times \langle\Delta_k\rangle, T) = \prod_{i=1}^{k} L(\langle\Delta_i\rangle, T).$$

If $R = \overline{\mathbb{Q}}_\ell$ this is [11, Theorem 2.7] and the case $\mathfrak{m}$ a banale multisegment was covered in [26, Theorem 3.1, Theorem 5.7]. Note that here we have to slightly depart from their strategy, however the main idea stays the same, thanks to the results of Section 3.3. From now on we thus assume that $R = \overline{\mathbb{F}}_\ell$.

Proof of Theorem 5.3.2. The idea of the proof is now to apply Proposition 3.3.3 together with Corollary 4.1.5. Indeed, note that this shows that

$$P(\mathcal{D}_{r,\rho,max}(\pi), T) | P(\pi, T) \tag{16}$$

for all $\square$-irreducible cuspidal unramified representations ρ. Finally, note that by Theorem 5.2.2 we have a precise description of $\mathcal{D}_{r,\rho,max}(\langle\mathfrak{m}\rangle)$ in terms of the multisegment $\mathfrak{m}$.

We start by proving the claim that

$$L(\langle[0,b]_\rho\rangle, T) = \frac{1}{1 - \chi(\varpi_{\mathrm{F}}) q^{-db + \frac{1-d}{2}} T}$$

if ρ is unramified and $\square$-irreducible and 1 otherwise. We proceed by induction on $l([0,b]_\rho)$. Note that the base-case is [26, Theorem 3.1] and if ρ is either ramified or not $\square$-irreducible, Lemma 2.10.4 implies

$$P(\langle[0,b]_\rho\rangle, T) | P(\rho^{b+1}, T) \overset{(Lemma\ 5.3.1)}{=} 1,$$

which proves the claim. Thus we assume ρ unramified and $\square$-irreducible. Then the claim follows from the induction-hypothesis since

$$\mathcal{D}_{r,\rho,max}(\langle[0,b]_\rho\rangle) = \langle[1,b]_\rho\rangle$$

by Theorem 5.2.2 and Lemma 2.6.3 and hence by (16)

$$1 - \rho(\varpi_{\mathrm{F}}) q^{-db + \frac{1-d}{2}} T | P(\langle[0,b]_\rho\rangle, T).$$

On the other hand, since ρ is supercuspidal, we can find a lift $[0,b]_{\tilde{\rho}}$ with $\tilde{\rho}$ unramified and hence $\mathrm{r}_\ell(\langle[0,b]_{\tilde{\rho}}\rangle)$ contains $[0,b]_\rho$. By Lemma 2.10.4 and Lemma 2.10.3 also

$$P(\langle[0,b]_\rho\rangle,T)|1-\rho(\varpi_{\mathrm{F}})q^{-db+\frac{1-d}{2}}T,$$

proving the claim for a segment.

Next we compute the L-function of $\langle\mathfrak{m}\rangle$. Let us argue by induction on $\deg(\mathfrak{m})$ and assume that we have proven the claim already for all multisegments of degree lesser than $\deg(\mathfrak{m})$, the base case being [26, Theorem 3.1]. We can write $\langle\mathfrak{m}\rangle=\pi_1\times\pi_2$, where π_1 has $\square$-irreducible unramified cuspidal support and π_2 has $\square$-reducible or ramified cuspidal support by Lemma 2.4.1. Since a representation induced from $\square$-reducible cuspidal representations or ramified cuspidal support has trivial L-factor by Lemma 5.3.1, $L(\pi_2,T)=1$ by Lemma 2.10.4. Again by Lemma 5.3.1

$$L(\langle\mathfrak{m}\rangle,T)=L(\pi_1,T)L(\pi_2,T)=L(\pi_1,T)$$

and hence we can assume that $\langle\mathfrak{m}\rangle$ has $\square$-irreducible cuspidal unramified support.

From the $\overline{\mathbb{Q}}_\ell$-case we obtain that for any lift $\widetilde{\mathfrak{m}}$ of $\mathfrak{m}$

$$\prod_{i=1}^{k}P(\langle\Delta_i\rangle,T)=\mathrm{r}_\ell(P(\langle\mathfrak{m}\rangle),T),$$

since we assumed $\langle\mathfrak{m}\rangle$ to have $\square$-irreducible unramified cuspidal support. From (16) and Theorem 4.2.8 we know that

$$P(\mathcal{D}_{r,\rho,max}(\langle\mathfrak{m}\rangle),T)|P(\langle\mathfrak{m}\rangle,T)$$

and from Lemma 2.10.4 and Lemma 5.3.1 it follows that

$$P(\langle\mathfrak{m}\rangle,T)|\prod_{i=1}^{k}P(\langle\Delta_i\rangle,T).$$

Fix now a $\square$-irreducible cuspidal unramified character ρ such that $\mathcal{D}_{r,\rho}(\pi)\neq 0$. By Theorem 4.2.8, Theorem 5.2.1 and the induction-hypothesis we then have that

$$\prod_{i=1}^{k}P(\langle\Delta_i\rangle,T)P(\mathcal{D}_{r,\rho,max}(\langle\mathfrak{m}\rangle),T)^{-1}=P(\rho,T)^{-k},$$

where k is the multiplicity of $[0,0]_\rho$ in $\mathfrak{m}^{\rho,f}$. Similarly, for a fixed $a\in\mathbb{Z}$ such that $\mathcal{D}_{\rho^\vee v_\rho^{-a},r}(\langle\mathfrak{m}^\vee\rangle)\neq 0$, we let k' be the multiplicity of $[-a,-a]_{\rho^\vee}$ in $(\mathfrak{m}^\vee)^{\rho^\vee v_\rho^{-a},f}$. We thus obtain that

$$P(\langle\mathfrak{m}\rangle,T)P(\rho,T)^{l}=\prod_{i=1}^{k}P(\langle\Delta_i\rangle,T)$$

for some $l\leq k$ and

$$P(\langle\mathfrak{m}^\vee\rangle,T)P(\rho^\vee v_\rho^{-a},T)^{l'}=\prod_{i=1}^{k}P(\langle\Delta_i^\vee\rangle,T)$$

for $l' \leq k'$. We will now show that $l = l' = 0$. By Lemma 2.10.3(2)

$$\frac{\prod_{i=1}^{k} P(\langle\Delta_i\rangle, T)}{\prod_{i=1}^{k} P(\langle\Delta_i^{\vee}\rangle, q^{-1}T^{-1})} \mathrm{r}_\ell(\epsilon(T, \langle\widetilde{\mathfrak{m}}\rangle, \widetilde{\psi})) =$$

$$= \frac{\mathrm{r}_\ell(L(\langle\widetilde{\mathfrak{m}}^{\vee}\rangle, q^{-1}T^{-1}))}{\mathrm{r}_\ell((L(\langle\widetilde{\mathfrak{m}}\rangle, T))} \mathrm{r}_\ell(\epsilon(T, \langle\widetilde{\mathfrak{m}}\rangle, \widetilde{\psi})) = \frac{L(\langle\mathfrak{m}^{\vee}\rangle, q^{-1}T^{-1})}{L(\langle\mathfrak{m}\rangle, T)} \epsilon(T, \langle\mathfrak{m}\rangle, \psi).$$

Rewriting, we obtain that

$$\frac{P(\rho, T)^{l}}{P(\rho^{\vee} v_\rho^{-a}, q^{-1}T^{-1})^{l'}} = \frac{\epsilon(T, \langle\mathfrak{m}\rangle, \psi)}{\mathrm{r}_\ell(\epsilon(T, \langle\widetilde{\mathfrak{m}}\rangle, \widetilde{\psi}))}.$$

Recall now that the right side is a unit in $\overline{\mathbb{F}}_\ell[T, T^{-1}]$ and hence so has to be left side. But

$$\frac{P(\rho, T)^{l}}{P(\rho^{\vee} v_\rho^{-a}, q^{-1}T^{-1})^{l'}} = \frac{(1 - \rho(\varpi_{\mathrm{F}}) q^{\frac{1-d}{2}} T)^{l}}{(1 - \rho(\varpi_{\mathrm{F}}^{-1}) q^{ad + \frac{1-d}{2} - 1} T^{-1})^{l'}}$$

and in order for this to be a unit either $l = l' = 0$, in which case we are done, or $l = l' > 0$ and $-a + 1 = 0 \mod o(\rho)$. We let now m be the multiplicity of $[0,0]_\rho$ and m' be the multiplicity $[1,1]_\rho$ in $\mathfrak{m}$. By Theorem 4.2.8 and Theorem 5.2.1 we obtain that if $[0,0]_\rho$ appears with non-zero multiplicity in $\mathfrak{m}^{\rho,f}$, then there have to exist more copies of $[0,0]_\rho$ in $\mathfrak{m}$ than copies of $[1,1]_\rho$. Thus if $k \geq l > 0$, we have $m' < m$. But on the other hand, the same argument shows that if $[-1,-1]_{\rho^\vee} = [-a,-a]_{\rho^\vee}$ appears with non-zero multiplicity in $(\mathfrak{m}^{\vee})^{\rho^{\vee} v_\rho^{-1}, f}$, then there have to exist more copies of $[-1,-1]_{\rho^\vee}$ in $\mathfrak{m}^{\vee}$ than copies of $[0,0]_{\rho^\vee}$. Thus if $k' \geq l' > 0$, we have $m < m'$. But since we already showed that $l = l'$, the only possibility is $l = l' = 0$. ☐

We obtain as a corollary the following.

Corollary 5.3.3. *Let* $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ *be an aperiodic multisegment. Then*

$$\epsilon(T, \langle\mathfrak{m}\rangle, \psi) = \prod_{i=1}^{k} \epsilon(T, \langle\Delta_i\rangle, \psi)$$

and

$$\gamma(T, \langle\mathfrak{m}\rangle, \psi) = \prod_{i=1}^{k} \gamma(T, \langle\Delta_i\rangle, \psi).$$

Proof. Notice that if π is a subquotient of π', $\gamma(T, \pi, \psi) = \gamma(T, \pi', \psi)$ and hence the claim for the γ-factor follows from Lemma 5.3.1. By the functional equation, the claim also follows for the ϵ-factors. ☐

5.3.1 LLC and local factors In this subsection assume D = F. We recall the map

$$\mathrm{C}\colon \mathfrak{Irr}_n \overset{\cong}{\to} \{\mathrm{C}-\text{parameters of length } n\}$$

defined in [18]. Let us introduce the main actors of this story. We denote by W_{F} the Weil group of F, I_{F} the inertia sugroup and by ν the unique unramified character of W_{F} acting on the Frobenius by $\nu(\mathrm{Frob}) = q^{-1}$. A Deligne R-representation of W_{F} is a pair (Φ, U), where Φ is a finite-dimensional, smooth representation of W_{F} over R and U an element in $\mathrm{Hom}_{W_{\mathrm{F}}}(\nu\Phi, \Phi)$. We call (Φ, U) *semi-simple* if Φ is semi-simple and if $R = \overline{\mathbb{F}}_\ell$, let $o(\Phi)$ be the smallest natural number k such that

$$\Phi \cong \nu^k\Phi.$$

Let

$$\mathbb{Z}[\Phi] := \begin{cases} \{\Phi, \dots, \nu^{o(\Phi)-1}\Phi\} & R = \overline{\mathbb{F}}_\ell, \\ \{\nu^k\Phi : k \in \mathbb{Z}\} & R = \overline{\mathbb{Q}}_\ell. \end{cases}$$

We say a semi-simple R-Deligne representation is supported on $\mathbb{Z}[\Phi]$ if the underlying W_{F}-representation is the direct sum of elements in $\mathbb{Z}[\Phi]$. We say two indecomposable representations (Φ, U) and (Φ', U') are equivalent if $(\Phi', U') \cong (\Phi, \lambda U)$ for some non-zero scalar $\lambda \in R$ and two general semi-simple R-Deligne representations are equivalent if their respective indecomposable parts are equivalent. We denote by $\mathrm{Rep}_{ss}(W_{\mathrm{F}}, R)$ the set of equivalence classes of semi-simple R-Deligne representations and by $\mathrm{Rep}_{ss,n}(W_{\mathrm{F}}, R)$ the subset of representations of length n, where the length is the dimension of the underlying W_{F}-representation. We call an R-Deligne representation (Φ, U) *nilpotent* if U is nilpotent and recall that over $\overline{\mathbb{Q}}_\ell$, the equivalence classes of semi-simple representations are in bijections with equivalence classes of nilpotent semi-simple representations. We denote the latter set by $\mathrm{Nil}_{ss}(W_{\mathrm{F}}, R)$ and by $\mathrm{Nil}_{ss,n}(W_{\mathrm{F}}, R)$ its subset consisting of representations of length n.

For $r \in \mathbb{Z}_{\geq 1}$, we let $[0, r-1]$ be the R-Deligne representation whose underlying W_{F}-representation is

$$\bigoplus_{i=0}^{r-1} \nu^k$$

and U is defined by

$$U(x_0, \dots, x_{r-1}) = (0, x_0, \dots, x_{r-2}).$$

For Φ a general irreducible representation of W_{F} we can then define

$$[0, r-1] \otimes \Phi$$

and these parametrize all nilpotent semi-simple R-Deligne representations up to equivalence. The local Langlands correspondence gives then a well-known canonical bijection

$$\mathrm{LLC}\colon \mathrm{Nil}_{ss,n}(W_{\mathrm{F}}, \overline{\mathbb{Q}}_\ell) \overset{\cong}{\to} \mathfrak{Irr}_n(\overline{\mathbb{Q}}_\ell).$$

Over $\overline{\mathbb{F}}_\ell$ there exists a second class of indecomposable semi-simple representations of the following form. Let Φ be an irreducible representation of W_{F}, denote

$$\Psi(\Phi) := \bigoplus_{k=0}^{o(\Phi)-1} \nu^k \Phi$$

and pick an isomorphism $I\colon \nu^{o(\Phi)}\Phi \xrightarrow{\cong} \Phi$. We then define $C(I,\Phi) = C(\Phi)$ as the $\overline{\mathbb{F}}_\ell$-Deligne representation with underlying W_{F}-representation $\Psi(\Phi)$ and

$$U(x_0,\dots x_{o(\Phi)-1}) = (I(x_{o(\Phi)-1}), x_0, \dots, x_{o(\Phi)-2}).$$

Then $C(\Phi)$ depends up to equivalence only on $\mathbb{Z}[\Phi]$ and the underlying morphism U of $C(\phi)$ is a bijection. The indecomposable semi-simple $\overline{\mathbb{F}}_\ell$-Deligne representations are then classified up to equivalence by

$$[0,r-1] \otimes \Phi \text{ and } [0,r-1] \otimes C(\Phi),$$

see [18, Main Theorem 1].

We call an semi-simple $\overline{\mathbb{Q}}_\ell$-representation Φ of W_{F} *integral* if it admits a W_{F}-stable $\overline{\mathbb{Z}}_\ell$-lattice L, which generates Φ. One can then define its reduction mod ℓ $\mathrm{r}_\ell(\Phi)$ by taking the semi-simplification of $L \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell$, which is independent of the chosen lattice. We let $\mathrm{Nil}^e_{ss}(W_{\mathrm{F}}, \overline{\mathbb{Q}}_\ell)$ be the semi-simple nilpotent representations whose underlying W_{F}-representation is integral. One obtains hence a map

$$\mathrm{r}_\ell\colon \mathrm{Nil}^e_{ss}(W_{\mathrm{F}}, \overline{\mathbb{Q}}_\ell) \to \mathrm{Nil}_{ss}(W_{\mathrm{F}}, \overline{\mathbb{F}}_\ell)$$

sending

$$[0,r-1] \otimes \Phi \mapsto [0,r-1] \otimes \mathrm{r}_\ell(\Phi).$$

We also recall the following surjection introduced in [37, I.8.6]. We denote by $\mathfrak{Irr}_n(\overline{\mathbb{Q}}_\ell)_e$ the isomorphism classes of integral irreducible representations. For π in this set, choose α the maximal ordered partition such that π is α-degenerate. Then map π to the unique irreducible constituent in $[\mathrm{r}_\ell(\pi)]$ which is α-degenerate. The so-constructed map is surjective and denoted by

$$\mathrm{J}_\ell\colon \mathfrak{Irr}_n(\overline{\mathbb{Q}}_\ell)_e \to \mathfrak{Irr}_n(\overline{\mathbb{F}}_\ell).$$

Theorem 5.3.4 ([37, Theorem 1.6, Theorem 1.8.5])**.** *There exists a bijection*

$$\mathrm{V}\colon \mathfrak{Irr}_n(\overline{\mathbb{F}}_\ell) \xrightarrow{\cong} \mathrm{Nil}_{ss,n}(W_{\mathrm{F}}, \overline{\mathbb{F}}_\ell),$$

which is uniquely characterized by the equality

$$\mathrm{V}(\mathrm{J}_\ell(\mathrm{LLC}(\Phi,U)^*)^*) = \mathrm{r}_\ell(\Phi,U).$$

We recall next the map

$$\mathrm{CV}\colon \mathrm{Nil}_{ss}(W_{\mathrm{F}},\overline{\mathbb{F}}_\ell) \to \mathrm{Rep}_{ss}(W_{\mathrm{F}},\overline{\mathbb{F}}_\ell),$$

cf. [18, §6.3]. A nilpotent $\overline{\mathbb{F}}_\ell$-Deligne representation supported in $\mathbb{Z}[\Phi]$ can always be written in the form

$$\Phi_{ap} \oplus \Phi_{cyc}$$

with

$$\Phi_{ap} = \bigoplus_{i\geq 1} \bigoplus_{k=1}^{o(\Phi)-1} c_{i,k}[0,i-1] \otimes \nu^k \Phi,$$

$c_{i,k} \in \mathbb{Z}_{\geq 0}$, $c_{i,k} = 0$ for $i >> 0$ and for each i there exists at least one k such that $c_{i,k} = 0$. Furthermore, Φ_{cyc} is of the form

$$\Phi_{cyc} = \bigoplus_{i\geq 1} a_i[0,i-1] \otimes \bigoplus_{k=1}^{o(\Phi)-1} \nu^k \Phi$$

with $a_i \in \mathbb{Z}_{\geq 0}$, $a_i = 0$ for $i >> 0$. Then

$$\mathrm{CV}(\Phi_{ap} \oplus \Phi_{cyc}) = \Phi_{ap} \oplus \bigoplus_{i\geq 1} a_i[0,i-1] \otimes C(\Phi).$$

Finally, for a nilpotent $\overline{\mathbb{F}}_\ell$-Deligne representation $(\Phi_1,U_1) \oplus \cdots \oplus (\Phi_k,U_k)$ with each of the summands supported in another $\mathbb{Z}[\Phi']$,

$$\mathrm{CV}((\Phi_1,U_1) \oplus \cdots \oplus (\Phi_k,U_k)) = \mathrm{CV}(\Phi_1,U_1) \oplus \cdots \oplus \mathrm{CV}(\Phi_k,U_k).$$

One can then finally define the map

$$\mathrm{C} \coloneqq \mathrm{CV} \circ \mathrm{V}\colon \mathfrak{Irr}_n(\overline{\mathbb{F}}_\ell) \to \mathrm{Rep}_{ss}(W_{\mathrm{F}},\overline{\mathbb{F}}_\ell).$$

We note the following properties of C, see [18, Examples 6.5]. If ρ is supercuspidal, $\mathrm{C}(\rho)$ is banal if and only $\rho \in \mathfrak{C}^\square$ and in this case $\mathrm{C}(\rho)$ is unramified if and only if ρ is. On the other hand, if ρ is $\square$-reducible, $\mathrm{C}(\rho) = (\Phi, I_\Phi)$ for some bijective map I_Φ. More generally, if $\pi = \langle \mathfrak{m} \rangle \in \mathfrak{Irr}_n$ for an aperiodic multisegment $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ with $\Delta_i = [a_i,b_i]_{\rho_i}$ we have

$$\mathrm{C}(\pi) = \bigoplus_{i=1}^{k} [0, b_i - a_i] \otimes \nu^{a_i} \mathrm{C}(\rho_i).$$

For $(\Phi,U) \in \mathrm{Rep}_{ss}(W_{\mathrm{F}},R)$ one can associate a local L-factor

$$L((\Phi,U),T) \coloneqq \det(1 - T\Phi(\mathrm{Frob})|_{\mathrm{Ker}(U)^{I_{\mathrm{F}}}})^{-1} \in R(T)$$

and an ϵ-factor $\epsilon(T,(\Phi,U),\psi) \in R[T,T^{-1}]^*$ and γ-factor $\gamma(T,(\Phi,U),\psi) \in R(T)$ to (Φ,U), *cf.* [18, §5]. Moreover, the γ-factor does only depend on Φ, all three factors are multiplicative with respect to $\oplus$ and for $(\widetilde{\Phi},U) \in \mathrm{Nil}_{ss}^{e}(W_{\mathrm{F}},\overline{\mathbb{Q}}_\ell)$,

$$\mathrm{r}_\ell(\gamma(T,(\widetilde{\Phi},U),\widetilde{\Psi})) = \gamma(T,\mathrm{r}_\ell(\widetilde{\Phi},U),\psi),$$

see [18, Proposition 5.6, Proposition 5.11].

Theorem 5.3.5. *The map* C *respects the local factors,* i.e. *for* $\pi \in \mathfrak{Irr}_n$

$$L(\pi, T) = L(\mathrm{C}(\pi), T),\ \epsilon(T, \pi, \psi) = \epsilon(T, \mathrm{C}(\pi), \psi),\ \gamma(T, \pi, \psi) = \gamma(T, \mathrm{C}(\pi), \psi).$$

Proof. We start with the equality of the γ-factors. Since the γ-factor of π does only depend on the supercuspidal support of π by Lemma 2.10.4 and the γ factor of $\mathrm{C}(\pi)$ depends only on the underlying W_{F}-representation, the multiplicativity of the γ-factor of $\mathrm{Rep}_{ss}(W_{\mathrm{F}}, \overline{\mathbb{F}}_\ell)$ and Lemma 5.3.1 show that it is enough to show the equality for a supercuspidal representations ρ. But by picking a lift $\tilde{\rho}$ of ρ, we see that

$$\gamma(T, \rho, \psi) = \mathrm{r}_\ell(\gamma(T, \tilde{\rho}, \widetilde{\Psi})) = \mathrm{r}_\ell(\gamma(T, \mathrm{LLC}^{-1}(\tilde{\rho}), \widetilde{\Psi})) = \gamma(T, \mathrm{C}(\rho), \psi).$$

For the L-factor, note that by Theorem 5.3.2 both sides are multiplicative and hence it is enough to show the claim for a segment $\pi = \langle [a, b]_\rho \rangle$. It is now easy to check from the definition and the discussion above the theorem, that

$$L([0, b-a] \otimes \mathrm{C}(\rho v_\rho^a), T) = \begin{cases} 1 & \text{if } o(\rho) = 1 \text{ or } \rho \text{ is ramified,} \\ \frac{1}{1-\rho(\varpi_{\mathrm{F}})q^{-db+\frac{1-d}{2}}T} & \text{if } \rho \in \mathfrak{C}^{\square} \text{ and } \rho \text{ is unramified.} \end{cases}$$

By Theorem 5.3.2 this concise with $L(\langle [a, b]_\rho \rangle, T)$. Finally, the equality of ϵ-factors follows from the functional equation [18, Definition 5.5]. ☐

6 Quiver varieties

In this section we will describe certain quiver varieties and explain how they relate to the representation theory of G_n over $\overline{\mathbb{F}}_\ell$. We start by setting the following geometric stage.

6.1 Quiver varieties

We fix a supercuspidal representation ρ. Let $Q = A^+_{o(\rho)-1}$ be the affine Dynkin quiver with $o(\rho)$ vertices numbered $\{0, \dots, o(\rho)-1\}$ and an arrow from i to j if $j = i + 1 \mod o(\rho)$, *i.e.* of Q is of the form

$$\begin{tikzcd} & \dots \arrow[dr, bend left] & \\ 0 \arrow[ur, bend left] & & o(\rho)-2 \arrow[dl, bend left] \\ & o(\rho)-1 \arrow[ul, bend left] & \end{tikzcd}$$

If $o(\rho) = \infty$, we let $A_\infty = A^+_\infty$

$$\begin{tikzcd} \dots \arrow[r] & i-1 \arrow[r] & i \arrow[r] & i+1 \arrow[r] & \dots \end{tikzcd}$$

A finite dimensional representation of Q is a graded vector space

$$V = \bigoplus_{i=0}^{o(\rho)-1} V_i$$

over some algebraically closed and complete field K (e.g. $\mathbb{C}$) and a linear map $T_{i\to j} \in \mathrm{Hom}(V_i, V_j)$ for each edge $i \to j$. Fixing V we denote by $E(V)$ the affine space of representations of Q with underlying vector space V and equip it with the natural topology coming from K. The group

$$\mathrm{GL}(V) = \mathrm{GL}(V_0) \times \ldots \times \mathrm{GL}(V_{o(\rho)-1})$$

acts on $E(V)$ by

$$(g_0, \ldots, g_{o(\rho)-1}) \cdot (x_0, \ldots, x_{o(\rho)-1}) \coloneqq (g_1 x_0 g_0^{-1}, g_2 x_1 g_1^{-1}, \ldots, g_0 x_{o(\rho)-1} g_{o(\rho)-1}^{-1}).$$

We denote by $N(V) \subseteq E(V)$ the space of nilpotent representations, *i.e.* the collection of all $T \in E(V)$ such that for large enough $N \in \mathbb{N}$, $T^N = 0$, where T is seen as an element of $\mathrm{Hom}(V, V)$. Set

$$\mathfrak{s}_V \coloneqq (\dim_K V_0) \cdot [\rho] + \ldots + (\dim_K V_{o(\rho)-1}) \cdot [\rho v^{o(\rho)-1}].$$

The $\mathrm{GL}(V)$-orbits of $N(V)$ are parametrized by multisegments with cuspidal support $\mathfrak{s}_V$ as follows, see [23, §15].

First assign to a segment $\Delta = [a, b]_\rho$ with cuspidal support

$$\mathrm{cusp}_{\mathcal{MS}}(\Delta) = d_0 \cdot [\rho] + \ldots + d_{o(\rho)-1} \cdot [\rho v^{o(\rho)-1}]$$

the vector space

$$V(\Delta) \coloneqq \bigoplus_{i=0}^{o(\rho)-1} K^{d_i}$$

together with a basis $e_a, e_{a+1}, \ldots, e_b$, $e_i \in V_i$, where the index of e_i is seen modulo $o(\rho)$. The linear map $\lambda(\Delta)$ associated to Δ sends e_i to e_{i+1} for $a \leq i < b$ and e_b to 0.

To a multisegment $\mathfrak{m} = \Delta_1 + \ldots + \Delta_n$ we then associate the linear maps $\lambda(\mathfrak{m}) \coloneqq \lambda(\Delta_1) \oplus \ldots \oplus \lambda(\Delta_n)$ of $V(\mathfrak{m}) \coloneqq V(\Delta_1) \oplus \ldots \oplus V(\Delta_n)$. It is easy to see that if $\mathrm{cusp}(\mathfrak{m}) = \mathfrak{s}_V$ then $V(\mathfrak{m}) \cong V$ as graded vector spaces and that $\lambda(\mathfrak{m})$ and $\lambda(\mathfrak{m}')$ lie in the same orbit of G_V if and only if $\mathfrak{m}$ and $\mathfrak{m}'$ are equivalent. Denote the orbit of $\lambda(\mathfrak{m})$ in V as $X_\mathfrak{m}$. We define an order on the set of multisegments by setting $\mathfrak{n} \leq \mathfrak{m}$ if

$$X_\mathfrak{m} \subseteq \overline{X}_\mathfrak{n},$$

where $\overline{X}_\mathfrak{n}$ is the closure of $X_\mathfrak{n}$ in $N(V)$ with its topology being the one coming from $E(V)$.

6.2 To motivate the next definitions we recall first the case $R = \overline{\mathbb{Q}}_\ell$, which was treated in [38]. Assume for a moment that $R = \overline{\mathbb{Q}}_\ell$. Let ρ be a cuspidal representation of G_m and $\Delta = [a,b]_\rho$ and $\Delta' = [a',b']_\rho$ two segments in $\mathcal{S}(\rho)$ such that Δ precedes Δ', *i.e.*

$$a + 1 \leq a' \leq b + 1 \leq b'.$$

Define the union and intersection of Δ and Δ' as

$$\Delta \cup \Delta' := [a,b']_\rho,\ \Delta \cap \Delta' := [a',b]_\rho$$

and observe that they are unlinked. In [38, §4] the following decomposition in the Grothendieck group was proven.

$$[\mathrm{Z}(\Delta) \times \mathrm{Z}(\Delta')] = [\mathrm{Z}(\Delta \cup \Delta) \times \mathrm{Z}(\Delta \cap \Delta')] + [\mathrm{Z}(\Delta + \Delta')]. \tag{17}$$

Let $\mathfrak{m} \in \mathcal{MS}(\rho)$ be a multisegment. Then in [39] and [38, §7] the authors proved that in this case $[\mathrm{Z}(\mathfrak{n})] \leq \mathrm{I}(\mathfrak{m})$ if and only if $\mathfrak{n} \leq \mathfrak{m}$, see Section 6.1. In this section we are going to prove the analogous statement over $\overline{\mathbb{F}}_\ell$.

6.3 Elementary operations We recall the following combinatorial description of $X_\mathfrak{m} \subseteq \overline{X_\mathfrak{n}}$ in terms of the underlying multisegments. Fix a supercuspidal ρ, Q and $V \cong V(\mathfrak{m}) \cong V(\mathfrak{n})$ as in Section 6.1. We define the notation of an elementary operation on a multisegment as follows. Pick two ρ-segments Δ and Δ' in $\mathfrak{m}$, which are by definition only defined up to equivalence. Next pick two representatives for Δ and Δ'

$$[a,b]_\rho = (\rho v_\rho^a, \dots, \rho v_\rho^b),\ [a',b']_\rho = (\rho v_\rho^{a'}, \dots, \rho v_\rho^{b'})$$

such that

$$a + 1 \leq a' \leq b + 1 \leq b' \tag{18}$$

and define the elementary operation with respect to (Δ, Δ')

$$\mathfrak{m} \mapsto \mathfrak{m} - \Delta - \Delta' + [a,b']_\rho + [a',b]_\rho.$$

If there do not exist representatives satisfying (18), we cannot perform an elementary operation with respect to (Δ, Δ'). If we can perform an elementary operation we call the segments linked.

Lemma 6.3.1 ([16], [33, Theorem 3.12])**.** *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be two multisegments with the same support contained in* $\mathcal{MS}(\rho)$*. Then* $\mathfrak{n} \leq \mathfrak{m}$ *if and only if* $\mathfrak{n}$ *can be obtained from* $\mathfrak{m}$ *via finitely many elementary operations.*

The next lemmas will allow us to have a better grip on elementary operations and their proofs are rather straightforward.

Lemma 6.3.2. *Let* $\Delta_1, \Delta_2, \Gamma_1, \Gamma_2$ *be segments in* $\mathcal{S}(\rho)$*, where* Δ_1 *or* Δ_2 *are also allowed be the empty segment and* $\Delta_1 + \Delta_2 \leq \Gamma_1 + \Gamma_2$*. Then either*

$$\Delta_1 + \Delta_2^+ \leq \Gamma_1 + \Gamma_2^+ \textit{ and } (\Delta_1 + \Delta_2^+)^1 = (\Gamma_1 + \Gamma_2^+)^1 \textit{ or} \tag{19}$$

$$\Delta_1^+ + \Delta_2 \leq \Gamma_1 + \Gamma_2^+ \text{ and } (\Delta_1^+ + \Delta_2)^1 = (\Gamma_1 + \Gamma_2^+)^1. \tag{20}$$

If in the first case Δ_2 *is the empty segment, we mean by* Δ_2^+ *the segment* $[b+1, b+1]_\rho$, *where* b *is such that* $\Gamma_2 = [a,b]_\rho$ *and analogously in the second case for* Δ_1*. Moreover,*

$$\Delta_1^+ + \Delta_2^+ \leq \Gamma_1^+ + \Gamma_2^+ \text{ and } (\Delta_1^+ + \Delta_2^+)^1 = (\Gamma_1^+ + \Gamma_2^+)^1.$$

Proof. We will prove the first claim by induction on the number of elementary operations necessary to obtain $\Delta_1 + \Delta_2$ from $\Gamma_1 + \Gamma_2$. We can moreover assume without loss of generality that $l(\Delta_1) > l(\Delta_2)$. Assume first that they differ by one and write the representatives of Γ_1 and Γ_2 as $[a,b]_\rho$ and $[a',b']_\rho$ such that the elementary operation is performed by

$$\Gamma_1 + \Gamma_2 \mapsto [a,b']_\rho + [a',b]_\rho$$

and the representatives satisfy (18). Now, since $l(\Delta_1) > l(\Delta_2)$,

$$\Delta_1 = [a,b']_\rho,\ \Delta_2 = [a',b]_\rho.$$

We have $a+1 \leq a' \leq b+1 \leq b'+1$ and therefore

$$\Gamma_1 + \Gamma_2^+ \mapsto [a, b'+1]_\rho + [a',b]_\rho$$

is an elementary operation, implying (20). Next we assume that again they differ by one elementary operation but this time the representative of Γ_1 is $[a',b']_\rho$ and the representative of Γ_2 is $[a,b]_\rho$, again satisfying (18). Again by $l(\Delta_1) > l(\Delta_2)$,

$$\Delta_1 = [a,b']_\rho,\ \Delta_2 = [a',b]_\rho.$$

Note that $a+1 \leq a' \leq b+2$. If $b+2 \leq b'$, we are in the situation of (19), since

$$\Gamma_1 + \Gamma_2^+ \mapsto [a,b']_\rho + [a',b+1]_\rho$$

is an elementary operation. On the other hand, if $b+1 = b'$, then

$$\Gamma_1 + \Gamma_2^+ = \Delta_1 + \Delta_2^+$$

and we have (19).

If they differ by more than one elementary operation, choose $\mathfrak{n} = \Delta_1' + \Delta_2'$ such that $\mathfrak{n} \leq \Gamma_1 + \Gamma_2$ differ by one elementary operation and $\Delta_1 + \Delta_2 \leq \mathfrak{n}$. By the base case $\Delta_1' + \Delta_2'^+ \leq \Gamma_1 + \Gamma_2^+$ or $\Delta_1'^+ + \Delta_2' \leq \Gamma_1 + \Gamma_2^+$ and by the induction hypothesis $\Delta_1 + \Delta_2^+ \leq \Gamma_1 + \Gamma_2^+$ or $\Delta_1^+ + \Delta_2 \leq \Gamma_1 + \Gamma_2^+$. The claim regarding $(-)^1$ follows easily by the same induction argument.

For the second claim, we will also proceed on the number of elementary operations necessary to obtain $\Delta_1 + \Delta_2$ from $\Gamma_1 + \Gamma_2$. of Γ_1 and Γ_2 as $[a,b]_\rho$ and $[a',b']_\rho$ such that the elementary operation is performed by

$$\Gamma_1 + \Gamma_2 \mapsto [a,b']_\rho + [a',b]_\rho$$

and the representatives satisfy (18). Then $a+1 \leq a' \leq b+2 \leq b'+1$ and hence

$$\Gamma_1^+ + \Gamma_2^+ \mapsto [a, b'+1]_\rho + [a', b+1]_\rho$$

is an elementary operation. The induction step follows now as in the second claim. □

Next, we need the following lemma, which can already be found in [38] for complex representations. The argument presented there works analogously, however we will write it out for completeness.

Lemma 6.3.3. *Let* $\mathfrak{n}_1$, $\mathfrak{n}_2$, $\mathfrak{m}_1$, $\mathfrak{m}_2$ *be multisegments with* $\mathfrak{n}_1^1 = \mathfrak{m}_1^1$ *and* $\mathfrak{n}_1^- + \mathfrak{n}_2 \leq \mathfrak{m}_1^- + \mathfrak{m}_2$. *Then there exists* $\mathfrak{n}' = \mathfrak{n}_1' + \mathfrak{n}_2'$ *with* $\mathfrak{n}' \leq \mathfrak{m}_1 + \mathfrak{m}_2$, $\mathfrak{n}_1'^1 = \mathfrak{n}_1^1$ *and* $\mathfrak{n}_1'^- + \mathfrak{n}_2' = \mathfrak{n}_1^- + \mathfrak{n}_2$.

Proof. Assume first that $\mathfrak{n}_1^- + \mathfrak{n}_2$ and $\mathfrak{m}_1^- + \mathfrak{m}_2$ differ by one elementary operation. Let $\Delta_1 + \Delta_2$ resp. $\Gamma_1 + \Gamma_2$ be the multisegments contained in $\mathfrak{m}_1^- + \mathfrak{m}_2$ resp. $\mathfrak{n}_1^- + \mathfrak{n}_2$ involved in the elementary operation. Applying Lemma 6.3.2 gives multisegments $\Delta_1^* + \Delta_2^*$ resp. $\Gamma_1^* + \Gamma_2^*$ such that $\Gamma_1^* + \Gamma_2^*$ is contained in $\mathfrak{m}_1 + \mathfrak{m}_2$, $\Delta_1^* + \Delta_2^* \leq \Gamma_1^* + \Gamma_2^*$,

$$\Delta_1^* \in \{\Delta_1, \Delta_1^+\},\ \Delta_2^* \in \{\Delta_2, \Delta_2^+\},\ \Gamma_1^* \in \{\Gamma_1, \Gamma_1^+\},\ \Gamma_2^* \in \{\Gamma_2, \Gamma_2^+\}$$

and $(\epsilon_{\Gamma_1}\Gamma_1^* + \epsilon_{\Gamma_2}\Gamma_2^*)^1 = (\epsilon_{\Delta_1}\Delta_1^* + \epsilon_{\Delta_2}\Delta_2^*)^1$ with

$$\epsilon_\Lambda := \begin{cases} 1 & \text{if } \Lambda^* = \Lambda^+, \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$\mathfrak{n}' := \Delta_1^* + \Delta_2^* + \mathfrak{m}_1 + \mathfrak{m}_2 - \Gamma_1^* - \Gamma_2^*.$$

It is clear by construction that $\mathfrak{n}' \leq \mathfrak{m}_1 + \mathfrak{m}_2$ via one elementary operation. Write $\Delta_1^* + \Delta_2^* - \Gamma_1^* - \Gamma_2^*$ as $\mathfrak{k}_1 + \mathfrak{k}_2$, where we allow that the coefficients of the segments in $\mathfrak{k}_i$'s are not necessarily positive integers and

$$\mathfrak{k}_1 = \epsilon_{\Delta_1}\Delta^* + \epsilon_{\Delta_2}\Delta_2^* - \epsilon_{\Gamma_1}\Gamma_1^* - \epsilon_{\Gamma_2}\Gamma_2^*.$$

Note that $\mathfrak{k}_1^- + \mathfrak{k}_2 = \Gamma_1 + \Gamma_2 - \Delta_1 - \Delta_2$ and $\mathfrak{k}_1^1 = 0$, where we extended the operations $(-)^-$ and $(-)^1$ in the straightforward way to multisegments with integer coefficients.

We now define $\mathfrak{n}_1' := \mathfrak{m}_1 + \mathfrak{k}_1$ and $\mathfrak{n}_2' := \mathfrak{m}_2 + \mathfrak{k}_2$. By construction the coefficients of all segments in $\mathfrak{n}_1'$ and $\mathfrak{n}_2'$ are positive integers, because $\epsilon_{\Gamma_1} = 1$ if and only if Γ_1 was a segment of $\mathfrak{m}_1$ and similarly for Γ_2. Thus $\mathfrak{n}_1'$ and $\mathfrak{n}_2'$ are well defined multisegments. We therefore obtain

$$\mathfrak{n}_1'^- + \mathfrak{n}_2' = \mathfrak{m}_1^- + \mathfrak{m}_2 + \Delta_1 + \Delta_2 - \Gamma_1 - \Gamma_2 = \mathfrak{n}_1^- + \mathfrak{n}_2$$

since $\mathfrak{k}_1^- + \mathfrak{k}_2 = \Delta_1 + \Delta_2 - \Gamma_1 - \Gamma_2$ and $\mathfrak{n}_1'^1 = \mathfrak{m}_1^1 = \mathfrak{n}_1^1$, since $\mathfrak{k}_1^1 = 0$. Therefore we constructed $\mathfrak{n}' = \mathfrak{n}_1' + \mathfrak{n}_2'$ as desired.

If $\mathfrak{n}_1^- + \mathfrak{n}_2$ differs by more than one elementary operation from $\mathfrak{m}_1^- + \mathfrak{m}_2$ the claim follows by induction on the number of elementary operations. Indeed, pick $\mathfrak{k}$ such that $\mathfrak{m}_1^- + \mathfrak{m}_2$ is obtained from $\mathfrak{k}$ via one elementary operation and $\mathfrak{k} \leq \mathfrak{m}_1^- + \mathfrak{m}_2$. By the induction hypothesis we know that there exists $\mathfrak{k}' = \mathfrak{k}_1' + \mathfrak{k}_2'$, $\mathfrak{k}_1'^- + \mathfrak{k}_2' = \mathfrak{k}'$, $\mathfrak{k}_1'^- = \mathfrak{m}_1^1$ with $\mathfrak{k}' \leq \mathfrak{m}_1 + \mathfrak{m}_2$. Moreover, by applying the induction hypothesis a second time, we obtain $\mathfrak{n}' = \mathfrak{n}_1' + \mathfrak{n}_2'$, $\mathfrak{n}_1'^- + \mathfrak{n}_2' = \mathfrak{n}_1^- + \mathfrak{n}_2$, $\mathfrak{n}' \leq \mathfrak{k}' \leq \mathfrak{m}_1 + \mathfrak{m}_2$ and $\mathfrak{n}_1'^1 = \mathfrak{k}_1'^1 = \mathfrak{m}_1^1 = \mathfrak{n}_1^1$. □

6.4 We are now ready to prove the result announced at the beginning of the section.

Theorem 6.4.1. *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be two multisegments with the same cuspidal support contained in* $\mathcal{MS}(\rho)$*. Then* $\mathrm{Z}(\mathfrak{n})$ *appears in* $\mathrm{Z}(\mathfrak{m})$ *if and only if* $\mathfrak{n} \leq \mathfrak{m}$*.*

Proof. **First direction: $\mathfrak{n} \leq \mathfrak{m} \Rightarrow [\mathrm{Z}(\mathfrak{n})] \leq \mathrm{I}(\mathfrak{m})$**

By Lemma 6.3.1 and Theorem 2.6.6 it is enough to show that if $\mathfrak{n}$ is obtained from $\mathfrak{m}$ via one elementary operation, then $\mathrm{I}(\mathfrak{n}) \leq \mathrm{I}(\mathfrak{m})$. By the exactness of parabolic induction, it is thus enough that if $\Delta = [a,b]_\rho$ and $\Delta' = [a',b']_\rho$ satisfying (18), then

$$\mathrm{I}([a,b']_\rho + [a',b]_\rho) \leq \mathrm{I}(\Delta + \Delta').$$

If ρ is supercuspidal, we can use Theorem 2.8.1 to find $\tilde{\rho}$ such that $\mathrm{r}_\ell(\tilde{\rho}) = \rho$. By (17) we have that

$$\mathrm{I}([a,b']_{\tilde{\rho}} + [a',b]_{\tilde{\rho}}) \leq \mathrm{I}([a,b]_{\tilde{\rho}} + [a',b']_{\tilde{\rho}}).$$

Since parabolic reduction commutes with r_ℓ, we have thus that

$$\mathrm{I}([a,b']_\rho + [a',b]_\rho) = \mathrm{r}_\ell(\mathrm{I}([a,b']_{\tilde{\rho}} + [a',b]_{\tilde{\rho}})) \leq \mathrm{r}_\ell(\mathrm{I}([a,b]_{\tilde{\rho}} + [a',b']_{\tilde{\rho}})) = \mathrm{I}(\Delta + \Delta').$$

If ρ is not supercuspidal, $o(\rho) = 1$ and we argue as follows. Write $\mathfrak{n} = [a,b']_\rho + [a',b]_\rho$ and $\mathfrak{m} = \Delta + \Delta'$. Observe first if $\mathrm{Z}(\mathfrak{n})$ does not have its cuspidal support contained in $\mathbb{N}(\mathbb{Z}[\rho])$ then it follows by [31, Proposition 9.32] that $\mathfrak{n}$ consists of two segments of equal length and $\ell = 2$. For such an $\mathfrak{n}$, $[\mathrm{Z}(\mathfrak{n})] \leq \mathrm{I}(\mathfrak{n}) \leq \mathrm{I}(\mathfrak{m})$ implies therefore $\mathfrak{n} = \Delta + \Delta'$ by Theorem 2.6.6. It thus follows that it is enough to show that the multiplicity of any irreducible representation with cuspidal support in $\mathbb{N}(\mathbb{Z}[\rho])$ in $\mathrm{I}(\mathfrak{n})$ is lesser or equal than its multiplicity in $\mathrm{I}(\mathfrak{m})$. Let n be the length of $\mathfrak{m}$. We can therefore utilize the map $\xi_{\rho,n}$ between irreducible representations with cuspidal support $n \cdot [\rho]$ and irreducible modules of $\mathscr{H}(n,1)$. Note that this map can be upgraded in the following way, see [32, Proposition 4.19]. If $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a partition of n and $\pi = \pi_1 \otimes \ldots \otimes \pi_k$ is an irreducible representation of M_α with cuspidal support in $n \cdot [\rho]$, then there exists a bijection between between the irreducible constituents of

$$\mathrm{Hom}_{\mathscr{H}_{\mathbb{F}_\ell}(\alpha_1,1)\otimes\ldots\otimes\mathscr{H}_{\mathbb{F}_\ell}(\alpha_k,1)}(\mathscr{H}(n,1), \xi_{\rho,\alpha_1}(\pi_1) \otimes \ldots \otimes \xi_{\rho,\alpha_k}(\pi_k))$$

and the irreducible constituents of $\pi_1 \times \ldots \times \pi_k$ with cuspidal support $n \cdot [\rho]$. Moreover, the map respects multiplicities. Now for any irreducible representation with cuspidal support $n \cdot [\rho]$, the $\overline{\mathbb{F}}_\ell[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$-part of the Hecke-algebra acts trivially, see [24, Theorem 3.7]. Thus we can regard $\xi_{\rho,\alpha_i}(\pi_i)$ and $\xi_{\rho,n}(\pi)$ as $\overline{\mathbb{F}}_\ell[\mathrm{S}_n]$-modules and as well as the induced Hom-spaces. We are thus in the realm of the modular representation theory of S_n, which was already discussed in Section 2.7. The desired claim is then the last remark of this section.

Second direction: $\mathfrak{n} \leq \mathfrak{m} \Leftarrow [\mathrm{Z}(\mathfrak{n})] \leq \mathrm{I}(\mathfrak{m})$

We will proceed inductively and assume the statement to be true for all G_k with $k < \deg(\mathfrak{m})$. By Lemma 2.6.9

$$[r_{(\deg(\mathfrak{m}^-),\deg(\mathfrak{m}^1))}(\mathrm{I}(\mathfrak{m}))] \geq [r_{(\deg(\mathfrak{m}^-),\deg(\mathfrak{m}^1))}(\mathrm{Z}(\mathfrak{n}))] \geq [\mathrm{Z}(\mathfrak{n}^-) \otimes \mathrm{Z}(\mathfrak{n}^1)].$$

We apply the Geometric Lemma applied to $r_{(\deg(\mathfrak{m}^-),\deg(\mathfrak{m}^1))}(\mathrm{I}(\mathfrak{m}))$ and observe that Lemma 2.6.1 implies that there exist $\mathfrak{m}_1, \mathfrak{m}_2$ with $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$ such that

$$[\mathrm{Z}(\mathfrak{n}^-) \otimes \mathrm{Z}(\mathfrak{n}^1)] \leq \mathrm{I}(\mathfrak{m}_1^- + \mathfrak{m}_2) \otimes \mathrm{I}(\mathfrak{m}_1^1).$$

Thus $\mathfrak{m}_1^1 = \mathfrak{n}^1$ by Theorem 2.6.5 and by the induction hypothesis $\mathfrak{n}^- \leq \mathfrak{m}_1^- + \mathfrak{m}_2$. We apply Lemma 6.3.3 to the case $\mathfrak{n}_1 = \mathfrak{n}, \mathfrak{n}_2 = 0$ and obtain therefore the existence of $\mathfrak{n}' = \mathfrak{n}_1' + \mathfrak{n}_2'$ such that $\mathfrak{n}' \leq \mathfrak{m}$, ${\mathfrak{n}_1'}^1 = \mathfrak{n}^1$ and $\mathfrak{n}^- = {\mathfrak{n}_1'}^- + \mathfrak{n}_2'$. For any multisegment $\mathfrak{k}$ we will write $\mathfrak{k}[1]$ for the multisegment obtain from $\mathfrak{k}$ by replacing every segment Δ in it by ${}^-(\Delta^+)$ and $\mathfrak{k}^+$ for the multisegment obtain from $\mathfrak{k}$ by replacing every segment Δ in it by Δ^+. Note that $(\mathfrak{k}^+)^1 = \mathfrak{k}^1[1]$.

Because $\mathfrak{n}^- = {\mathfrak{n}_1'}^- + \mathfrak{n}_2'$, it follows that $\mathfrak{n}$ is of the form

$$\mathfrak{n} = (\mathfrak{n}_1'^-)^+ + (\mathfrak{n}_2')^+ + \mathfrak{k}',$$

where $\mathfrak{k}' \in \mathcal{MS}(\rho)$ is some multisegment consisting of segments of length 1. Thus $\mathfrak{n}^1$ is of the form

$$\mathfrak{n}^1 = \mathfrak{n}_1'^2[1] + \mathfrak{n}_2'^1[1] + \mathfrak{k}'.$$

But on the other hand $\mathfrak{n}^1 = \mathfrak{n}_1'^1$ and hence

$$\mathfrak{n}_1'^1 = \mathfrak{n}_1'^2[1] + \mathfrak{n}_2'^1[1] + \mathfrak{k}'.$$

Therefore

$$\mathfrak{n}_1' = (\mathfrak{n}_1'^-)^+ + \mathfrak{n}_2'^1[1] + \mathfrak{k}'.$$

Since $(\mathfrak{n}_2')^+ \leq \mathfrak{n}_2' + \mathfrak{n}_2'^1[1]$, it follows that

$$\mathfrak{n} = (\mathfrak{n}_1'^-)^+ + (\mathfrak{n}_2')^+ + \mathfrak{k}' \leq (\mathfrak{n}_1'^-)^+ + \mathfrak{n}_2' + \mathfrak{n}_2'^1[1] + \mathfrak{k}' = \mathfrak{n}'.$$

Thus $\mathfrak{n} \leq \mathfrak{n}' \leq \mathfrak{m}$. □

Corollary 6.4.2. *A multisegment* $\mathfrak{m}$ *in* $\mathcal{MS}(\rho)$ *is unlinked if and only if* $X_{\mathfrak{m}}$ *is an open orbit of* $N(V(\mathfrak{m}))$.

Proof. Since there are only finitely many multisegments $\mathfrak{n} \in \mathcal{MS}(\rho)$ with $\mathrm{cusp}_{\mathcal{MS}}(\mathfrak{n}) = \mathrm{cusp}_{\mathcal{MS}}(\mathfrak{m})$, we have

$$\overline{N(V)\backslash X_{\mathfrak{m}}} = \overline{\bigcup_{\mathfrak{n}\neq\mathfrak{m}} X_{\mathfrak{n}}} = \bigcup_{\mathfrak{n}\neq\mathfrak{m}} \overline{X_{\mathfrak{n}}},$$

$$\bigcup_{\mathfrak{n}\neq\mathfrak{m}} X_{\mathfrak{n}} = N(V)\backslash X_{\mathfrak{m}}.$$

Now $\bigcup_{\mathfrak{n}\neq\mathfrak{m}} \overline{X_{\mathfrak{n}}} = \bigcup_{\mathfrak{n}\neq\mathfrak{m}} X_{\mathfrak{n}}$ if and only if $X_{\mathfrak{m}}$ is not contained in $\overline{X_{\mathfrak{n}}}$ for any $\mathfrak{n} \neq \mathfrak{m}$. By Theorem 6.4.1 and Lemma 6.3.1 this is equivalent to $\mathfrak{m}$ being unlinked. ☐

Recall the following theorem of [27].

Theorem 6.4.3 ([27, Theorem 6.1]). *Let* ρ *be an unramified character of* G_1 *and* $\mathfrak{m}$ *an aperiodic multisegment in* $\mathcal{MS}(\rho)$. *Then* $\mathrm{Z}(\mathfrak{m})$ *is unramified if and only if* $\mathfrak{m}$ *is unlinked. Moreover, if* $\pi \in \mathfrak{Irr}(\Omega_{\rho',n})$ *is an unramified representation for some cuspidal* ρ', *then* ρ' *is an unramified character and there exists and an unlinked multisegment* $\mathfrak{m} \in \mathcal{MS}(\rho')$ *such that* $\mathrm{Z}(\mathfrak{m}) \cong \pi$.

Observe that an unlinked multisegment in $\mathcal{MS}(\rho)$ is automatically aperiodic. Therefore, if $\mathfrak{m} \in \mathcal{MS}(\rho)$ is unlinked, the cuspidal support of $\mathrm{Z}(\mathfrak{m})$ is contained in $\mathbb{N}(\mathbb{Z}[\rho])$ by [31, Proposition 9.34].

Corollary 6.4.4. *Let* $\pi \cong \mathrm{Z}(\mathfrak{m}) \in \mathfrak{Irr}(\Omega_{\rho,n})$ *be a representation. Then* π *is unramified if and only if the orbit* $X_{\mathfrak{m}}$ *is open and* ρ *is an unramified character.*

We will also use the opportunity to give a formula for the number of unramified representations with given cuspidal support. One has yet to find a use for it though.

Proposition 6.4.5. *Let* ρ *an unramified character,*

$$\mathfrak{s} = d_0[\rho] + \ldots + d_{o(\rho)-1}[\rho v^{o(\rho)-1}],\ d_i \in \mathbb{Z}_{\geq 0}$$

a fixed cuspidal support, $D := \min_i d_i$ *and* $m = \#\{i : d_i = D\}$. *The number of unramified irreducible representations* π *with cuspidal support* $\mathfrak{s}$, *or equivalently, the number of irreducible components of* $N(V(\mathfrak{s}))$, *is* m *if* $D > 0$ *and* 1 *if* $D = 0$.

Proof. If $k = 0$ the space $N(V) = E(V)$ has precisely one irreducible component and thus only one open orbit. Therefore there exists a unique unramified irreducible representations with cuspidal support $\mathfrak{s}$. We will now give a precise description of the corresponding multisegment $\mathfrak{m}_{\mathfrak{s}}$. Let $d_k = \max_i d_i$ and $j \leq k$ be such that

$$d_{j-1} < d_j = d_{j+1} = \ldots = d_k > d_{k+1}.$$

Set

$$\mathfrak{s}' := d_0[\rho] + \ldots + (d_j - 1)[\rho v^j] + \ldots + (d_k - 1)[\rho v^k] + \ldots + d_{o(\rho)-1}[\rho v^{o(\rho)-1}].$$

Then we can compute $\mathfrak{m}_\mathfrak{s}$ iteratively by noting that $\mathfrak{m}_\mathfrak{s} = \mathfrak{m}_{\mathfrak{s}'} + [j,k]$.

If $D > 0$ let $\mathfrak{m}_\mathfrak{s}$ be the set of irreducible unramified representations with cuspidal support $\mathfrak{s}$. Let

$$\mathfrak{s}^{+1} = (d_0 + 1) \cdot [\rho] + \ldots + (d_{o(\rho)-1} + 1) \cdot [\rho v^{o(\rho)-1}]$$

Then we construct an injective map $i\colon \mathfrak{m}_\mathfrak{s} \to \mathfrak{m}_{\mathfrak{s}^{+1}}$ by sending an unlinked segment $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ with $l(\Delta_1) \geq l(\Delta_i)$ for $i \in \{1, \ldots, k\}$ and $\Delta_1 = [a,b]$ to $[a, b + o(\rho)] + \Delta_2 + \ldots + \Delta_k$, which is unlinked since $\mathfrak{m}$ is.

There exists also a map $p\colon \mathfrak{m}_{\mathfrak{s}^{+1}} \to \mathfrak{m}_\mathfrak{s}$ constructed as follows. Note that for $\Delta_1 + \ldots + \Delta_k = \mathfrak{m} \in \mathfrak{m}_{\mathfrak{s}^{+1}}$ there exists a unique Δ_i with $l(\Delta_i) \geq o(\rho)$. Indeed, $\rho v^{o(\rho)-1}$ and ρ must be contained in the same segment, which is then of length greater then $o(\rho)$, since otherwise the two segments would be linked. Since any two segments of length greater than $o(\rho) - 1$ are linked, uniqueness follows. Assume without loss of generality that $l(\Delta_1) \geq l(\Delta_i)$ for $i \in \{1, \ldots, k\}$ and write $\Delta_1 = [a,b]$. The map then sends $\mathfrak{m}$ to $[a, b - o(\rho)] + \Delta_2 + \ldots + \Delta_k$, which is again unlinked. Because $D > 0$ it follows that p is injective.

Since both i and p are injective if $D > 0$ and $p \circ i = \mathrm{id}$, $\#\mathfrak{m}_\mathfrak{s} = \#\mathfrak{m}_{\mathfrak{s}^{+1}}$ if $D > 0$. Thus it suffices to prove the claim for $D = 1$. Write

$$\mathfrak{s} = \mathfrak{s}' + [\rho] + \ldots + [\rho v^{o(\rho)-1}]$$

We know from the case $D = 0$ that we can write $\mathfrak{m}_{\mathfrak{s}'} = \mathfrak{m}_1 + \ldots + \mathfrak{m}_k$, where $\mathfrak{m}_i = \Delta_i^1 + \ldots + \Delta_i^{l_i}$, $\Delta_i^j = [a_i^j, b_i^j]$ with $a_i^j \leq a_i^{j'} \leq b_i^{j'} \leq b_i^j$ for $j \leq j'$, $\Delta_i^j = \Delta_i^{j'}$ if and only if $j = j'$ and $b_i^1 + 1 < a_{i'}^1$ if $i < i'$.

We now construct m many unlinked segments with support $\mathfrak{s}$ as follows. For $c \in \{0, \ldots, o(\rho) - 1\}$ such that either $c \leq a_i^1 - 2$ or $c \geq b_i^1 + 2$ for all i it is easy to check that $\mathfrak{m}_\mathfrak{s} + [c, c + o(\rho) - 1]$ is unlinked. If $b_i + 1 = c$ or $c = a_i^1 - 1$ then the multisegment

$$\mathfrak{m}_1 + .. + \mathfrak{m}_{i-1} + [a_i^1, b_i^1 + o(\rho)] + [a_i^2, b_i^2] + \ldots + [a_i^{l_i}, b_i^{l_i}] + \mathfrak{m}_{i+1} + \ldots + \mathfrak{m}_k$$

is unlinked. Thus $\#\mathfrak{m}_\mathfrak{s} \geq m$. On the other hand if $\mathfrak{m} \in \mathfrak{m}_\mathfrak{s}$ then $p(\mathfrak{m}) = \mathfrak{m}_{\mathfrak{s}'}$ which shows that $\mathfrak{m}$ has to be of above form and hence showing $\#\mathfrak{m}_\mathfrak{s} \leq m$. □

6.5 □-irreducible representations revisited We will use the opportunity to extend [20, Conjecture 4.2] to representations with □-irreducible cuspidal support and give a conjectural classification of square-irreducible representations. Using Lemma 2.4.1 we can quickly reduce the classification to a classification for representations $\mathrm{Z}(\mathfrak{m})$ for $\mathfrak{m} \in \mathcal{MS}(\rho)_{ap}$ for some cuspidal representation ρ. By

Lemma 3.2.5 $o(\rho) > 1$ if $\mathrm{Z}(\mathfrak{m})$ has to have any chance of being square-irreducible. We fix for the rest of the section an aperiodic multisegment $\mathfrak{m} \in \mathcal{MS}(\rho)_{ap}$ for some $\rho \in \mathfrak{C}^{\square}$. We set $Q^+ := A^+_{o(\rho)-1}$. Moreover, we recall the quiver representation $(V(\mathfrak{m}), \lambda(\mathfrak{m}))$ over $\mathbb{C}$ of Q^+ as in Section 6.1. Next, we denote by Q^- the quiver obtained from Q by inverting the arrows. For a fixed graded vector space $V = \oplus_{i=1}^{o(\rho)-1} V_i$ over $\mathbb{C}$ as in Section 6.1 we write $E^+(V)$ respectively $E^-(V)$ for the representations of Q^+ respectively Q^- with underlying vector space V and by $N^+(V)$ and $N^-(V)$ the subvariety of nilpotent representations. The third quiver of this story is denoted by $\overline{Q}$ and of the form

$$\begin{tikzcd}
 & \dots \arrow[dr, bend left, leftrightarrow] & \\
0 \arrow[ur, bend left, leftrightarrow] & & o(\rho)-2 \\
 & o(\rho)-1 \arrow[ul, bend left, leftrightarrow] \arrow[ur, bend right, leftrightarrow] &
\end{tikzcd} .$$

We let $\overline{E}(V)$ be the affine variety of representations of Q with underlying vector space V. Again $\mathfrak{Rep}(V)$ admits an action of

$$\mathrm{GL}(V) = \mathrm{GL}(V_0) \times \ldots \times \mathrm{GL}(V_{o(\rho)-1}).$$

Moreover, there exist two G_V-equivariant maps

$$\begin{tikzcd} E^+(V) & \overline{E}(V) \arrow[l, two heads, "p_+"'] \arrow[r, two heads, "p_-"] & E^-(V). \end{tikzcd}$$

The preimage of $N^+(V) \times N^-(V)$ is denoted by $\overline{N}(V)$. Finally, we recall the moment map

$$[\cdot,\cdot] \colon \overline{N}(V) \to \mathfrak{gl}(V),$$

$$(X_{i\mapsto i+1}, Y_{i+1\mapsto i})_{i\in\{0,\ldots,o(\rho)-1\}} \mapsto (X_{i\mapsto i+1}Y_{i+1\mapsto i} - Y_{i\mapsto i-1}X_{i-1\mapsto i})_{i\in\{0,\ldots,o(\rho)-1\}}.$$

Let $\Lambda(V)$ be the kernel of $[\cdot,\cdot]$.

Recall now above fixed multisegment $\mathfrak{m}$ and $V := V(\mathfrak{m})$, $\lambda(\mathfrak{m})$ and $X_{\mathfrak{m}}$ from Section 6.1. To avoid confusion we denote the map $\lambda(\mathfrak{m})^+$ in $N^+(V(\mathfrak{m}))$ and $\lambda(\mathfrak{m})^-$ in $N^-(V(\mathfrak{m}))$ and similarly for $X^+_{\mathfrak{m}}$ and $X^-_{\mathfrak{m}}$. We now let $\mathcal{MS}(V)_{ap}$ be the aperiodic segments $\mathfrak{n} \in \mathcal{MS}(\rho)_{ap}$ with the same cuspidal support as $\mathfrak{m}$. For $\mathfrak{n} \in \mathcal{MS}(V)_{ap}$ denote the closure of $p_+^{-1}(X^+_{\mathfrak{n}}) \cap \Lambda(V)$ in $\Lambda(V)$ by $\mathcal{C}^+_{\mathfrak{n}}$ and the closure of $p_-^{-1}(X^-_{\mathfrak{n}}) \cap \Lambda(V)$ in $\Lambda(V)$ by $\mathcal{C}^-_{\mathfrak{n}}$.

Lemma 6.5.1 ([23, Proposition 15.5]). *The maps* $\mathfrak{n} \mapsto \mathcal{C}^+_{\mathfrak{n}}$ *respectively* $\mathfrak{n} \mapsto \mathcal{C}^-_{\mathfrak{n}}$ *induce a bijection between* $\mathcal{MS}(V)_{ap}$ *and the irreducible constituents of* $\Lambda(V)$.

Moreover, the interaction of $\mathcal{C}^+$ and $\mathcal{C}^-$ is governed by the Aubert-Zelevinsky duality.

Lemma 6.5.2 ([22, Propostion 5.2]). *For* $\mathfrak{n} \in \mathcal{MS}(V)_{ap}$

$$\mathcal{C}^+_{\mathfrak{n}} = \mathcal{C}^-_{\mathfrak{n}^*}.$$

We thus can generalize [20, Conjecture 4.2] as follows.

Conjecture 6.5.1. *Let* $\mathfrak{m} \in \mathcal{MS}(\rho)_{ap}$ *for some* $\rho \in \mathfrak{C}^\square$. *Then* $\mathrm{Z}(\mathfrak{m})$ *is square-irreducible if and only if* $\mathcal{C}^+_{\mathfrak{m}}$ *contains an open* $\mathrm{GL}(V)$*-orbit.*

At this point some remarks are in order.

Remark. (1). If $\mathfrak{m}$ is banale, *i.e.* the cuspidal support of $\mathfrak{m}$ does not contain $[\rho] + \ldots + [\rho v_\rho^{o(\rho)-1}]$, then above conjecture reduces to [20, Conjecture 4.2].

(2). The conjecture is true for segments. First of all note that $\mathrm{Z}(\Delta)$, $\Delta = [a,b]_\rho$, is $\square$-irreducible if and only if $b-a+1 < o(\rho)$. Thus if $b-a+1 < o(\rho)$, we know from the banale case that $\mathcal{C}^+_\Delta$ admits an open orbit. On the other hand, if $b-a+1 \geq o(\rho)$, we can reduce the claim by [1, Corollay 8.7] to the case $b-a+1 \geq o(\rho)$. Indeed, if we assume $b-a+1 > o(\rho)$, let in the Corollary $C_1 = \mathcal{C}^+_{\Delta^-}$ and $C_2 = \mathcal{C}^+_{[b,b]_\rho}$. In the language of the above Corollary, it is then not hard to see that $C_1 * C_2 = \mathcal{C}^+_\Delta$. Thus we can assume without loss of generality that $a = 0$, $b = o(\rho)-1$. Then the claim that $\mathcal{C}^+_\Delta$ contains not an open orbit is equivalent to the claim that

$$T^*_{\lambda(\Delta)} X_\Delta = \{y \in N^-(V(\Delta)) : [\lambda(\Delta), y] = 0\}$$

contains an open orbit of the centralizer of $\lambda(\Delta)$. But in this case the centralizer is equal to $\{(a,\ldots,a) \in (C^*)^{o(\rho)}\}$, which acts trivially on $T^*_{\lambda(\Delta)} X_\Delta$. However, $T^*_{\lambda(\Delta)} X_\Delta$ is one-dimensional and hence $\mathcal{C}^+_\Delta$ does not contain an open orbit.

Example. Let us give the following family of examples of non-banale $\square$-irreducible representations. Let $\rho \in \mathfrak{C}^\square_m$ and consider $\mathfrak{m} = [0, o(\rho)-1]_\rho + [0,0]_\rho$. We claim that $\mathrm{Z}(\mathfrak{m})$ is $\square$-irreducible and $\mathfrak{C}^+_{\mathfrak{m}}$ admits an open orbit.

Let us first check that $\mathfrak{C}^+_{\mathfrak{m}}$ admits an open orbit. Note that

$$\mathfrak{m}^* = [0,0]_\rho + \ldots + [o(\rho)-2, o(\rho)-2]_\rho + [o(\rho)-1, o(\rho)]_\rho$$

We set

$$x := ((1,0), 1, \cdots, 1) \in X^+_{\mathfrak{m}},\ x^* := ((0,1)^T, 0, \ldots, 0) \in X^-_{\mathfrak{m}^*}.$$

Here the i-th coordinate represents the map from $V_{i-1}(\mathfrak{m}) \to V_i(\mathfrak{m})$ respectively $V_i(\mathfrak{m}) \to V_{i-1}(\mathfrak{m})$. It suffices now to show that the stabilizer of x admits an open orbit in

$$T^*_x X_{\mathfrak{m}} = \{y \in N^-(V(\mathfrak{m})) : [x,y] = 0\}.$$

To see this, we observe that the stabilizer of x equals to

$$G_x := \{(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, a \ldots, a) \in \mathrm{GL}_2(\mathbb{C}) \times (\mathbb{C}^*)^{o(\rho)}\}.$$

Moreover, the set $\{(x,y)^T, 0, \dots, 0\} : y \neq 0\}$ is an open subset of $T_x^* X_{\mathfrak{m}}$, which is equal to $G_x \cdot x^*$, finishing the argument.

We continue by showing that $\mathrm{Z}(\mathfrak{m})$ is □-irreducible. To do so, we prove the following three lemmata.

Lemma 6.5.3. *Let* $\mathrm{Z}(\mathfrak{n}) \in \mathfrak{Irr}_n$*, such that all cuspidal representations appearing in the cuspidal support of* $\mathfrak{n}$ *are representations of* G_m*. Then* $\mathrm{Z}(\mathfrak{n})$ *is □-irreducible if and only if*

$$[r_{(n-m,m)}(\mathrm{Z}(\mathfrak{n}+\mathfrak{n}))] = [r_{(n-m,m)}(\mathrm{Z}(\mathfrak{n}) \times \mathrm{Z}(\mathfrak{n}))]$$

Proof. One direction is trivially true by Proposition 2.6.7. For the other one, assume that the above equality holds and $\mathrm{Z}(\mathfrak{n})$ is not □-irreducible. Then there exits either an irreducible quotient or subrepresentation π of $\mathrm{Z}(\mathfrak{n}) \times \mathrm{Z}(\mathfrak{n})$ different from $\mathrm{Z}(\mathfrak{n}+\mathfrak{n})$. Since π has the same cuspidal support as $\mathrm{Z}(\mathfrak{n}+\mathfrak{n})$, the claim follows from the exactness of parabolic reduction. □

Lemma 6.5.4. *Let* $a \in \{0, \dots, o(\rho) - 2\}$*. Then*

$$\mathrm{Z}([0,a]_\rho) \times \mathrm{Z}([0, o(\rho)-1]_\rho + [0,0]_\rho)$$

is irreducible.

Proof. We argue by induction on a and use Lemma 6.5.3. Theorem 4.2.8 and Lemma 4.1.3 allow us to compute

$$\begin{aligned}[]
[r_{(o(\rho)m+(a+1)m,m)}&(\mathrm{Z}([0,a]_\rho + [0,o(\rho)-1]_\rho + [0,0]_\rho))] = \\
&= [\mathrm{Z}([0,a-1]_\rho + [0,o(\rho)-1]_\rho + [0,0]_\rho) \otimes \rho v_\rho^a] + \\
&+ [\mathrm{Z}([0,a]_\rho + [0,o(\rho)-2]_\rho + [0,0]_\rho) \otimes \rho v_\rho^{-1}].
\end{aligned}$$

On the other hand, the Geometric Lemma gives that

$$\begin{aligned}[]
[r_{(o(\rho)m+(a+1)m,m)}&(\mathrm{Z}([0,a]_\rho) \times \mathrm{Z}([0,o(\rho)-1]_\rho + [0,0]_\rho))] = \\
&= [\mathrm{Z}([0,a-1]_\rho) \times \mathrm{Z}([0,o(\rho)-1]_\rho + [0,0]_\rho) \otimes \rho v_\rho^a] + \\
&+ [\mathrm{Z}([0,a]_\rho) \times \mathrm{Z}([0,o(\rho)-2]_\rho + [0,0]_\rho) \otimes \rho v_\rho^{-1}].
\end{aligned}$$

If $a = 0$, Theorem 2.6.2 implies

$$\rho \times \mathrm{Z}([0,o(\rho)-2]_\rho + [0,0]_\rho) \cong \rho^2 \times \mathrm{Z}([0,o(\rho)-2]_\rho) \cong \mathrm{Z}([0,0]_\rho + [0,o(\rho)-2]_\rho + [0,0]_\rho)$$

and hence the parabolic reductions of the two representations agree. If $a > 0$, the induction hypothesis implies that the parabolic restrictions of the two representations agree, hence the claim follows. □

Lemma 6.5.5. *Let $b \in \{0,\dots,o(\rho)-2\}$. Then*

$$\rho \times \mathrm{Z}([0,b]_\rho) \times \mathrm{Z}([0,o(\rho)-1]_\rho + [0,0]_\rho)$$

is irreducible.

Proof. The proof proceeds analogously to the one of Lemma 6.5.4 by showing inductively on b that

$$[r_{(o(\rho)m+(b+2)m,m)}(\rho \times \mathrm{Z}([0,b]_\rho) \times \mathrm{Z}([0,o(\rho)-1]_\rho + [0,0]_\rho))] =$$

$$= [r_{(o(\rho)m+(b+2)m,m)}(\mathrm{Z}([0,0]_\rho + [0,b]_\rho + [0,o(\rho)-1]_\rho + [0,0]_\rho))].$$

Indeed,

$$\begin{aligned}
&[r_{(o(\rho)m+(b+2)m,m)}(\rho \times \mathrm{Z}([0,b]_\rho) \times \mathrm{Z}([0,o(\rho)-1]_\rho + [0,0]_\rho))] = \\
&= [\mathrm{Z}([0,b]_\rho) \times \mathrm{Z}([0,o(\rho)-1]_\rho + [0,0]_\rho)) \otimes \rho] + \\
&+ [\rho \times \mathrm{Z}([0,b-1]_\rho) \times \mathrm{Z}([0,o(\rho)-1]_\rho + [0,0]_\rho) \otimes \rho v^{b}] + \\
&+ [\rho \times \mathrm{Z}([0,b]_\rho) \times \mathrm{Z}([0,o(\rho)-2]_\rho + [0,0]_\rho) \otimes \rho v_\rho^{-1}]
\end{aligned}$$

and

$$\begin{aligned}
&[r_{(o(\rho)m+(b+2)m,m)}(\mathrm{Z}([0,0]_\rho + [0,b]_\rho + [0,o(\rho)-1]_\rho + [0,0]_\rho))] = \\
&= [\mathrm{Z}([0,b]_\rho + [0,o(\rho)-1]_\rho + [0,0]_\rho)) \otimes \rho] + \\
&+ [\mathrm{Z}([0,b-1]_\rho + [0,o(\rho)-1]_\rho + [0,0]_\rho + [0,0]_\rho) \otimes \rho v^{b}] + \\
&+ [\mathrm{Z}([0,b]_\rho + [0,0]_\rho + [0,o(\rho)-2]_\rho + [0,0]_\rho) \otimes \rho v_\rho^{-1}]
\end{aligned}$$

If $b = 0$, we use Lemma 6.5.4 for $a = 0$ and Theorem 2.6.2 to show that the two representations agree. For $b > 0$, we use the induction hypothesis and Lemma 6.5.4. □

We now come finally to the □-irreducibility of $\mathrm{Z}(\mathfrak{m})$. Note that by Theorem 4.2.8 and Lemma 4.1.3

$$r_{(o(\rho)m,m)}(\mathrm{Z}(\mathfrak{m})) = \mathrm{Z}([0,o(\rho)-2] + [0,0]_\rho) \otimes \rho v_\rho^{-1}.$$

Thus the Geometric Lemma and Lemma 6.5.5 imply that

$$[r_{(o(\rho)m,m)}(\mathrm{Z}(\mathfrak{m}) \times \mathrm{Z}(\mathfrak{m}))] = 2[\mathrm{Z}([0,o(\rho)-2] + [0,0]_\rho + \mathfrak{m}) \otimes \rho v_\rho^{-1}].$$

If $\mathrm{Z}(\mathfrak{m}) \times \mathrm{Z}(\mathfrak{m})$ is not irreducible, it has an irreducible subquotient π different from $\mathrm{Z}(\mathfrak{m}+\mathfrak{m})$ with the same cuspidal support. But by above computation and Frobenius reciprocity, we conclude that $\mathcal{D}_{r,\rho v_\rho^{-1}}(\pi) \cong \mathcal{D}_{r,\rho v_\rho^{-1}}(\mathrm{Z}(\mathfrak{m}+\mathfrak{m})) \neq 0$ and hence $\pi \cong \mathrm{Z}(\mathfrak{m}+\mathfrak{m})$. Since $\mathrm{Z}(\mathfrak{m}+\mathfrak{m})$ appears only with multiplicity 1 in $\mathrm{Z}(\mathfrak{m})\times\mathrm{Z}(\mathfrak{m})$, the claim follows.

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