D-branes in non-abelian gauged linear sigma models

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Outline

CYs and GLSMs

D-branes in GLSMs

Conclusions
A Calabi-Yau space (CY) can be realized as the low energy configuration of a supersymmetric $\mathcal{N} = (2, 2)$ gauge theory in two dimensions - the gauged linear sigma model (GLSM). [Witten 93]

- The choice of gauge groups and matter content determines the CY.
- Hypersurfaces and complete intersections in toric ambient spaces are realized by GLSMs with gauge groups $G = U(1)^k$.
- Non-abelian gauge groups lead to exotic CYs (e.g. Pfaffian CYs).
- Phase: by tuning coupling constants (FI parameters) in the gauge theory, we can probe the (enlarged) Kähler moduli space of a CY.
GLSM Data

• Gauge group \( G = \frac{U(1)^l \times H}{\{\pm 1, \pm 1\}} \).
  
  • This talk: \( H = USp(k) \) or \( H = O(k) \), \( l = 1 \).

• Matter
  
  • Chiral fields
    • \( p^i, i = 1, \ldots, M, U(1)^l \)-charges \( Q^l_{p^i} \), trivial rep. of \( H \)
    • \( x^a_i, i = 1, \ldots, N, a = 1, \ldots, k, U(1)^l \)-charges \( Q^l_{x^a_i} \), fundamental rep. of \( H \)

  • Twisted chiral fields
    • \( \sigma_k \), take values in the maximal torus of \( G \).

• Superpotential \( W \)

\[
H = SU(2) : \quad W = \sum_{i,j=1}^{N} A^{ij}(p)[x_i x_j] = \sum_{i,j=1}^{N} \sum_{a,b=1}^{2} A^{ij}(p)x_i^ax_b\epsilon_{ab}x_j^b
\]

\[
H = O(2) : \quad W = \sum_{i,j=1}^{N} S^{ij}(p)(x_i x_j) = \sum_{i,j=1}^{N} \sum_{a,b=1}^{2} S^{ij}(p)x_i^a\delta_{ab}x_j^b
\]
Equations of motion I

- The vacua, i.e. solutions to the equations of motion, determine the CY.
- On the Higgs branch $\sigma = 0$.
- D-terms

\[
U(1)^I : \sum_{i=1}^{M} Q^I_{p^i}|p^i|^2 + \sum_{j=1}^{N} Q^I_{x_j}||x_j||^2 = r_I
\]

\[
SU(2) : \quad xx^\dagger - \frac{1}{2}||x||^2 1_2 = 0
\]

\[
O(2) : \quad xx^\dagger - (xx^\dagger)^T = 0.
\]

- $r_I \in \mathbb{R}$ are the FI parameters. Together with the $\theta$-angle they can be identified with the complexified Kähler modulus of the CY: $t = r - i\theta$.
- By changing the value of the $r_I$ we can probe the Kähler moduli space of the CY.
Equations of motion II

- E.o.ms continued...
  - F-terms

\[
\frac{\partial W}{\partial p^i} = 0 \quad \frac{\partial W}{\partial x^a_i} = 0.
\]

- Phases of the GLSM
  - Different FI parameter regions have different solutions of the e.o.ms.
  - Gauge group gets broken to a subgroup in different phases.
    ⇒ strong coupling effects if broken to continuous subgroup
  - CYs in phases are not necessarily birational, yet share the same Kähler moduli space.
  - Conjecture: The derived categories associated to the CYs in the various phases are equivalent.
CY data

- CY condition

\[
\sum_{i=1}^{M} Q^i_{p} + k \sum_{j=1}^{N} Q^j_{x} = 0 \quad \text{rk} H = k
\]

- Dimension:

\[
\text{dim} = M - 1 - \frac{k(k \pm 1)}{2} + \ldots \text{USp}(k), \ldots \text{O}(k)
\]

- Hodge number estimate
  - \(h^{1,1} \Leftrightarrow \) number of FI-parameters
  - \(h^{2,1} \Leftrightarrow \) number of monomials in \(W\) modulo reparametrization

- Quantum corrections on the Coulomb branch determine the conifold point(s).
Non-abelian Duality

• There is a duality between models with different non-abelian gauge groups. [Hori 11]

<table>
<thead>
<tr>
<th></th>
<th>original</th>
<th>dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td>$G = U(1)^l \times USp(k)$</td>
<td>$\tilde{G} = U(1)^l \times USp(N - k - 1)$</td>
</tr>
<tr>
<td>fields</td>
<td>$p^i \ p^j \ x_i \ x_j$</td>
<td>$\tilde{p}^i \ \tilde{p}^j \ a_{ij} = -a_{ji}$</td>
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<tr>
<td></td>
<td>$Q_{p^i} \ Q_{p^j} \ Q_{x_i} \ Q_{x_j}$</td>
<td>$Q_{\tilde{p}^i} \ -Q_{\tilde{p}^j} \ Q_{x_i} + Q_{x_j}$</td>
</tr>
<tr>
<td>$W$</td>
<td>$W = \sum_{i,j=1}^{N} A^{ij}(p)[x_i x_j]$</td>
<td>$\tilde{W} = \sum_{i,j=1}^{N} [\tilde{x}^i \tilde{x}^j]a_{ij} + A^{ij}(p)a_{ij}$</td>
</tr>
</tbody>
</table>

• Similar structure for orthogonal gauge groups with

\[
O(k) \leftrightarrow SO(N - k + 1)
\]

\[
s_{ij} = s_{ji}
\]

• Strong/weak coupling duality: Certain properties of the CY are more easily derived in the dual theory.
Non-abelian examples

- **Notation**: Models $A^k_q$ and $S^{k,q}$
  - $A^k_q$: $H = USp(k)$, with charges $q$
  - $S^{k,q}$: $H = SO(k)$ ($\bullet = 0$) or $H = O(k) \simeq SO(k) \times \mathbb{Z}_2$ ($\bullet = \pm$) with charges $q$

- **Rodland Model** $A^2_{-2^7,1^7}$
  - $r \gg 0$: Complete intersection of codim. 7 in $G(2, 7)$
  - $r \ll 0$: Pfaffian CY $rkA(p) = 4$ (strongly coupled)

- **Hosono-Takagi** $S^{2^2,1^{2^5},1^{1^5}}$[Hosono-Takagi][Hori '11]
  - $r \gg 0$: Complete intersection of codim. 5 in $(\mathbb{P}^4 \times \mathbb{P}^4)/\mathbb{Z}_2$
  - $r \ll 0$: Determinantal variety $rkS(p) \leq 4$ (strongly coupled)

- **Further interesting new one-parameter examples**.[Hori-JK, '13]
  - CYs with different Hodge numbers $h^{2,1}$ but same Kähler moduli space.
Matrix Factorizations

- D-branes are gauge-invariant, R-invariant matrix factorizations of the GLSM potential.
  - Take a square matrix $Q$ with polynomial entries such that
    \[ Q^2 = W1 \]
  - Gauge invariance:
    \[ \rho(g)^{-1}Q(g\phi)\rho(g) = Q(\phi) \]
  - R-invariance:
    \[ \lambda^{r^*}Q(\lambda^R\phi)\lambda^{-r^*} = \lambda Q(\phi) \]
- GLSM branes map to (trivial or non-trivial) branes in the individual phases.
Elementary matrix factorizations I

- To construct matrix factorizations for the Rodland and HT-examples we start with a **toy model** with gauge group $U(2)$.
  - **Superpotential**
    \[ W = p[x_1 x_2] = p(x_1^1 x_2^2 - x_1^2 x_2^1). \]
  - **Gauge transformations** with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2)$:
    \[
    \begin{align*}
    p & \to (ad - bc)^{-1} p \\
    x_i^1 & \to ax_i^1 + bx_i^2 \\
    x_i^2 & \to cx_i^1 + dx_i^2.
    \end{align*}
    \]
  - **R-charge**: can choose $[p] = 2$, $[x_i^a] = 0$. 
Elementary matrix factorizations II

- Simplest one:

\[
Q = \begin{pmatrix}
0 & \rho \\
-x_1 x_2^2 - x_1^2 x_1^1 & 0
\end{pmatrix}
\quad \quad \rho = \begin{pmatrix}
1 & 0 \\
0 & \det g
\end{pmatrix}
\]

- Using a basis of 4 × 4 Clifford matrices \( \{ \eta_i, \bar{\eta}_j \} = \delta_{ij} \):

\[
Q = x_1^1 \eta_1 + px_2^2 \bar{\eta}_1 + x_1^2 \eta_2 - px_2^1 \bar{\eta}_2
\]

\[
Q = \begin{pmatrix}
0 & 0 & x_1^1 & x_1^2 \\
0 & 0 & -px_2^1 & -px_2^2 \\
px_2^2 & x_1^2 & 0 & 0 \\
-px_2^1 & -x_1^1 & 0 & 0
\end{pmatrix}
\quad \rho = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1/\det g & 0 & 0 \\
0 & 0 & 1/\det g & d/\det g \\
0 & 0 & -1/\det g & c/\det g
\end{pmatrix}
\]

- For the models of interest we can obtain matrix factorizations by taking tensor products of these two elementary constructions.
D-brane charges

- Quantum corrected D-brane central charges are computed by the hemisphere partition function. [Honda-Okuda,Hori-Romo '13]

\[ Z_{D^2}(B) = C \int d l^G \tau \prod_{\alpha > 0} \alpha(\tau) \sinh(i\pi \alpha(\tau)) \prod_i \Gamma \left( Q_i(\tau) + \frac{R_i}{2} \right) e^{t_R(\tau)} f_B(\tau) \]

- $\alpha > 0$ positive roots
- $\tau = \in \mathfrak{t}_C$ with $\mathfrak{t}_C = \text{Lie}(T_G) \otimes \mathbb{C}$ (maximal torus)
- $l_G = \text{rk}(G)$
- $R_i \ldots R$-charges, $Q_i \ldots$ weights under the maximal torus
- $t_R = r - i\theta \ldots$ complexified FI (Kähler) parameter
- $\gamma \ldots$ integration contour (s.t. integral is convergent)

**Brane factor**

\[ f_B(\tau) = \text{tr}_M \left( (-1)^{r^*} e^{-2\pi i \rho_*(\tau)} \right) \]

- $M \ldots$ Chan-Paton space
- The brane input is obtained by restricting the matrices $\rho(g)$ and $\lambda^{r*}$ to the maximal torus.
Structure Sheaf

- In geometric phases $Z_{D^2}$ reduces to

\[ Z_{D^2}(\mathcal{B}) = \int_X \hat{\Gamma}(X) e^{B + \frac{i}{2\pi} \omega \text{ch}(\mathcal{B})}. \]

- The structure sheaf encodes the topological data of the CY.

\[
Z(\mathcal{O}_X) = \frac{H^3}{3!} \left( \frac{it}{2\pi} \right)^3 + \left( \frac{it}{2\pi} \right) \frac{c_2 H}{24} + i \frac{\zeta(3)}{(2\pi)^3} \chi(X) + O(e^{-t})
\]

- Different topological data for different (geometric) phases in the GLSM.
- Identify the brane factor/matrix factorization associated to the structure sheaf.
Structure Sheaf II

- For Rodland and HT matrix factorizations discussed above lead to a brane factors of type

\[ f_B(a, b) = (1 - e^{-2\pi i(\tau_1 + \tau_2)})^a (1 - e^{-2\pi i\tau_1})^b (1 - e^{-2\pi i\tau_2})^b \quad a, b \in \mathbb{Z}_{\geq 0} \]

- In \( r \gg 0 \)-phases the structure sheaf is given by the matrix factorization with brane factor \( f(N, 0) \)

\[ Q(N, 0) = \sum_{i=1}^{N} p^i \eta_i + \frac{\partial W}{\partial p^i} \bar{\eta}_i \]

- In the strongly-coupled \( r \ll 0 \)-phases no simple matrix factorization has been found. However,
  - There are brane factors that give the right charges, e.g. for Rodland:

\[ f_B = 2f(5, 1) - 2f(3, 2) + f(6, 1) - 2f(4, 2) - f(5, 2) \]

- In the weakly-coupled dual theory the structure sheaf is described by a simple matrix factorization.
Conclusions

• We constructed matrix factorizations in examples of one-parameter non-abelian GLSMs.
• Using the hemisphere partition function we computed quantum-corrected D-brane charges.
  • We identified the structure sheaf.
  • We found many examples of lower-dimensional branes in all phases.
• These methods are very general.
  • Not restricted to hypersurfaces/complete intersections in toric spaces.
  • No mirror symmetry required.
  • No Landau-Ginzburg point required for matrix factorizations.
Outlook

- D-brane transport
  - By choosing integration contours such that $Z_{D^2}$ converges one can derive a grade restriction rule. \[ \text{[HHP '08]} \]
  - This reproduces mathematical results. \[ \text{[Addington-Donovan-Segal '14]} \]
  - For strongly coupled phases, branes have to be grade restricted deep inside the phase.

- How do construct matrix factorizations for the structure sheaf in strongly coupled phases?
- How to classify matrix factorizations in (non-)abelian GLSMs?
- Interesting phenomenology on these CYs?