Some remarks on the paper „Hankel determinants for some common lattice paths“ by R.A. Sulanke and G. Xin.

by
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In [1] and in my unpublished survey paper [2] I have calculated Hankel determinants for the coefficients of two classes of generating functions which are associated with certain $q-$exponential functions. It turns out that the methods used in [3] give simpler proofs of these facts. Since the proofs are simple extensions of the proofs in [3] I shall only sketch the main ideas.

1) Let $F(z,x,y,q)=\sum_{n\geq 0} w(n,x,y,q)z^n$ satisfy

$$F(z,x,y,q)=1+(x+y)zF(z,x,y,q)+qxyz^2F(z,x,y,q)F(qz,x,y,q). \tag{1}$$

Then

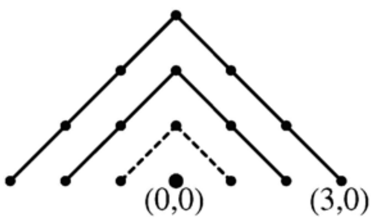
$$\det(w(i+j,x,y,q))_{i,j=0}^{n-1}=(xy)^{\binom{n}{2}}q^{\sum_{k=0}^{n-1}k^2}=(xy)^{\binom{n}{2}}q^{\frac{n(n-1)(2n-1)}{6}} \tag{2}$$

and

$$\det(w(i+j+1,x,y,q))_{i,j=0}^{n-1}=(qxy)^{\binom{n}{2}}\frac{x^{n+1}-y^{n+1}}{x-y}q^{\frac{n(n-1)(2n-1)}{6}}. \tag{3}$$

Here $w(n,x,y,q)$ is the weight of the nonnegative paths form $(0,0)$ to $(n,0)$ with steps U,D and $H=(1,0),$ where the weight is defined by $\omega(U)=1$, $\omega(H)=q^i(x+y)$ if H is on the line $y=i$ and $\omega(D)=q^{2i+1}xy$ if the endpoint of D has height i. The weight of a path is the product of the weights of all its steps.

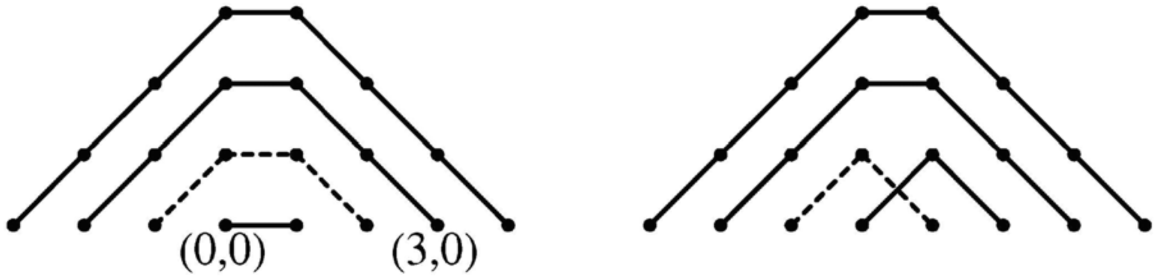
The Gessel-Viennot-Lindström method immediately gives the first Hankel determinant. I use the terminology of [3]. The only difference is that I use other weights. In [3], Proposition 1.1, is shown that each nonintersecting path is a sequence of $U-$steps followed by a sequence of $D-$steps as in the following figure.



These sequences have the weights $(xy)^i q^{i^2}$ for $i=0,\cdots,n-1.$ This implies (2).

If we set $c(n,x,y)=(xy)^{\binom{n}{2}}q^{\frac{n(n-1)(2n-1)}{6}}$, from the figure we also conclude that $c(n,x,y)=(qxy)^{n-1}c(n-1,qx,qy).$

The second Hankel determinant $d(n,x,y)=\det(w(i+j+1,x,y,q))_{i,j=0}^{n-1}$ follows from [3],Proposition 2.2. As shown there the nonintersecting $n-$tuples belong to one of two types as in the following figure.



This gives
$d(n,x,y)=(qxy)^{n-1}(x+y)d(n-1,qx,qy)-(qxy)^2(qxyq^3xy)^{n-2}d(n-2,q^2x,q^2y)$
with
$d(1,x,y)=x+y$ and $d(2,x,y)=q^2xy(x^2+xy+y^2).$
Now it is easily verified that
$$d(n,x,y)=(qxy)^{\binom{n}{2}}\frac{x^{n+1}-y^{n+1}}{x-y}q^{\frac{n(n-1)(2n-1)}{6}}=q^{\binom{n}{2}}\frac{x^{n+1}-y^{n+1}}{x-y}\det(H_n),$$
since this sequence satisfies the same recurrence and initial values.

For $a(n,x,y)=\dfrac{x^{n+1}-y^{n+1}}{x-y}$
satisfies
$a(n,x,y)=(x+y)a(n-1,x,y)-xya(n-2,x,y)$.
Therefore
$$b(n,x,y)=q^{\binom{n}{2}}a(n,x,y)=q^{\binom{n}{2}}\left((x+y)a(n-1,x,y)-xya(n-2,x,y)\right)$$
$$=(x+y)q^{\binom{n-1}{2}}a(n-1,qx,qy)-qxyq^{\binom{n-2}{2}}a(n-2,q^2x,q^2y)$$
$$=(x+y)b(n-1,qx,qy)-qxyb(n-2,q^2x,q^2y)$$
and
$$d(n,x,y)=c(n,x,y)b(n,x,y)=(qxy)^{n-1}c(n-1,qx,qy)\left((x+y)b(n-1,qx,qy)-qxyb(n-2,q^2x,q^2y)\right)$$
$$=(qxy)^{n-1}(x+y)d(n-1,qx,qy)-(qxy)^n(q^3xy)^{n-2}d(n-2,q^2x,q^2y)$$

2) Let
$$f(z,x,y)=1+xzf(z,x,y)+yzf(z,x,y)f(qz,x,y)=\sum_{n\geq 0}c(n,x,y,q)z^n. \tag{4}$$
Then
$$\det\left(c(i+j,x,y,q)\right)_{i,j=0}^{n-1}=q^{\frac{n(n-1)^2}{2}}y^{\binom{n}{2}}\prod_{j=0}^{n-2}(x+q^jy)^{n-1-j} \tag{5}$$
and
$$\det\left(\left(c(i+j+1,x,y,q)\right)_{i,j=0}^{n-1}\right)=q^{\frac{n^2(n-1)}{2}}y^{\binom{n}{2}}(x+y)^n(x+qy)^{n-1}\cdots(x+q^{n-2}y)(x+q^{n-1}y). \tag{6}$$

A modification of (4) gives

$$g(z,x,y)=1+xz^2g(z,x,y)+yz^2g(z,x,y)g(qz,x,y)=\sum_{n\geq 0}c(n,x,y,q^2)z^{2n}. \qquad (7)$$

The corresponding lattice paths are the nonnegative paths with steps $U,D,H=(2,0)$ where the weights are given by $\omega(H)=xq^{2i}z^2$ if H is on height i, $\omega(U)=q^{i-1}z$ if the endpoint of U is on height i and $\omega(D)=yq^{i-1}z$ if the initial point of D is on height i.

The Hankel determinant $\det\left(c(i+j,x,y,q^2)z^{2(i+j)}\right)_{i,j=0}^{n-1}$ corresponds to I-T-CONFIG of [3], Lemma 2.2. The proof of [3], Lemma 2.2, shows that the $i-$th path of a nonintersecting $n-$tuple begins with $i-1$ $U-$steps from $(-2i+2,0)$ to $(-i+1,i-1)$ and ends with $i-1$ $D-$steps from $(i-1,i-1)$ to $(2i-2,0).$ From [3], Lemma 2.1, we see therefore that the Hankel determinant equals $\det\left(q^{\binom{i-1}{2}+\binom{j-1}{2}}y^{j-1}z^{i+j-2}\omega(P'_{i,j})\right)$, where $\omega(P'_{i,j})$ denotes the weight of all paths in I-T-CONFIG-NEW of [3].

Let $A(z)$ denote the $n\times n-$matrix with $A(z)_{i,j}=\omega(P'_{i,j}).$

Classifying according to the first step we see that

$$A(z)_{i,j}=q^{i-2}yzA(z)_{i-1,j}+q^{2i-2}xz^2A(qz)_{i-1,j-1}+q^{i-1}zA(qz)_{i,j-1}$$

for $i>1$ and $j>1$ with $A(z)_{1,j}=q^{\binom{j-1}{2}}z^{j-1}$ and $A(z)_{i,1}=q^{\binom{i-1}{2}}(yz)^{i-1}.$

Define the $n\times n-$matrix $M^{(n)}(0,z)$ by

$$M^{(n)}(0,z)_{i,j}:=q^{\binom{i-1}{2}+\binom{j-1}{2}}y^{j-1}z^{i+j-2}A(z)_{i,j}$$

and the $n\times n-$matrices $M^{(n)}(k,z)$ by

$$M^{(n)}(k,z)_{i,j}=\begin{cases}M^{(n)}(k-1,z)_{i,j} & \text{for} \quad 1\leq i\leq k\\ M^{(n)}(k-1,z)_{i,j}-yz^2q^{2k+2i-6}M^{(n)}(k-1,z)_{i-1,j} & \textit{for} \quad k+1\leq i\leq n\end{cases}$$

Then with the same reasoning as in [3] $M^{(n)}(n-1,z)$ is upper triangular with

$$M^{(n)}(n-1,z)_{i,i}=q^{(i-1)(3i-4)}y^{i-1}z^{4i-4}\prod_{j=0}^{i-2}(x+q^{2j}y).$$

First observe that

$$M^{(n)}(0,z)_{i,j}=q^{2i-4}yz^2M^{(n)}(0,z)_{i-1,j}+q^{2i-2}xyz^4M^{(n)}(0,qz)_{i-1,j-1}+yz^2M^{(n)}(0,qz)_{i,j-1}$$

with

$$M^{(n)}(0,z)_{1,j}=q^{2\binom{j-1}{2}}y^{j-1}z^{2j-2} \text{ and } M^{(n)}(0,z)_{i,1}=q^{2\binom{i-1}{2}}y^{i-1}z^{2i-2}.$$

By induction we see that

a) $M^{(n)}(k,z)_{i,j}=q^{2i+2k-4}yz^2M^{(n)}(k,z)_{i-1,j}+q^{2i-2}xyz^4M^{(n)}(k,qz)_{i-1,j-1}+yz^2M^{(n)}(k,qz)_{i,j-1}$
for $i,j>k,$

b) $M^{(n)}(k,z)_{i,i}=q^{(i-1)(3i-4)}y^{i-1}z^{4i-4}\prod_{j=0}^{i-2}(x+q^{2j}y)$ for $i\le k,$

c) $M^{(n)}(k,z)_{i,j}=0$ for $i>j$ and $j\le k,$ and

d) $M^{(n)}(k,z)_{i,k+1}=q^{(i-1)(i+2k-2)}y^{i-1}z^{2i+2k-2}\prod_{j=0}^{k-1}(x+q^{2j}y)$ for $i\ge k+1.$

To prove a) let
$w(i,j,k+1,z)=M^{(n)}(k,z)_{i,j}-yz^2q^{2k+2i-4}M^{(n)}(k,z)_{i-1,j}.$
Then

$$w(i,j,k+1,z)-(q^{2i+2k-2}yz^2w(i-1,j,z)+q^{2i-2}xyz^4w(i-1,j-1,qz)+yz^2w(i,j-1,qz))$$
$$=M^{(n)}(k,z)_{i,j}-yz^2q^{2k+2i-4}M^{(n)}(k,z)_{i-1,j}-q^{2i-2}xyz^4M^{(n)}(k,qz)_{i-1,j-1}-yz^2M^{(n)}(k,qz)_{i,j-1}$$
$$-yz^2q^{2i+2k-2}M^{(n)}(k,z)_{i-1,j}+yz^2q^{2i+2k-2}yz^2q^{2k+2i-6}M^{(n)}(k,z)_{i-2,j}$$
$$+q^{2i-2}xyz^4yq^2z^2q^{2k+2i-6}M^{(n)}(k,qz)_{i-2,j-1}+yz^2yq^2z^2q^{2k+2i-4}M^{(n)}(k,qz)_{i-1,j-2}$$
$$=-yz^2q^{2i+2k-2}\left(M^{(n)}(k,z)_{i-1,j}-yz^2q^{2k+2i-6}M^{(n)}(k,z)_{i-2,j}-q^{2i-4}xyz^4M^{(n)}(k,qz)_{i-2,j-1}-yz^2M^{(n)}(k,qz)_{i-1,j-2}\right)=0.$$

For $k=0$ b) and c) are not defined and d) is true.
For $k=1$ a) is true. b) is true because $M^{(n)}(1,z)_{1,1}=M^{(n)}(0,z)_{1,1}=1,$
c) follows from

$$M^{(n)}(1,z)_{i,1}=M^{(n)}(0,z)_{i,1}-yz^2q^{2i-4}M^{(n)}(0,z)_{i-1,1}=q^{2\binom{i-1}{2}}y^{i-1}z^{2i-2}-yz^2q^{2i-4}q^{2\binom{i-2}{2}}y^{i-2}z^{2i-4}=0$$
for $i>1.$
d) says that for $i\ge 2$ we have
$M^{(n)}(1,z)_{i,2}=q^{(i-1)i}y^{i-1}z^{2i}(x+y).$
Now we have by definition
$M^{(n)}(1,z)_{i,2}=M^{(n)}(0,z)_{i,2}-yz^2q^{2i-4}M^{(n)}(0,z)_{i-1,2}$
and by a) for $k=0$
$M^{(n)}(1,z)_{i,2}=M^{(n)}(0,z)_{i,2}-yz^2q^{2i-4}M^{(n)}(0,z)_{i-1,2}=q^{2i-2}xyz^4M^{(n)}(0,qz)_{i-1,1}+yz^2M^{(n)}(0,qz)_{i,1}.$
By property d) for $k=0$ the right hand side is

$$q^{2i-2}xyz^4M^{(n)}(0,qz)_{i-1,1}+yz^2M^{(n)}(0,qz)_{i,1}=q^{2i-2}xyz^4q^{2\binom{i-2}{2}}y^{i-2}(qz)^{2i-4}+yz^2q^{2\binom{i-1}{2}}y^{i-1}(qz)^{2i-2}$$
$$=q^{(i-1)i}y^{i-1}z^{2i}(x+y).$$

Therefore for $k=1$ all properties are true. Now suppose them to hold for $k-1.$
Then for k we have that a) is true.

In order to show b) we have to show that $M^{(n)}(k,z)_{k,k} = q^{(k-1)(3k-4)} y^{k-1} z^{4k-4} \prod_{j=0}^{k-2} (x+q^{2j}y).$

But $M^{(n)}(k,z)_{k,k} = M^{(n)}(k-1,z)_{k,k} = q^{(k-1)(k+2k-4)} y^{k-1} z^{2k+2k-4} \prod_{j=0}^{k-2} (x+q^{2j}y)$ follows from d) for $k-1.$

For c) we have to show that $M^{(n)}(k,z)_{i,k} = 0$ for $i > k.$

Here we use d) for $k-1$:

$$M^{(n)}(k,z)_{i,k} = M^{(n)}(k-1,z)_{i,k} - yz^2 q^{2k+2i-6} M^{(n)}(k-1,z)_{i-1,k}$$
$$= q^{(i-1)(i+2k-4)} y^{i-1} z^{2i+2k-4} \prod_{j=0}^{k-2} (x+q^{2j}y) - yz^2 q^{2k+2i-6} q^{(i-2)(i+2k-5)} y^{i-2} z^{2i+2k-6} \prod_{j=0}^{k-2} (x+q^{2j}y) = 0.$$

Finally we have to show d)

$$M^{(n)}(k,z)_{i,k+1} = q^{(i-1)(i+2k-2)} y^{i-1} z^{2i+2k-2} \prod_{j=0}^{k-1} (x+q^{2j}y) \text{ for } i \geq k+1.$$

By definition

$$M^{(n)}(k,z)_{i,k+1} = M^{(n)}(k-1,z)_{i,k+1} - yz^2 q^{2k+2i-6} M^{(n)}(k-1,z)_{i-1,k+1}.$$

Using a) we get

$$M^{(n)}(k,z)_{i,k+1} = q^{2i-2} xyz^4 M^{(n)}(k-1,qz)_{i-1,k} + yz^2 M^{(n)}(k-1,qz)_{i,k}.$$

By d) for $k-1$ this equals

$$q^{2i-2} xyz^4 q^{(i-2)(i+2k-5)} y^{i-2} (qz)^{2i+2k-6} \prod_{j=0}^{k-2} (x+q^{2j}y) + yz^2 q^{(i-1)(i+2k-4)} y^{i-1} (qz)^{2i+2k-4} \prod_{j=0}^{k-2} (x+q^{2j}y)$$
$$= y^{i-1} z^{2i+2k-2} q^{(i-1)(i+2k-2)} (x+q^{2k-2}y) \prod_{j=0}^{k-2} (x+q^{2j}y).$$

Since all $M^{(n)}(j,z)$ have the same determinant we get

$$\det M^{(n)}(0,z) = \det M^{(n)}(n-1,z) = q^{n(n-1)^2} y^{\binom{n}{2}} z^{2n^2-2n} \prod_{j=0}^{n-2} (x+q^{2j}y)^{n-1-j}.$$

By the Gessel-Viennot-Lindström theorem
$\det\left(c(i+j,x,y,q^2) z^{2i+2j}\right)_{i,j=0}^{n-1}$ equals the sum of the signed weights of the nonintersecting $n-$tuples in the original configuration. Since all such $n-$tuples belong to the new configuration we see that

$$\det\left(c(i+j,x,y,q^2) z^{2(i+j)}\right)_{i,j=0}^{n-1} = \det M^{(n)}(0,z) = \det M^{(n)}(n-1,z) = q^{n(n-1)^2} y^{\binom{n}{2}} z^{2n^2-2n} \prod_{j=0}^{n-2} (x+q^{2j}y)^{n-1-j}.$$

By choosing $z=1$ and $q^2 \to q$ we get (5).

For the second Hankel determinant consider the initial points $(-2n+1,0),\cdots,(-1,0)$ and the terminal points $(1,0),(3,0),\cdots,(2n-1,0).$

There are again two classes of nonintersecting $n-$tuples. The first class contains the horizontal step H from -1 to $1,$ and the second class contains there instead the sequence UD.

But precisely these steps also occur if we prepend a $U-$step and a $D-$step to every path. Therefore the weight of all such paths is

$$\frac{\det M^{(n+1)}(n,\frac{z}{q})}{(zq^{-1})^{2n}y^n}=\frac{M^{(n+1)}(n,z)}{z^{2n}q^{2n^2}y^n}.$$

This implies (6).

The special case $x=0, y=1$ of (4) gives

$$f(z,0,1)=1+zf(z,0,1)f(qz,0,1)=\sum_{n\geq 0}C_n(q)z^n.$$

In this case it is easily verified that $M^{(n)}(0,1)_{i,j}=q^{\binom{i-1}{2}+\binom{j-1}{2}}\begin{bmatrix} i+j-2 \\ j-1 \end{bmatrix}$.

The polynomials $C_n(q)$ are the $q-$Catalan numbers of Carlitz.

In this case $\det\left(C_{i+j}(q)\right)_{i,j=0}^{n-1}=\det\left(q^{\binom{i}{2}+\binom{j}{2}}\begin{bmatrix} i+j \\ j \end{bmatrix}\right)_{i,j=0}^{n-1}$ is an interesting relationship between $q-$binomial coefficients and $q-$Catalan numbers.

References

[1] J. Cigler, $q-$Catalan numbers and $q-$Narayana polynomials, arXiv math. CO/ 0507225 (some newer versions can be found in http://homepage.univie.ac.at/johann.cigler/)

[2] J. Cigler, Mathematische Randbemerkungen 7: Catalanzahlen und verwandte Zahlenfolgen, http://homepage.univie.ac.at/johann.cigler/

[3] R.A. Sulanke & G. Xin, Hankel determinants for some common lattice paths, Advances Appl. Math. 40(2008), 149-167