How to guess and prove explicit formulas for some Hankel determinants

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Abstract
In many papers all hints how the results have been found are eliminated. In this note I want to reverse this procedure and sketch how in some cases explicit formulas for Hankel determinants can be guessed and proved.

1. Theoretical background
Let \((a(n))\) be a sequence of real numbers with \(a(0) = 1\).
It is often easy to guess the Hankel determinants \(\det(a(i + j))_{i,j=0}^{n-1}\). Fortunately in some cases it is even possible to prove the guessed results by guessing some more results.

We make use of the following results:
1) If all Hankel determinants \(\det(a(i + j))_{i,j=0}^{n-1} \neq 0\) then the polynomials

\[
p(n, x) = \frac{1}{\det(a(i + j))_{i,j=0}^{n-1}} \det \begin{pmatrix} a(0) & a(1) & \cdots & a(n-1) & 1 \\ a(1) & a(2) & \cdots & a(n) & x \\ a(2) & a(3) & \cdots & a(n+1) & x^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a(n) & a(n+1) & \cdots & a(2n-1) & x^n \end{pmatrix}
\]

(1.1)

are orthogonal with respect to the linear functional \(F\) defined by

\[
F(x^n) = a(n).
\]

(1.2)

This means that \(F\left(p(n, x)p(m, x)\right) = 0\) if \(m \neq n\) and \(F\left(p(n, x)^2\right) \neq 0\).

In particular for \(m = 0\) we get

\[
F\left(p(n, x)\right) = [n = 0].
\]

(1.3)

These identities also characterize the linear functional \(F\).

By Favard’s theorem there exist numbers \(s(n), t(n)\) such that

\[
p(n, x) = (x - s(n-1))p(n-1, x) - t(n-2)p(n-2, x).
\]

(1.4)
2) If for given sequences $s(n)$ and $t(n)$ we define $a(n, j)$ by

\[
\begin{align*}
a(0, j) &= [j = 0] \\
a(n, 0) &= s(0)a(n-1, 0) + t(0)a(n-1, 1) \\
a(n, j) &= a(n-1, j-1) + s(j)a(n-1, j) + t(j)a(n-1, j+1)
\end{align*}
\]

(1.5)

then the Hankel determinant $\det (a(i + j, 0))_{i,j=0}^{n-1}$ is given by

\[
\det (a(i + j, 0))_{i,j=0}^{n-1} = \prod_{i=0}^{n-1} \prod_{j=0}^{n-1} t(j).
\]

(1.6)

So the Hankel determinant $\det (a(i + j, 0))_{i,j=0}^{n-1}$ only depends on the sequence $(t(n))$.

Thus if we start with $(a(n))$ and guess all $s(n), t(n)$ and $a(n, j)$, then our guesses lead to an exact proof, if (1.5) holds and $a(n, 0) = a(n)$. In this case we also have

\[
\sum_{k=0}^{n} a(n,k) p(k, x) = x^n.
\]

(1.7)

This situation is well-known (cf. e.g. [3] and the literature cited there). It is especially useful if for a given sequence $(a(n))_{n \geq 0}$ closed expressions for $s(n), t(n)$ and $a(n, j)$ can be found.

2. A simple example

Let us consider the sequence

\[
a(n) = \binom{2n}{n}.
\]

(2.1)

After computing (with Mathematica) the first Hankel determinants

\[
\text{Table}[	ext{Det}[	ext{Table}[	ext{Binomial}[2 \times 2 + j, i + j], \{i, 0, n - 1\}, \{j, 0, n - 1\}\}], \{n, 1, 8\}] \{1, 2, 4, 8, 16, 32, 64, 128\}
\]

we guess that

\[
\det \left( \binom{2i + 2j}{i + j} \right)_{i,j=0}^{n-1} = 2^{n-1}.
\]

First we compute the orthogonal polynomials associated with $a(n) = \binom{2n}{n}$. We get

\[
\]
Then we compute the corresponding $s(n), t(n)$ and get

\[
\text{Table}[p[n, x], \{n, 0, 5\}],
\]

\[(1, -2 + x, 2 - 4 x + x^2, -2 + 9 x - 6 x^2 + x^3, 2 - 16 x + 20 x^2 - 8 x^3 + x^4, -2 + 25 x - 50 x^2 + 35 x^3 - 10 x^4 + x^5)\]

Thus our guess is that $s(n) = 2, \ t(0) = 2$ and $t(n) = 1$ for $n > 0$.

Therefore the polynomials $p(n, x)$ satisfy the recursion

\[
p(n, x) = (x - 2)p(n - 1, x) - p(n - 2, x)
\]

(2.2)

for $n > 2$.

But how can we be sure that our guesses are correct?

To this end we must compute $a(n, j)$ defined by (1.5).

This gives

\[
a[n, j] :=
\]

\[
\text{Table}[\text{Factor}[\text{PolynomialQuotient}[p[n + 1, x], p[n, x], x]], \{n, 0, 5\}],
\]

\[
\{-2 + x, -2 + x, -2 + x, -2 + x, -2 + x, -2 + x\}
\]

\[
\text{Table}[t[n], \{n, 0, 7\}],
\]

\[
\{2, 1, 1, 1, 1, 1, 1, 1\}
\]

\[
\text{Table}[\text{Factor}[\text{PolynomialRemainder}[p[n + 2, x], p[n + 1, x], x] / p[n, x]], \{n, 0, 5\}],
\]

\[
\text{Table}[f[n], \{n, 0, 7\}],
\]

\[
\{2, 1, 1, 1, 1, 1, 1, 1\}
\]

\[
\text{Table}[\text{Factor}[\text{PolynomialRemainder}[p[n + 2, x], p[n + 1, x], x] / p[n, x]], \{n, 0, 5\}],
\]

\[
\text{Table}[f[n], \{n, 0, 7\}],
\]

\[
\{2, 1, 1, 1, 1, 1, 1, 1\}
\]

Thus our guess is that $s(n) = 2, \ t(0) = 2$ and $t(n) = 1$ for $n > 0$.

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But how can we be sure that our guesses are correct?

To this end we must compute $a(n, j)$ defined by (1.5).

This gives

\[
a[n, j] :=
\]

\[
\text{If}[n = 0, \text{If}[j = 0, 1, 0], \text{If}[j = 0, 2 a[n - 1, 0] + 2 a[n - 1, 1], a[n - 1, j - 1] + 2 a[n - 1, j] + a[n - 1, j + 1]]]
\]
It is now easy to guess that
\[ a(n, j) = \frac{2n}{n-j}. \quad (2.3) \]
If you are not so familiar with binomial coefficients then it suffices to look into The Online Encyclopedia of Integer Sequences (OEIS) \[7\]. There this array is listed under A094527.

Now using the recursion for the binomial coefficients it is immediately verified that with
\[ a(n, j) = \frac{2n}{n-j} \]
the equations (1.5) are satisfied.
Therefore \( a(n, 0) = a(n) \) and our guesses are correct.

The coefficients of the corresponding orthogonal polynomials \( p(n, x) \) can also be guessed or found in OEIS \[7\], A110162 and A127677. A simpler method will be given in the next paragraph.
They are given by \( p(0, x) = 1 \) and
\[
p(n, x) = (-1)^n \left( 2 + \sum_{j=1}^{n} (-1)^j \frac{n}{j} \left( \frac{n+j-1}{2} \right) x^j \right) \quad (2.4)
\]
for \( n > 0 \).
This can be written in hypergeometric form
\[
p(n, x) = 2(-1)^n \sum_{j=0}^{n} \frac{(-n)_j (n)_j}{j!} x^j = 2(-1)^n {}_2 F_1 \left( \frac{-n,n}{1/2}; \frac{x}{4} \right). \quad (2.5)
\]
Here \( (x)_n \) is defined by \( (x)_n = x(x+1) \cdots (x+n-1) \). Details about notation and other standard properties of hypergeometric polynomials can be found in \[1\] and \[6\].
Remark

Guessing can often be simplified when all $s(n) = 0$. In this case identity (1.5) reduces to

$$A(0, j) = [j = 0]$$
$$A(n, 0) = T(0)A(n-1, 1)$$
$$A(n, j) = A(n-1, j-1) + T(j)A(n-1, j+1). \tag{2.6}$$

There we have $A(2n, 2j+1) = A(2n+1, 2j) = 0$ for all $n, j$.

A useful identity is

$$\sum_{k=0}^{n} (-1)^k A(2n, 2k) \prod_{j=0}^{k-1} T(2j) = [n = 0]. \tag{2.7}$$

For the proof let $T(-1) = 0$ and observe that $A(2n, -2) = A(2n, 2n + 2) = 0$. We have

$$A(2n, 2k) = A(2n-1, 2k-1) + T(2k)A(2n-1, 2k+1)$$
$$= A(2n-2, 2k-2) + T(2k-1)A(2n-2, 2k) + T(2k)A(2n-2, 2k) + T(2k)T(2k+1)A(n-2, 2k+2)$$

Therefore we get

$$\sum_{k=0}^{n} (-1)^k A(2n+2, 2k) \prod_{j=0}^{k-1} T(2j)$$
$$= \sum_{k=0}^{n} (-1)^k \left( A(2n, 2k-2) + (T(2k-1) + T(2k))A(2n, 2k) + T(2k)T(2k+1)a(2n, 2k+2) \prod_{j=0}^{k-1} T(2j) \right)$$
$$= \sum_{k=0}^{n} (-1)^k A(2n, 2k) \prod_{j=0}^{k-1} T(2j)T(2k-1) + T(2k)T(2k) - T(2k)T(2k-1) = 0.$$

If we define

$$a(n, j) = A(2n, 2j), \tag{2.8}$$

then it is easily verified that (1.5) holds with

$$s(0) = T(0),$$
$$s(n) = T(2n-1) + T(2n),$$
$$t(n) = T(2n)T(2n+1). \tag{2.9}$$
Furthermore we have
\[ p(n, x) = P(2n, \sqrt{x}). \tag{2.10} \]

This follows immediately from
\[
P(2n, x) = xP(2n - 1, x) - T(2n - 2)P(2n - 2, x)
= (x^2 - T(2n - 2))P(2n - 2, x) - xT(2n - 3)P(2n - 3, x),
P(2n - 2, x) = xP(2n - 3, x) - T(2n - 4)P(2n - 4, x)
\]

Eliminating \(xP(2n - 3, x)\) we get
\[
P(2n, x) = xP(2n - 1, x) - T(2n - 2)P(2n - 2, x)
= (x^2 - T(2n - 2) - T(2n - 3))P(2n - 2, x) - T(2n - 3)T(2n - 4)P(2n - 4, x)
= (x^2 - s(n - 1))P(2n - 2, x) - t(n - 2)P(2n - 4, x).
\]

For the above example we set
\[
A(2n) = \binom{2n}{n}, A(2n + 1) = 0. \tag{2.11}
\]

Then we get \(T(0) = 2\) and \(T(n) = 1\) for \(n > 0\).

Therefore the Hankel determinants are \(\det (A(i + j, 0))_{i,j=0}^{n-1} = 2^{n-1}\).

The table \(A(n, j)\) is given by OEIS, A108044,

<p>| | | | | | | | | |</p>
<table>
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<td>28</td>
<td>0</td>
<td>8</td>
<td>0</td>
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</tbody>
</table>

Here we have \(A(2n, 2k) = \binom{2n}{n-k}, A(2n + 1, 2k + 1) = \binom{2n + 1}{n-k}\).

Here again (2.6) can be easily verified.
The corresponding orthogonal polynomials satisfy

\[ P(n, x) = xP(n - 1, x) - P(n - 2, x) \]

with initial values \( P(0, x) = 1 \), \( P(1, x) = x \) and \( P(2, x) = x^2 - 2 \).

Recall that the Lucas polynomials

\[ L(n, x, s) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} s^k \]  

(2.12)

are characterized by \( L(0, x, s) = 2 \), \( L(1, x, s) = x \) and the recursion

\[ L(n, x, s) = xL(n - 1, x, s) + sL(n - 2, x, s). \]  

(2.13)

The sequence \( L(n, x, -1) \) begins with

\[ \text{Expand[Table[1[n, x, -1], {n, 0, 6}]]} \]

\{ 2, x, -2 + x^2, -3 x + x^3, 2 - 4 x^2 + x^4, 5 x - 5 x^3 + x^5, -2 + 9 x^2 - 6 x^4 + x^6 \}

Since the Lucas polynomials \( L(n, x, -1) \) satisfy the same recurrence and initial values \( L(1, x, -1) = x \) and \( L(2, x, -1) = x^2 - 1 \) we conclude that for \( n > 0 \) these polynomials coincide with the Lucas polynomials

\[ P(n, x) = L(n, x, -1) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-k} \binom{n-k}{k} (-1)^k x^{n-2k}. \]  

(2.14)

Equation (1.7) reduces in this case to

\[ \sum_{k=0}^{n} \binom{2n}{n-k} L(2k, x, -1) = x^{2n}, \]

\[ \sum_{k=0}^{n} \binom{2n+1}{n-k} L(2k + 1, x, -1) = x^{2n+1}. \]  

(2.15)

The linear functional \( F \) defined by

\[ F(x^n) = A(n) \]

(2.16)

is also uniquely determined by

\[ F(P(n, x)) = F(L(n, x, -1)) = 0 \]  

(2.17)

for \( n > 1 \).
Now we can also express the orthogonal polynomials (2.4) in terms of the Lucas polynomials. For (2.10) gives that

\[ p(n, x) = L\left(2n, \sqrt{x}, -1\right). \]  

(2.18)

Thus

\[
p(n, x) = \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} x^{n-k} = (-1)^n \sum_{k=0}^{n} (-1)^k \frac{2n}{n+k} \binom{n+k}{n-k} x^k
\]

\[
= (-1)^n \left(2 + \sum_{k=1}^{n} (-1)^k \frac{n+k-1}{2k-1} x^k\right).
\]

3. Central trinomial coefficients

As a slight generalization we consider the generalized trinomial coefficients

\[ a(n) = [x^n]\left(1 + ax + bx^2\right)^n \]  

(3.1)

which for \(a = 2, b = 1\) reduce to \(\binom{2n}{n}\).

From

\[ d[n_] := \text{Det}\left[\text{Table}\left[\text{Coefficient}\left\{\left(1 + ax + bx^2\right)^n, \{i, 0, n-1\}, \{j, 0, n-1\}\right}\right]^{(i+j)} x, \{i, j\}\right]\right]
\]

\[
\text{Table}\left[d[n], \{n, 1, 5\}\right]
\]

\{1, 2 b, 4 b^3, 8 b^6, 16 b^{10}\}

we guess that

\[ d(n) = 2^{n-1} b^{\frac{n(n+1)}{2}}. \]  

(3.2)

We have

\[
\left(1 + ax + bx^2\right)^n = \sum_{j=0}^{n} \binom{n}{j} (ax)^j \sum_{k=0}^{j} \binom{j}{k} \left(\frac{b}{a}\right)^k = \sum_{j,k} \binom{n}{j} \binom{j}{k} a^{j-k} b^k x^{k+j}.
\]
Therefore we get

\[ [x^{n+m}](1+ax+bx^2)^n = \sum_{j+k=n+m} \binom{n}{j} \binom{j}{k} a^{j-k} b^k = \sum_k \frac{n!a^{n+m-2k}b^k}{(k-m)!k!(n+m-2k)!} \]

This gives

\[ a(n) = [x^n](1+ax+bx^2)^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} a^{n-2k} b^k \]

and

\[ [x^{n+1}](1+ax+bx^2)^n = \sum_k \frac{n!a^{n+1-2k}b^k}{(k-1)!k!(n+1-2k)!} \]

\[ [x^{n-1}](1+ax+bx^2)^n = \sum_k \frac{n!a^{n-1-2k}b^k}{(k+1)!k!(n-1-2k)!} = \sum_k \frac{n!a^{n+1-2k}b^{k-1}}{(k-1)!k!(n+1-2k)!} \]

which implies that

\[ [x^{n+1}](1+ax+bx^2)^n = b[x^{n-1}](1+ax+bx^2)^n. \quad (3.3) \]

We compute the corresponding \( s(n), t(n) \).

\[ h[n_] := \text{Coefficient } [(1 + ax + bx^2)^n, x, n] \]
\[ d[n_] := \text{Factor } [\text{Det}[	ext{Table}[h[i+j], \{i, 0, n-1\}, \{j, 0, n-1\}]]]; d[0] := 1 \]
\[ a[n_, i_, j_] := \text{If} [j < n, h[i+j], x^i] \]
\[ p[n_] := \text{Factor } [\text{Det}[	ext{Table}[a[n, i, j], \{i, 0, n\}, \{j, 0, n\}]] / d[n]] \]
\[ p[0] := 1 \]

\[ \text{Table}[\text{Factor } \text{PolynomialQuotient } [p[n+1], p[n], x]], \{n, 0, 5\}] \]
\[ \{-a+x, -a+x, -a+x, -a+x, -a+x, -a+x\} \]
\[ t[n_] := -\text{Factor } \text{PolynomialRemainder } [p[n+2], p[n+1], x] / p[n] \]
\[ \text{Table}[t[n], \{n, 0, 6\}] \]
\[ \{2 b, b, b, b, b, b, b\} \]

Therefore we guess that
If this guess is correct we immediately get (3.2).
Define now \( a(n,k) \) by (1.5).
We get

\[
\begin{align*}
\text{TableForm}[\text{Expand}[\text{Table}[a[n,k], \{n,0,4\}, \{k,0,n\}]]] \\
\end{align*}
\]

\[
\begin{array}{cccc}
1 & a & 1 \\
a^2 + 2b & 2a & 1 \\
a^3 + 6ab & 3a^2 + 3b & 3a & 1 \\
a^4 + 12a^2b + 6b^2 & 4a^3 + 12ab & 6a^2 + 4b & 4a & 1 \\
\end{array}
\]

Now we have also

\[
\begin{align*}
\text{TableForm}[\text{Table}[\text{Coefficient}[(1 + ax + bx^2)^n, x, n-k], \{n,0,4\}, \{k,0,n\}]] \\
\end{align*}
\]

\[
\begin{array}{cccc}
1 & a & 1 \\
a^2 + 2b & 2a & 1 \\
a^3 + 6ab & 3a^2 + 3b & 3a & 1 \\
a^4 + 12a^2b + 6b^2 & 4a^3 + 12ab & 6a^2 + 4b & 4a & 1 \\
\end{array}
\]

Therefore we guess that \( a(n,k) = [x^{n-k}](1 + ax + bx^2)^n \).

This can again easily be verified:
For \( k \geq 1 \) we have

\[
\begin{align*}
a(n,k) &= [x^{n-k}](1 + ax + bx^2)^n = [x^{n-k}](1 + ax + bx^2)(1 + ax + bx^2)^{n-1} \\
&= [x^{n-1-(k-1)}](1 + ax + bx^2)^{n-1} + a[x^{n-1-k}](1 + ax + bx^2)_{n-1} + b[x^{n-1-(k+1)}](1 + ax + bx^2)^{n-1} \\
&= a(n-1,k-1) + aa(n-1,k) + ba(n-1,k+1). \\
\end{align*}
\]

For \( k = 0 \) we get

\[
\begin{align*}
\end{align*}
\]
This completes the proof.

It should be mentioned that for this example the sequence \( A(n) \) with \( A(2n) = a(n), A(2n + 1) = 0 \) gives no simplification.

For in this case we get from (2.9) and (3.4) that

\[
T(2n) = a - T(2n - 1), \quad T(2n + 1) = \frac{b}{a - T(2n - 1)}
\]

with initial values \( T(0) = a \) and \( T(1) = \frac{2b}{a} \).

For the central trinomial coefficients \((a = b = 1)\) we therefore get for

\[
D(n) = \det(A(i + j)))_{i,j=0}^{n-1}
\]

the rather curious sequence of determinants

\[
D(1) = D(2) = D(3) = 1, \quad D(4) = 2, \quad D(6n + 5) = (-1)^{n-1}2^{6n+2},
\]

\[
D(6n + 6) = (-1)^{n-1}2^{6n+3}, \quad D(6n + 7) = (-1)^{n-1}2^{6n+5},
\]

\[
D(6n + 8) = D(6n + 9) = (-1)^{n-1}2^{6n+6}, \quad D(6n + 10) = 2^{6n+7}.
\]

4. Our main example

In [3] I have considered in some detail the example \( a(n) = \frac{(b;q)_n}{(a;q)_n} \) where \((x;q)_n = \prod_{j=0}^{n-1}(1 - q^j x)\).

After completion of [3] the paper [5] appeared where analogous results have been derived with other methods. In that paper the authors considered the slightly modified sequence

\[
a(n) = \frac{(aq;q)_n}{(abq^2;q)_n}.
\]

This has the advantage that the corresponding orthogonal polynomials are the little \( q - \) Jacobi polynomials in the usual notation.

Since my aim in this paper is to show how you can guess Hankel determinants I want to sketch the above method in this case in some detail. I am using Mathematica for the computations. I also use the \( q - \) notations \([n] = \frac{1 - q^n}{1 - q}\), \([n]! = \prod_{j=0}^{n-1}[j] \) and

\[
\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \frac{[n]!}{[k]![n-k]!} \quad \text{for } 0 \leq k \leq n \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix} = 0 \quad \text{for } k < 0 \text{ and } k > n.
\]

For later use we observe that
\[
\lim_{q \to 1} \frac{(q^{\alpha+1}; q)_n}{(q^{\alpha+\beta+2}; q)_n} = \frac{(\alpha + 1)_n}{(\alpha + \beta + 2)_n}. \quad (4.1)
\]

Let
\[
a(n) = \frac{(aq; q)_n}{(abq^2; q)_n}. \quad (4.2)
\]

Define \( A(2n) = a(n), A(2n + 1) = 0 \). Then we compute the corresponding first polynomials \( P(n, x, a, b) \) and the corresponding Favard recurrences. This gives the first values of \( T(n, a, b, q) \):

\[
\begin{align*}
  A_0 &= \frac{1-aq}{1-abq^2}, \\
  A_n &= T(2n, a, b, q) = \frac{q^n(1-q^{-n+1}a)(1-q^{-n+1}ab)}{(1-q^{-2n+1}ab)(1-q^{-2n+2}ab)}, \\
  C_n &= T(2n-1, a, b, q) = \frac{aq^n(1-q^{-n})(1-q^{-n}b)}{(1-q^{-2n+1}ab)(1-q^{-2n+2}ab)}. 
\end{align*}
\]

By (2.9) the orthogonal polynomials \( p(n, x, a, b) \) corresponding to the original sequence \( (a(n)) \) satisfy the recurrence

\[
xp(n, x, a, b) = p(n + 1, x, a, b) + (A_n + C_n)p(n, x, a, b) + A_{n-1}C_np(n - 1, x, a, b). \quad (4.5)
\]

It turns out that this is the normalized recurrence relation [5], (14.12.4), for the little \( q \)–Jacobi polynomials (cf. [5], (14.12.1)).
\[ p_n(x;a,b|q) = 2\phi \left( \frac{q^{-n}, abq^{n+1}}{aq}; q, qx \right) = \sum_{j=0}^{n} \frac{(q^{-n};q)_j}{(aq;q)_j} (abq^{n+1};q)_j q^{j} x^{j}. \quad (4.6) \]

We could of course also have guessed this result directly.

Let \( r(n,j) = \frac{[x^{j+1}] p(n,x,a,b)}{[x^j] p(n,x,a,b)} \) where \([x^j] p(x)\) denotes the coefficient of \( x^j \) in \( p(x) \).

\[
\begin{align*}
\text{w[4]} &= \left( -\frac{(1+q)(1+q^2)(-1+abq^5)}{q^2(-1+aq)} \right), \quad \left( -\frac{1+q+q^2+q^4}{q^2(1+q)} \right), \quad \left( -\frac{1+q}{q(1+q+q^2)} \right) \\
\text{w[5]} &= \left( -\frac{1+q+q^2+q^4}{q^2(1+q^2)} \right), \quad \left( -\frac{1+q^2+q^4}{q^2(1+q^2)} \right), \quad \left( -\frac{1+q+q^2+q^4}{q^2(1+q^2+q^3+q^4)} \right)
\end{align*}
\]

From this we guess that

\[ r(n,j) = \frac{q(1-q^{-n})(1-abq^{j+1})}{(1-q^{j+1}a)(1-q^{j+1})}. \]

Therefore we conclude that \( p(n,x,a,b) \) is the normalization of (4.6).

If our guesses are correct this would be another proof of the known fact that the moments for the linear functional \( F \) defined by \( F(p(n,x,a,b)) = [n=0] \) for the normalized little \( q - \) Jacobi polynomials

\[ p(n,x,a,b) = (-1)^n q^{\binom{n+1}{2}} \frac{(aq;q)_n}{(abq^{n+1};q)_n} \phi \left( \frac{q^{-n}, abq^{n+1}}{aq}; q, qx \right) \quad (4.7) \]

are given by

\[ F(x^n) = \frac{(aq;q)_n}{(abq^2;q)_n}. \quad (4.8) \]
All that remains to be done in order to prove these results is to define \( A(n, j) \) by (2.6) and to verify that
\[
A(2n, 0) = \frac{(aq; q)_n}{(abq^2; q)_n}.
\]

To this end we consider again some small values:

\[
\begin{align*}
a[n_] & := q^n (1 - a q^n (n + 1)) (1 - a b q^n (n + 1)) / (1 - a b q^n (2 n + 1)) / (1 - a b q^n (2 n + 2)) \\
c[n_] & := q^n a (1 - q^n (n + 1)) (1 - a b q^n (n + 1)) / (1 - a b q^n (2 n + 1)) / (1 - a b q^n (2 n)) \\
t[n_] & := \text{If}[\text{EvenQ}[n], a[n/2], c[(n + 1)/2]] \\
A[0, j_] & := \text{If}[j = 0, 1, 0];
\end{align*}
\]

\[
\text{Factor}[\text{Table}[A[2 n, 0], \{n, 0, 3\}]] = \{1, \frac{-1 + a q}{-1 + a b q^2}, \frac{(-1 + a q) (-1 + a q^2)}{(-1 + a b q^2) (-1 + a b q^3)}, \frac{(-1 + a q) (-1 + a q^2) (-1 + a q^3)}{(-1 + a b q^2) (-1 + a b q^3) (-1 + a b q^4)}\}
\]

\[
\text{Factor}[\text{Table}[A[2 n, 2], \{n, 0, 4\}]] = \{0, 1, \frac{1 + q (-1 + a q^2)}{-1 + a b q^4}, \frac{(1 + q) (-1 + a q^2) (-1 + a q^4)}{(-1 + a b q^4) (-1 + a b q^5)}, \frac{(1 + q) (1 + q^3) (-1 + a q^4) (-1 + a q^6)}{(-1 + a b q^5) (-1 + a b q^6) (-1 + a b q^7)}\}
\]

\[
\text{Factor}[\text{Table}[A[2 n, 4], \{n, 0, 4\}]] = \{0, 0, 1, \frac{1 + q + q^2 (-1 + a q^3)}{-1 + a b q^6}, \frac{(1 + q^2) (1 + q + q^2) (-1 + a q^3) (-1 + a q^4)}{(-1 + a b q^6) (-1 + a b q^7)}\}
\]

We guess that
\[
A(2n, 2k) = \left[\begin{array}{c} n \\ k \end{array}\right] \frac{(aq^{k+1}; q)_{n-k}}{(abq^{2k+2}; q)_{n-k}}.
\]

In the same way we guess that
\[
A(2n + 1, 2k + 1) = \left[\begin{array}{c} n \\ k \end{array}\right] \frac{(aq^{k+2}; q)_{n-k}}{(abq^{2k+3}; q)_{n-k}}.
\]

It remains to verify (2.6).
\( A(2n,0) = T(0,a,b,q)A(2n-1,1) \) reduces to the trivial identity

\[
\frac{(aq;q)_n}{(abq^2;q)_n} = \frac{(1-aq)}{(1-abq^2)} \frac{(aq^2;q)_{n-1}}{(abq^3;q)_{n-1}}.
\]

\( A(2n,2j) = A(2n-1,2j-1) + T(2j)A(2n-1,2j+1) \) means

\[
\begin{align*}
&\left[ n \right] \left( \frac{aq^{j+1};q}{(abq^{2j};q)} \right)_{n-j} = \left[ n-1 \right] \left( \frac{aq^{j+1};q}{(abq^{2j+2};q)} \right)_{n-j} \\
&+ q^j \left[ 1-aq^{j+1} \right] \left( 1-q^{j+1}ab \right) \left( \frac{n-j}{1-abq^{2j+1}} \right) \left( \frac{n-j}{1-abq^{2j+1}} \right) \\
&+ \left[ n-j \right] \left[ j \right] (1-aq^{j+1})(1-q^{j+1}ab) \left( \frac{n-j}{1-abq^{2j+1}} \right) = \left[ n \right] \left( \frac{aq^{j+1};q}{(abq^{2j+2};q)} \right)_{n-j}.
\end{align*}
\]

The right-hand side equals

\[
\begin{align*}
&\left[ n-1 \right] \left( \frac{aq^{j+1};q}{(abq^{2j+2};q)} \right)_{n-j} = \left[ n-1 \right] \left( \frac{aq^{j+1};q}{(abq^{2j+2};q)} \right)_{n-j} \frac{(n-j)}{(1-abq^{2j+1})} \left[ j \right] (1-abq^{2j+1}) \\
&= \left[ n-j \right] \left( \frac{aq^{j+1};q}{(abq^{2j+2};q)} \right)_{n-j} \frac{(n-j)}{(1-abq^{2j+1})} \left( \frac{n-j}{1-abq^{2j+1}} \right) = \left[ n \right] \left( \frac{aq^{j+1};q}{(abq^{2j+2};q)} \right)_{n-j}.
\end{align*}
\]

In the same way we see that \( A(2n+1,2j-1) = A(2n,2j-2) + T(2j-1)A(2n,2j) \):

\[
\begin{align*}
&A(2n,2j-2) + T(2j-1)A(2n,2j) = \left[ n \right] \left( \frac{aq^{j+1};q}{(abq^{2j+2};q)} \right)_{n-j} \\
&+ \left[ n-j \right] \left( \frac{aq^{j+1};q}{(abq^{2j+2};q)} \right)_{n-j} \frac{(n-j)}{(1-abq^{2j+1})} \left[ j \right] (1-abq^{2j+1}) \\
&= \left[ n \right] \left( \frac{aq^{j+1};q}{(abq^{2j+2};q)} \right)_{n-j} \frac{(n-j)}{(1-abq^{2j+1})} \left( \frac{n-j}{1-abq^{2j+1}} \right) = \left[ n \right] \left( \frac{aq^{j+1};q}{(abq^{2j+2};q)} \right)_{n-j} = A(2n+1,2j-1).
\end{align*}
\]

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Later we will need the concrete form of (2.7) for this case. By (4.3) and (4.4) we get
\[ \prod_{j=0}^{k-1} T(2j) = q^{(k) \choose 2} \frac{(aq;q)_k}{(abq^2;q)_{2k-1}} = q^{(k) \choose 2} \frac{(aq;q)_k}{(abq^{k+1};q)_k} \]

Thus (2.7) implies
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} q^{(k) \choose 2} \frac{(aq^{k+1};q)_{n-k}}{(abq^{2k+2};q)_{n-k}} = [n = 0] \]
or equivalently
\[ \sum_{k=0}^{n} (-1)^k q^{(k) \choose 2} \binom{n}{k} \frac{(1-abq^{2k+1})(aq;q)_{n-k}}{(abq^{k+1};q)_{n+1}} = [n = 0] \]

Dividing by \((a;q)_n\) and replacing \(ab \rightarrow c^{-1}q^{-1}\) we get the following useful well-known result:
\[ \sum_{k=0}^{n} (-1)^k q^{(k) \choose 2} \binom{n}{k} \frac{1-cq^{2k}}{(cq^k;q)_{n+1}} = [n = 0]. \] (4.11)

For the sequence \(a(n) = \frac{(aq;q)_n}{(abq^2;q)_n}\) we get
\[ t(n,a,b,q) = A_{n+1} C_{n+1} = \frac{aq^{2n+1}(1-q^{n+1})(1-q^{n+1}b)(1-q^{n+1}a)(1-q^{n+1}ab)}{(1-q^{2n+1}ab)(1-q^{2n+2}ab)^2(1-q^{2n+3}ab)}. \] (4.12)

For \(n = 0\) this reduces to
\[ t(0,a,b,q) = \frac{aq(1-q)(1-qb)(1-qa)}{(1-q^2ab)^2(1-q^3ab)}. \] (4.13)

Using (1.6) we can compute the corresponding Hankel determinants. This has been done in [3] and [5].
5. A simple special case

By changing $q \to q^2, a = b \to q^{-1}$ we get

$$a(n) = \frac{(q; q^2)_n}{(q^2; q^2)_n} = \frac{(1-q)(1-q^3)\cdots(1-q^{2^{n-1}})}{(1-q^2)(1-q^4)\cdots(1-q^{2^n})} = \frac{1}{\prod_{j=1}^{n} (1+q^j)^2} \left[ \begin{array}{c} 2n \\ n \end{array} \right] = (-1)^n \left[ \begin{array}{c} -\frac{1}{2} \\ n \end{array} \right]_{q \to q^2}. \quad (5.1)$$

If we let $q$ tend to 1 we get $a(n) = \frac{1}{4^n} \binom{2n}{n}$.\[8pt]

Now observe that if we change the sequence $a(n) \to r^n a(n)$, then the following functions change too: $s(n) \to rs(n), t(n) \to r^2 t(n), a(n, k) \to r^{n-k} a(n, k)$ and $p(n, x) \to r^n p \left( n, \frac{x}{r} \right)$.\[8pt]

Therefore we get again the results about $\binom{2n}{n}$ which we obtained in the beginning.\[8pt]

In this special case direct guessing is simpler:\[8pt]

Consider $A(2n) = \frac{(q; q^2)_n}{(q^2; q^2)_n}, A(2n+1) = 0$.\[8pt]

In this case we get

$$T(0) = \frac{1}{1+q} \quad (5.2)$$

and for $n > 0$

$$T(n) = \frac{q^n}{(1+q^n)(1+q^{n+1})}. \quad (5.3)$$

This gives

$$\det \left( A(i+j) \right)_{i,j=0}^{n-1} = \frac{q^{\binom{n}{3}}}{\prod_{k=1}^{n} (1+q^k)^{2n-2k-1}}. \quad (5.4)$$

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The orthogonal polynomials are given by

\[ P(n, x) = xP(n - 1, x) - T(n - 2)P(n - 2, x) \]

with initial values \( P(0, x) = 1, P(1, x) = x \) and \( P(2, x) = x^2 - \frac{1}{1 + q}. \)

We compute these polynomials and get

\[ t[0] := 1 / (1 + q) \]
\[ t[n_] := q^n / (1 + q^n) / (1 + q^{n + 1}) \]

Since for \( q \to 1 \) we get the Lucas polynomials we compare the coefficients of \( p(n, x) \) with the natural \( q \) – analogue of the coefficients of the Lucas polynomials.

\[ c[n_, k_] := \text{Factor}[\text{Coefficient}[p[n], x, n - 2 k]] \]
\[ cc[n_, k_] := \text{qbin}[n - k, k, q] \text{qbin}[n, 1, q] / \text{qbin}[n - k, 1, q] \]

\[ \text{Factor}[\text{Table}[c[n, 1] / cc[n, 1], \{n, 2, 5\}]] \]

\[ \left\{ -\frac{1}{(1 + q)^2}, -\frac{1}{(1 + q)} \left( 1 + q^2 \right), -\frac{1}{(1 + q)^2 \left( 1 - q + q^4 \right)}, -\frac{1}{(1 + q) \left( 1 + q^4 \right)} \right\} \]

\[ \text{Factor}[\text{Table}[c[n, 2] / cc[n, 2], \{n, 4, 6\}]] \]

\[ \left\{ \frac{q^2}{(1 + q)^2 \left( 1 - q + q^4 \right)}, \frac{q^2}{(1 + q)^2 \left( 1 - q + q^4 \right)}, \frac{q^2}{(1 + q)^2 \left( 1 - q^2 + q^3 + q^6 \right)} \right\} \]

\[ \text{Factor}[\text{Table}[c[n, 3] / cc[n, 3], \{n, 6, 8\}]] \]

\[ \left\{ \frac{q^6}{(1 + q)^4 \left( 1 - q + q^2 \right) \left( 1 - q^2 + q^4 \right) \left( 1 - q^2 - q^3 + q^6 \right)}, \right. \]
\[ \frac{q^6}{(1 + q)^3 \left( 1 + q^2 \right) \left( 1 - q + q^2 \right) \left( 1 - q^2 + q^4 \right) \left( 1 - q^2 - q^3 + q^4 \right) \right.} \]
\[ \frac{q^6}{(1 + q)^4 \left( 1 + q^2 \right) \left( 1 - q + q^2 \right) \left( 1 - q^2 + q^4 \right) \left( 1 - q + q^2 - q^3 + q^4 \right) \left( 1 - q + q^2 - q^3 + q^4 - q^5 + q^6 \right)} \]

This leads to the conjecture
\[ P(n, x) = \sum_{k=0}^{n-2} (-1)^k q^k \binom{n-k}{k} \frac{[n]}{[n-k]} \frac{x^{n-2k}}{\prod_{j=1}^{k} (1 + q^j)(1 + q^{n-j})}, \tag{5.5} \]

which can be easily verified.

For the sequence

\[ a(n) = \frac{1}{\prod_{j=1}^{n} (1 + q^j)^2} \binom{2n}{n} \]

we get in the same way as above

\[ \det \left( a(i + j) \right)_{i,j=0}^{n-1} = \frac{q^{6n-6}}{\prod_{i=1}^{n-1} (1 + q^i)^{2n-i}}. \tag{5.7} \]

**Remark**

It is natural to ask for analogous results for the sequence \( A(n \mid q) \) defined by

\[ A(2n \mid q) = \binom{2n}{n}, A(2n + 1 \mid q) = 0. \tag{5.8} \]

But unfortunately here we get no simple formulas for the Hankel determinants. This can already be seen by computing the first values of \( I(n) \) which are

\[
\begin{align*}
\{1 + q, & \frac{q (-1 + q + q^2 + q^3)}{1 + q}, \frac{q (1 + q^2) (-2 - q + q^2 + q^3 + 2 q^4 + q^5)}{(1 + q) (-1 + q + q^2 + q^3)}, \\
& \frac{q (1 + q) (-2 - 2 q - 2 q^2 + q^3 + 2 q^4 + q^5 + q^6 - 2 q^7 - 3 q^8 - 4 q^9 + q^{11} + 4 q^{12} + 3 q^{13} + 3 q^{14} + 2 q^{15} + q^{16}}{(1 + q^2) (-1 + q + q^2 + q^3)} \}
\end{align*}
\]

There are also no simple formulas for the corresponding orthogonal polynomials.

But there is an interesting \( q \) – analogue of (2.15).

Let
We want to compute the uniquely determined polynomials \( P(n, x \mid q) \) which satisfy

\[
\sum_{k=0}^{n} A(n, k \mid q) P(k, x \mid q) = x^n. \tag{5.10}
\]

These polynomials are also \( q \)– analogues of the Lucas polynomials. With the same reasoning as above we guess that

\[
P(n, x \mid q) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{\lfloor n/2 \rfloor} \frac{[n]}{[n-k]} \frac{n-k}{k} x^{n-2k} \tag{5.11}
\]

for \( n > 0 \). For \( n > 0 \) these are the \( q \)– Lucas polynomials \( \text{Luc}_n(x, -1) \) which I have studied in [2].

To prove (5.10) we have to show that

\[
\sum_{k=0}^{n} \begin{bmatrix} 2n \backslash n-k \end{bmatrix} P(2k, x \mid q) = x^{2n} \tag{5.12}
\]

and

\[
\sum_{k=0}^{n} \begin{bmatrix} 2n+1 \backslash n-k \end{bmatrix} P(2k+1, x \mid q) = x^{2n+1}. \tag{5.13}
\]

Equation (5.12) is true because by (4.11) the coefficient of \( x^{2m} \) is
\[
\sum_{j=0}^{n-m} (-1)^j q^{j\binom{j}{2}} \frac{[2n]![2m+2j][2m+j-1]!}{[n-m-j]![n+m+j]![j]![2m]!} \\
= \frac{[2n]!}{[2m]![n-m]!} \sum_{j=0}^{n-m} (-1)^j q^{j\binom{j}{2}} [2m+2j] \left[ \begin{array}{l} n-m \\ j \end{array} \right] \frac{[2m+j-1]!}{[n-m+2m+j]!} \\
= \frac{[2n]!(1-q)^{n-m}}{[2m]![n-m]!} \sum_{j=0}^{n-m} (-1)^j q^{j\binom{j}{2}} (1-q^2)^{j\left[\binom{j}{2}+m\right]} \frac{1}{(q^j q^{2m};q)_{n-m+1}} = [m = n].
\]

In the same way we get for the coefficient of \(x^{2m+1}\) in (5.13)
\[
\sum_{j=0}^{n-m} (-1)^j q^{j\binom{j}{2}} \frac{[2n+1]![2m+2j+1][2m+j]!}{[n-m-j]![n+m+j+1]![j]![2m+1]!} \\
= \frac{[2n+1]!}{[2m+1]![n-m]!} \sum_{j=0}^{n-m} (-1)^j q^{j\binom{j}{2}} [2m+2j+1] \left[ \begin{array}{l} n-m \\ j \end{array} \right] \frac{[2m+j]!}{[n-m+2m+j+1]!} \\
= \frac{[2n+1]!(1-q)^{n-m}}{[2m+1]![n-m]!} \sum_{j=0}^{n-m} (-1)^j q^{j\binom{j}{2}} (1-q^2)^{j\left[\binom{j}{2}+m\right]} \frac{1}{(q^j q^{2m+1};q)_{n-m+1}} = [m = n].
\]

Identity (2.17) has the following \(q\) – analogue.
Define a linear functional on the polynomials by \(F(x^n) = A(n \mid q)\).
Then we have
\[
F(P(n, x \mid q)) = [n = 0]. \tag{5.14}
\]
Note however that these polynomials are not orthogonal with respect to \(F\), since e.g.
\[
F(P(1, x, q)P(3, x, q)) = F(x(x^3 - [3]x)) = F(x^4 - [3]F(x^2) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - [3] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = q^4 - q \neq 0.
\]
To prove (5.14) we use the identities (5.12) and (5.13). It is clear that \(F\left(P(2n+1, x \mid q)\right) = 0\), since it contains only odd powers of \(x\).

By induction we get from (5.12)
\[
\begin{bmatrix} \frac{2n}{n} \end{bmatrix} F \left( P(0, x | q) \right) + \begin{bmatrix} \frac{2n}{0} \end{bmatrix} F \left( P(2n, x | q) \right) = F \left( x^{2n} \right) = \begin{bmatrix} \frac{2n}{n} \end{bmatrix}
\]
and therefore

\[ F \left( P(2n, x | q) \right) = 0 \text{ for } n > 0. \]

6. Generalized central binomial coefficients and Catalan numbers

As examples I want to consider the sequences \( \left( \frac{k!(2n)!}{n!(n+k)!} \right)_{k \geq 0} \) for \( k \in \mathbb{N} \), which give a common generalization of the central binomial coefficients \( (k=0) \) and the Catalan numbers \( (k=1) \).

In general \( f(n,k) = \frac{k!(2n)!}{n!(n+k)!} \) is not an integer but \( f(0,k) = f(k,k) = 1 \).

E.g. \( (f(n,2)) = \left( \frac{2}{3}, \frac{14}{3}, 12, 33, \frac{286}{3}, \ldots \right) \) or
\( (f(n,7)) = \left( \frac{1}{4}, \frac{1}{6}, \frac{7}{33}, \frac{1}{13}, 1, \frac{17}{4}, \ldots \right) \)

More generally let

\[ a(n,k,\ell) = \frac{k!(2n+2k)!(2k+\ell)!}{(n+2k+\ell)!(n+k)!(2k)!} = 4^n \left( \frac{k + \frac{1}{2}}{2} \right)_n \]

(6.1)

By (4.1) this is the limit for \( q \rightarrow 1 \) of

\[ a(n,k,\ell,q) = \frac{(q^{2k+1}; q^2)_n}{(q^{4k+2+\ell}; q^2)_n} = \frac{1}{\prod_{j=2k+1}^{2n+2k} (1+q^j)^{[[k]]}[[2n+2k]]![[2k+\ell]]!} \]

\[ \prod_{j=1}^{n} \left[ j \right]_q^2 \text{ with } \left[ j \right]_q^2 = \frac{1-q^{2j}}{1-q^2}. \]

Here we use \( [[n]]! \) to denote \( \prod_{j=1}^{n} \left[ j \right]_q^2 \).

In this case we get by (4.4)

\[ A_n := T(2n, q^{2k-1}, q^{2k+2r-1}, q^2) = \frac{q^{2n}(1-q^{2n+2k-1})(1-q^{2n+4k+2r})}{(1-q^{4n+4k+2r})(1-q^{4n+4k+2r+2})}, \]
\[ C_n := T(2n-1, a, b, q) = \frac{q^{2n}(1-q^{2n})(1-q^{2n+2k+2r-1})}{(1-q^{4n+4k+2r})(1-q^{4n+4k+2r+2})}. \]

(6.2)
For $n = 0$ we get
\[ A_0 := T(0, q^{2k-1}, q^{2k+2^i-1}, q^2) = \frac{(1 - q^{2^i})}{(1 - q^{2^{k+2^i}})}. \]

If we let $q \to 1$ we see that for $a(n, k, \ell)$ we get
\begin{align*}
T(0, k, \ell) &= \frac{2(2k + 1)}{(2k + \ell + 1)}, \quad (6.3) \\
T(2n, k, \ell) &= \frac{2(2n + 2k + 1)(n + 2k + \ell)}{(2n + 2k + \ell + 1)(2n + 2k + \ell)} \quad (6.4)
\end{align*}

and
\begin{align*}
T(2n - 1, k, \ell) &= \frac{2n(2n + 2k + 2\ell - 1)}{(2n + 2k + \ell)(2n + 2k + \ell - 1)}. \quad (6.5)
\end{align*}

**Theorem 1**

For each integer $k \geq 1$
\[
\det \left( \frac{(2k - 1)! (2i + j + 2k)!}{(k - 1)! (i + j + k)! (i + j + 2k)!} \right)_{i,j=0}^{n-1} = \prod_{j=0}^{k-2} \binom{n+k+j}{k} \prod_{j=0}^{k-2} \binom{k+j}{k}.
\]

**Proof**

To prove this we consider for integers $k \geq 0$ the sequence
\[ Z_{n,k} = a(n, k, 0) = \frac{k!(2n+2k)!}{(n+2k)!(n+k)!}, \quad (6.7) \]

depending on the parameter $k$. Notice that $Z_{0,k} = 1$.

For $k = 0$ we get the central binomial coefficients $Z_{n,0} = \binom{2n}{n}$, for $k = 1$ the Catalan numbers $C_{n+1} = \frac{1}{n+2} \binom{2n+2}{n+1}$. For general $k$ we have $Z_{n,k} = \frac{k!}{(2k)!} S(n+k,k)$, where $S(m,n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}$ are the super Catalan numbers considered in [4].

In this case (6.4) and (6.5) reduce to
\[ T(2j,k) = \frac{j + 2k}{j + k}, \quad T(2j - 1,k) = \frac{j}{j + k}. \] \hspace{1cm} (6.8)

For the sequence \( \{Z_{nk}\} \) we get by (2.9)
\[ s(j,k) = 2, \quad t(j,k) = \frac{j + 1}{j + 1 + k} - \frac{j + 2k}{j + k}. \] \hspace{1cm} (6.9)

From this we derive for \( k \geq 1 \)
\[ d(n,k) = \text{det}(a(i + j,k,0))_{i,j=0}^{n-1} = \prod_{i=1}^{n-1} \prod_{j=0}^{n-1} t(j,k) = \prod_{i=1}^{n-1} \prod_{j=0}^{n-1} \frac{j + 1}{j + 1 + k} - \frac{j + 2k}{j + k}. \]

This can be simplified to give
\[ \text{det}(Z_{nk})_{i,j=0}^{n-1} = \frac{1}{2k - 1} \prod_{i=0}^{k-2} \frac{n + k + i}{k - k + i}. \] \hspace{1cm} (6.10)

If instead of \( Z_{nk} \) we consider the integers \( \binom{2k - 1}{k} \) \( Z_{nk} \) we get (6.6).

For the sequence \( \{b(n,k,\ell)\} \) we get by (2.9)
\[ s(0,k,l) = T(0,k,l) = a(1,k,\ell) = \frac{2(2k + 1)}{2k + 1 + \ell}, \]
\[ s(n,k,l) = T(2n - 1,k,l) + T(2n,k,l) = 2 + \frac{2\ell(2k + \ell - 1)}{(2n + 2k + \ell)(2n + 2k + \ell + 1)}, \] \hspace{1cm} (6.11)
\[ t(n,k,l) = T(2n,k,l)T(2n + 1,k,l) = \frac{4(n + 1)(2k + n + \ell)(2k + 2n + 1)(2k + 2n + 2\ell + 1)}{(2k + 2n + \ell + 1)^2(2k + 2n + \ell)(2k + 2n + \ell + 2)}. \]

Let
\[ d(n,k,\ell) := \text{det}(a(i + j,k,\ell))_{i,j=0}^{n-1}. \] \hspace{1cm} (6.12)

Now consider
\[ a(n,0,k) = \frac{k!(2n)!}{n!(n+k)!}. \] \hspace{1cm} (6.13)
In this case we get the simpler formula

\[ T(j,0,k) = \frac{(j+1)(j+2k)}{(j+k)(j+k+1)}. \]  
(6.14)

We know that

\[
\frac{d(n,0,\ell+1)}{d(n,0,\ell)} = \prod_{i=1}^{n-1} \prod_{j=0}^{\ell-1} \frac{t(j,0,\ell+1)}{t(j,0,\ell)}.
\]

By (6.11) we have

\[
\prod_{j=0}^{\ell-1} t(j,0,\ell+1) = \prod_{j=0}^{\ell-1} \frac{(j+\ell+1)(2j+2\ell+3)}{(2j+\ell+2)^2} \frac{(2j+\ell+1)(2j+\ell+2)}{(j+\ell)(2j+2\ell+1)}
\]

\[
= \frac{(\ell+\ell+i)(2i+2\ell+1)(\ell+1)}{(2\ell+1)(2i+\ell+1)(2i+\ell)}
\]

Thus

\[
\prod_{i=1}^{n-1} \frac{t(j,0,\ell+1)}{t(j,0,\ell)} = \frac{n!}{2^n} \frac{(\ell+1)!}{(\ell+\ell+i-1)!} \frac{(2n+2\ell-i)!}{(\ell+\ell+i)!} 
\]

\[
\frac{(\ell+1)}{(2\ell+1)} \frac{(\ell+n-1)!}{(\ell+1)!} \frac{(2n+2\ell)!}{(\ell+2\ell+1)!} 
\]

\[
= \frac{(\ell+1)}{(2\ell+1)} \frac{(\ell-1)!}{(2\ell-1)!} \frac{(2n+2\ell-1)!}{(2\ell-1)!}
\]

We have therefore proved that

\[
\frac{d(n,0,\ell+1)}{d(n,0,\ell)} = \frac{1}{2^n} \frac{(\ell+1)}{(2\ell+1)} \frac{(2n+2\ell-1)!}{(\ell-1)!} \frac{(\ell+n-1)!}{(\ell+1)!} 
\]

\[
\frac{(2n+2\ell-1)!}{(\ell-1)!} \frac{(2n+\ell-1)!}{(2\ell-1)!}.
\]

Since \( d(n,0,1) = 1 \) we get

\[
d(n,0,r+1) = \prod_{i=1}^{r} \frac{1}{2^n} \frac{(\ell+1)}{(2\ell+1)} \frac{(2n+2\ell-i)!}{(\ell-1)!} 
\]

\[
\frac{(2n+2\ell-i)!}{(\ell-1)!} \frac{(2n+\ell-i)!}{(2\ell-1)!}
\]

\[
= \frac{1}{2^n} \left( \frac{2^{r+1}}{2r+2} \right)^n \prod_{i=1}^{r} \frac{(2n+2\ell-i)!}{(\ell-1)!} \frac{(2n+\ell-i)!}{(2\ell-1)!} = \frac{1}{\left( \frac{2r+2}{r+1} \right)^n} \prod_{i=1}^{r} \frac{(2n+2\ell-i)!}{(\ell-1)!} \frac{(2n+\ell-i)!}{(2\ell-1)!}
\]

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Consider now the sequences

$$C_{n,r} = \binom{2r+1}{r} a(n,0,r+1) = \frac{(2r+1)!}{r!} \frac{(2n)!}{n!(n+r+1)!}$$

(6.15)

which have been studied by Gessel [4].

Here we get

**Theorem 2**

$$\det(C_{i+j,r})_{i,j=0}^{r-1} = \prod_{\ell=1}^{r} \frac{(2n+2\ell-1)!(\ell-1)!}{(2n+\ell-1)!(2\ell-1)!} = \prod_{i=0}^{2r-1} \frac{2n+2r-1-2i}{2n+2i} \frac{2r-1-2i}{2i}.$$  

(6.16)

More generally with the same method we find that

$$\frac{d(n,k,r)}{d(n,k,0)} = \left( \frac{2k+r}{k} \right)^n \prod_{\ell=0}^{r-1} \frac{(k+\ell-1)!(2k+2n+2\ell-1)!(2k+\ell+n-1)!}{(2k+2n+\ell-1)!(2k+2\ell-1)!(k+n+\ell-1)!}.$$  

(6.17)

**References**


