How to guess and prove explicit formulas for some Hankel determinants

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Abstract

In many papers all hints how the results have been found are eliminated. In this note I want to reverse this procedure and sketch how in some cases explicit formulas for Hankel determinants can be guessed and proved.

1. Theoretical background

Let $(a(n))$ be a sequence of real numbers with $a(0)=1.$
It is often easy to guess the Hankel determinants $\det(a(i+j))_{i,j=0}^{n-1}.$ Fortunately in some cases it is even possible to prove the guessed results by guessing some more results.

We make use of the following results:
1) If all Hankel determinants $\det(a(i+j))_{i,j=0}^{n-1}\neq 0$ then the polynomials

$$p(n,x)=\frac{1}{\det(a(i+j))_{i,j=0}^{n-1}}\det\begin{pmatrix} a(0) & a(1) & \cdots & a(n-1) & 1 \\ a(1) & a(2) & \cdots & a(n) & x \\ a(2) & a(3) & \cdots & a(n+1) & x^2 \\ \vdots & & & & \vdots \\ a(n) & a(n+1) & \cdots & a(2n-1) & x^n \end{pmatrix} \tag{1.1}$$

are orthogonal with respect to the linear functional F defined by

$$F(x^n)=a(n). \tag{1.2}$$

This means that $F\left(p(n,x)p(m,x)\right)=0$ if $m\neq n$ and $F\left(p(n,x)^2\right)\neq 0.$
In particular for $m=0$ we get

$$F\left(p(n,x)\right)=[n=0]. \tag{1.3}$$

These identities also characterize the linear functional $F.$

By Favard's theorem there exist numbers $s(n),t(n)$ such that

$$p(n,x)=(x-s(n-1))p(n-1,x)-t(n-2)p(n-2,x). \tag{1.4}$$

2) If for given sequences $s(n)$ and $t(n)$ we define $a(n,j)$ by

$$\begin{aligned} &a(0,j)=[j=0] \\ &a(n,0)=s(0)a(n-1,0)+t(0)a(n-1,1) \\ &a(n,j)=a(n-1,j-1)+s(j)a(n-1,j)+t(j)a(n-1,j+1) \end{aligned} \tag{1.5}$$

then the Hankel determinant $\det\left(a(i+j,0)\right)_{i,j=0}^{n-1}$ is given by

$$\det\left(a(i+j)\right)_{i,j=0}^{n-1}=\prod_{i=1}^{n-1}\prod_{j=0}^{i-1}t(j). \tag{1.6}$$

So the Hankel determinant $\det\left(a(i+j,0)\right)_{i,j=0}^{n-1}$ only depends on the sequence $\left(t(n)\right)$.

Thus if we start with $\left(a(n)\right)$ and guess all $s(n),t(n)$ and $a(n,j)$, then our guesses lead to an exact proof, if (1.5) holds and $a(n,0)=a(n)$. In this case we also have

$$\sum_{k=0}^{n}a(n,k)p(k,x)=x^n. \tag{1.7}$$

This situation is well- known (cf. e.g. [3] and the literature cited there). It is especially useful if for a given sequence $\left(a(n)\right)_{n\geq 0}$ closed expressions for $s(n),t(n)$ and $a(n,j)$ can be found.

2. A simple example

Let us consider the sequence

$$a(n)=\binom{2n}{n}. \tag{2.1}$$

After computing (with Mathematica) the first Hankel determinants

```
Table[Det[Table[Binomial[2 i + 2 j, i + j], {i, 0, n - 1}, {j, 0, n - 1}]], {n, 1, 8}]
{1, 2, 4, 8, 16, 32, 64, 128}
```

we guess that $\det\left(\binom{2i+2j}{i+j}\right)_{i,j=0}^{n-1}=2^{n-1}.$

First we compute the orthogonal polynomials associated with $a(n)=\binom{2n}{n}.$ We get

```
p[n_, x_] :=
 Expand[Det[Table[If[j < n, Binomial[2 i + 2 j, i + j], x^i], {i, 0, n}, {j, 0, n}]] /
   Det[Table[Binomial[2 i + 2 j, i + j], {i, 0, n - 1}, {j, 0, n - 1}]]];
p[0, x_] := 1
```

```
Table[p[n, x], {n, 0, 5}]
```

$\{1, -2+x, 2-4x+x^2, -2+9x-6x^2+x^3, 2-16x+20x^2-8x^3+x^4, -2+25x-50x^2+35x^3-10x^4+x^5\}$

Then we compute the corresponding $s(n), t(n)$ and get

```
Table[Factor[PolynomialQuotient[p[n + 1, x], p[n, x], x]], {n, 0, 5}]
```

$\{-2+x, -2+x, -2+x, -2+x, -2+x, -2+x\}$

```
t[n_] := -Factor[PolynomialRemainder[p[n + 2, x], p[n + 1, x], x] / p[n, x]]
```

```
Table[t[n], {n, 0, 7}]
```

$\{2, 1, 1, 1, 1, 1, 1, 1\}$

Thus our guess is that $s(n) = 2$, $t(0) = 2$ and $t(n) = 1$ for $n > 0$.

Therefore the polynomials $p(n, x)$ satisfy the recursion

$$p(n,x) = (x-2)p(n-1,x) - p(n-2,x) \tag{2.2}$$

for $n > 2$.

But how can we be sure that our guesses are correct?

To this end we must compute $a(n, j)$ defined by (1.5).

This gives

```
a[n_, j_] :=
 If[n == 0, If[j == 0, 1, 0], If[j == 0, 2 a[n - 1, 0] + 2 a[n - 1, 1], a[n - 1, j - 1] + 2 a[n - 1, j] + a[n - 1, j + 1]]]
```

```
TableForm[Table[a[n, j], {n, 0, 6}, {j, 0, n}]]
```

1						
2	1					
6	4	1				
20	15	6	1			
70	56	28	8	1		
252	210	120	45	10	1	
924	792	495	220	66	12	1

It is now easy to guess that

$$a(n, j) = \binom{2n}{n-j}. \qquad (2.3)$$

If you are not so familiar with binomial coefficients then it suffices to look into The Online Encyclopedia of Integer Sequences (OEIS) [7]. There this array is listed under A094527.

Now using the recursion for the binomial coefficients it is immediately verified that with $a(n, j) = \binom{2n}{n-j}$ the equations (1.5) are satisfied.

Therefore $a(n,0) = a(n)$ and our guesses are correct.

The coefficients of the corresponding orthogonal polynomials $p(n,x)$ can also be guessed or found in OEIS [7], A110162 and A127677. A simpler method will be given in the next paragraph. They are given by $p(0,x) = 1$ and

$$p(n,x) = (-1)^n \left(2 + \sum_{j=1}^{n} (-1)^j \frac{n}{j} \binom{n+j-1}{2j-1} x^j \right) \qquad (2.4)$$

for $n > 0.$

This can be written in hypergeometric form

$$p(n,x) = 2(-1)^n \sum_{j=0}^{n} \frac{(-n)_j (n)_j}{j! \left(\frac{1}{2}\right)_j 4^j} x^j = 2(-1)^n \, {}_2F_1\left(\begin{matrix} -n, n \\ \frac{1}{2} \end{matrix} ; \frac{x}{4} \right). \qquad (2.5)$$

Here $(x)_n$ is defined by $(x)_n = x(x+1)\cdots(x+n-1).$ Details about notation and other standard properties of hypergeometric polynomials can be found in [1] and [6].

Remark

Guessing can often be simplified when all $s(n)=0$. In this case identity (1.5) reduces to

$$\begin{aligned} &A(0,j)=[j=0]\\ &A(n,0)=T(0)A(n-1,1)\\ &A(n,j)=A(n-1,j-1)+T(j)A(n-1,j+1). \end{aligned} \tag{2.6}$$

There we have $A(2n,2j+1)=A(2n+1,2j)=0$ for all n,j.

A useful identity is

$$\sum_{k=0}^{n}(-1)^k A(2n,2k)\prod_{j=0}^{k-1}T(2j)=[n=0]. \tag{2.7}$$

For the proof let $T(-1)=0$ and observe that $A(2n,-2)=A(2n,2n+2)=0$. We have

$$A(2n,2k)=A(2n-1,2k-1)+T(2k)A(2n-1,2k+1)$$

$$=A(2n-2,2k-2)+T(2k-1)A(2n-2,2k)+T(2k)A(2n-2,2k)+T(2k)T(2k+1)A(2n-2,2k+2)$$

Therefore we get

$$\sum_{k=0}^{n+1}(-1)^k A(2n+2,2k)\prod_{j=0}^{k-1}T(2j)$$

$$=\sum_{k=0}^{n+1}(-1)^k\left(A(2n,2k-2)+\left(T(2k-1)+T(2k)\right)A(2n,2k)+T(2k)T(2k+1)a(2n,2k+2)\prod_{j=0}^{k-1}T(2j)\right.$$

$$=\sum_{k=0}^{n}(-1)^k A(2n,2k)\prod_{j=0}^{k-1}T(2j)\left(T(2k-1)+T(2k)-T(2k)-T(2k-1)\right)=0.$$

If we define

$$a(n,j)=A(2n,2j), \tag{2.8}$$

then it is easily verified that (1.5) holds with

$$\begin{aligned} &s(0)=T(0),\\ &s(n)=T(2n-1)+T(2n),\\ &t(n)=T(2n)T(2n+1). \end{aligned} \tag{2.9}$$

Furthermore we have

$$p(n,x) = P(2n,\sqrt{x}). \tag{2.10}$$

This follows immediately from

$$P(2n,x) = xP(2n-1,x) - T(2n-2)P(2n-2,x)$$
$$= (x^2 - T(2n-2))P(2n-2,x) - xT(2n-3)P(2n-3,x),$$
$$P(2n-2,x) = xP(2n-3,x) - T(2n-4)P(2n-4,x)$$

Eliminating $xP(2n-3,x)$ we get

$$P(2n,x) = xP(2n-1,x) - T(2n-2)P(2n-2,x)$$
$$= (x^2 - T(2n-2) - T(2n-3))P(2n-2,x) - T(2n-3)T(2n-4)P(2n-4,x)$$
$$= (x^2 - s(n-1))P(2n-2,x) - t(n-2)P(2n-4,x).$$

For the above example we set

$$A(2n) = \binom{2n}{n}, A(2n+1) = 0. \tag{2.11}$$

Then we get $T(0) = 2$ and $T(n) = 1$ for $n > 0$.

Therefore the Hankel determinants are $\det\left(A(i+j,0)\right)_{i,j=0}^{n-1} = 2^{n-1}$.

The table $\left(A(n,j)\right)$ is given by OEIS, A108044,

1								
0	1							
2	0	1						
0	3	0	1					
6	0	4	0	1				
0	10	0	5	0	1			
20	0	15	0	6	0	1		
0	35	0	21	0	7	0	1	
70	0	56	0	28	0	8	0	1

Here we have $A(2n,2k) = \binom{2n}{n-k}, A(2n+1,2k+1) = \binom{2n+1}{n-k}.$

Here again (2.6) can be easily verified.

The corresponding orthogonal polynomials satisfy

$P(n,x) = xP(n-1,x) - P(n-2,x)$
with initial values $P(0,x)=1$, $P(1,x)=x$ and $P(2,x)=x^2-2$.

Recall that the Lucas polynomials

$$L(n,x,s) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} s^k \tag{2.12}$$

are characterized by $L(0,x,s)=2, L(1,x,s)=x$ and the recursion

$$L(n,x,s) = xL(n-1,x,s) + sL(n-2,x,s). \tag{2.13}$$

The sequence $L(n,x,-1)$ begins with

```
Expand[Table[l[n, x, -1], {n, 0, 6}]]
{2, x, -2 + x^2, -3 x + x^3, 2 - 4 x^2 + x^4, 5 x - 5 x^3 + x^5, -2 + 9 x^2 - 6 x^4 + x^6}
```

Since the Lucas polynomials $L(n,x,-1)$ satisfy the same recurrence and initial values $L(1,x,-1)=x$ and $L(2,x,-1)=x^2-1$ we conclude that for $n>0$ these polynomials coincide with the Lucas polynomials

$$P(n,x) = L(n,x,-1) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-k} \binom{n-k}{k} (-1)^k x^{n-2k}. \tag{2.14}$$

Equation (1.7) reduces in this case to

$$\begin{aligned} \sum_{k=0}^{n} \binom{2n}{n-k} L(2k,x,-1) &= x^{2n}, \\ \sum_{k=0}^{n} \binom{2n+1}{n-k} L(2k+1,x,-1) &= x^{2n+1}. \end{aligned} \tag{2.15}$$

The linear functional F defined by

$$F(x^n) = A(n) \tag{2.16}$$

is also uniquely determined by

$$F(P(n,x)) = F(L(n,x,-1)) = 0 \tag{2.17}$$

for $n>1$.

Now we can also express the orthogonal polynomials (2.4) in terms of the Lucas polynomials. For (2.10) gives that

$$p(n,x)=L\left(2n,\sqrt{x},-1\right). \tag{2.18}$$

Thus

$$p(n,x)=\sum_{k=0}^{n}(-1)^k\frac{2n}{2n-k}\binom{2n-k}{k}x^{n-k}=(-1)^n\sum_{k=0}^{n}(-1)^k\frac{2n}{n+k}\binom{n+k}{n-k}x^k$$
$$=(-1)^n\left(2+\sum_{k=1}^{n}(-1)^k\frac{n}{k}\binom{n+k-1}{2k-1}x^k\right).$$

3. Central trinomial coefficients

As a slight generalization we consider the generalized trinomial coefficients

$$a(n)=[x^n]\left(1+ax+bx^2\right)^n \tag{3.1}$$

which for $a=2, b=1$ reduce to $\binom{2n}{n}$.

From

```
d[n_] := Det[Table[Coefficient[(1 + a x + b x^2)^(i + j), x, i + j], {i, 0, n - 1}, {j, 0, n - 1}]]
Table[d[n], {n, 1, 5}]
{1, 2 b, 4 b^3, 8 b^6, 16 b^10}
```

we guess that

$$d(n)=2^{n-1}b^{\binom{n}{2}}. \tag{3.2}$$

We have

$$\left(1+ax+bx^2\right)^n=\sum_{j=0}^{n}\binom{n}{j}(ax)^j\sum_{k=0}^{j}\binom{j}{k}\left(\frac{b}{a}x\right)^k=\sum_{j,k}\binom{n}{j}\binom{j}{k}a^{j-k}b^kx^{k+j}.$$

Therefore we get

$$[x^{n+m}]\left(1+ax+bx^2\right)^n=\sum_{j+k=n+m}\binom{n}{j}\binom{j}{k}a^{j-k}b^k=\sum_k\frac{n!a^{n+m-2k}b^k}{(k-m)!k!(n+m-2k)!}$$

This gives

$$a(n)=[x^n]\left(1+ax+bx^2\right)^n=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}\binom{2k}{k}a^{n-2k}b^k$$

and

$$[x^{n+1}]\left(1+ax+bx^2\right)^n=\sum_k\frac{n!a^{n+1-2k}b^k}{(k-1)!k!(n+1-2k)!}$$

$$[x^{n-1}]\left(1+ax+bx^2\right)^n=\sum_k\frac{n!a^{n-1-2k}b^k}{(k+1)!k!(n-1-2k)!}=\sum_k\frac{n!a^{n+1-2k}b^{k-1}}{(k-1)!k!(n+1-2k)!}$$

which implies that

$$[x^{n+1}]\left(1+ax+bx^2\right)^n=b[x^{n-1}]\left(1+ax+bx^2\right)^n. \tag{3.3}$$

We compute the corresponding $s(n), t(n)$.

```
h[n_] := Coefficient[(1 + a x + b x^2)^n, x, n]
d[n_] := Factor[Det[Table[h[i + j], {i, 0, n - 1}, {j, 0, n - 1}]]]; d[0] := 1
a[n_, i_, j_] := If[j < n, h[i + j], x^i]
p[n_] := Factor[Det[Table[a[n, i, j], {i, 0, n}, {j, 0, n}]] / d[n]]
p[0] := 1

Table[Factor[PolynomialQuotient[p[n + 1], p[n], x]], {n, 0, 5}]
{-a + x, -a + x, -a + x, -a + x, -a + x, -a + x}
t[n_] := -Factor[PolynomialRemainder[p[n + 2], p[n + 1], x] / p[n]]
Table[t[n], {n, 0, 6}]
{2 b, b, b, b, b, b, b}
```

Therefore we guess that

$$s(n) = a$$
$$t(0) = 2b, t(n) = b \tag{3.4}$$

If this guess is correct we immediately get (3.2).
Define now $a(n,k)$ by (1.5).
We get

```
a[n_, k_] :=
 If[n == 0, If[k == 0, 1, 0], If[k == 0, a a[n - 1, 0] + 2 b a[n - 1, 1], a[n - 1, k - 1] + a a[n - 1, k] + b a[n - 1, k + 1]]]
```

```
TableForm[Expand[Table[a[n, k], {n, 0, 4}, {k, 0, n}]]]
```

1				
a	1			
$a^2 + 2b$	2 a	1		
$a^3 + 6ab$	$3a^2 + 3b$	3 a	1	
$a^4 + 12a^2b + 6b^2$	$4a^3 + 12ab$	$6a^2 + 4b$	4 a	1

Now we have also

```
TableForm[Table[Coefficient[(1 + a x + b x^2)^n, x, n - k], {n, 0, 4}, {k, 0, n}]]
```

1				
a	1			
$a^2 + 2b$	2 a	1		
$a^3 + 6ab$	$3a^2 + 3b$	3 a	1	
$a^4 + 12a^2b + 6b^2$	$4a^3 + 12ab$	$6a^2 + 4b$	4 a	1

Therefore we guess that $a(n,k) = [x^{n-k}](1+ax+bx^2)^n$.

This can again easily be verified:
For $k \geq 1$ we have

$$a(n,k) = [x^{n-k}](1+ax+bx^2)^n = [x^{n-k}]\left(1+ax+bx^2\right)(1+ax+bx^2)^{n-1}$$
$$= [x^{n-1-(k-1)}](1+ax+bx^2)^{n-1} + a[x^{n-1-k}](1+ax+bx^2)^{n-1} + b[x^{n-1-(k+1)}](1+ax+bx^2)^{n-1}$$
$$= a(n-1,k-1) + aa(n-1,k) + ba(n-1,k+1).$$

For $k = 0$ we get

$$a(n,0)=[x^{n-1-(-1)}](1+ax+bx^2)^{n-1}+a[x^{n-1-0}](1+ax+bx^2)^{n-1}+b[x^{n-1-(1)}](1+ax+bx^2)^{n-1}$$
$$=ba(n-1,1)+aa(n-1,0)+ba(n-1,1)=aa(n-1,0)+2ba(n-1,1).$$

This completes the proof.

It should be mentioned that for this example the sequence $A(n)$ with $A(2n)=a(n), A(2n+1)=0$ gives no simplification.
For in this case we get from (2.9) and (3.4) that

$$T(2n)=a-T(2n-1), T(2n+1)=\frac{b}{a-T(2n-1)}$$

with initial values $T(0)=a$ and $T(1)=\frac{2b}{a}.$

For the central trinomial coefficients $(a=b=1)$ we therefore get for

$$D(n)=\det\left(A(i+j)\right)_{i,j=0}^{n-1}$$

the rather curious sequence of determinants

$$D(1)=D(2)=D(3)=1, D(4)=2, D(6n+5)=(-1)^{n-1}2^{6n+2}, D(6n+6)=(-1)^{n-1}2^{6n+3},$$
$$D(6n+7)=(-1)^{n-1}2^{6n+5}, D(6n+8)=D(6n+9)=(-1)^{n-1}2^{6n+6}, D(6n+10)=2^{6n+7}.$$

4. Our main example

In [3] I have considered in some detail the example $a(n)=\frac{(b;q)_n}{(a;q)_n}$ where $(x;q)_n=\prod_{j=0}^{n-1}(1-q^jx).$

After completion of [3] the paper [5] appeared where analogous results have been derived with other methods. In that paper the authors considered the slightly modified sequence $a(n)=\frac{(aq;q)_n}{(abq^2;q)_n}$. This has the advantage that the corresponding orthogonal polynomials are the little $q-$Jacobi polynomials in the usual notation.

Since my aim in this paper is to show how you can guess Hankel determinants I want to sketch the above method in this case in some detail. I am using Mathematica for the computations. I also use the $q-$notations $[n]=\frac{1-q^n}{1-q}$, $[n]!=[1][2]\cdots[n]$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}=\frac{[n]!}{[k]![n-k]!} \text{ for } 0\le k\le n \text{ and } \begin{bmatrix} n \\ k \end{bmatrix}=0 \text{ for } k<0 \text{ and } k>n.$$

For later use we observe that

$$\lim_{q\to 1}\frac{(q^{\alpha+1};q)_n}{(q^{\alpha+\beta+2};q)_n}=\frac{(\alpha+1)_n}{(\alpha+\beta+2)_n}. \tag{4.1}$$

Let

$$a(n)=\frac{(aq;q)_n}{(abq^2;q)_n}. \tag{4.2}$$

Define $A(2n)=a(n), A(2n+1)=0.$ Then we compute the corresponding first polynomials $P(n,x,a,b)$ and the corresponding Favard recurrences. This gives the first values of $T(n,a,b,q)$:

$$\frac{-1+a\,q}{-1+a\,b\,q^2},\ \frac{a\,(-1+q)\,q\,(-1+b\,q)}{(-1+a\,b\,q^2)\,(-1+a\,b\,q^3)},\ \frac{q\,(-1+a\,q^2)\,(-1+a\,b\,q^2)}{(-1+a\,b\,q^3)\,(-1+a\,b\,q^4)},$$
$$\frac{a\,(-1+q)\,q^2\,(1+q)\,(-1+b\,q^2)}{(-1+a\,b\,q^4)\,(-1+a\,b\,q^5)},\ \frac{q^2\,(-1+a\,q^3)\,(-1+a\,b\,q^3)}{(-1+a\,b\,q^5)\,(-1+a\,b\,q^6)},$$
$$\frac{a\,(-1+q)\,q^3\,(1+q+q^2)\,(-1+b\,q^3)}{(-1+a\,b\,q^6)\,(-1+a\,b\,q^7)}$$

We guess that

$$A_0=\frac{1-aq}{1-abq^2} \tag{4.3}$$

and for $n>0$

$$\begin{aligned} A_n &:= T(2n,a,b,q)=\frac{q^n(1-q^{n+1}a)(1-q^{n+1}ab)}{(1-q^{2n+1}ab)(1-q^{2n+2}ab)},\\ C_n &:= T(2n-1,a,b,q)=\frac{aq^n(1-q^n)(1-q^nb)}{(1-q^{2n+1}ab)(1-q^{2n}ab)}. \end{aligned} \tag{4.4}$$

By (2.9) the orthogonal polynomials $p(n,x,a,b)$ corresponding to the original sequence $(a(n))$ satisfy the recurrence

$$xp(n,x,a,b)=p(n+1,x,a,b)+(A_n+C_n)p(n,x,a,b)+A_{n-1}C_n p(n-1,x,a,b). \tag{4.5}$$

It turns out that this is the normalized recurrence relation [5], (14.12.4), for the little $q-$Jacobi polynomials (cf. [5], (14.12.1))

$$p_n(x;a,b|q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx\right) = \sum_{j=0}^{n} \frac{\left(q^{-n};q\right)_j \left(abq^{n+1};q\right)_j}{\left(aq;q\right)_j \left(q;q\right)_j} q^j x^j. \quad (4.6)$$

We could of course also have guessed this result directly.

Let $r(n,j) = \dfrac{[x^{j+1}]p(n,x,a,b)}{[x^j]p(n,x,a,b)}$ where $[x^j]p(x)$ denotes the coefficient of x^j in $p(x)$.

```
r[n_, j_] := Coefficient[p[n], x, j + 1] / Coefficient[p[n], x, j]
w[n_] := Factor[Table[r[n, j], {j, 0, n - 1}]]
```

```
w[4]
```

$$\left\{-\frac{(1+q)(1+q^2)(-1+abq^5)}{q^3(-1+aq)}, -\frac{(1+q+q^2)(-1+abq^6)}{q^2(1+q)(-1+aq^2)}, -\frac{(1+q)(-1+abq^7)}{q(1+q+q^2)(-1+aq^3)}, -\frac{-1+abq^8}{(1+q)(1+q^2)(-1+aq^4)}\right\}$$

```
w[5]
```

$$\left\{-\frac{(1+q+q^2+q^3+q^4)(-1+abq^6)}{q^4(-1+aq)}, -\frac{(1+q^2)(-1+abq^7)}{q^3(-1+aq^2)}, -\frac{-1+abq^8}{q^2(-1+aq^3)}, -\frac{-1+abq^9}{q(1+q^2)(-1+aq^4)}, -\frac{-1+abq^{10}}{(1+q+q^2+q^3+q^4)(-1+aq^5)}\right\}$$

From this we guess that

$$r(n,j) = \frac{q(1-q^{j-n})(1-abq^{j+n+1})}{(1-q^{j+1}a)(1-q^{j+1})}.$$

Therefore we conclude that $p(n,x,a,b)$ is the normalization of (4.6).

If our guesses are correct this would be another proof of the known fact that the moments for the linear functional F defined by $F\left(p(n,x,a,b)\right) = [n=0]$ for the normalized little $q-$Jacobi polynomials

$$p(n,x,a,b) = (-1)^n q^{\binom{n+1}{2}} \frac{(aq;q)_n}{\left(abq^{n+1};q\right)_n} {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx\right) \quad (4.7)$$

are given by

$$F\left(x^n\right) = \frac{(aq;q)_n}{(abq^2;q)_n}. \quad (4.8)$$

All that remains to be done in order to prove these results is to define $A(n, j)$ by (2.6) and to verify that $A(2n,0)=\frac{(aq;q)_n}{(abq^2;q)_n}$.

To this end we consider again some small values:

```
a[n_] := q^n (1 - a q^(n + 1)) (1 - a b q^(n + 1)) / (1 - a b q^(2 n + 1)) / (1 - a b q^(2 n + 2))
c[n_] := q^n a (1 - q^n) (1 - q^n b) / (1 - a b q^(2 n + 1)) / (1 - a b q^(2 n))
t[n_] := If[EvenQ[n], a[n / 2], c[(n + 1) / 2]]
A[n_, j_] := If[j == 0, t[0] A[n - 1, 1], A[n - 1, j - 1] + t[j] A[n - 1, j + 1]];
A[0, j_] := If[j == 0, 1, 0];
```

```
Factor[Table[A[2 n, 0], {n, 0, 3}]]
```

$$\left\{1, \frac{-1+a q}{-1+a b q^2}, \frac{(-1+a q)(-1+a q^2)}{(-1+a b q^2)(-1+a b q^3)}, \frac{(-1+a q)(-1+a q^2)(-1+a q^3)}{(-1+a b q^2)(-1+a b q^3)(-1+a b q^4)}\right\}$$

```
Factor[Table[A[2 n, 2], {n, 0, 4}]]
```

$$\left\{0, 1, \frac{(1+q)(-1+a q^2)}{-1+a b q^4}, \frac{(1+q+q^2)(-1+a q^2)(-1+a q^3)}{(-1+a b q^4)(-1+a b q^5)}, \frac{(1+q)(1+q^2)(-1+a q^2)(-1+a q^3)(-1+a q^4)}{(-1+a b q^4)(-1+a b q^5)(-1+a b q^6)}\right\}$$

```
Factor[Table[A[2 n, 4], {n, 0, 4}]]
```

$$\left\{0, 0, 1, \frac{(1+q+q^2)(-1+a q^3)}{-1+a b q^6}, \frac{(1+q^2)(1+q+q^2)(-1+a q^3)(-1+a q^4)}{(-1+a b q^6)(-1+a b q^7)}\right\}$$

We guess that

$$A(2n,2k)=\begin{bmatrix} n \\ k \end{bmatrix}\frac{\left(aq^{k+1};q\right)_{n-k}}{\left(abq^{2k+2};q\right)_{n-k}}. \tag{4.9}$$

In the same way we guess that

$$A(2n+1,2k+1)=\begin{bmatrix} n \\ k \end{bmatrix}\frac{\left(aq^{k+2};q\right)_{n-k}}{\left(abq^{2k+3};q\right)_{n-k}}. \tag{4.10}$$

It remains to verify (2.6).

$A(2n,0)=T(0,a,b,q)A(2n-1,1)$ reduces to the trivial identity

$$\frac{\left(aq;q\right)_n}{\left(abq^2;q\right)_n}=\frac{(1-aq)}{(1-abq^2)}\frac{\left(aq^2;q\right)_{n-1}}{\left(abq^3;q\right)_{n-1}}.$$

$A(2n,2j)=A(2n-1,2j-1)+T(2j)A(2n-1,2j+1)$ means

$$\begin{bmatrix}n\\j\end{bmatrix}\frac{\left(aq^{j+1};q\right)_{n-j}}{\left(abq^{2j+2};q\right)_{n-j}}=\begin{bmatrix}n-1\\j-1\end{bmatrix}\frac{\left(aq^{j+1};q\right)_{n-j}}{\left(abq^{2j+1};q\right)_{n-j}}$$
$$+\frac{q^j(1-q^{j+1}a)(1-q^{j+1}ab)}{(1-q^{2j+1}ab)(1-q^{2j+2}ab)}\begin{bmatrix}n-1\\j\end{bmatrix}\frac{\left(aq^{j+2};q\right)_{n-1-j}}{\left(abq^{2j+3};q\right)_{n-1-j}}$$

The right-hand side equals

$$\begin{bmatrix}n-1\\j-1\end{bmatrix}\frac{\left(aq^{j+1};q\right)_{n-j}}{\left(abq^{2j+2};q\right)_{n-j}}\left(\frac{(1-abq^{n+j+1})}{(1-abq^{2j+1})}+\frac{[n-j]q^j(1-q^{j+1}ab)}{[j](1-abq^{2j+1})}\right)$$
$$=\begin{bmatrix}n-1\\j-1\end{bmatrix}\frac{\left(aq^{j+1};q\right)_{n-j}}{\left(abq^{2j+2};q\right)_{n-j}}\frac{[n]-abq^{2j+1}\left([j]q^{n-j}+[n-j]\right)}{[j](1-abq^{2j+1})}=\begin{bmatrix}n\\j\end{bmatrix}\frac{\left(aq^{j+1};q\right)_{n-j}}{\left(abq^{2j+2};q\right)_{n-j}}.$$

In the same way we see that $A(2n+1,2j-1)=A(2n,2j-2)+T(2j-1)A(2n,2j)$:

$$A(2n,2j-2)+T(2j-1)A(2n,2j)=\begin{bmatrix}n\\j-1\end{bmatrix}\frac{\left(aq^j;q\right)_{n-j+1}}{\left(abq^{2j};q\right)_{n-j+1}}$$
$$+\frac{aq^j(1-q^j)(1-q^jb)}{(1-q^{2j+1}ab)(1-q^{2j}ab)}\begin{bmatrix}n\\j\end{bmatrix}\frac{\left(aq^{j+1};q\right)_{n-j}}{\left(abq^{2j+2};q\right)_{n-j}}$$
$$=\begin{bmatrix}n\\j-1\end{bmatrix}\frac{\left(aq^{j+1};q\right)_{n-j+1}}{\left(abq^{2j+1};q\right)_{n-j+1}}\left(\frac{(1-aq^j)(1-abq^{n+j+1})}{(1-aq^{n+1})(1-abq^{2j})}+\frac{aq^j(1-q^{n-j+1})(1-bq^j)}{(1-aq^{n+1})(1-abq^{2j})}\right)$$
$$=\begin{bmatrix}n\\j-1\end{bmatrix}\frac{\left(aq^{j+1};q\right)_{n-j+1}}{\left(abq^{2j+1};q\right)_{n-j+1}}=A(2n+1,2j-1).$$

Later we will need the concrete form of (2.7) for this case.
By (4.3) und (4.4) we get

$$\prod_{j=0}^{k-1} T(2j) = q^{\binom{k}{2}} \frac{(aq;q)_k (abq^2;q)_{k-1}}{(abq^2;q)_{2k-1}} = q^{\binom{k}{2}} \frac{(aq;q)_k}{(abq^{k+1};q)_k}$$

Thus (2.7) implies

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{(aq^{k+1};q)_{n-k}}{(abq^{2k+2};q)_{n-k}} q^{\binom{k}{2}} \frac{(aq;q)_k}{(abq^{k+1};q)_k} = [n=0]$$

or equivalently

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(1-abq^{2k+1})(aq;q)_n}{(abq^{k+1};q)_{n+1}} = [n=0]$$

Dividing by $(a;q)_n$ and replacing $ab \to cq^{-1}$ we get the following useful well-known result:

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(1-cq^{2k})}{(cq^k;q)_{n+1}} = [n=0]. \tag{4.11}$$

For the sequence $a(n) = \dfrac{(aq;q)_n}{(abq^2;q)_n}$ we get

$$t(n,a,b,q) = A_n C_{n+1} = \frac{aq^{2n+1}(1-q^{n+1})(1-q^{n+1}b)(1-q^{n+1}a)(1-q^{n+1}ab)}{(1-q^{2n+1}ab)(1-q^{2n+2}ab)^2(1-q^{2n+3}ab)}. \tag{4.12}$$

For $n=0$ this reduces to

$$t(0,a,b,q) = \frac{aq(1-q)(1-qb)(1-qa)}{(1-q^2ab)^2(1-q^3ab)}. \tag{4.13}$$

Using (1.6) we can compute the corresponding Hankel determinants. This has been done in [3] and [5].

5. A simple special case

By changing $q \to q^2, a = b \to q^{-1}$ we get

$$a(n) = \frac{\left(q;q^2\right)_n}{\left(q^2;q^2\right)_n} = \frac{(1-q)(1-q^3)\cdots(1-q^{2n-1})}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} = \frac{1}{\prod_{j=1}^{n}(1+q^j)^2}\begin{bmatrix}2n\\ n\end{bmatrix} = (-1)^n \begin{bmatrix}-\frac{1}{2}\\ n\end{bmatrix}_{q\to q^2}. \tag{5.1}$$

If we let q tend to 1 we get $a(n) = \frac{1}{4^n}\binom{2n}{n}.$

Now observe that if we change the sequence $a(n) \to r^n a(n),$ then the following functions change too: $s(n) \to rs(n), t(n) \to r^2 t(n), a(n,k) \to r^{n-k} a(n,k)$ and $p(n,x) \to r^n p\left(n, \frac{x}{r}\right).$

Therefore we get again the results about $\binom{2n}{n}$ which we obtained in the beginning.

In this special case direct guessing is simpler:

Consider $A(2n) = \frac{\left(q;q^2\right)_n}{\left(q^2;q^2\right)_n}, A(2n+1) = 0.$

In this case we get

$$T(0) = \frac{1}{1+q} \tag{5.2}$$

and for $n > 0$

$$T(n) = \frac{q^n}{(1+q^n)(1+q^{n+1})}. \tag{5.3}$$

This gives

$$\det\left(A(i+j)\right)_{i,j=0}^{n-1} = \frac{q^{\binom{n}{3}}}{\prod_{k=1}^{n}(1+q^k)^{2n-2k-1}}. \tag{5.4}$$

The orthogonal polynomials are given by

$$P(n,x) = xP(n-1,x) - T(n-2)P(n-2,x)$$

with initial values $P(0,x)=1$, $P(1,x)=x$ and $P(2,x)=x^2-\frac{1}{1+q}$.

We compute these polynomials and get

```
t[0] := 1 / (1 + q)

t[n_] := q^n / (1 + q^n) / (1 + q^(n + 1))

p[n_] := p[n] = x p[n - 1] - t[n - 2] p[n - 2]; p[0] := 1; p[1] := x
```

Since for $q \to 1$ we get the Lucas polynomials we compare the coefficients of $p(n,x)$ with the natural $q-$analogue of the coefficients of the Lucas polynomials.

```
c[n_, k_] := Factor[Coefficient[p[n], x, n - 2 k]]

cc[n_, k_] := qbin[n - k, k, q] qbin[n, 1, q] / qbin[n - k, 1, q]


Factor[Table[c[n, 1] / cc[n, 1], {n, 2, 5}]]
```

$$\left\{-\frac{1}{(1+q)^2}, -\frac{1}{(1+q)(1+q^2)}, -\frac{1}{(1+q)^2(1-q+q^2)}, -\frac{1}{(1+q)(1+q^4)}\right\}$$

```
Factor[Table[c[n, 2] / cc[n, 2], {n, 4, 6}]]
```

$$\left\{\frac{q^2}{(1+q)^2(1+q^2)^2(1-q+q^2)}, \frac{q^2}{(1+q)^2(1+q^2)(1-q+q^2)(1+q^4)}, \frac{q^2}{(1+q)^2(1+q^2)(1+q^4)(1-q+q^2-q^3+q^4)}\right\}$$

```
Factor[Table[c[n, 3] / cc[n, 3], {n, 6, 8}]]
```

$$\left\{-\frac{q^6}{(1+q)^4(1+q^2)(1-q+q^2)^2(1+q^4)(1-q+q^2-q^3+q^4)},\right.$$
$$-\frac{q^6}{(1+q)^3(1+q^2)^2(1-q+q^2)(1+q^4)(1-q^2+q^4)(1-q+q^2-q^3+q^4)},$$
$$\left.-\frac{q^6}{(1+q)^4(1+q^2)^2(1-q+q^2)(1-q^2+q^4)(1-q+q^2-q^3+q^4)(1-q+q^2-q^3+q^4-q^5+q^6)}\right\}$$

This leads to the conjecture

$$P(n,x)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^k q^{2\binom{k}{2}}\frac{[n]}{[n-k]}\begin{bmatrix} n-k \\ k \end{bmatrix}\frac{x^{n-2k}}{\prod_{j=1}^{k}(1+q^j)(1+q^{n-j})}, \tag{5.5}$$

which can be easily verified.

For the sequence

$$a(n)=\frac{1}{\prod_{j=1}^{n}(1+q^j)^2}\begin{bmatrix} 2n \\ n \end{bmatrix} \tag{5.6}$$

we get in the same way as above

$$\det\left(a(i+j)\right)_{i,j=0}^{n-1}=\frac{q^{\frac{n(n-1)(4n-5)}{6}}}{\prod_{k=1}^{n-1}(1+q^k)^{2n-k-1}}. \tag{5.7}$$

Remark

It is natural to ask for analogous results for the sequence $A(n\,|\,q)$ defined by

$$A(2n\,|\,q)=\begin{bmatrix} 2n \\ n \end{bmatrix}, A(2n+1\,|\,q)=0. \tag{5.8}$$

But unfortunately here we get no simple formulas for the Hankel determinants. This can already be seen by computing the first values of $t(n)$ which are

$$\left\{1+q,\ \frac{q\,(-1+q+q^2+q^3)}{1+q},\ \frac{q\,(1+q^2)\,(-2-q+q^3+q^4+2\,q^5+q^6)}{(1+q)\,(-1+q+q^2+q^3)},\right.$$
$$\left.\frac{q\,(1+q)\,(-2-2\,q-2\,q^2+q^3+2\,q^4+q^5+q^6-2\,q^7-3\,q^8-4\,q^9+q^{11}+4\,q^{12}+3\,q^{13}+3\,q^{14}+2\,q^{15}+q^{16})}{(1+q^2)\,(-1+q+q^2+q^3)\,(-2-q+q^3+q^4+2\,q^5+q^6)}\right\}$$

There are also no simple formulas for the corresponding orthogonal polynomials.

But there is an interesting $q-$analogue of (2.15).

Let

$$A(2n,2k\,|\,q)=\begin{bmatrix}2n\\n-k\end{bmatrix}, A(2n+1,2k\,|\,q)=0,$$
$$A(2n+1,2k+1\,|\,q)=\begin{bmatrix}2n+1\\n-k\end{bmatrix}, A(2n,2k+1\,|\,q)=0. \tag{5.9}$$

We want to compute the uniquely determined polynomials $P(n,x\,|\,q)$ which satisfy

$$\sum_{k=0}^{n} A(n,k\,|\,q)P(k,x\,|\,q)=x^n. \tag{5.10}$$

```
a[n_, k_, q_] := If[EvenQ[n - k], qbin[n, (n - k) / 2, q], 0]

p[n_, x_, q_] := x^n - Sum[a[n, k, q] p[k, x, q], {k, 0, n - 1}]; p[0, x_, q_] := 1

Expand[Table[p[n, x, q], {n, 0, 4}]]
{1, x, -1 - q + x^2, -x - q x - q^2 x + x^3, q + q^3 - x^2 - q x^2 - q^2 x^2 - q^3 x^2 + x^4}
```

These polynomials are also $q-$analogues of the Lucas polynomials. With the same reasoning as above we guess that

$$P(n,x\,|\,q)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^k q^{\binom{k}{2}}\frac{[n]}{[n-k]}\begin{bmatrix}n-k\\k\end{bmatrix}x^{n-2k} \tag{5.11}$$

for $n>0$. For $n>0$ these are the $q-$Lucas polynomials $Luc_n(x,-1)$ which I have studied in [2].

To prove (5.10) we have to show that

$$\sum_{k=0}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}P(2k,x\,|\,q)=x^{2n} \tag{5.12}$$

and

$$\sum_{k=0}^{n}\begin{bmatrix}2n+1\\n-k\end{bmatrix}P(2k+1,x\,|\,q)=x^{2n+1}. \tag{5.13}$$

Equation (5.12) is true because by (4.11) the coefficient of x^{2m} is

$$\sum_{j=0}^{n-m}(-1)^j q^{\binom{j}{2}} \frac{[2n]![2m+2j][2m+j-1]!}{[n-m-j]![n+m+j]![j]![2m]!}$$

$$= \frac{[2n]!}{[2m]![n-m]!}\sum_{j=0}^{n-m}(-1)^j q^{\binom{j}{2}}[2m+2j]\begin{bmatrix} n-m \\ j \end{bmatrix}\frac{[2m+j-1]!}{[n-m+2m+j]!}$$

$$= \frac{[2n]!(1-q)^{n-m}}{[2m]![n-m]!}\sum_{j=0}^{n-m}(-1)^j q^{\binom{j}{2}}(1-q^{2j}q^{2m}\begin{bmatrix} n-m \\ j \end{bmatrix}\frac{1}{\left(q^j q^{2m};q\right)_{n-m+1}} = [m=n].$$

In the same way we get for the coefficient of x^{2m+1} in (5.13)

$$\sum_{j=0}^{n-m}(-1)^j q^{\binom{j}{2}} \frac{[2n+1]![2m+2j+1][2m+j]!}{[n-m-j]![n+m+j+1]![j]![2m+1]!}$$

$$= \frac{[2n+1]!}{[2m+1]![n-m]!}\sum_{j=0}^{n-m}(-1)^j q^{\binom{j}{2}}[2m+2j+1]\begin{bmatrix} n-m \\ j \end{bmatrix}\frac{[2m+j]!}{[n-m+2m+j+1]!}$$

$$= \frac{[2n+1]!(1-q)^{n-m}}{[2m+1]![n-m]!}\sum_{j=0}^{n-m}(-1)^j q^{\binom{j}{2}}(1-q^{2j}q^{2m+1}\begin{bmatrix} n-m \\ j \end{bmatrix}\frac{1}{\left(q^j q^{2m+1};q\right)_{n-m+1}} = [m=n].$$

Identity (2.17) has the following $q-$analogue.

Define a linear functional on the polynomials by $F(x^n) = A(n\,|\,q).$

Then we have

$$F(P(n,x\,|\,q)) = \left[n=0\right]. \qquad (5.14)$$

Note however that these polynomials are not orthogonal with respect to $F,$ since e.g.

$$F(P(1,x,q)P(3,x,q)) = F(x(x^3-[3]x)) = F(x^4)-[3]F(x^2) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - [3]\begin{bmatrix} 2 \\ 1 \end{bmatrix} = q^4 - q \neq 0.$$

To prove (5.14) we use the identities (5.12) and (5.13). It is clear that $F\left(P(2n+1,x\,|\,q)\right) = 0,$ since it contains only odd powers of $x.$

By induction we get from (5.12)

$$\begin{bmatrix}2n\\n\end{bmatrix}F\left(P(0,x\,|\,q)\right)+\begin{bmatrix}2n\\0\end{bmatrix}F\left(P(2n,x\,|\,q)\right)=F\left(x^{2n}\right)=\begin{bmatrix}2n\\n\end{bmatrix}$$

and therefore

$$F\left(P(2n,x\,|\,q)\right)=0 \text{ for } n>0.$$

6. Generalized central binomial coefficients and Catalan numbers

As examples I want to consider the sequences $\left(\frac{k!(2n)!}{n!(n+k)!}\right)_{n\geq 0}$ for $k\in\mathbb{N}$, which give a common generalization of the central binomial coefficients ($k=0$) and the Catalan numbers ($k=1$).
In general $f(n,k)=\frac{k!(2n)!}{n!(n+k)!}$ is not an integer but $f(0,k)=f(k,k)=1$.
E.g. $\left(f(n,2)\right)=\left(1,\frac{2}{3},1,2,\frac{14}{3},12,33,\frac{286}{3},\cdots\right)$ or
$\left(f(n,7)\right)=\left(1,\frac{1}{4},\frac{1}{6},\frac{1}{6},\frac{7}{33},\frac{7}{22},\frac{7}{13},1,2,\frac{17}{4},\cdots\right)$

More generally let

$$a(n,k,\ell)=\frac{k!(2n+2k)!(2k+\ell)!}{(n+2k+\ell)!(n+k)!(2k)!}=4^n\frac{\left(k+\frac{1}{2}\right)_n}{(2k+\ell+1)_n} \tag{6.1}$$

By (4.1) this is the limit for $q\to 1$ of

$$a(n,k,\ell,q)=\frac{(q^{2k+1};q^2)_n}{(q^{4k+2\ell+2};q^2)_n}=\frac{1}{\prod\limits_{j=2k+1}^{2n+2k}(1+q^j)}\frac{[[k]]![[2n+2k]]![[2k+\ell]]!}{[[n+2k+\ell]]![[n+k]]![[2k]]!}$$

Here we use $[[n]]!$ to denote $\prod\limits_{j=1}^{n}[j]_{q^2}$ with $[j]_{q^2}=\frac{1-q^{2j}}{1-q^2}$.

In this case we get by (4.4)

$$\begin{aligned}A_n&:=T(2n,q^{2k-1},q^{2k+2\ell-1},q^2)=\frac{q^{2n}(1-q^{2n+2k+1})(1-q^{2n+4k+2\ell})}{(1-q^{4n+4k+2\ell})(1-q^{4n+2+4k+2\ell})},\\ C_n&:=T(2n-1,a,b,q)=\frac{q^{2n+2k-1}(1-q^{2n})(1-q^{2n+2k+2\ell-1})}{(1-q^{4n+4k+2\ell})(1-q^{4n+2+4k+2\ell})}.\end{aligned} \tag{6.2}$$

For $n=0$ we get

$A_0 := T(0, q^{2k-1}, q^{2k+2\ell-1}, q^2) = \frac{(1-q^{2k+1})}{(1-q^{2+4k+2\ell})}.$

If we let $q \to 1$ we see that for $a(n,k,\ell)$ we get

$$T(0,k,\ell) = \frac{2(2k+1)}{(2k+\ell+1)}, \tag{6.3}$$

$$T(2n,k,\ell) = \frac{2(2n+2k+1)(n+2k+\ell)}{(2n+2k+\ell+1)(2n+2k+\ell)} \tag{6.4}$$

and

$$T(2n-1,k,\ell) = \frac{2n(2n+2k+2\ell-1)}{(2n+2k+\ell)(2n+2k+\ell-1)}. \tag{6.5}$$

Theorem 1

For each integer $k \geq 1$

$$\det\left(\frac{(2k-1)!(2i+2j+2k)!}{(k-1)!(i+j+k)!(i+j+2k)!}\right)_{i,j=0}^{n-1} = \frac{\prod_{j=0}^{k-2}\binom{n+k+j}{k}}{\prod_{j=0}^{k-2}\binom{k+j}{k}}. \tag{6.6}$$

Proof

To prove this we consider for integers $k \geq 0$ the sequence

$$Z_{n,k} = a(n,k,0) = \frac{k!(2n+2k)!}{(n+2k)!(n+k)!} \tag{6.7}$$

depending on the parameter k. Notice that $Z_{0,k} = 1$.

For $k=0$ we get the central binomial coefficients $Z_{n,0} = \binom{2n}{n}$, for $k=1$ the Catalan numbers $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$. For general k we have $Z_{n,k} = \frac{k!}{(2k)!}S(n+k,k)$, where $S(m,n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}$ are the super Catalan numbers considered in [4].

In this case (6.4) and (6.5) reduce to

$$T(2j,k)=\frac{j+2k}{j+k}, T(2j-1,k)=\frac{j}{j+k}. \tag{6.8}$$

For the sequence $\left(Z_{n,k}\right)$ we get by (2.9)

$$s(j,k)=2, t(j,k)=\frac{j+1}{j+1+k}\frac{j+2k}{j+k} \tag{6.9}$$

From this we derive for $k\geq 1$

$$d(n,k)=\det\left(a(i+j,k,0)\right)_{i,j=0}^{n-1}=\prod_{i=1}^{n-1}\prod_{j=0}^{i-1}t(j,k)=\prod_{i=1}^{n-1}\prod_{j=0}^{i-1}\frac{j+1}{j+1+k}\frac{j+2k}{j+k}$$

$$=\prod_{i=1}^{n-1}\frac{i!k!(2k+i-1)!(k-1)!}{(k+i)!(2k-1)!(k+i-1)!}=\prod_{i=1}^{n-1}\frac{\binom{2k+i-1}{k}}{\binom{k+i}{k}}\frac{k!(k-1)!}{(2k-1)!}=\frac{1}{\binom{2k-1}{k}^{n-1}}\prod_{i=1}^{n-1}\frac{\binom{2k+i-1}{k}}{\binom{k+i}{k}}.$$

This can be simplified to give

$$\det\left(Z_{i+j,k}\right)_{i,j=0}^{n-1}=\frac{1}{\binom{2k-1}{k}^{n}}\prod_{i=0}^{k-2}\frac{\binom{n+k+i}{k}}{\binom{k+i}{k}}. \tag{6.10}$$

If instead of $Z_{n,k}$ we consider the integers $\binom{2k-1}{k}Z_{n,k}$ we get (6.6).

For the sequence $\left(b(n,k,\ell)\right)$ we get by (2.9)

$$s(0,k,l)=T(0,k,l)=a(1,k,\ell)=\frac{2(2k+1)}{2k+1+\ell}$$

$$s(n,k,l)=T(2n-1,k,l)+T(2n,k,l)=2+\frac{2\ell(2k+\ell-1)}{(2n+2k+\ell-1)(2n+2k+\ell+1)} \tag{6.11}$$

$$t(n,k,l)=T(2n,k,l)T(2n+1,k,l)=\frac{4(n+1)(2k+n+\ell)(2k+2n+1)(2k+2n+2\ell+1)}{(2k+2n+\ell+1)^2(2k+2n+\ell)(2k+2n+\ell+2)}$$

Let

$$d(n,k,\ell):=\det\left(a(i+j,k,\ell)\right)_{i,j=0}^{n-1}. \tag{6.12}$$

Now consider

$$a(n,0,k)=\frac{k!(2n)!}{n!(n+k)!}. \tag{6.13}$$

In this case we get the simpler formula

$$T(j,0,k)=\frac{(j+1)(j+2k)}{(j+k)(j+k+1)}. \tag{6.14}$$

We know that

$$\frac{d(n,0,\ell+1)}{d(n,0,\ell)}=\prod_{i=1}^{n-1}\prod_{j=0}^{i-1}\frac{t(j,0,\ell+1)}{t(j,0,\ell)}.$$

By (6.11) we have

$$\prod_{j=0}^{i-1}\frac{t(j,0,\ell+1)}{t(j,0,\ell)}=\prod_{j=0}^{i-1}\frac{(j+\ell+1)(2j+2\ell+3)}{(2j+\ell+2)^2(2j+\ell+3)}\frac{(2j+\ell+1)(2j+\ell)(2j+\ell+2)}{(j+\ell)(2j+2\ell+1)}$$
$$=\frac{(\ell+i)(2i+2\ell+1)(\ell+1)}{(2\ell+1)(2i+\ell+1)(2i+\ell)}$$

Thus

$$\prod_{i=1}^{n-1}\frac{(\ell+i)(2i+2\ell+1)(\ell+1)}{(2\ell+1)(2i+\ell+1)(2i+\ell)}=\prod_{i=1}^{n-1}\frac{(\ell+i)(2i+2\ell+1)(\ell+1)}{(2\ell+1)(2i+\ell+1)(2i+\ell)}$$
$$=\left(\frac{\ell+1}{2\ell+1}\right)^{n-1}\frac{(\ell+n-1)!}{(\ell)!}\frac{(\ell+1)!(2n+2\ell)!2^{\ell+1}(\ell+1)!}{(\ell+2n-1)!(2\ell+2)!2^{n+\ell}(n+\ell)!}$$
$$=\left(\frac{\ell+1}{2\ell+1}\right)^{n}\frac{(\ell-1)!(2n+2\ell-1)!}{2^n(2n+\ell-1)!(2\ell-1)!}$$

We have therefore proved that

$$\frac{d(n,0,\ell+1)}{d(n,0,\ell)}=\frac{1}{2^n}\left(\frac{\ell+1}{2\ell+1}\right)^{n}\frac{(2n+2\ell-1)!(\ell-1)!}{(2n+\ell-1)!(2\ell-1)!}.$$

Since $d(n,0,1)=1$ we get

$$d(n,0,r+1)=\prod_{\ell=1}^{r}\frac{1}{2^n}\left(\frac{\ell+1}{2\ell+1}\right)^{n}\frac{(2n+2\ell-1)!(\ell-1)!}{(2n+\ell-1)!(2\ell-1)!}$$
$$=\frac{1}{2^{nr}}\left(\frac{2^{r+1}}{\binom{2r+2}{r+1}}\right)^{n}\prod_{\ell=1}^{r}\frac{(2n+2\ell-1)!(\ell-1)!}{(2n+\ell-1)!(2\ell-1)!}=\frac{1}{\binom{2r+1}{r}^{n}}\prod_{\ell=1}^{r}\frac{(2n+2\ell-1)!(\ell-1)!}{(2n+\ell-1)!(2\ell-1)!}$$

Consider now the sequences

$$C_{n,r} = \binom{2r+1}{r} a(n,0,r+1) = \frac{(2r+1)!}{r!} \frac{(2n)!}{n!(n+r+1)!} \qquad (6.15)$$

which have been studied by Gessel [4].
Here we get

Theorem 2

$$\det\left(C_{i+j,r}\right)_{i,j=0}^{n-1} = \prod_{\ell=1}^{r} \frac{(2n+2\ell-1)!(\ell-1)!}{(2n+\ell-1)!(2\ell-1)!} = \prod_{i=0}^{\left\lfloor \frac{2r-1}{4} \right\rfloor} \frac{\binom{2n+2r-1-2i}{2n+2i}}{\binom{2r-1-2i}{2i}}. \qquad (6.16)$$

More generally with the same method we find that

$$\frac{d(n,k,r)}{d(n,k,0)} = \left(\frac{\binom{2k+r}{k}}{\binom{2k+2r}{k+r}} \right)^{n} \prod_{\ell=0}^{r-1} \frac{(k+\ell-1)!(2k+2n+2\ell-1)!(2k+\ell+n-1)!}{(2k+2n+\ell-1)!(2k+2\ell-1)!(k+n+\ell-1)!}. \qquad (6.17)$$

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