

# Ramanujan's $q$ -continued fractions and Schröder-like numbers

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## Abstract

In a recent paper G. Bhatnagar has given simple proofs of some of Ramanujan's continued fractions. In this note we show that some variants of these continued fractions are generating functions of  $q$ -Schröder-like numbers.

## 1. Introduction

In a recent "tutorial" Gaurav Bhatnagar [3] has given simple proofs of some of Ramanujan's (convergent)  $q$ -continued fractions by using an elementary method of Euler. I had not been aware of these continued fractions before but came across similar formulae in the study of formal power series which are generating functions of Schröder-like numbers and their  $q$ -analogues (cf. [4]). The purpose of this note is to call attention to these connections and to give simple proofs of the corresponding continued fractions from this point of view.

A well-known example of the following story is the sequence of little Schröder numbers  $(s_n)_{n \geq 0} = (1, 1, 3, 11, 45, 197, \dots)$  (cf. [6], A001003) whose generating function

$$f(z) = \sum_{n \geq 0} s_n z^n \tag{1.1}$$

satisfies

$$f(z) = 1 - zf(z) + 2zf(z)^2. \tag{1.2}$$

The computation of Hankel determinants for Schröder numbers leads to the continued fraction

$$f(z) = \frac{1}{1-z} - \frac{2z^2}{1-3z} - \frac{2z^2}{1-3z} - \frac{2z^2}{1-3z} - \dots \tag{1.3}$$

but there are also other interesting continued fractions for the little Schröder numbers (cf. [6], A001003):

$$f(z) = \frac{1}{1+z} - \frac{2z}{1+z} - \frac{2z}{1+z} - \frac{2z}{1+z} - \dots, \tag{1.4}$$

$$f(z) = \frac{1}{1 - \frac{z}{1 - \frac{z}{1 - \frac{z}{1 - \dots}}}} \quad (1.5)$$

and

$$f(z) = \frac{1}{1 - \frac{z}{1 - \frac{2z}{1 - \frac{z}{1 - \frac{2z}{1 - \frac{z}{1 - \dots}}}}}}. \quad (1.6)$$

These will appear as special cases of the following considerations.

## 2. Generating functions of $q$ -Schröder-like numbers

Some of the following results have been obtained in [4]. We repeat them in order to make the exposition self-contained.

Let  $x, y$  be real or complex numbers and  $z$  an indeterminate.

Let the formal power series

$$F(z) = F(z, x, y) = \sum_{n \geq 0} A(n, x, y) z^n \quad (2.1)$$

satisfy the identity

$$F(z) = 1 + xzF(z) + yzF(z)F(qz). \quad (2.2)$$

This implies that

$$A(n, x, y) = xA(n-1, x, y) + y \sum_{k=0}^{n-1} A(k, x, y) q^k A(n-1-k, x, y) \quad \text{with } A(0, x, y) = 1.$$

We are mainly interested in the series

$$f(z) = f(z, x, y) = \sum_{n \geq 0} a(n, x, y) z^n = \frac{x + yF(z, x, y)}{x + y}. \quad (2.3)$$

It is easily verified that it satisfies the equation

$$f(z, x, y) = 1 - xzf(qz, x, y) + (x + y)zf(z, x, y)f(qz, x, y). \quad (2.4)$$

Its coefficients are given by

$$a(n, x, y) = -q^{n-1}xa(n-1, x, y) + (x + y) \sum_{k=0}^{n-1} a(k, x, y) q^k a(n-1-k, x, y) \quad \text{with } a(0, x, y) = 1.$$

The sequence  $(A(n, 1, q))$  is a  $q$ -analogue of the (large) Schröder numbers and the sequence  $(a(n, 1, q))$  is a  $q$ -analogue of the little Schröder numbers. The numbers  $A(n, 1, q)$  have been studied from a combinatorial point of view in [1]. We call  $A(n, x, y)$  and  $a(n, x, y)$   $q$ -Schröder-like numbers. They are polynomials in  $x$  and  $y$ .

The numbers  $A(n, 0, 1) = a(n, 0, 1) = C_n(q)$  are the Carlitz  $q$ -Catalan numbers. For  $q = 1$  they

reduce to the Catalan numbers  $C_n = \frac{1}{n+1} \binom{2n}{n}$ .

Equation (2.2) implies immediately the expansion of the formal power series  $F(z, x, y)$  into a continued fraction

$$F(z, x, y) = \frac{1}{1 - xz - yzF(qz, x, y)} = \frac{1}{1 - xz - \frac{yz}{1 - qxz - \frac{qyz}{1 - q^2xz - \dots}}}. \quad (2.5)$$

Using (2.3) it is easily verified that

$$F(z, x, y) = 1 + (x + y)zF(z, x, y)f(qz, x, y). \quad (2.6)$$

Consider the uniquely determined series  $h(z) = h(z, x, y) = 1 + \sum_{n \geq 1} h_n(x, y)z^n$  which satisfies

$$F(z, x, y) = \frac{h(qz, x, y)}{h(z, x, y)}. \quad (2.7)$$

From the defining equation for  $F(z)$  we get  $\frac{h(qz)}{h(z)} = 1 + xz \frac{h(qz)}{h(z)} + yz \frac{h(qz)}{h(z)} \frac{h(q^2z)}{h(qz)}$

and therefore  $h(qz) = h(z) + xzh(qz) + yzh(q^2z)$ .

Comparing coefficients we get

$$(q^n - 1)h_n = q^{n-1}(x + q^{n-1}y)h_{n-1}.$$

This implies

$$h_k(x, y) = q^{\binom{k}{2}} \frac{(x + y)(x + qy) \cdots (x + q^{k-1}y)}{(q - 1)(q^2 - 1) \cdots (q^k - 1)} \quad (2.8)$$

and thus

$$h(z, x, y) = \sum_{k \geq 0} q^{\binom{k}{2}} \frac{(x + y)(x + qy) \cdots (x + q^{k-1}y)}{(1 - q)(1 - q^2) \cdots (1 - q^k)} (-z)^k. \quad (2.9)$$

The identity

$$(x + y)h(z, x, y) = xh(z, x, y) + yh(qz, x, y),$$

implies

$$f(z, x, y) = \frac{x + yF(z, x, y)}{x + y} = \frac{h(z, x, y)}{h(z, x, y)}. \quad (2.10)$$

Observing that

$$\begin{aligned} F(z, x, y)f(qz, x, y) &= \frac{h(qz, x, y)}{h(z, x, y)} \frac{h(qz, x, qy)}{h(qz, x, y)} = \frac{h(qz, x, qy)}{h(z, x, y)} \\ &= \frac{h(z, x, qy)}{h(z, x, y)} \frac{h(qz, x, qy)}{h(z, x, qy)} = f(z, x, y)F(z, x, qy) \end{aligned} \quad (2.11)$$

we see that (2.4) and (2.6) can be written in the form

$$F(z, x, y) = 1 + (x + y)zF(z, x, y)f(qz, x, y) \quad (2.12)$$

and

$$f(z, x, y) = 1 + yzf(z, x, y)F(z, x, qy). \quad (2.13)$$

The last line follows from (2.3) and (2.11):

$$f(z, x, y) = \frac{x + yF(z, x, y)}{x + y} = \frac{x + y + y(F(z, x, y) - 1)}{x + y} = 1 + yzF(z, x, y)f(qz, x, y).$$

We shall also need the formula

$$h(z, x, y) = \left(1 + \frac{xz}{q}\right)h(z, x, qy) - \frac{z}{q}(x + qy)h\left(z, x, q^2y\right). \quad (2.14)$$

For the proof it suffices to show that

$$h_k(x, y) = (-1)^k q^{\binom{k}{2}} \frac{(x + y)(x + qy) \cdots (x + q^{k-1}y)}{(1 - q)(1 - q^2) \cdots (1 - q^k)} \quad (2.15)$$

satisfies

$$h_k(x, y) = h_k(x, qy) + \frac{x}{q}h_{k-1}(x, qy) - (x + qy)\frac{1}{q}h_{k-1}(x, q^2y). \quad (2.16)$$

This is equivalent with

$$\begin{aligned} q^{\binom{k}{2}} \frac{(x + y)(x + qy) \cdots (x + q^{k-1}y)}{(1 - q)(1 - q^2) \cdots (1 - q^k)} &= q^{\binom{k}{2}} \frac{(x + qy)(x + qy) \cdots (x + q^k y)}{(1 - q)(1 - q^2) \cdots (1 - q^k)} \\ -q^{\binom{k-1}{2}} \frac{x}{q} \frac{(x + qy)(x + q^2y) \cdots (x + q^{k-1}y)}{(1 - q)(1 - q^2) \cdots (1 - q^{k-1})} &+ (x + qy)q^{\binom{k-1}{2}-1} \frac{(x + q^2y)(x + q^3y) \cdots (x + q^k y)}{(1 - q)(1 - q^2) \cdots (1 - q^{k-1})} \end{aligned}$$

or equivalently

$$q^k(x + y) = q^k(x + q^k y) + x(q^k - 1) - (x + q^k y)(q^k - 1),$$

which is obviously true.

Let now  $y$  also be an indeterminate. Then we can give another characterization of  $f(z, x, y)$ :

$$f(z, x, y) = \frac{\sum_{k \geq 0} \frac{q^{k^2}}{(q; q)_k (xz; q)_k} (-yz)^k}{\sum_{k \geq 0} \frac{q^{k^2-k}}{(q; q)_k (xz; q)_k} (-yz)^k}. \quad (2.17)$$

Here as usual  $(xz; q)_k = (1 - xz)(1 - qxz) \cdots (1 - q^{k-1}xz)$ .

To prove this observe that (2.14) implies

$$f(z, x, y) = \frac{h(z, x, qy)}{h(z, x, y)} = \frac{1}{1 + \frac{xz}{q} - \frac{z}{q}(x + qy)} \frac{h(z, x, q^2 y)}{h(z, x, qy)} = \frac{1}{1 + \frac{xz}{q} - \frac{z}{q}(x + qy)} f(z, x, qy) \quad (2.18)$$

and thus also

$$f(z, x, y) = 1 - \frac{xz}{q} f(z, x, y) + \frac{z}{q}(x + qy) f(z, x, y) f(z, x, qy). \quad (2.19)$$

If we set in this equation

$$f(z, x, y) = \frac{H(z, x, qy)}{H(z, x, y)} \quad (2.20)$$

with a formal power series

$$H(z, x, y) = \sum_n H_n(x, z) y^n \quad (2.21)$$

(2.19) implies

$$H(z, x, qy) = H(z, x, y) - \frac{xz}{q} H(z, x, qy) + \frac{z}{q}(x + qy) H(z, x, q^2 y). \quad (2.22)$$

Comparing coefficients of  $y^n$  we get

$$q^n H_n = H_n - q^{n-1} xz H_n + q^{2n-1} xz H_n + q^{2n-2} z H_{n-1}$$

or

$$H_n(x, z) = - \frac{q^{2n-2} z}{(1 - q^n)(1 - q^{n-1} xz)} H_{n-1}(x, z).$$

This gives

$$H(z, x, y) = \sum_{k \geq 0} \frac{q^{k^2-k}}{(q; q)_k (xz; q)_k} (-yz)^k \quad (2.23)$$

as a formal power series in  $y$  whose coefficients are formal power series in  $z$ .

$$\text{Since } \frac{1}{(xz; q)_k} = \sum_{j \geq 0} \begin{bmatrix} k+j-1 \\ j \end{bmatrix} z^j \text{ we get } H(z, x, y) = \sum_{n \geq 0} z^n \sum_{j+k=n} \begin{bmatrix} n-1 \\ j \end{bmatrix} (-1)^k \frac{q^{k^2-k}}{(q; q)_k} y^k.$$

Here  $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$  is a  $q$ -binomial coefficient.

Thus  $H(z, x, y)$  is a formal power series in  $z$  whose coefficients are polynomials in  $y$ .

Therefore the right-hand side of (2.17) is also a formal power series in  $z$  whose coefficients are polynomials in  $y$ . Therefore it is possible in (2.17) to replace the indeterminate  $y$  by a real or complex number.

Comparing (2.20) with (2.10) we see that

$$f(z, x, y) = \frac{h(z, x, qy)}{h(z, x, y)} = \frac{H(z, x, qy)}{H(z, x, y)}. \quad (2.24)$$

This implies

$$\frac{h(z, x, qy)}{H(z, x, qy)} = \frac{h(z, x, y)}{H(z, x, y)}. \quad (2.25)$$

Since  $\frac{h(z, x, y)}{H(z, x, y)} = \sum_n c_n(x, z) y^n$  is a formal power series in  $y$  whose coefficients are formal power series in  $z$  the equation  $\sum_n c_n(x, z) y^n = \sum_n c_n(x, z) q^n y^n$  implies  $c_n(x, z) = 0$  for  $n > 0$ .

Thus

$$\frac{h(z, x, y)}{H(z, x, y)} = \frac{h(z, x, 0)}{H(z, x, 0)} = \frac{1}{e(xz)}. \quad (2.26)$$

Here  $e(z) = \sum_{n \geq 0} \frac{z^n}{(q; q)_n}$  denotes the  $q$ -exponential series which satisfies

$$\frac{1}{e(z)} = \sum_{n \geq 0} (-1)^n q^{\binom{n}{2}} \frac{z^n}{(q; q)_n}.$$

Formula (2.26) is a formal power series version of [2], Entry 9.

For  $(x, y) = (q, -q)$  we have  $h(z, q, -q) = 1$  and  $H(z, q, -q) = \sum_{k \geq 0} \frac{q^{k^2} z^k}{(q; q)_k (qz; q)_k}$ .

In this case (2.26) reduces to a well-known identity of Cauchy.

### 3. Associated continued fractions

From the results of [4], (3.20) (there is a typo; it should read  $s(n) = q^{n-1}(x + q^n(1+q)y)$ ) we can deduce the Jacobi type continued fraction for  $f(z, x, y)$  which we state without proof:

$$f(z, x, y) = \frac{1}{1 - yz} - \frac{y(x + qy)z^2}{1 - (x + q(1+q)y)z} - \frac{q^3 y(x + q^2 y)z^2}{1 - q(x + q^2(1+q)y)z} - \frac{q^6 y(x + q^3 y)z^2}{1 - q^2(x + q^3(1+q)y)z} - \dots$$

But here we are interested in other continued fractions.

a) From (2.12) and (2.13) we get

$$F(z, x, y) = \frac{1}{1 - (x + y)zf(qz, x, y)} \quad (3.1)$$

and

$$f(z, x, y) = \frac{1}{1 - yzF(z, x, qy)}. \quad (3.2)$$

This gives the following continued fraction for  $f(z, x, y)$ :

$$f(z, x, y) = \frac{1}{1 - \frac{yz}{1 - \frac{(x + qy)z}{1 - \frac{q^2 yz}{1 - \frac{q(x + q^2 y)z}{1 - \frac{q^4 yz}{1 - \dots}}}}}} \quad (3.3)$$

which generalizes (1.6).

#### Remark

This and the other results about continued fractions are of course well known and due to Ramanujan who essentially developed the right-hand side of (2.24) into a convergent continued fraction of the form (3.3). The only new fact if anything in our approach is the connection with  $q$ -analogues of Schröder numbers. We are not interested in convergence questions and use only formal power series in  $z$  instead of convergent power series in  $q$ . In this sense (3.3) can also be found in [2], (13.5) and [3], (6.5), where in our notation  $f(z, qx, -qy)$  instead of  $f(z, x, y)$  has been used.

**b)** Another continued fraction for  $f(z)$  which is related to [3], (7.1) is

$$f(z, x, y) = \frac{1}{1 + \frac{xz}{q} - \frac{\frac{z}{q}(x + qy)}{1 + \frac{xz}{q} - \frac{\frac{z}{q}(x + q^2y)}{1 + \frac{xz}{q} - \dots}} \quad (3.4)$$

which generalizes (1.4).

This is an immediate consequence of (2.18).

### Remark

Note that (3.4) is essentially Touchard's continued fraction which has been studied by Helmut Prodinger in [5]. We get

$$f(z, q, -q) = \frac{1}{1 + z - \frac{(1-q)z}{1 + z - \frac{(1-q^2)z}{1 + z - \dots}}} \quad (3.5)$$

By (2.4) or by (2.10) we derive that

$$f(z, q, -q) = h(z, q, -q^2) = \sum_n (-1)^n q^{\binom{n+1}{2}} z^n. \quad (3.6)$$

Prodinger has given a direct proof that

$$\frac{H(z, q, -q^{i+2})}{H(z, q, -q)} = \sum_{k \geq 0} q^{\binom{k+1}{2}} \begin{bmatrix} k+i \\ k \end{bmatrix} (-z)^k. \quad (3.7)$$

In our setting this follows from (2.26) since

$$\frac{H(z, q, -q^{i+2})}{H(z, q, -q)} = \frac{h(z, q, -q^{i+2})}{h(z, q, -q)} = \sum_{k \geq 0} q^{\binom{k+1}{2}} \frac{(1-q^{i+1})(1-q^{i+2}) \cdots (1-q^{i+k}y)}{(1-q)(1-q^2) \cdots (1-q^k)} (-z)^k.$$

c) Finally we derive the analogue of (1.5) (see [2], Entry 15, or [3], (5.3))

$$f(z, x, y) = \frac{1}{1 - \frac{yz}{1 - xz} - \frac{qyz}{1 - qxz} - \dots} \quad (3.8)$$

By (2.12) we have

$$f(z, x, y) = \frac{1}{1 - yzF(z, x, qy)}.$$

By (2.2)

$$F(z, x, qy) - xzF(z, x, qy) - qyzF(qz, x, qy)F(z, x, qy)$$

and therefore

$$F(z, x, qy) = \frac{1}{1 - xz - qyzF(qz, x, qy)} = \frac{1}{1 - xz - \frac{qyz}{1 - qxz} - \frac{q^2yz}{1 - q^2xz} - \dots}.$$

This implies (3.8).

As a special case we get

$$\frac{1}{1 + \frac{z}{1 - z} + \frac{qz}{1 - qz} + \frac{q^2z}{1 - q^2z} + \dots} = \sum_{n \geq 0} (-1)^n q^{\binom{n}{2}} z^n. \quad (3.9)$$

Note that for the famous Rogers-Ramanujan continued fraction

$$f(z, 0, -q) = \frac{1}{1 + \frac{qz}{1 + \frac{q^2z}{1 + \frac{q^3z}{1 + \dots}}}} \quad \text{both formulae (2.10) and (2.17) coincide.}$$

For the little  $q$ -Schröder numbers the corresponding continued fractions are

$$f(z, 1, q) = \frac{1}{1 + \frac{z/q}{1 - \frac{(1+q^2)z/q}{1 + \frac{(1+q^3)z/q}{1 + \dots}}}}, \quad (3.10)$$

$$f(z, 1, q) = \frac{1}{1 - \frac{qz}{1 - z} - \frac{q^2z}{1 - qz} - \dots}, \quad (3.11)$$

and

$$f(z, 1, q) = \frac{1}{1 - \frac{qz}{1 - \frac{(1+q^2)z}{1 - \frac{q^3z}{1 - \frac{(q+q^4)z}{1 - \dots}}}}}. \quad (3.12)$$

## References

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