Some results and conjectures about recurrence relations for certain sequences of binomial sums.

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Abstract

In a previous paper [1] I have obtained some results about recurrences of certain binomial sums. The aim was to find an explanation of the remarkable identities $\sum_{k\in\mathbb{Z}}(-1)^k\binom{n}{\left\lfloor\frac{n+5k+2}{2}\right\rfloor}=F_n$ *and* $\sum_{k\in\mathbb{Z}}(-1)^k\binom{n}{\left\lfloor\frac{n+5k}{2}\right\rfloor}=F_{n+1}$ *for Fibonacci numbers* F_n *by putting these identities into a more general context. These identities have first been obtained by I. Schur [4] in his proof of the famous Rogers-Ramanujan identities. In the meantime computer experiments have led to several conjectures about the concrete form of these recurrences, some of which are proved in this paper.*

1. Some general results

We recall the following known fact (cf. [1]):

Theorem 1

Let $m\geq 2, i\geq 1$ *be integers,* $n\in\mathbb{N}$ *and* $\ell\in\mathbb{Z}$.
The sequences

$$a(n,i,\ell,m,z)=\sum_{k\in\mathbb{Z}}\binom{n}{\left\lfloor\frac{n+ik+\ell}{m}\right\rfloor}z^k\in\mathbb{Q}\left[z,z^{-1}\right] \qquad (1)$$

satisfy linear recurrences of order i *with constant coefficients. More precisely there exist uniquely determined polynomials*

$$p_1(n,m,x,s)=x^n+\sum_{j=1}^{\left\lfloor\frac{n}{m}\right\rfloor}c_1(n,m,j)x^{n-mj}s^j \qquad (2)$$

and

$$p_k(n,m,x,s) = \sum_{j=1+\left\lfloor \frac{(k-1)n}{m} \right\rfloor}^{\left\lfloor \frac{kn}{m} \right\rfloor} c_k(n,m,j) x^{kn-mj} s^j, \tag{3}$$

with integer coefficients such that

$$\sum_{k=1}^{m-1} z^{k-1} p_k(i,m,E,1) a(n,i,\ell,m,z) = \left(\frac{1}{z} + (-1)^{m(i-1)} z^{m-1} \right) a(n,i,\ell,m,z), \tag{4}$$

where E *denotes the shift operator defined by* $E^k f(n) = f(n+k)$.

Sketch of the proof

We observe as in [3] that the recurrence relations for the binomial coefficients imply that $t(m,n,k) = \binom{n}{\left\lfloor \frac{n+k}{m} \right\rfloor}$ satisfies the recurrence relation

$$t(m,n,k) = t(m,n-1,k-m+1) + t(m,n-1,k+1). \tag{5}$$

If we introduce the shift operator K defined by $K^j f(k) = f(k-j)$, then (5) can be written in the form

$$t(m,n,k) = Et(m,n-1,k) = (K^{m-1} + K^{-1}) t(m,n-1,k) = (K^{m-1} + K^{-1})^n t(m,0,k). \tag{6}$$

This implies

$$p_j(i,m,E,1) t(m,n,k) = p_j(i,m,K^{m-1} + K^{-1},1) t(m,n,k). \tag{7}$$

It turns out that in order to prove (4) it suffices to show

$$\sum_{j=1}^{m-1} x^{(j-1)n} p_j(i,m,x^{m-1} + \frac{1}{x},1) = \frac{1}{x^i} + (-1)^{m(i-1)} x^{(m-1)i}. \tag{8}$$

For then we get from (7)

$$\sum_{j=1}^{m-1} p_j(i,m,E,1) t(m,n,k-(j-1)i) = t(m,n,k+i) + (-1)^{m(i-1)} t(m,n,k-(m-1)i) \tag{9}$$

for all $n \in \mathbb{N}$.

This means that

$$\sum_{j=1}^{m-1} p_j(i,m,E,1)\binom{n}{\left\lfloor \frac{n+k+\ell-(j-1)i}{m} \right\rfloor} = \binom{n}{\left\lfloor \frac{n+k+i+\ell}{m} \right\rfloor} + (-1)^{m(i-1)}\binom{n}{\left\lfloor \frac{n+k-(m-1)i+\ell}{m} \right\rfloor}.$$

If we replace k by ik, multiply each side by z^k and sum over all $k \in \mathbb{Z}$ we get

$$\sum_{j=1}^{m-1} z^{j-1} p_j(i,m,E,1) a(n,i,\ell,m,z) = \left(\frac{1}{z} + (-1)^{m(i-1)} z^{m-1}\right) a(n,i,\ell,m,z),$$

Identity (8) is equivalent with

$$\sum_{k=1}^{m-1} p_k(n,m,x+1,x) = 1 + (-1)^{m(n-1)} x^n. \tag{10}$$

This follows from

$$\sum_{k=1}^{m-1} x^{kn} p_k(n,m,x^{m-1}+\frac{1}{x},1) = \sum_{k=1}^{m-1}\sum_j c_k(n,m,j) x^{kn}\left(x^{m-1}+\frac{1}{x}\right)^{kn-mj}$$
$$= \sum_{k=1}^{m-1}\sum_j c_k(n,m,j) x^{mj}\left(x^m+1\right)^{kn-mj} = \sum_{k=1}^{m-1} p_k(n,m,x^m+1,x^m) = 1 + (-1)^{m(n-1)} x^{mn}.$$

For $n=0$ we set $p_k(0,m,x,s) = (-1)^{k-1}\binom{m}{k}$. Then (10) remains true for $n=0$.

It is easy to verify that the polynomials $p_k(n,m,x,s)$ of the form (3) which satisfy (10) are uniquely determined.

Let now $b(n,m,j) = (-1)^{m(n-1)-1} c_k(n,m,j)$ for $\left\lfloor \frac{(k-1)n}{m} \right\rfloor < j \le \frac{kn}{m}$.

Then (10) may be reformulated as

$$(-1)^{m(n-1)} x^n + (-1)^{m(n-1)} \sum_{k=1}^{m-1} \sum_{j=1+\left\lfloor \frac{(k-1)n}{m} \right\rfloor}^{\left\lfloor \frac{kn}{m} \right\rfloor} b(n,m,j)(x+1)^{kn-mj} x^j = (1+x)^n - 1. \tag{11}$$

If we consider the homomorphism which sends $1+x$ to a primitive root of unity ζ_n of order n, then it has been shown in [1] that (11) implies

$$\sum_{j=1}^{n} b(n,m,j)x^j = \prod_{k=1}^{n}\left(x-\zeta_n^{-mk}(\zeta_n^k-1)\right). \tag{12}$$

Let s_j be the j's elementary symmetric function of $\zeta_n^{-mk}(\zeta_n^k-1), \quad 1\le j<n,$ and π_j the j's power sum of those numbers. Let $k(\operatorname{mod} n)$ be the remainder modulo n with $0\le k(\operatorname{mod} n)<n.$

Then $\pi_j = \sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}\sum_{k=1}^{n}\zeta_n^{ki-mkj} = (-1)^{j-mj(\operatorname{mod} m)}n\binom{j}{mj(\operatorname{mod} n)}.$

Newton’s formula gives

$\sum_{i=0}^{j-1}(-1)^i\pi_{j-i}s_i = (-1)^{j-1}js_j.$

Since $s_j = (-1)^j b_{n,m,n-j}$ we get

Theorem 2

Let $d(n,m,j)=(-1)^{(j-mj(\operatorname{mod} n))}\binom{j}{mj(\operatorname{mod} n)}n$ *and let* $b(n,m,n-j)$ *be defined by*

$b(n,m,n-j) = -\frac{1}{j}\sum_{i=0}^{j-1}d(n,m,j-i)b(n,m,n-i)$ *with* $b(n,m,n)=1.$

Then for $n\ge 1$ *we have*

$$p_1(n,m,x,s) = x^n+\sum_j c_1(n,m,j)x^{n-mj}s^j = x^n-(-1)^{m(n-1)}\sum_{j=1}^{\left\lfloor \frac{n}{m}\right\rfloor} b(n,m,j)x^{n-mj}s^j \tag{13}$$

and

$$p_k(n,m,x,s) = \sum_j c_k(n,m,j)x^{kn-mj}s^j = -(-1)^{m(n-1)}\sum_{j=1+\left\lfloor \frac{(k-1)n}{m}\right\rfloor}^{\left\lfloor \frac{kn}{m}\right\rfloor} b(n,m,j)x^{kn-mj}s^j. \tag{14}$$

2. Concrete evaluations

Now we want to obtain more concrete information about the polynomials $p_k(n,m,x,s)$.
First I derive some known results (cf. [1]) from the present point of view.

2.1. The case $m=2$

For $m=2$ formula (10) tells us that there are polynomials $p_1(n,2,x,s)$ of the form $\sum_j c_j x^{n-2j}s^j$ which satisfy

$$p_1(n,2,x+1,x)=x^n+1. \tag{15}$$

It is easy to see that the sequence $\left(p_1(n,2,x,1)\right)_{n\geq 1}$ begins with
$x, x^2-2, x^3-3x, x^4-4x^2+2, x^5-5x^3+5x, x^6-6x^4+9x^2-2,\cdots.$

Everyone familiar with the Lucas polynomials $L_n(x,s)=\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n-j}{j}\frac{n}{n-j}x^{n-2j}s^j$ will immediately guess that

$$p_1(n,2,x,1)=\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^j\binom{n-j}{j}\frac{n}{n-j}x^{n-2j} \tag{16}$$

and thus
$p_1(n,2,x,s)=L_n(x,-s)$ holds.
It is well known that the Lucas polynomials satisfy the recurrence
$L_n(x,s)=xL_{n-1}(x,s)+sL_{n-2}(x,s)$ with initial values $L_0(x,s)=2$ and $L_1(x,s)=x.$

Therefore it remains only to show that $p_1(n,2,x,s)=L_n(x,-s)$ satisfies (15), i.e.

$$L_n(x+1,-x)=x^n+1. \tag{17}$$

This is of course a well-known result. It may be proved in the following way:
The polynomials $L_n(x+1,-x)$ satisfy the recurrence
$L_{n+2}(x+1,-x)-(x+1)L_{n+1}(x+1,-x)+xL_n(x+1,-x)=\left(E^2-(x+1)E+x\right)L_n(x+1,-x)=0.$
If $p(E)f(n)=0$ we call $p(z)$ the characteristic polynomial of the sequence $f(n).$ Thus the characteristic polynomial of the sequence $L_n(x+1,-x)$ is $z^2-(x+1)z+x=(z-1)(z-x).$
This implies that the constant sequence 1 and the sequence (x^n) satisfy the same recurrence as $\left(L_n(x+1,-x)\right).$
Therefore we get
$\left(E^2-(x+1)E+x\right)\left(L_n(x+1,-x)-1-x^n\right)=0$
with initial values $L_0(x+1,-x)-1-x^0=2-1-1=0$ and
$L_1(x+1,-x)-1-x=(x+1)-1-x=0.$

This implies that $L_n(x+1,-x)-1-x^n=0$ for all $n\in\mathbb{N}$ as asserted.

Another way to derive this result is by considering the generating function $\sum_{n\geq 0} p_1(n,2,x+1,x)z^n=\sum_{n\geq 0}(1+x^n)z^n=\frac{1}{1-z}+\frac{1}{1-xz}=\frac{2-(x+1)z}{1-(x+1)z+xz^2}$. Since the polynomials are uniquely determined we get the generating function of the Lucas polynomials $\sum_{n\geq 0} p_1(n,2,x,s)z^n=\frac{2-xz}{1-xz+sz^2}$.

We have thus shown that

$$L_i(E,-1)a(n,i,\ell,2,z)=\left(z+\frac{1}{z}\right)a(n,i,\ell,2,z) \tag{18}$$

holds.

E.g. for $i=5$ we get $L_5(E,-1)=E^5-5E^3+5E$. Therefore $(E^5-5E^3+5E+2)a(n,5,\ell,2,-1)=0.$ Now we have $(E^5-5E^3+5E+2)=(E+2)(E^2-E-1)^2.$ The Fibonacci numbers satisfy the same recurrence. Considering the initial values we get $\sum_{k\in\mathbb{Z}}(-1)^k\binom{n}{\left\lfloor\frac{n+5k+2}{2}\right\rfloor}=F_n$ and $\sum_{k\in\mathbb{Z}}(-1)^k\binom{n}{\left\lfloor\frac{n+5k}{2}\right\rfloor}=F_{n+1}$.

In [3] we have also obtained the generating function

$$\sum_{n\geq 0} a(n,i,0,2,z)x^n=\frac{\frac{x^{i-1}}{z}+F_i(1,-x^2)+xF_{i-1}(1,-x^2)}{L_i(1,-x^2)-x^i\left(z+\frac{1}{z}\right)}, \tag{19}$$

where $F_n(x,s)=\sum_{k=0}^{n-1}\binom{n-1-k}{k}s^k x^{n-2k-1}$ are the Fibonacci polynomials.

As shown in [1] and [3] for $z=\pm 1$ there are always simpler recurrences. The formulas are different depending whether i is odd or even. For example

$$\sum_{n\geq 0} a(n,2m+1,0,2,-1)x^n=\frac{F_{2m+1}(1,-x^2)+xF_{2m}(1,-x^2)-x^{2m}}{L_{2m+1}(1,-x^2)+2x^{2m+1}}=\frac{F_m(1,-x^2)}{F_{m+1}\left(1,-x^2\right)-xF_m\left(1,-x^2\right)}.$$

Remark

It should be noted that $a(n,2m+1,0,2,-1)$ has a simple combinatorial interpretation (cf. [2]). It is the number of the set $A(n,m)$ of all lattice paths in $\mathbb{R}^2$ of length n which are contained in the strip $-m-1<y<m$, start at the origin and consist of $\left\lfloor \frac{n}{2} \right\rfloor$ northeast steps $(1,1)$ and $\left\lfloor \frac{n+1}{2} \right\rfloor$ southeast steps $(1,-1)$. Define a peak as a vertex preceded by a northeast step and followed by a southeast step, and a valley as a vertex preceded by a southeast step and followed by a northeast step. The height of a vertex is its $y-$coordinate. The peaks with height at least 1 and the valleys with height at most -2 are called extremal points. Define the weight of a path v by $w(v,t)=t^{d(v)}$ where $d(v)$ is the number of extremal points of the path v. The weight of a set of paths is the sum of the weights of all paths of the set.

It has been shown in [2] that if we set $\binom{n}{k}=0$ for $n<0$ then the weight of the set $A(n,m)$ is

$$w(A(n,m),t)=\sum_{j\in\mathbb{Z}}(-1)^j\sum_{\ell\geq|j|}\binom{\left\lfloor \frac{n+(2m-3)j}{2} \right\rfloor}{\ell-j}\binom{\left\lfloor \frac{n+1-(2m-3)j}{2} \right\rfloor}{\ell+j}t^{\ell}.$$

For $t=1$ this reduces by Vandermonde's formula to $\sum_{j\in\mathbb{Z}}(-1)^j\binom{n}{\left\lfloor \frac{n+(2m+1)j}{2} \right\rfloor}$.

For example $w(A(n,2),t)=\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor}\binom{n-2k}{k}t^k=F_{n+1}(1,t)$.

As shown in [2] with other methods for $m\geq 2$ the polynomial $w(A(n,m),t)$ satisfies the recurrence relation

$$\left(F_m(E^2+1-t,-E^2)-(1+E)F_{m-1}(E^2+1-t,-E^2)+EF_{m-2}(E^2+1-t,-E^2)\right)w(A(n,m),t)=0$$

of order $2m-2$ which for $t=1$ reduces to the recurrence relation

$$\left(F_{m+1}(E,-1)-F_m(E,-1)\right)w(A(n,m),1)=\left(F_{m+1}(E,-1)-F_m(E,-1)\right)a(n,2m+1,0,2,-1)=0$$

of order m.

It should also be noted that $\left(a(n,5,0,2,-1)\right)_{n\geq 0}$, $\left(a(n,7,0,2,-1)\right)_{n\geq 0}$ and $\left(a(n,9,0,2,-1)\right)_{n\geq 0}$ are the sequences A000045, A028495 and A061551 of The On-Line Encyclopedia of Integer Sequences.

2.2. The case $m=3$.

In [1] I have obtained explicit formulas for the recurrence relations of the sums

$$a(n,i,\ell,3,z)=\sum_{k\in\mathbb{Z}}\binom{n}{\left\lfloor \frac{n+ik+\ell)}{3}\right\rfloor}z^k.$$

From Theorem 1 we know that there are polynomials $v_i(x)$ and $w_i(x)$ such that

$$\left(v_i(E)-zw_i(E)-\left(\frac{1}{z}+(-1)^{i-1}z^2\right)\right)a(n,i,\ell,z)=0.$$

Computer experiments led to the table

	1	2	3	4	5	6	7
$v_i(x)$	x	x^2	x^3-3	x^4-4x	x^5-5x^2	x^6-6x^3+3	x^7-7x^4+7x
$w_i(x)$	0	$2x$	3	$2x^2$	$5x$	$3+2x^3$	$7x^2$

An inspection of these polynomials led to the conjecture that they satisfy the recurrences

$$v_n(x)=xv_{n-1}(x)-v_{n-3}(x) \tag{20}$$

with initial values

$$v_0(x)=3, v_1(x)=x, v_2(x)=x^2 \tag{21}$$

and

$$w_n(x)=xw_{n-2}(x)+w_{n-3}(x) \tag{22}$$

with initial values

$$w_0(x)=3, w_1(x)=0, w_2(x)=2x. \tag{23}$$

It is not difficult to determine these polynomials explicitly: For $n>0$ we have

$$v_n(x)=\sum_{3j\le n}(-1)^j\binom{n-2j}{j}\frac{n}{n-2j}x^{n-3j} \tag{24}$$

and

$$w_n(x)=\sum_{3j\le 2n}\binom{n-j}{2n-3j}\frac{n}{n-j}x^{2n-3j}. \tag{25}$$

For the recurrences are easily verified and the initial values coincide for $n=1,2,3$.

Thus we have experimental evidence for

Theorem 3
For $i \geq 1$ *the sequence*

$$a(n,i,\ell,3,z) := \sum_{k\in\mathbb{Z}} \binom{n}{\left\lfloor \frac{n+ik+\ell}{3} \right\rfloor} z^k \tag{26}$$

satisfies the recurrence

$$(v_i(E) - zw_i(E))a(n,i,\ell,z) - (\frac{1}{z} + (-1)^{i-1}z^2)a(n,i,\ell,z) = 0 \tag{27}$$

for all $n \in \mathbb{N}.$

Remark
In [1] unfortunately some typos corrupted the formulation of this theorem.

With the notations introduced above we have only to verify that the polynomials $p_i(n,3,x,s)$ are given by $v_n(x,s)$ and $w_n(x,s),$ i.e.

$$p_1(n,3,x,s) = v_n(x,s) = \sum_{3j\leq n} (-1)^j \binom{n-2j}{j} \frac{n}{n-2j} x^{n-3j} s^j \tag{28}$$

and

$$-p_2(n,3,x,s) = w_n(x,s) = \sum_{3j\leq 2n} \binom{n-j}{2n-3j} \frac{n}{n-j} x^{2n-3j} s^j. \tag{29}$$

Note that $v_n(x) = v_n(x,1)$ and $w_n(x) = w_n(x,1).$

With other words we have to show that

$$v_n(1+x,x) - w_n(1+x,x) - 1 + (-x)^n = 0. \tag{30}$$

These polynomials satisfy the recurrences
$v_n(x,s) = xv_{n-1}(x,s) - sv_{n-3}(x)$ and $w_n(x,s) = xsw_{n-2}(x) + s^2 w_{n-3}(x)$ for all $n \in \mathbb{N}$ if we set $v_0(x,s) = w_0(x,s) = 3.$
The sequences $v_n(1+x,x)$ and the constant sequence 1 satisfy the recurrence $(E^3 - (1+x)E^2 + x)v_n(1+x,x) = 0$ and $(E^3 - (1+x)E^2 + x)1 = 0$ and the sequences

$w_n(1+x,x)$ and $(-x)^n$ satisfy $(E^3-(1+x)xE-x^2)w_n(1+x,x)=0$ and $(E^3-(1+x)xE-x^2)(-x)^n=0$.

Therefore $f_n(x)=v_n(1+x,x)-w_n(1+x,x)-1+(-x)^n$ satisfies the recurrence $(E^3-(1+x)E^2+x)(E^3-(1+x)xE-x^2)f_n(x)=0$ of order 6. Since the first 6 initial values are 0 we get $f_n(x)=0$ for all $n\in\mathbb{N}$.

It is again instructive to consider the generating functions

$$\sum_{n\ge0}p_1(n,3,x,s)z^n=\frac{3-2xz}{1-xz+sz^3} \text{ and } \sum_{n\ge0}p_2(n,3,x,s)z^n=-\frac{3-xsz^2}{1-xsz^2-s^2z^3}.$$

This gives $\sum_{n\ge0}p_1(n,3,x+1,x)z^n=\frac{3-2(x+1)z}{1-(x+1)z+xz^3}=\frac{1}{1-z}+\frac{2-xz}{1-xz-xz^2}$

and $\sum_{n\ge0}p_2(n,3,x+1,x)z^n=-\frac{3-(x+1)xz^2}{1-(x+1)xz^2-x^2z^3}=\frac{-1}{1+xz}-\frac{2-xz}{1-xz-xz^2}$.

These again imply

$$\sum_{n\ge0}(p_1(n,3,x+1,x)+p_2(n,3,x+1,x))z^n=\frac{1}{1-z}-\frac{1}{1+xz}=\sum_{n\ge0}(1+(-1)^{n-1}x^n)z^n.$$

In addition to Theorem 3 we compute the generating function

$$\sum_{n\ge0}a(n,i,0,3,z)x^n=\frac{b_i(x)+c_i(x)z+(-1)^{i-1}x^{i-1}z^2}{v_i\left(1,x^3\right)-\frac{z}{x^i}w_i\left(1,x^3\right)-x^i\left(\frac{1}{z}+(-1)^{i-1}z^2\right)}. \tag{31}$$

Computer experiments suggest that $b_i(x)$ satisfies the same recurrence as $v_i\left(1,x^3\right)$, i.e. $b_i(x)=b_{i-1}(x)-x^3b_{i-3}(x)$.

The initial values are $b_1(x)=1, b_2(x)=1+x, b_3(x)=1+x+x^2$.

The generating function of these polynomials is

$$\sum_{i\ge0}b_{i+1}(x)t^i=\frac{1+xt+x^2t^2}{1-t+x^3t^3}.$$

Let $\frac{1}{1-t+x^3t^3}=\sum_{n\ge0}r_n(x)t^n$. Then it is easily verified that $r_n(x)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n-2k}{k}x^{3k}$. The first values of the sequence $\left(r_n(x)\right)_{n\ge0}$ are $1,1,1,1-x^3,1-2x^3,1-3x^3,1-4x^3+x^6,\cdots$.

This gives $b_{i+1}(x)=r_i(x)+xr_{i-1}(x)+x^2r_{i-2}(x)$.

The polynomials $c_i(x)$ satisfy the same recurrence as $\frac{w_i(1,x^3)}{x^i}$, i.e.

$c_i(x) = xc_{i-2}(x) + x^3 c_{i-3}(x)$.

The initial values are $c_1(x) = 1, c_2(x) = 1-x, c_3(x) = x+2x^2$.

The generating function of these polynomials is

$$\sum_{i\geq 0} c_{i+1}(x)t^i = \frac{1+(1-x)t+2x^2t^2}{1-xt^2-x^3t^3}.$$

Let $\frac{1}{1-xt^2-x^3t^3} = \sum_{n\geq 0} s_n(x)t^n$. Then $s_n(x) = \sum_{j=0}^{\left\lfloor \frac{n+1}{3} \right\rfloor} \binom{\left\lfloor \frac{n+1}{3} \right\rfloor + j}{3j-\varepsilon(n)} x^{n-3j+\varepsilon(n)}$,

where $\varepsilon(n) \equiv n (\bmod 3)$ with $\varepsilon(n) \in \{-1,0,1\}$. This can easily be verified by considering each of the cases $s_{3n}(x), s_{3n-1}(x), s_{3n+1}(x)$ separately. The first values of the sequence $(s_n(x))_{n\geq 0}$ are $1, 0, x, x^3, x^2, 2x^4, x^3+x^6, 3x^5, x^4+3x^7, \cdots$.

Another representation is $s_n(x) = \sum_{k=0}^{n} \binom{\frac{n+k}{3}}{k} x^{n-k} [n+k \equiv 0 (\bmod 3)]$.

Finally we get $c_{i+1}(x) = s_i(x) + (1-x)s_{i-1}(x) + 2x^2 s_{i-2}(x)$.

In order to prove (31) we note first that

$$\left(v_i\left(1,x^3\right) - \frac{z}{x^i} w_i\left(1,x^3\right) - x^i\left(\frac{1}{z} + (-1)^{i-1}z^2\right)\right) \sum_{n\geq 0} a(n,i,0,3,z)x^n$$

is a polynomial $d_i(x,z)$ in x of degree $< i$. For a polynomial $p(x) = \sum_k p_k x^k$ let

$$P_i(p(x)) = \sum_{k=0}^{i-1} p_k x^k.$$

Therefore

$$d_i(x,z) = P_i\left(\left(v_i\left(1,x^3\right) - \frac{z}{x^i} w_i\left(1,x^3\right)\right) \sum_{n\geq 0} a(n,i,0,3,z)x^n \right).$$

Since no factor contains negative powers of z we see that the sum of the first i terms of

$$P_i\left(v_i\left(1,x^3\right) \sum_{n=0}^{i-1} a(n,i,0,3,0)x^n \right) = d_i(x,0).$$

Let $a_i(x,z) = \sum_{n=0}^{i-1} a(n,i,0,3,z)x^n$.

Then

$$a_i(x,z)=\sum_{n=0}^{i-1}\binom{n}{\left\lfloor \frac{n}{3}\right\rfloor}x^n+z\sum_{n=\left\lfloor \frac{i-2}{2}\right\rfloor}^{i-1}\binom{n}{\left\lfloor \frac{n+i}{3}\right\rfloor}x^n+z^2x^{i-1}. \tag{32}$$

This implies

$$d_i(x,0)-d_{i-1}(x,0)+x^3d_{i-3}(x,0)=P_i\left(v_{i-1}(1,x^3)\left(a_i(x,0)-a_{i-1}(x,0)\right)+x^3v_{i-3}(1,x^3)\left(a_i(x,0)-a_{i-3}(x,0)\right)\right)=0.$$

Since the initial values coincide we get $d_i(x,0)=b_i(x).$

Next we determine the coefficient of z^2 of $d_i(x,z).$ From (32) we get

$$\left[z^2\right]\left(v_i\left(1,x^3\right)\sum_{n=0}^{i-1}a(n,i,0,3,0)x^n\right)=x^{i-1}.$$

On the other hand we have

$$-\frac{z}{x^{2i}}w_{2i}\left(1,x^3\right)z\sum_{n=\left\lfloor \frac{2i-2}{2}\right\rfloor}^{2i-1}\binom{n}{\left\lfloor \frac{n+2i}{3}\right\rfloor}x^n=-z^2\left(2x^i+\cdots\right)\left(x^{i-1}+\cdots\right)=-2x^{2i-1}z^2+\cdots.$$

This implies $\left[z^2\right]d_{2i}(x,z)=-x^{2i-1}.$

For odd indices we get

$$-\frac{z}{x^{2i+1}}w_{2i+1}\left(1,x^3\right)z\sum_{n=\left\lfloor \frac{2i+1-2}{2}\right\rfloor}^{2i}\binom{n}{\left\lfloor \frac{n+2i+1}{3}\right\rfloor}x^n=-z^2x^{i+2}(\cdots)\left(x^{i-1}+\cdots\right)=-x^{2i+1}z^2(\cdots).$$

This implies $\left[z^2\right]d_{2i+1}(x,z)=x^{2i-1}.$

As above we verify that

$$p_i(x)=P_i\left(\frac{1}{x^i}w_i\left(1,x^3\right)\sum_{n=0}^{i-1}\binom{n}{\left\lfloor \frac{n}{3}\right\rfloor}x^n\right)$$

satisfies the recurrence $p_i(x)-xp_{i-2}(x)-x^3p_{i-3}(x)=0,$ i.e the same recurrence as $\frac{1}{x^i}w_i\left(1,x^3\right).$ In the same way we get that $q_i(x)=P_i\left(v_i\left(1,x^3\right)\sum_{n=\left\lfloor \frac{i-2}{2}\right\rfloor}^{i-1}\binom{n}{\left\lfloor \frac{n+i}{3}\right\rfloor}x^n\right)$ satisfies the same recurrence.

Writing $d_i(x,z)=b_i(x)+c_i(x)z+(-1)^{i-1}x^{i-1}z^2$ we see therefore that $c_i(x)-xc_{i-2}(x)-x^3c_{i-3}(x)=0,$ from which equation (31) follows.

2.3. Some general observations.

As a special case of Theorem 1 we see that

$$a(n,1,0,m,z)=\sum_{k\in\mathbb{Z}}\binom{n}{\left\lfloor \frac{n+k}{m}\right\rfloor}z^k=\frac{1-z^m}{1-z}\left(\frac{1+z^m}{z}\right)^n.$$

From

$p_1(n,m,x+1,x)=x^n+\sum_{j=1}^{\left\lfloor \frac{n}{m}\right\rfloor}c_1(n,m,j)(x+1)^{n-mj}x^j$ and the fact that

$p_k(n,m,x+1,x)=x^{1+\left\lfloor \frac{(k-1)n}{m}\right\rfloor}(\cdots)$ for $k\ge 2$ we see that (10) implies that all coefficients $\left[x^j\right]p_1(n,m,x+1,x), 1\le j\le \frac{n}{m}$, must vanish, i.e. $P_{i+\left\lfloor \frac{n}{m}\right\rfloor}\left(p_1(n,m,x+1,x)\right)=1.$ By these conditions $p_1(n,m,x,s)$ is uniquely determined.

This observation implies that

$$p_1(n,m,x,s)=\sum_{j=0}^{\left\lfloor \frac{n}{m}\right\rfloor}(-1)^j\binom{n-(m-1)j}{j}\frac{n}{n-(m-1)j}x^{n-mj}s^j. \tag{33}$$

For

$$p_1(n,m,x+1,x)=\sum_{j=0}^{\left\lfloor \frac{n}{m}\right\rfloor}(-1)^j\binom{n-(m-1)j}{j}\frac{n}{n-(m-1)j}(x+1)^{n-mj}x^j$$

and therefore the coefficient of x^r is given by

$$\sum_{j=0}^{r}(-1)^j\binom{n-(m-1)j}{j}\frac{n}{n-(m-1)j}\binom{n-mj}{r-j}=\frac{n}{r!}\sum_{j=0}^{r}(-1)^j\binom{r}{j}(n-(m-1)j-1)\cdots(n-(m-1)j-r+1).$$

for all r such that $r\le\left\lfloor \frac{n}{m}\right\rfloor$.

Let $q(n)=(n-1)_{r-1}=(n-1)(n-2)\cdots(n-r+1)$. This is a polynomial in n of degree $r-1$.

With this notation the coefficient of x^r is $\frac{n}{r!}(1-E^{m-1})^r q(n).$ Since

$(1-E^{m-1})n^k=n^k-(n+m-1)^k=k(m-1)n^{k-1}+\cdots$ the operator $1-E^{m-1}$ decreases the degree of a polynomial by 1. Therefore $\frac{n}{r!}(1-E^{m-1})^r q(n)=0$ and the assertion is proved.

It is easily verified that (33) satisfies the recurrence relation

$$\left(E^m - xE^{m-1} - s\right) p_1(n,m,x,s) = 0.$$

Now we observe that

$$p_{m-1}(n,m,x,s) = \sum_{j=0}^{\left\lfloor \frac{n}{m} \right\rfloor} \binom{n-j}{(m-1)n-mj} \frac{n}{n-j} x^{(m-1)n-mj} s^j.$$

We know that

$$d(n,m,j) = (-1)^{(j-mj(\operatorname{mod} n))} \binom{j}{mj(\operatorname{mod} n)} n.$$

This implies that $d(n,m,0) = n$ and $d(n,m,j) = 0$ for $j \le \frac{n}{m}$. For in this case we have $mj(\operatorname{mod} m) = mj$ and therefore $\binom{j}{mj} = 0.$

Thus also

$$b(n,m,n-j) = -\frac{1}{j} \sum_{i=0}^{j-1} d(n,m,j-i) b(n,m,n-i) = 0.$$

Let now $\frac{n}{m} < j \le \frac{2n}{m}$.

From

$$-jb(n,m,n-j) = \sum_{i=0}^{j-1} d(n,m,j-i) b(n,m,n-i)$$

$$= d(n,m,j) b(n,m,n) + \sum_{i=1}^{\left\lfloor \frac{m}{n} \right\rfloor} d(n,m,j-i) b(n,m,n-i) + \sum_{i=1+\left\lfloor \frac{m}{n} \right\rfloor}^{j-1} d(n,m,j-i) b(n,m,n-i)$$

we deduce that $-jb(n,m,n-j) = d(n,m,j) = (-1)^{(j-mj(\operatorname{mod} n))} \binom{j}{mj(\operatorname{mod} n)} n,$

because the first sum vanishes since $b(n,m,n-i) = 0$ and the second sum vanishes because $d(n,m,j-i) = 0.$

Therefore for $\frac{(m-2)n}{m} < j \le \frac{(m-1)n}{m}$ we get

$$b(n,m,j) = -(-1)^{(n-j+m(n-j)(\operatorname{mod} n))} \binom{n-j}{m(n-j)(\operatorname{mod} n)} \frac{n}{n-j} = -(-1)^{n+j+(m-1)n-mj} \binom{n-j}{(m-1)n-mj} \frac{n}{n-j}.$$

Therefore we get

$$p_{m-1}(n,m,x,s)=(-1)^m \sum_{j=1+\left\lfloor \frac{(m-2)n}{m}\right\rfloor}^{\left\lfloor \frac{(m-1)n}{m}\right\rfloor} \binom{n-j}{(m-1)n-mj}\frac{n}{n-j}x^{(m-1)n-mj}((-1)^{m-1}s)^j. \tag{34}$$

In order to give a simple description of these results we introduce polynomials

$$v_n(m,x,s)=\sum_{j\in\mathbb{N}}(-1)^j\binom{n-(m-1)j}{j}\frac{n}{n-(m-1)j}x^{n-mj}s^j \tag{35}$$

and

$$w_n(m,x,s)=\sum_{j\in\mathbb{N}}\binom{n-j}{(m-1)n-mj}\frac{n}{n-j}x^{(m-1)n-mj}s^j, \tag{36}$$

where as always in this paper $\binom{n}{k}=0$ for $n<0.$

These polynomials satisfy the recurrence relations

$$v_{n+m}(m,x,s)=xv_{n+m-1}(m,x,s)-sv_n(m,x,s) \tag{37}$$

with initial values $v_0(m,x,s)=m$ and $v_i(m,x,s)=x^i$ for $1\le i\le m-1$
and

$$w_{n+m}(m,x,s)=xs^{m-2}w_{n+1}(m,x,s)+s^{m-1}w_n(m,x,s) \tag{38}$$

with initial values $w_0(m,x,s)=m$, $w_i(m,x,s)=0$ for $1\le i\le m-2,$ and $w_{m-1}(m,x,s)=(m-1)s^{m-2}x.$

Let $c_{m,k}(x,s,z)$ be the characteristic polynomial of (the recurrence of) the sequence $\left(p_k(n,m,x,s)\right)_{n\ge 0}$, so that $c_{m,k}(x,s,E)p_k(n,m,x,s)=0.$

The characteristic polynomial of $v_n(m,x+1,x)$ is

$$c_{m,1}(x+1,x,z)=z^{m-1}(z-1)-x(z^{m-1}-1)=(z-1)(z^{m-1}-xz^{m-2}-xz^{m-3}-\cdots-xz-x) \tag{39}$$

and that of $w_n(m,x+1,(-1)^{m-1}x)$ is

$$c_{m,m-1}(x+1,x,z)=z^m-(x+1)x^{m-2}z-(-1)^{m-1}x^{m-1}=\left(x-(-1)^m z\right)\left(z^{m-1}+xz^{m-2}+\cdots+x^{m-2}z-x^{m-2}\right).$$

It is easily verified that

$$c_{m,m-1}(x+1,x,z)=\frac{z^m}{x}c_{m,1}(x+1,x,(-1)^m\frac{x}{z}). \tag{40}$$

This is in accord with the fact that the right hand side of (10) has the characteristic polynomial $c(z)=(z-1)\left(z-(-1)^m x\right)$ satisfies $c\left((-1)^m\frac{x}{z}\right)=\frac{(-1)^m x}{z^2}c(z).$

These results can be formulated as

Theorem 4

For each $m\geq 2$ *we have*

$$p_1(n,m,x,s)=v_n(m,x,s) \tag{41}$$

and

$$p_{m-1}(n,m,x,s)=(-1)^m w_n(m,x,(-1)^{m-1}s). \tag{42}$$

2.4. The case $m=4$.

Now we want to consider the case $m=4$ in more detail.
We already know that

$$p_1(n,4,x,s)=\sum_{j=0}^{\left\lfloor\frac{n-1}{3}\right\rfloor}(-1)^j\binom{n-3j}{j}\frac{n}{n-3j}x^{n-4j}s^j$$

and

$$p_3(n,4,x,s)=\sum_{j=\left\lfloor\frac{2n}{3}\right\rfloor}^{n-1}(-1)^j\binom{n-j}{3n-4j}\frac{n}{n-j}x^{3n-4j}s^j.$$

The polynomials $p_2(n,4,x,s)$ turn out to satisfy a recursion of order 6 instead of order 4. It is given by

$$p_2(n,4,x,s)=sp_2(n-2,4,x,s)+sx^2p_2(n-3,4,x,s)+s^2p_2(n-4,4,x,s)-s^3p_2(n-6,4,x,s) \tag{43}$$

with initial values
$p_2(0,4,x,s)=-6,\ p_2(1,4,x,s)=0,\ p_2(2,4,x,s)=-2s,\ p_2(3,4,x,s)=-3sx^2,$
$p_2(4,4,x,s)=-6s^2,\ p_2(5,4,x,s)=-5s^2x^2.$

To prove this we observe that we know already the generating functions
$\sum_{n\ge 0} p_1(n,4,x,s)z^n=\frac{4-3xz}{1-xz+sz^4}$ and $\sum_{n\ge 0} p_3(n,4,x,s)z^n=\frac{4-xs^2z^3}{1-xs^2z^3+s^3z^4}.$
This gives $\sum_{n\ge 0} p_1(n,4,x+1,x)z^n=\frac{4-3(x+1)z}{1-(x+1)z+xz^4}=\frac{1}{1-z}+\frac{3-2xz-xz^2}{1-xz-xz^2-xz^3}$ and
$\sum_{n\ge 0} p_3(n,4,x+1,x)z^n=\frac{4-(x+1)x^2z^3}{1-(x+1)x^2z^3+x^3z^4}=\frac{1}{1-xz}+\frac{3+2xz+x^2z^2}{1+xz+x^2z^2-x^2z^3}.$

Furthermore we know that $p_2(n,4,x+1,x)=1+x^n-p_1(n,4,x+1,x)-p_3(n,4,x+1,x)$.
This gives the generating function

$$\sum_{n\ge 0} p_2(n,4,x+1,x)z^n=-\frac{3-2xz-xz^2}{1-xz-xz^2-xz^3}-\frac{3+2xz+x^2z^2}{1+xz+x^2z^2-x^2z^3}$$
$$=\frac{-6+4xz^2+3x(x+1)^2z^3+2x^2z^4}{1-xz^2-x(x+1)^2z^3-x^2z^4+x^3z^6}$$
and thus also
$$\sum_{n\ge 0} p_2(n,4,x,s)z^n=\frac{-6+4sz^2+3sx^2z^3+2s^2z^4}{1-sz^2-sx^2z^3-s^2z^4+s^3z^6}.$$

We did not find a simple expression for the polynomials $p_2(n,4,x,s)$ themselves.

2.5. The case $m=5$.

For $m=5$ we have found the following characteristic polynomials for $p_k(n,5,x+1,x)$:
$c_{5,1}(x+1,x,z)=z^5-(1+x)z^4+x=(z-1)(z^4-xz^3-xz^2-xz-x),$

$c_{5,2}(x+1,x,z)=z^{10}-x(1+x)z^7-x(1+x)^3z^6-2x^2z^5-x^2(1+x)^2z^4+x^3(1+x)z^2+x^4$
$=(z^4-xz^3-xz^2-xz-x)(z^6+xz^5+x(1+x)z^4+x^2(1+x)z^3-x^2(1+x)z^2+x^3z-x^3),$

$c_{5,3}(x+1,x,z)=z^{10}+x(1+x)z^8-x^2(1+x)^2z^6+2x^3z^5-x^3(1+x)^3z^4+x^4(1+x)z^3+x^6$
$=(z^4-xz^3+x^2z^2-x^3z-x^3)(z^6+xz^5+x(1+x)z^4+x^2(1+x)z^3-x^2(1+x)z^2+x^3z-x^3),$

$c_{5,4}(x+1,x,z)=z^5-(1+x)x^3z-x^4=(z+x)(z^4-xz^3+x^2z^2-x^3z-x^3).$

Once found these identities can be verified as above by proving (10).

3. Some open problems

Extensive computer experiments suggest the following facts:

$\deg c_{m,k}(x+1,x,z)=\binom{m}{k}$; for $k<\frac{m}{2}$ the polynomial $c_{m,k}(x+1,x,z)$ factors in the form $c_{m,k}(x+1,x,z)=v_{m,k-1}v_{m,k}$ where $v_{m,0}=z-1$; for $k>\frac{m}{2}$ the factorization is $c_{m,k}(x+1,x,z)=v_{m,k}v_{m,k+1}$; for $k=\frac{m}{2}$ we get $c_{m,k}(x+1,x,z)=v_{m,k-1}v_{m,k+1}$; $\deg v_{m,k}=\binom{m-1}{k}$; for m odd we have $v_{m,\left\lfloor\frac{m}{2}\right\rfloor}=v_{m,\left\lceil\frac{m}{2}\right\rceil}$; $v_{m,m}=z+(-1)^{m-1}x.$ Furthermore it seems that as for $k=0$ and $k=1$ also in general $c_{m,m-k}(x+1,x,z)$ is the monic polynomial proportional to $c_{m,k}(x+1,x,(-1)^m\frac{x}{z})$. So it would suffice to compute $v_{m,k}$ for $k<\frac{m}{2}$.

But till now I could not prove these general results.

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