Some elementary results and conjectures  
about q-Newton binomials  

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Abstract  
This paper gives an elementary presentation of some $q$–analogues of the binomial theorem with unusual $q$–powers and collects some results and conjectures about variants of Gauss’s identity.  

0. Introduction  
This paper is concerned with elementary aspects of $q$–Newton binomials of the form  
$$\sum_{j=0}^{n} q^{r(j)} \binom{n}{j}.$$  
It is self-contained and makes no use of the theory of basic hypergeometric series. After some background material we show that for each $m \in \mathbb{Z}$ the general $q$–Newton polynomials  
$$\sum_{k=0}^{n} q^{(m+1)\binom{k}{2}} \binom{n}{k} x^{n-k}$$  
satisfy a recurrence of order $|m|+1$. Then motivated by  

Gauss’s identity  
$$\sum_{k=0}^{2n} (-1)^{k} \binom{2n}{k} = (q;q^2)_n$$  
and its counterpart  
$$\sum_{k=0}^{n} q^{k} \binom{n}{k} = (-q;q)_n$$  
we study some $q$–Newton binomials which for $q \to \pm 1$ converge to 0. Computer experiments show that many of them have factorizations $a(q)b(q)$ into a non-trivial “nice” part $a(q)$ which has a closed expression and an “ugly” part $b(q)$ which is a polynomial with integer coefficients, but has nice values for $q = \pm 1$. We formulate some conjectures and provide proofs for some special cases. I want to thank Ole Warnaar for reference [1] and some useful comments.  

1. Some background material  
In [3] I tried to study polynomials of the form  
$$\sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k} x^{k}$$  
where $m$ is an arbitrary integer. Similar efforts were made by Boris Kupershmidt in [6] and in my paper [4]. In the present essay I resume these topics with some new results and conjectures.
In the context of the general $q$–binomial theorem (1.3) I mostly consider polynomials
\[ p(x,s) = \sum_{k=0}^{n} q^{n(k)} \binom{n}{k} s^k x^{n-k} \] in two variables $x$ and $s$. But since $p(x,s) = x^n p\left(1, \frac{s}{x}\right)$ for real numbers $x, s$, it suffices in this case to consider the univariate polynomials $p(1,x)$.

We let $q$ be a real number and will mostly assume that $|q|<1$. As usual we set
\[ [x] = [x]_q = \frac{1-q^x}{1-q} \] for real numbers $x$ and let $[n]! = \prod_{j=1}^{n} [j]$ for $n \in \mathbb{N}$. The $q$–binomials $\binom{x}{k}$ are defined as
\[ \binom{x}{k} = \binom{x}{k}_q = \frac{\prod_{j=0}^{k-1} (1-q^{x-j})}{\prod_{j=0}^{k-1} (1-q^{k-j})} \]. We shall also use the $q$–Pochhammer symbols
\[ (x; q)_n = \prod_{j=0}^{n-1} (1-q^j x) \] and \[ (x; q)_\infty = \prod_{j=0}^{\infty} (1-q^j x) \].

The $q$–binomial coefficients satisfy two recurrence relations
\[ \binom{x+1}{k} = \binom{x}{k} + \binom{x}{k-1} \] \[ (1.1) \]
and
\[ \binom{x+1}{k} = \binom{x}{k} + q^{x+1-k} \binom{x}{k-1} \] \[ (1.2) \]

My first attempts in this direction led to my paper [2], where I rediscovered Schützenberger’s general $q$–binomial theorem [8] in the following form:

Let $q$ be a real number and let $A$ and $B$ be linear operators on $\mathbb{C}[x,s]$ which $q$–commute, i.e. satisfy $BA=qAB$, then
\[ (A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \] \[ (1.3) \]
where $\binom{n}{k}$ are $q$–binomial coefficients.

The proof is almost trivial since
\[ (A+B)(A+B)^{n-1} = (A+B) \sum_k \binom{n}{k} A^k B^{n-k} = \sum_k \binom{n}{k-1} + q^{k-1} \binom{n}{k} A^k B^{n-k} = \sum_k \binom{n+1}{k} A^k B^{n-k} \].
Let me first recall some well-known facts from this point of view. Let \( x \) and \( s \) be the multiplication operators with \( x \) and \( s \) respectively, and let \( e \) be the augmenting operator defined by \( e f(x, s) = f(x, qs) \) for \( f \in \mathbb{C}[x, s] \).

The operators \( A = s e \) and \( B = x e \) are \( q \)-commuting because
\[
BAf(x, s) = x e s e f(x, s) = q s x f(x, q^2 s) = q s e x e f(x, s) = q B A f(x, s).
\]

This implies that \( (s e + x e)^n = \sum_{k=0}^{n} \binom{n}{k} (s e)^k (x e)^{n-k} \). By applying this identity to the constant polynomial \( f(x, s) = 1 \) we get the **Euler polynomials**
\[
\sum_{k=0}^{n} q^{\binom{n}{k}} \binom{n}{k} s^k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (s e)^k (x e)^{n-k} 1 = (s e + x e)^n 1 = (x + s)(x + qs)\cdots(x + q^{n-1}s).
\]

(1.4)

The \( q \)-commuting operators \( A = s \) and \( B = x e \) in the same way give the **Rogers-Szegö polynomials**
\[
\sum_{k=0}^{n} \binom{n}{k} s^k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (s e)^k (x e)^{n-k} 1 = (s + x e)^n 1.
\]

(1.5)

By comparing coefficients the general \( q \)-binomial theorem is equivalent with the \( q \)-exponential law
\[
\exp((A + B)z) = \exp(Az) \exp(Bz)
\]

(1.6)

for the **first \( q \)-exponential series**
\[
\exp(z) = \exp_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]!}
\]

(1.7)

All considered series will be **formal power series**.

There is also a **second \( q \)-exponential series**
\[
\Exp(z) = \Exp_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]!}
\]

(1.8)

Note that
\[
\exp(z) \Exp(-z) = 1.
\]

(1.9)
To show this let $A = s$ and $B = -s\varepsilon$. These operators are again $q$–commuting. Therefore 
\[ \exp(sz)\exp(-sz)\varepsilon = \exp(s(1-\varepsilon)z). \]
Since $s(1-\varepsilon)l = 0$ we get 
\[ l = \exp(s(1-\varepsilon)z)l = \exp(sz)\exp(-sz)\varepsilon l = \exp(sz)\sum_{n=0}^{\infty}(-1)^n s^n z^n \frac{[n]!}{[n]!} = \exp(sz)\exp(-sz). \]

Instead of $\exp(z)$ we will also consider the variant 
\[ e(z) = e_q(z) = \exp\left(\frac{z}{1-q}\right) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n}, \]
which is the most convenient one in the context of basic hypergeometric series.

From our point of view the $q$–analogue 
\[ \frac{\exp(z)-\exp(qz)}{(1-q)z} = \exp(z) \quad \text{of} \quad \frac{d}{dx} e^x = e^x \]
implies the well-known results 
\[ e_q(qz) = (1-z)e_q(z) \quad (1.10) \]
and by iteration 
\[ e_q(z) = \frac{1}{(z;q)_\infty}. \quad (1.11) \]

Therefore 
\[ e_q(z^2) = \prod_{j=0}^{\infty} (1-q^{2j}z^2) = \prod_{j=0}^{\infty} (1+q^j z)\prod_{j=0}^{\infty} (1-q^j z) = e_q(-z)e_q(z). \quad (1.12) \]

Later we will need the following $q$–analogue of $e^{(x+y)^2} = e^xe^y$. 

**Lemma 1.1 ([5], Corollary 3.3)**

*Let $A$ and $B$ be linear operators on $\mathbb{C}[x,s]$ which satisfy $BA = qAB$.*

*Then $\exp_{q^2}((A+B)^2) = \exp_{q^2}(A^2)\exp_{q^2}((1+q)AB)\exp_{q^2}(B^2)$ or equivalently*

\[ e_{q^2}((A+B)^2) = e_{q^2}(A^2)e_{q^2}(AB)e_{q^2}(B^2). \quad (1.13) \]

**Proof**

Since $B^A A = Aq^A B^A$ we get $e_q(B) A = Ae_q(qB)$ and therefore 
\[ e_q(B) A e_q(B)^{-1} = Ae_q(qB)e_q(B)^{-1} = A(1-B)e_q(B)e_q(B)^{-1} = A(1-B). \]

Thus 
\[ e_q(B) A^* e_q(B)^{-1} = (A(1-B))^* \quad \text{and} \quad e_q(B) e_q(A)e_q(B)^{-1} = e_q(A(1-B)). \]

This implies
\[ e_q(B)e_q(A) = e_q(A - AB)e_q(B) = e_q(A)e_q(-AB)e_q(B). \] (1.14)

Therefore
\[ e_q((A + B)^2) = e_q((-A + B))e_q((A + B)) = e_q((-A)e_q(-B)e_q(A)e_q(B) \]
\[ = e_q(-A)e_q(A)e_q(AB)e_q(-B)e_q(B) = e_q(A^2)e_q(AB)e_q(B^2). \]

Let us denote by \( r_n(x, s, m) = r_n(x, s, m, q) \) the general \( q \)-Newton binomials
\[ r_n(x, s, m, q) = \sum_{k=0}^{n} \binom{n}{k}_q s^k x^{n-k} = \sum_{k=0}^{n} \binom{n+1}{k}_q s^k x^{n-k}. \] (1.15)

Note that for \( m = 0 \) we get the Euler polynomials and for \( m = -1 \) the Rogers-Szegö polynomials. Let us also note the well-known generating functions of these special polynomials.

The generating function of the Rogers-Szegö polynomials is
\[ \sum_{n=0}^\infty r_n(x, s, -1) \frac{z^n}{[n]!} = \exp(xz)\exp(sz). \] (1.16)

This can be proved by comparing coefficients or by the exponential version of (1.5).

An equivalent version is
\[ \sum_{n=0}^\infty r_n(1, s, -1) \frac{z^n}{[q]_n} = \frac{1}{(z; q)_\infty (sz; q)_\infty}. \] (1.17)

The generating function of the Euler polynomials is
\[ \sum_{n=0}^\infty r_n(x, s, 0) \frac{z^n}{[n]!} = \text{Exp}(sz)\exp(xz) = \frac{\exp(xz)}{\exp(-sz)}. \] (1.18)

It is equivalent with
\[ \sum_{n=0}^\infty r_n(1, -s, 0) \frac{s^n}{[q]_n} \frac{(s; q)_\infty}{(z; q)_\infty} = \sum_{n=0}^\infty \frac{(s; q)_n}{(q; q)_n} \frac{z^n}{[n]!} = \frac{(sz; q)_\infty}{(z; q)_\infty}. \] (1.19)

which plays the role of the \( q - \text{binomial theorem} \) within the theory of basic hypergeometric series.

The general \( q - \text{Newton binomials} \) \( r_n(x, s, m, q) \) satisfy the recurrence relation
\[ r_n(x, s, m, q) = xr_{n-1}(x, qs, m, q) + sr_{n-1}(x, q^{m+1}s, m, q). \] (1.20)
This can be seen by comparing coefficients of \( s^k \) which gives the identity

\[
q^{\binom{m+1}{2}} \sum_{k=0}^{n} \binom{n}{k} x^{n-k} = q^{\binom{k}{2}} \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} + q^{\binom{k}{2}} \sum_{k=0}^{n-1} \binom{n-1}{k-1} x^{n-k}
\]

which is equivalent with

\[
\binom{n}{k} = q^k \binom{n-1}{k} + \binom{n-1}{k-1}.
\]

From (1.20) we deduce the well known recurrence

\[
r_n(x,s,-1) = (x + s) r_{n-1}(x,s,-1) + (q^{n-1} - 1) s r_{n-2}(x,s,-1).
\]  

(1.21)

For

\[
r_n(x,s,-1) = s r_{n-1}(x,q s,-1) + s r_{n-1}(x,s,-1) = (x + s) r_{n-1}(x,s,-1) + s \left( r_{n-1}(x,q s,-1) - r_{n-1}(x,s,-1) \right)
\]

= \( (x + s) r_{n-1}(x,s,-1) + (q^{n-1} - 1) s r_{n-2}(x,s,-1) \).

The last identity follows from

\[
r_{n-1}(x,q s,-1) - r_{n-1}(x,s,-1) = \sum_{k=0}^{n-1} \binom{n-1}{k} q^k x^{n-k} = \left( q^{n-1} - 1 \right) \sum_{k=0}^{n-2} \binom{n-2}{k} q^k x^{n-k} = s r_{n-2}(x,s,-1)
\]

by noting that

\[
\binom{n}{k} = \frac{1 - q^n}{1 - q^k} \binom{n-1}{k-1}.
\]

For later use let us mention another recurrence which we state for the univariate polynomials

\[
f(n,x,q) = r_n(1,x,-1,q) = \sum_{j=0}^{n} x^j \binom{n}{j + q}.
\]  

(1.22)

They satisfy

\[
f(n,x,q) = \left( 1 + (1 + q) q^{n-2} x + x^2 \right) f(n-2,x,q) - \left( 1 - q^{n-3} \right) \left( 1 - q^{n-2} \right) x^2 f(n-4,x,q).
\]  

(1.23)

**Proof**

By (1.21) we have

\[
f(n,x,q) = (1 + x) f(n-1,x,q) + (q^{n-1} - 1) xf(n-2,x,q).
\]

Therefore we get
\[ f(n, x, q) = (1 + x) f(n-1, x, q) + \left(q^{n-1} - 1\right) x f(n-2, x, q), \]
\[ f(n-1, x, q) = (1 + x) f(n-2, x, q) + \left(q^{n-2} - 1\right) x f(n-3, x, q), \]
\[ (1 + x) f(n-3, x, q) = f(n-2, x, q) + \left(1 - q^{n-3}\right) x f(n-4, x, q). \]

This implies
\[ f(n, x, q) = (1 + x) \left((1 + x) f(n-2, x, q) + \left(q^{n-2} - 1\right) x f(n-3, x, q)\right) + \left(q^{n-1} - 1\right) x f(n-2, x, q) \]
\[ = \left((1 + x)^2 + \left(q^{n-1} - 1\right) x\right) f(n-2, x, q) + \left(q^{n-2} - 1\right) x \left( f(n-2, x, q) + \left(1 - q^{n-3}\right) x f(n-4, x, q)\right) \]
\[ = \left(1 + (1 + q)q^{n-2}x + x^2\right) f(n-2, x, q) - \left(1 - q^{n-3}\right)(1 - q^{n-2}) x^2 f(n-4, x, q). \]

For \( x = 1 \) and \( s = -1 \) (1.21) reduces to
\[ r_n(1, -1, -1) = \left(1 - q^{n-1}\right) r_{n-2}(1, -1, -1). \] Since \( r_0(1, -1, -1) = 1 \) and \( r_1(1, -1, -1) = 0 \)
and thus leads to

**Gauss’s identity**
\[ \sum_{k=0}^{2n+1} (-1)^k \left[\begin{array}{c} 2n+1 \\ k \end{array}\right] = 0, \]
\[ \sum_{k=0}^{2n} (-1)^k \left[\begin{array}{c} 2n \\ k \end{array}\right] = \left(q; q^2\right)_n. \]

This identity is equivalent with
\[ \exp(z)\exp(-z) = \sum_{n\geq0} \left(q; q^2\right)_n \frac{z^{2n}}{[2n]!}, \]
\[ \text{(1.25)} \]
or equivalently with (1.12).

Let us also note the following \( q \) – analogue of \( e^x e^x = e^x \):
\[ \exp_q \left(\frac{z}{[2]}\right) \exp_q \left(\frac{qz}{[2]}\right) = \exp_q (z). \]
\[ \text{(1.26)} \]
or equivalently
\[ e_q \cdot (qz)e_q \cdot (z) = e_q (z) \]
\[ \text{(1.27)} \]
which is obvious by (1.11).
By comparing coefficients this is equivalent with

$$\sum_{k=0}^{n} q^k \begin{bmatrix} n \\ k \end{bmatrix} = (1 + q)(1 + q^2) \cdots (1 + q^n) = (-q; q)_n. \quad (1.28)$$

More generally we get

**Theorem 1.1**

*For each* $m \in \mathbb{N}$ *we get*

$$\sum_{k=0}^{2^m - 1} q^{(2m+1)k} \begin{bmatrix} n \\ k \end{bmatrix} = (-q; q)_n a(m,n,q) \quad (1.29)$$

*with*

$$a(m,n,q) = \sum_{j=0}^{n} (-1)^j q^j \begin{bmatrix} m \\ j \end{bmatrix} \frac{(q; q)_n}{(q; q)_{n-j}} \in \mathbb{Z}[q]. \quad (1.30)$$

*The polynomial* $a(m,n,q)$ *satisfies*

$$a(m,n,1) = 1, \quad a(m,2n,-1) = 1, \quad a(m,2n+1,-1) = 2m + 1. \quad (1.31)$$

**Proof**

Comparing coefficients in

$$\sum_{n \geq 0} \sum_{k=0}^{n} q^{(2m+1)k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{z^n}{(q^2; q^2)_n} = e_q (q^2; q^2) z e_q (z) = (qz; q^2)_m e_q (qz) e_q (z) = (qz; q^2)_m e_q (z)$$

we get

$$\sum_{k=0}^{n} q^{(2m+1)k} \begin{bmatrix} n \\ k \end{bmatrix} = (q^2; q^2)_n \sum_{j=0}^{n} (-1)^j q^j \begin{bmatrix} m \\ j \end{bmatrix} \frac{1}{(q; q)_{n-j}} = (-q; q)_n \sum_{j=0}^{n} (-1)^j q^j \begin{bmatrix} m \\ j \end{bmatrix} \frac{(q; q)_n}{(q; q)_{n-j}}. \quad (1.32)$$

**Remark**

Kupershmidt [6] has given another equivalent form of this theorem which has also been considered in [4]. Note that there is no such factorization for $\sum_{k=0}^{n} q^{2mk} \begin{bmatrix} n \\ k \end{bmatrix}$.  

Boris A. Kupershmidt [6] has also shown that

$$r_n(x,s,-1) = \sum_{k=0}^{\left\lfloor \frac{n}{2k} \right\rfloor} q^k \begin{bmatrix} n \\ 2k \end{bmatrix} s^{2k} r_{n-2k} (x,s,0). \quad (1.32)$$
In fact he used another notation. To compare his version with the present one note that
\[
\binom{2n+1}{2k}(q^2;q^2)_k = \binom{n}{k} q^{2n+1} \left(\frac{1}{q^2}\right)_k \quad \text{and} \quad \binom{2n}{2k}(q;q^2)_k = \binom{n}{k} q^{2n-1} \left(\frac{1}{q^2}\right)_k.
\]

He gave three proofs of this fact. Let us recall another proof which we have given in [4].

We use the generating functions.

\[
\sum_{n\geq 0} r_k(x,s,-1) \frac{z^n}{[n]!} = \exp(xz)\exp(sz) \left(\exp(sz)\exp(-sz)\right) \frac{\exp(xz)}{\exp(-sz)} = \sum_{j\geq 0} r_j(s,s,-1) \frac{z^j}{[j]!} \sum_{k\geq 0} r_k(x,s,0) \frac{z^k}{[k]!}
\]
\[
= \sum_{j\geq 0} \left(q;q^2\right)_j s^{2j} \sum_{k\geq 0} r_k(x,s,0) \frac{z^k}{[k]!}.
\]

Comparing coefficients we get the above formula.

A. Berkovich and S.O. Warnaar [1] have given several other results for the Rogers-Szegő polynomials. I will recall them with a different proof.

**Lemma 1.2 ([1], (8.9) and (8.11))**

\[
\sum_{n\geq 0} r_{2n}(1,s,-1,q) \frac{z^n}{(q^2;q^2)_n} = \frac{1}{e_q(-sz)} e_q(s^2 z)e_q(z)
\]

(1.33)

\[
\sum_{n\geq 0} r_{2n+1}(1,s,-1,q) \frac{z^n}{(q^2;q^2)_n} = (1+s) \frac{1}{e_q(-qs z)} e_q(s^2 z)e_q(z).
\]

(1.34)

**Proof**

By (1.13) and (1.16) we get

\[
\sum_{n\geq 0} r_{2n}(1,s,-1,q) \frac{z^n}{(q^2;q^2)_n} = e_q(s^2 z) e_q(s^2 z) = e_q(s^2 z) e_q(z) = e_q(s^2 z) \frac{1}{e_q(-sz)} e_q(z).
\]

For the second identity observe that by (1.10)

\[
\sum_{n\geq 0} r_{2n+1}(1,s,-1,q) \frac{z^n}{(q^2;q^2)_n} = (s+\varepsilon) e_q(s^2 z) e_q(s^2 z) e_q(\varepsilon^2 z) = (s+\varepsilon) e_q(s^2 z) e_q(s^2 z) e_q(z)
\]

\[
= s(1+sz) e_q(s^2 z) \frac{1}{e_q(-qs z)} e_q(z) + (1-s^2 z) e_q(s^2 z) \frac{1}{e_q(-qs z)} e_q(z) = (1+s) \frac{e_q(s^2 z) e_q(z)}{e_q(-qs z)}.
\]
Comparing coefficients and observing (1.16) we get

**Corollary 1.1 ([1], (8.8))**

\[
r_{2n}(1,s,-1,q) = \sum_{k=0}^{n} (-q;q)_k q^{\left(\frac{k}{2}\right)} s^{\left[\begin{array}{c} n \\ k \end{array}\right]} r_{n-k}(1,s^2,-1,q^2) \tag{1.35}
\]

and

\[
r_{2n+1}(1,s,-1,q) = (1+s) \sum_{k=0}^{n} (-q;q)_k q^{\left(\frac{k+1}{2}\right)} s^{\left[\begin{array}{c} n \\ k \end{array}\right]} r_{n-k}(1,s^2,-1,q^2). \tag{1.36}
\]

Another result which is a \(q\) – analogue of \((1+s)^{2n} = \sum_{k=0}^{n} \left(\frac{n}{k}\right) s^{2k} \left(1+\frac{1}{s}\right)^k (1+s)^{n-k}\) is

**Corollary 1.2 (A. Berkovich and S.O. Warnaar [1], Theorem 8.1)**

\[
r_{2n}(1,s,-1,q) = \sum_{k=0}^{n} s^{2k} r_k \left(1,\frac{q}{s},0,q^2\right) r_{n-k}(1,s,0,q^2) \tag{1.37}
\]

and

\[
r_{2n+1}(1,s,-1,q) = (1+s) \sum_{k=0}^{n} s^{2k} r_k \left(1,\frac{q}{s},0,q^2\right) r_{n-k}(1,q^2s,0,q^2) \tag{1.38}
\]

**Proof**

By comparing coefficients (1.37) is equivalent with

\[
\sum_{n \geq 0} r_{2n}(1,s,-1,q) z^n = \sum_{n \geq 0} z^n \sum_{n \geq 0} s^{2n} r_n \left(1,\frac{q}{s},0,q^2\right) \sum_{n \geq 0} r_n(1,s,0,q^2) z^n \frac{z^n}{(q^2;q^2)_n} \\
 = e_q(z) e_q(s^2z) e_q(0) e_q(z) e_q(-sz) \frac{e_q(s^2z) e_q(z)} {e_q(-sz)}
\]

and (1.38) with

\[
\sum_{n \geq 0} r_{2n+1}(1,s,-1,q) z^n = (1+s) \sum_{n \geq 0} z^n \sum_{n \geq 0} s^{2n} r_n \left(1,\frac{q}{s},0,q^2\right) \sum_{n \geq 0} r_n(1,q^2s,0,q^2) z^n \frac{z^n}{(q^2;q^2)_n} \\
 = (1+s) e_q(z) e_q(s^2z) e_q(0) e_q(z) e_q(-sz) \frac{e_q(s^2z) e_q(z)} {e_q(-sz)}.
\]
2. Recurrence relations for the general $q$–Newton binomials

For each integer $m$ the operators $A = x e^m$ and $B = x e$ are $q$–commuting:

$$(x e)(x e^m) = q(x e^m)(x e).$$

For

$$(x e)(x e^m)f(x, s) = x e sf(x, q^m s) = x q sf(x, q^{m+1} s) = q(x e^m)xf(x, q s) = q(x e^m)(x e)f(x, s).$$

The identity (1.20) can be written as

$$r_n(x, s, m) = (x e + x e^{m+1})r(n-1, x, s, m).$$

By iteration we finally get

$$r_n(x, s, m) = (x e + x e^{m+1})^n 1,$$

$$i.e. r_n(x, s, m) \text{ can be obtained by applying the operator } (x e + x e^{m+1})^n \text{ to the constant function } f(x, s) = 1.$$

We now recall the recurrence relation of the sequence $(r_n(x, s, m))_{n \geq 0}$ for fixed $x$ and $s$, which has already been obtained in [3].

We show that for each $m \in \mathbb{Z}$ the sequence $(r_n(x, s, m))_{n \geq 0}$ satisfies a recurrence of order $|m|+1$. This is the reason for choosing $(m+1) \binom{k}{2}$ instead of $m \binom{k}{2}$ in (1.15).

**Theorem 2.1**

For $m \geq 0$ the sequence $(r_n(x, s, m))_{n \geq 0}$ satisfies the recurrence

$$r_n(x, s, m) = q^{(m+1)(n-1)} s r_{n-1}(x, s, m) - \sum_{k=1}^{m+1} (-1)^k \binom{k}{2} \binom{m+1}{k} q^{(n-k+1) \binom{m+1}{k}} - q^m \binom{m}{k} x^k r_{n-k}(x, s, m)$$

(2.2)

of order $m+1$.

**Proof**

In order to formulate the following results in a simple way consider the algebra $\mathbb{C}(q)[N]$ of all polynomials in $N$ whose coefficients are rational functions of $q$. 11
Consider the following polynomials \( C_k(N) = \frac{\prod_{j=0}^{k-1} (1-q^{-j}N)}{(q;q)_k} \) which satisfy \( C_k(q^n) = \binom{n}{k} \). It is clear that each polynomial has a unique representation of the form \( (q;q)_n \). Let \( E \) be the linear operator defined by \( Ef(N) = f(qN) \) and \( \Delta \) the operator defined by \( \Delta f(N) = \frac{f(qN)-f(N)}{N} = \frac{1}{N} (E-1) f(N) \). \( \Delta \) operates on \( \mathbb{C}(q)[N] \) since \( \Delta N^k = (q^k-1) N^{k-1} \).

It can also be characterized by \( \Delta q^{-\frac{k}{2}} C_k(N) = q^{-k} C_{k-1}(N) \).

By definition we have \( E = 1 + N \Delta \).

Let \( Z \) be the linear operator defined by \( Z C_k(N) = q^k C_k(N) \). Then \( E = Z (1 + \Delta) \).

For \( Z (1 + \Delta) C_k(N) = Z \left( C_k(N) + q^{1+k} C_{k-1}(N) \right) = q^k C_k(N) + C_{k-1}(N) = \frac{C_{k-1}(N)}{1-q^k} \left( q^k \left( 1 - \frac{N}{q^{k-1}} \right) + 1 - q^k \right) \)

\[ = \frac{C_{k-1}(N)}{1-q^k} (1 - qN) = C_k(qN) = EC_k(N). \]

Therefore we can write \( Z^{-1} = (1 + \Delta) E^{-1} = \left( 1 + \frac{1}{N} (E-1) \right) E^{-1} = \frac{1}{N} + \left( 1 - \frac{1}{N} \right) E^{-1} = \frac{1}{N} \left( 1 + (N-1) E^{-1} \right) \).

Note that \( Z \) and \( Z^{-1} \) are operators on \( \mathbb{C}(q)[N] \), whereas the multiplication operators \( \frac{1}{N} \) and the operator \( \frac{N-1}{N} \) are only defined on the larger space \( \mathbb{C}(q)[N, N^{-1}] \).

The operators \( A = \frac{1}{N} \) and \( B = \frac{N-1}{N} E^{-1} \) satisfy \( BA = qAB \) and therefore by the general \( q \)-binomial theorem we get for \( \ell \geq 0 \)

\[ Z^{-\ell} = \sum_{k=0}^{\ell} \binom{\ell}{k} \frac{1}{N^{\ell-k}} \left( \frac{N-1}{N} E^{-1} \right)^k = \frac{1}{N^\ell} \sum_{k=0}^{\ell} \binom{\ell}{k} \prod_{j=0}^{k-1} (N-q^j) E^{-k}. \quad (2.3) \]

Note further that

\[ Z^{-(\ell-1)} \Delta q^{\frac{\ell}{2}} C_k(N) = Z^{-(\ell-1)} q^{\frac{\ell}{2}-(k-1)} C_{k-1}(N) = q^{\frac{\ell}{2}-(k-1)-(\ell-1)(k-1)} C_{k-1}(N) = q^{\frac{k-1}{2}} C_{k-1}(N). \]
Consider now the polynomials in $N$

$$S_n(s,m,N) = \sum_{k=0}^{n} q^{\binom{k}{2}} C_k(N)s^k. \quad (2.4)$$

Then for $\ell \geq n$ we get $S_{\ell}(s,m,q^n) = r_{\ell}(1,s,m)$.

Now observe that

$$\frac{S_{n+1}(s,m,qN) - S_{n+1}(s,m,N)}{N} = \Delta S_{n+1}(s,m,N) = \sum_{k=0}^{n+1} q^{\binom{k}{2}} \Delta q^{\binom{k}{2}} C_k(N)s^k$$

$$= \sum_{k=0}^{n+1} q^{\binom{k}{2}} q^{k-1} C_{k-1}(N)s^k = \sum_{k=0}^{n} q^{\binom{k}{2}} q^{k-1} C_{k-1}(N)s^k = sZ^mS_n(s,m,N) = sS_n(q^m,s,m,N).$$

Thus

$$Z^{-m}\Delta S_{n+1}(s,m,N) = sS_n(s,m,N). \quad (2.5)$$

We have $NE^{-k}\Delta = q^kE^{-k}N\Delta = q^k(E^{1-k} - E^{-k})$ and therefore for $m \geq 0$

$$N^{m+1}Z^{-m}\Delta = \sum_{k=0}^{m} \left[ \sum_{k=0}^{m} \prod_{j=0}^{k-1} (N-q^j)q^kE^{-k} \right] (E-1)$$

$$= E + \sum_{k=0}^{m} \left[ \sum_{k=0}^{m} \prod_{j=0}^{k-1} (N-q^j)q^{k+1}E^{-k} - \sum_{k=0}^{m} \prod_{j=0}^{k-1} (N-q^j)q^kE^{-2} \right]$$

$$= E + \sum_{k=0}^{m} \left[ \sum_{k=0}^{m} \prod_{j=0}^{k-1} (N-q^j)(qN-q^{k+1})q^{k+1}E^{-k} - \sum_{k=0}^{m} \prod_{j=0}^{k-1} (N-q^j)q^{k+1}E^{-2} \right]$$

$$= E + \sum_{k=0}^{m} \left[ \prod_{k=0}^{m} \left[ \prod_{j=0}^{k-1} (N-q^j)q^{k+1}E^{-k} \right] \prod_{j=0}^{k-1} (N-q^j)q^{k+1}E^{-k} \right]$$

$$= E + \sum_{k=0}^{m} \left[ \prod_{k=0}^{m} \left[ \prod_{j=0}^{k-1} (N-q^j)q^{k+1}E^{-k} \right] \prod_{j=0}^{k-1} (N-q^j)q^{k+1}E^{-k} \right].$$

Therefore

$$sN^{m+1}S_n(s,m,N) = N^{m+1}Z^{-m}\Delta S_{n+1}(s,m,N)$$

$$= S_{n+1}(s,m,qN) + \sum_{k=0}^{m} \left[ \prod_{k=0}^{m} \left[ \prod_{j=0}^{k-1} (N-q^j)q^{k+1}E^{-k} \right] \prod_{j=0}^{k-1} (N-q^j)q^{k+1}E^{-k} \right].$$
By choosing \( N = q^{n-1} \) this reduces to
\[
q^{(m+1)(n-1)} s_{r_{n-1}}(1,s,m) = r_n(1,s,m)
\]
\[
+ \sum_{k=0}^{m} \left[ \frac{m}{k+1} \right] q^n - \left[ \frac{m+1}{k+1} \right] \prod_{j=0}^{k-1} (q^n - q^{j+1}) r_{n-k-1}(1,s,m)
\]
\[
= r_n(1,s,m) + \sum_{k=1}^{m+1} \left[ \frac{m}{k} \right] q^n - \left[ \frac{m+1}{k} \right] \prod_{j=0}^{k-2} (q^n - q^{j+1}) r_{n-k}(1,s,m)
\]
\[
= r_n(1,s,m) + \sum_{k=1}^{m+1} \left[ \frac{m}{k} \right] q^n - \left[ \frac{m+1}{k} \right] \prod_{j=0}^{k-2} (q^n - q^{j+1}) r_{n-k}(1,s,m)
\]
or
\[
r_n(1,s,m) = q^{(m+1)(n-1)} s_{r_{n-1}}(1,s,m) - \sum_{k=1}^{m+1} (-1)^k q^{\frac{k}{2}} (q^{n-k+1}; q)_{k-1} \left[ \frac{m+1}{k} \right] q^{\frac{m}{k}} r_{n-k}(1,s,m)
\]

Now observe that
\[
r_n(x,s,m) = x^n r_n \left( 1 - \frac{s}{x}, m \right)
\]
and therefore
\[
r_n(x,s,m) = q^{(m+1)(n-1)} s_{r_{n-1}}(x,s,m) - \sum_{k=1}^{m+1} (-1)^k q^{\frac{k}{2}} (q^{n-k+1}; q)_{k-1} \left[ \frac{m+1}{k} \right] q^{\frac{m}{k}} x^k r_{n-k}(x,s,m)
\]

For \( m = 0 \) we get
\[
r_{n+1}(x,s,0) = xr_n(x,s,0) = q^n s_{r_n}(x,s,0)
\]
and therefore
\[
r_n(x,s,0) = (x+s)(x+qs) \cdots (x+q^{n-1}s).
\]

For \( m = 1 \) we have
\[
r_n(x,s,1) = q^{2(1-n)} s_{r_{n-1}}(x,s,1) - \sum_{k=1}^{2} (-1)^k q^{\frac{k}{2}} (q^{n-k+1}; q)_{k-1} \left[ \frac{2}{k} \right] q^{\frac{1}{k}} x^k r_{n-k}(x,s,1)
\]
\[
= (q^{2(n-1)} s + (1 + q - q^n x) r_{n-1}(x,s,1) - q) (1 - q^{n-1}) x^2 r_{n-2}(x,s,1)
\]

**Theorem 2.2**

The sequence \( \left( r_n(x,s,-m) \right)_{m \geq 0} \) with \( m > 0 \) satisfies the recurrence relation
\[
r_n(x,s,-m) - x r_{n-1}(x,s,-m) = \frac{s}{q^{(m+1)(n-1)}} \sum_{k=0}^{m} \left[ \frac{m}{k} \right] (-1)^k q^{\frac{k}{2}} (q^{n-1}; \frac{1}{q})_{k-1} x^k r_{n-k-1}(x,s,-m).
\]
Proof

We know already that

\[ S_{n+1}(s, -m, qN) - S_{n+1}(s, -m, N) = sNZ^{-m}S_n(s, m, N) \]

and that

\[ sNZ^{-m}S_n(s, m, N) = sN \frac{1}{N^m} \sum_{k=0}^{m} \left[ m \choose k \right] \prod_{j=0}^{k-1} (N - q^j) E^{-k} S_n(s, m, N) \]

Therefore we get

\[ S_{n+1}(s, -m, qN) - S_{n+1}(s, -m, N) = s \frac{1}{N^{m+1}} \sum_{k=0}^{m} \left[ m \choose k \right] \prod_{j=0}^{k-1} (N - q^j) S_n(s, m, qN) \]

For \( N = q^{n-1} \) this reduces to

\[ r_n(1, s, -m) - r_{n-1}(1, s, -m) = \frac{s}{q^{(n-1)(n-2)}} \sum_{k=0}^{m} \left[ m \choose k \right] (-1)^k q^{\frac{k^2}{2}} \left( q^{n-1}, \frac{1}{q} \right)_k r_{n-k-1}(1, s, -m) \]

which implies (2.6).

As a special case we get again

\[ r_n(x, s, -1) - xr_{n-1}(x, s, -1) = s r_{n-1}(x, s, -1) - s(1 - q^{n-1})xr_{n-2}(x, -1) \]

Let us also mention

**Proposition 2.1**

The polynomials \( \sigma_n(m, s, q) = \sum_{j=0}^{n} q^{m^2(j^2)} \left[ n \choose mj \right] s^j \) satisfy the recurrence

\[ \sum_{k=0}^{m} (-1)^k \left[ m \choose k \right] q^{\frac{k^2}{2}} \sigma_{n+m-k}(m, s, q) = q^{nm} s \sigma_n(m, s, q). \]  

(2.7)

Proof

\[ \Delta^m \left( \begin{array}{c} \frac{m k}{2} \\ j \end{array} \right) C_{n k}(N) = \left( \begin{array}{c} \frac{m(k-1)}{2} \\ j \end{array} \right) C_{m(k-1)}(N) \] implies \( q^{m \left( \begin{array}{c} m^2 \frac{2}{2} \\ j \end{array} \right)} C_{m(j-1)}(N) = q^{m \left( \begin{array}{c} m^2 \frac{2}{2} \\ j \end{array} \right)} C_{m(j-1)}(N) \)

and therefore

\[ q^{m \left( \begin{array}{c} m^2 \frac{2}{2} \\ j \end{array} \right)} \sum_{j=0}^{n} q^{m^2 \left( \begin{array}{c} j^2 \frac{2}{2} \end{array} \right)} C_{n j}(N)s^j = \sum_{j=0}^{n} q^{m \left( \begin{array}{c} m^2 \frac{2}{2} \\ j \end{array} \right)} C_{n(j-1)}(N)s^j = \sum_{j=0}^{n-1} q^{m \left( \begin{array}{c} m^2 \frac{2}{2} \\ j \end{array} \right)} C_{n j}(N)s^j. \]

Now observe that
\[ \Delta^m = \frac{1}{q^{2m}} \binom{m}{2}(E-1)(E-q) \cdots (E-q^{m-1}) = \frac{1}{q^{2m}} \sum_{k=0}^{m} (-1)^k \binom{m}{k} q^{2k} E^{m-k}. \]

Therefore we get
\[ (q^{2m}) \Delta^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} q^{2k} E^{m-k}. \]
\[ \sum_{k=0}^{m} (-1)^k \binom{m}{k} q^{2k} \sum_{j=0}^{m^2} q^{m^2(j+2)} C_{m^2}(q^{m-k} N) s^j = N^{m^2} \sum_{j=0}^{m^2} q^{m^2(j+2)} C_{m^2}(N) s^j. \]

For \( N = q^n \) this reduces to (2.7).

For \( q = 1 \) and a prime number \( p \) we get the recurrence
\[ \sigma_n(p,-1,1) = \sum_{k=1}^{p-1} (-1)^{k-1} \binom{p}{k} \sigma_{n-k}(p,-1,1). \]

Since all \( \binom{p}{k} \) are multiples of \( p \) we see that \( p^{n/p} \) is a factor of \( \sigma_n(p,-1,1) \).

In the general case we get the recurrence
\[ \sigma_n(p,-1,q) = (q^{(p-1)/2} - q^{(p-1)/p}) \sigma_{n-p}(p,-1,q) - \sum_{k=1}^{p-1} (-1)^k \binom{p}{k} \sigma_{n-k}(p,-1,q). \]

All coefficients are multiples of \( \lceil p \rceil \) since \( q^{(p-1)/2} - q^{(p-1)/p} = \lceil p \rceil (1-q) q^{(p-1)/2} \left( n - \frac{p-1}{2} \right). \)

This gives
**Proposition 2.2**

For a prime number \( p \) we have
\[ \sigma_n(p,-1,q) = \lceil p \rceil^{n/p} c(n, p, q) \] (2.8)
with \( c(n, p, q) \in \mathbb{Z}[q] \).
3. Some generalizations of Gauss’s identity and related facts

**Theorem 3.1**

For each integer \( m \geq 0 \)

\[
\sum_{j=0}^{n} (-1)^{j} q^{jm} \begin{pmatrix} n \\ j \end{pmatrix} = \left( q; q^2 \right)_{n+1}^{-1} c_n(m, q) \tag{3.1}
\]

where \( c_n(m, q) \in \mathbb{Z}[q] \) is given by

\[
c_{2n}(m, q) = \sum_{r=0}^{n} q^{\binom{2n-2r}{2}} \binom{m}{2n-2r} \left( q^{2r+2}; q^{2} \right)_{n-r}
\]

and

\[
c_{2n+1}(m, q) = \sum_{r=0}^{n} q^{\binom{2n+1-2r}{2}} \binom{m}{2n+1-2r} \left( q^{2r+2}; q^{2} \right)_{n-r}.
\]

For \( m = 0 \) we get Gauss’s identity since \( c_{2n}(0, q) = 1 \) and \( c_{2n+1}(0, q) = 0 \). Other simple examples are

\[
\sum_{j=0}^{n} (-1)^{j} q^{j} \begin{pmatrix} n \\ j \end{pmatrix} = \left( q; q^2 \right)_{n+1}^{-1},
\]

\[
\sum_{j=0}^{2n} (-1)^{j} q^{2j} \begin{pmatrix} 2n \\ j \end{pmatrix} = \left( q; q^2 \right)_{n+1} (1 + q - q^{2n+1})
\]

and

\[
\sum_{j=0}^{2n+1} (-1)^{j} q^{2j} \begin{pmatrix} 2n+1 \\ j \end{pmatrix} = \left( q; q^2 \right)_{n+1} (1 + q).
\]

Note that \( c_n(m, 0) = 1 \) and \( c_{2n}(m, 1) = 1 \) and \( c_{2n+1}(m, 1) = m \).

**Proof**

By (1.10) we have

\[
\sum_{n, j=0}^{m} (-1)^{j} q^{jm} \begin{pmatrix} n \\ j \end{pmatrix} z^{n} (q; q)_{n}^{-1} = e_{q}(-q^{-n} z) e_{q}(z) = \left( -z; q \right)_{m} e_{q}(-z) e_{q}(z) = \sum_{j=0}^{m} q^{\binom{j}{2}} \begin{pmatrix} m \\ j \end{pmatrix} z^{j} \sum_{\ell \geq 0} \left( q^{2} z^{-2} \right)^{j}.
\]

Comparing coefficients gives
\[
\sum_{j=0}^{n} (-1)^j q^{jm} \left[ \frac{n}{j} \right] = (q; q)_n \sum_{j=2j'=n} q^{j'} \left[ \frac{m}{j'} \right] \frac{1}{(q^2; q^2)_\ell}.
\]

This implies
\[
\sum_{j=0}^{2n} (-1)^j q^{jm} \left[ \frac{2n}{j} \right] = (q; q)_{2n} \sum_{j=2j'=2n} q^{j'} \left[ \frac{m}{j'} \right] \frac{1}{(q^2; q^2)_\ell} = (q; q^2)_n \sum_{\ell=0}^{2n-2} q^{2n-2\ell} \left[ \frac{m}{2n-2\ell} \right] \frac{(q^2; q^2)_n}{(q^2; q^2)_\ell}
\]
and
\[
\sum_{j=0}^{2n+1} (-1)^j q^{jm} \left[ \frac{2n+1}{j} \right] = (q; q)_{2n+1} \sum_{j=2j'=2n+1} q^{j'} \left[ \frac{m}{j'} \right] \frac{1}{(q^2; q^2)_\ell} = (q; q^2)_{n+1} \sum_{\ell=0}^{2n+1-2\ell} q^{2n+1-2\ell} \left[ \frac{m}{2n+1-2\ell} \right] \frac{(q^2; q^2)_n}{(q^2; q^2)_\ell}.
\]

**Theorem 3.2**

For each integer \( m \geq 0 \),
\[
\sum_{j=0}^{n} (-1)^j q^{(2m+1)j} \left[ \frac{n}{j} \right] = (q; -q)_n c_n(m, q) \tag{3.4}
\]
with \( c_n(m, q) \in \mathbb{Z}[q] \).

For \( m = 0 \) we get
\[
\sum_{j=0}^{n} (-1)^j q^{j} \left[ \frac{n}{j} \right] = (q; -q)_n = (q; q^2)_{n+1} \left[ \frac{2n}{2} \right] (-q^2; q^2)_{\frac{n+1}{2}}. \tag{3.5}
\]

**Proof**

By replacing \( q \to -q \) we can use the same argument as in Theorem 1.1.

Computer experiments lead to

**Conjecture 3.1**

For integers \( r \geq 0, m \geq 0, k > 0 \)
\[
\sum_{j=0}^{n} (-1)^j q^{j^2+mk} \left[ \frac{n}{j} \right] = (q; q^2)_{n+1} \left[ \frac{2n+1}{2} \right] c_n(r, m, k, q) \tag{3.6}
\]
with \( c_n(r, m, k, q) \in \mathbb{Z}[q] \). For \( r = 0 \) and \( m = 1 \) all coefficients of \( c_n(0,1,k,q) \) are non-negative.
Let us now consider how fast \( \sum_{j=0}^{n} (-1)^j q^j \binom{n}{j} \) converges to 0 for \( q \to 1 \).

**Theorem 3.3**

\[
\lim_{q \to 1} \frac{\sum_{j=0}^{n} (-1)^j q^j \binom{2n}{j} \left( \frac{2n}{q} \right)^j}{(q; q^2)^n} = k^n
\]

and

\[
\lim_{q \to 1} \frac{\sum_{j=0}^{n+1} (-1)^j q^j \binom{2n+1}{j} \left( \frac{2n+1}{q} \right)^j}{(q; q^2)^{n+1}} = mk^n.
\]

**Proof**

Let \( f(n, x, q) = \sum_{j=0}^{n} x^j \binom{n}{j} q^{j} \).

Then \( f(2n, -q^m, q^k) = \sum_{j=0}^{2n} (-1)^j q^j \binom{2n}{j} \)

by (1.23) satisfies the recurrence

\[
f(2n, -q^m, q^k) = \left( 1 - (1 + q^k)q^{2k(n-1)+m} + q^{2m} \right)f(2n - 2, -q^m, q^k) - \left( 1 - q^{k(2n-3)} \right)\left( 1 - q^{2k(n-1)} \right)q^{2m}f(2n - 4, -q^m, q^k).
\]

Therefore

\[
h(n, q) = \frac{f(2n, -q^m, q^k)}{(q; q^2)_n}
\]

satisfies the recurrence

\[
h(n, q) = \frac{1 - (1 + q^k)q^{2k(n-1)+m} + q^{2m}}{1 - q^{2n-1}} h(n - 1, q) - \frac{\left( 1 - q^{k(2n-3)} \right)\left( 1 - q^{2k(n-1)} \right)q^{2m}}{(1 - q^{2n-1})(1 - q^{2n-3})} h(n - 2, q)
\]

with initial values

\[
h(0, q) = 1 \quad \text{and} \quad h(1, q) = [m] - q^{m+k}[m-k].
\]

For \( q \to 1 \) we get

\[
h(n, 1) = \frac{k(4n-3)}{2n-1} h(n-1, 1) - \frac{2k^2(n-1)}{2n-1} h(n-2, 1)
\]

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with initial values \( h(0,1) = 1 \) and \( h(1,1) = k \).

This implies \( h(n,1) = k^n \) and thus (3.7).

In the same way \( f(2n+1,-q^m, q^k) = \sum_{j=0}^{2n+1} (-1)^j q^{mj} \left[ \frac{2n+1}{j} \right] \) satisfies the recurrence
\[
f(2n+1,-q^m, q^k) = (1 - (1 + q^k) q^{k(2n-1)+m} + q^{2m}) f(2n-1,-q^m, q^k)
- (1-q^{k(2n-2)})(1-q^{k(2n-1)}) q^{2m} f(2n-3,-q^m, q^k).
\]

Therefore
\[
H(n,q) = \frac{f(2n+1,-q^m, q^k)}{(q;q^2)_{n+1}}
\]
satisfies the recurrence
\[
H(n,q) = \frac{1-(1+q^k)q^{k(2n-1)+m}+q^{2m}}{1-q^{2n+1}} H(n-1,q) - \frac{(1-q^{k(2n-1)})(1-q^{2k(n-1)}) q^{2m}}{(1-q^{2n-1})(1-q^{2n+1})} H(n-2,q)
\]
with initial values
\[
H(0,q) = [m] \quad \text{and} \quad H(1,q) = [m] \frac{m+k+q^m[2k-m]}{[3]}
\]

For \( q \to 1 \) this gives
\[
H(n,1) = \frac{k(4n-1)}{2n+1} H(n-1,1) - \frac{2k^2 (n-1)}{2n+1} H(n-2,1)
\]
with initial values \( H(0,1) = m \) and \( H(1,1) = km \). This gives (3.8).

The same argument gives

**Corollary 3.1**

\[
\lim_{q \to 1} \sum_{j=0}^{n} q^{(2n+1)j} \left[ \frac{2n}{j} \right] \frac{2n}{j} q^{2j} = (2k)^n
\]
and

\[
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\]
More generally let us consider the same question for

\[
\sum_{j=0}^{n} (-1)^j q^{j^2 + mj} \left\lfloor \frac{n}{j} \right\rfloor \left(\frac{q^{j^2}}{q^2}\right)_{n+1}.
\]

In a first draft of this paper I stated two conjectures, one of which I posted as question in MathOverflow. This has been proved by Will Sawin [7].

Lemma 3.1 (W. Sawin [7])

For all \( r, m \in \mathbb{Z} \)

\[
\sum_{j=0}^{2n} (-1)^j q^{j^2 + mj} \left\lfloor \frac{2n}{j} \right\rfloor = (k - 2r)^n,
\]

Proof

For \( k = 2r \) we get from (1.4)

\[
\sum_{j=0}^{n} (-1)^j q^{j^2 + mj} \left\lfloor \frac{n}{j} \right\rfloor = \left(1 - q^{k^2}ight) \left(1 - q^{k^2 + m}ight) \cdots \left(1 - q^{(2n-1)k^2 + m}\right) = \left(\frac{q^{k^2 + m}}{q^k} ; q^k\right)_n.
\]

Therefore

\[
\lim_{q \to 1} \frac{\sum_{j=0}^{2n} (-1)^j q^{j^2 + mj} \left\lfloor \frac{2n}{j} \right\rfloor}{\left(\frac{q^{k^2 + m}}{q^k} ; q^k\right)_{2n}} = \lim_{q \to 1} \frac{\left(\frac{q^{k^2 + m}}{q^k} ; q^k\right)_n}{\left(q; q^2\right)_n} = [n = 0]
\]

and

\[
\lim_{q \to 1} \frac{\sum_{j=0}^{2n+1} (-1)^j q^{j^2 + mj} \left\lfloor \frac{2n+1}{j} \right\rfloor}{\left(\frac{q^{k^2 + m}}{q^k} ; q^k\right)_{2n+1}} = \lim_{q \to 1} \frac{\left(\frac{q^{k^2 + m}}{q^k} ; q^k\right)_n}{\left(q; q^2\right)_{n+1}} = \left(\frac{k}{2} + m\right)[n = 0].
\]

Thus in this case the Lemma is true.

Let \( f(n, r, m, k) = \sum_{j=0}^{n} (-1)^j q^{j^2 + mj} \left\lfloor \frac{n}{j} \right\rfloor \).

To obtain the limit (3.11) we observe that the denominator has the root \( q = 1 \) with multiplicity \( n \). If the limit exists then the nominator also has the root \( q = 1 \) with multiplicity at least \( n \).
Sawin’s trick was to use de L’Hospital’s rule combined with Leibniz’s rule by writing \( f(n,r,m,k) \) in the form

\[
f(n,r,m,k) = \sum_{j=0}^{n} (-1)^j q^{a+j} \left[ n \right] \cdot \sum_{j=0}^{n} (-1)^j q^{k+j} \left[ j \right].
\]

This gives

\[
\frac{\partial^j f(2n,r,m,k)}{\partial q^j} (1) = \sum_{a=0}^{i} \left( \sum_{j=0}^{2n} (-1)^j q^{a+j} \frac{\partial^a q}{\partial q^a} (1) \frac{\partial^{i-a} q^{k+j}}{\partial q^{i-a}} (1) \right).
\]

Now \( \frac{\partial^{i-a} q^{k+j}}{\partial q^{i-a}} (1) \) is a polynomial of degree \( 2a \) in \( j \). Therefore the right-hand side can be written as a linear combination of the terms

\[
F(b,c,2n) = \sum_{j=0}^{2n} (-1)^j j^b \frac{\partial^{c} q^{j}}{\partial q^{c}} (1) \quad (3.13)
\]

where \( b + c \leq 2a + 2(i-a) = 2i \).

Let us now consider the known case \( f \left( 2n, \frac{k}{2}, m, k \right) \).

\[
\frac{\partial^i f \left( 2n, \frac{k}{2}, m, k \right)}{\partial q^i} (1) = \frac{\partial^i \left( \sum_{j=0}^{2n} (-1)^j q^{k+j} \left[ 2n \right] j^i q^{j} \right)}{\partial q^i} (1)
\]

\[
= \sum_{a=0}^{i} \left( \sum_{j=0}^{2n} (-1)^j q^{a+j} \left[ 2n \right] j^i q^{j} \right) \frac{\partial^{i-a} q^{k+j}}{\partial q^{i-a}} (1)
\]

\[
= \sum_{a=0}^{i} \left( \sum_{j=0}^{2n} (-1)^j (mj)(mj-1) \cdots (mj-a+1) \frac{\partial^{i-a} q^{k+j}}{\partial q^{i-a}} (1) \right)
\]

\[
= \sum_{a=0}^{i} \left( \sum_{j=0}^{2n} [f^b](mj)(mj-1) \cdots (mj-a+1) F(b,i-a,2n) \right).
\]

It is clear that \( F(0,0,2n) = [n = 0] \).

Now let us assume by induction that

\( F(b,c,2n) = 0 \) for \( b + c < i \) for some fixed \( i \leq 2n \).

Then we get
By (3.12) this polynomial is identically 0 for $i < 2n$.

Therefore all coefficients $F(a,i-a,2n)$ vanish. Thus $F(b,c,2n) = 0$ for $b + c = i$.

This implies that $F(b,c,2n) = 0$ for $b + 2c < 2n$ and therefore the first $n - 1$ partial derivatives vanish.

The only $F(b,c,2n)$ with $b + 2c \leq 2n$ which we did not yet compute is

$$F(2n,0,2n) = \sum_{j=0}^{2n} (-1)^j j^{2n} q^j \left[ \binom{2n}{j} \right] (1) = \sum_{j=0}^{2n} (-1)^j j^{2n} \binom{2n}{j} = (2n)!.$$ 

Since $j^{2n}$ only occurs for $a = n$ we get

$$\frac{\partial^n f(2n,r,m,k)}{\partial q^n} (1) = \sum_{a=0}^{n} \left( \binom{n}{a} \sum_{j=0}^{2n} (-1)^j j^{2n} q^j \right) \frac{\partial^n q^{\frac{r-k}{2}j^2 + mj}}{\partial q^n} (1) = F(2n,0,2n)[j^{2n}] \frac{\partial^n q^{\frac{r-k}{2}j^2 + mj}}{\partial q^n} (1).$$

Now

$$\left[ j^{2n} \right] \frac{\partial^n q^{\frac{r-k}{2}j^2 + mj}}{\partial q^n} (1) = \left[ j^{2n} \right] \left( r - \frac{k}{2} \right)^n.$$

Therefore we finally get

$$\frac{\partial^n f(2n,r,m,k)}{\partial q^n} (1) = (2n)! \left( r - \frac{k}{2} \right)^n.$$ 

By Gauss’s theorem we have $(q; q^2)_n = f(2n,0,0,1)$.

Thus
\[
\frac{\partial^n f(2n,0,0,1)}{\partial q^n}(1) = (2n)! \left( -\frac{1}{2} \right)^n.
\]

This implies
\[
\lim_{q \to 1} \left( \frac{\sum_{j=0}^{2n} (-1)^j q^{j^2+mj}}{(q; q^2)_n} \right) = \lim_{q \to 1} \left( \frac{\sum_{j=0}^{2n} (-1)^j q^{j^2+mj} [2n]_q}{\sum_{j=0}^{2n} (-1)^j [2n]_q} \right) = \left( \frac{r - k}{2} \right)^n = (k - 2r)^n.
\]

Let us now consider binomials of the form \( \sum_{j=0}^{n} q^{j^2+mj} [n]_q \).

It is easy to check that \( \lim_{q \to 1} \sum_{j=0}^{n} q^{j^2+mj} [n]_q = 0 \) for \( n > 0 \) if and only if \( r + m \equiv 1 \mod 2 \) and \( k \equiv 0 \mod 2 \).

Therefore we get

**Corollary 3.2**

*If* \( r + m \equiv 1 \mod 2 \) *then*

\[
\lim_{q \to 1} \left( \frac{\sum_{j=0}^{2n} q^{j^2+mj} [2n]}{(q; q^2)_{2n}} \right) = (k - r)^n.
\]  

This follows from \( (-q)^{j^2+mj} = (-1)^j q^{j^2+mj} \) and \( \lim_{q \to 1} \left( \frac{(q; q^2)_{2n}}{(q; q^2)_n} \right) = 2^n \).

The analogous results for \( 2n + 1 \) are stated as

**Conjecture 3.2**

\[
\lim_{q \to 1} \left( \frac{\sum_{j=0}^{2n+1} (-1)^j q^{j^2+mj} [2n+1]}{(q; q^2)_{n+1}} \right) = ((2n+1)r + m)(k - 2r)^n
\]  

*and for* \( r + m \equiv 1 \mod 2 \)
\[
\lim_{q \to 1} \left( \frac{2n+1}{(2n+1)!} \sum_{j=0}^{n} (-1)^j q^{j^2 + mj} \left\lfloor \frac{n}{j} \right\rfloor \right) = (2n+1)(k-r)^n. \tag{3.16}
\]

**Conjecture 3.3**

For a prime number \( p \) let \( v_p(x) \) denote the \( p \)-adic valuation of \( x \), which is the largest non-negative integer \( m \) such that \( p^m \) divides \( x \) and let \( V_p(x) = v_p(x) \). For integers \( m, r \geq 0 \) we get

\[
\sum_{j=0}^{n} (-1)^j q^{j^2 + mj} \left\lfloor \frac{n}{j} \right\rfloor = c_a(r,m,p,q) \prod_{j=1}^{\frac{n+1}{2}} \left( 1 - q^{2j-1} \right) \tag{3.17}
\]

with \( c_a(r,m,p,q) \in \mathbb{Z}[q] \).

Finally let us state the following

**Conjecture 3.4**

For any odd prime \( p \) and any \( m \in \mathbb{N} \) there exists a factorization

\[
A_p(m,n,q) = \sum_{j=0}^{n} q^{(2m+1)j} \left\lfloor \frac{n}{j} \right\rfloor = \prod_{k=1}^{\frac{n+1}{2}} \left( 1 + q^{k} \right) c_p(m,n,q) \tag{3.18}
\]

with \( c_p(m,n,q) \in \mathbb{Z}[q] \).

This polynomial satisfies \( c_p(m,n,1) = 1 \) and

\[
c_p(0, n, -1) = p^{b(n,p)} \tag{3.19}
\]

with

\[
b(n, p) = \left\lfloor \frac{n}{2} \right\rfloor + v_p(n!) - v_p\left(\left\lfloor \frac{n}{2} \right\rfloor !\right). \tag{3.20}
\]

**Example**

Let us consider for example \( A_3(2, n, q) = \sum_{j=0}^{n} q^{j^2} \left\lfloor \frac{n}{j} \right\rfloor \).

From \( \left( \frac{k}{V_3(k)} \right)_{k=1} = (1, 2, 1, 4, 2, 7, 8, 1, \ldots) \)
we find

\[ A_1(2,0,q) = 1, \quad A_1(2,1,q) = (1 + q) A_1(2,0,q), \quad A_1(2,2,q) = (1 + q^2) A_1(2,0,q), \]

\[ A_1(2,3,q) = (1 + q^3) A_1(2,0,q), \quad A_1(2,4,q) = (1 + q^4) A_1(2,0,q), \]

\[ A_1(2,5,q) = (1 + q^5) A_1(2,0,q), \cdots. \]

The first terms of the sequence \((c_3(2,n,q))_{n \geq 0}\) are

\[ 1, 1 - q + q^2 - q^3 + q^4, \quad 1 - q + q^2 - q^3 + q^4 (1 - q + q^2 - q^3 + q^4)(1 - q + q^2 - q^3 + q^4)(1 - q + q^2 - q^3 + q^4)(1 - q + q^2 - q^3 + q^4)(1 - q + q^2 - q^3 + q^4), \cdots \]

It is clear that \(c_3(2,n,1) = 1\).

For \(q = -1\) we get \((c_3(2,n, -1))_{n \geq 0} = (1, 5, 3 \cdot 3^2, 3^3, 3 \cdot 3^3, 3^4, 3 \cdot 3^4, 3^5, 3 \cdot 3^5, \cdots)\).

In order to compute \(c_p(m,n,-1)\) we observe that by Corollary 3.1 we have

\[ \lim_{q \to 1} \frac{\sum_{j=0}^{n} q^{(2m+1)n} \binom{2n}{j} q^{j^2}}{(-q; q)_{2n}^{2m+1}} = k^n, \quad (3.21) \]

and

\[ \lim_{q \to 1} \frac{\sum_{j=0}^{n} q^{(2m+1)n} \binom{2n+1}{j} q^{j^2}}{(-q; q)_{2n+1}^{2m+1}} = k^n(2m+1). \quad (3.22) \]

It remains to compute \(\lim_{q \to 1} \prod_{j=1}^{n} \frac{(-q; q)_n}{1 + q^{2^j p^{(j)}}} \cdot \prod_{j=1}^{n} \frac{1 + q^{2^j p^{(j)}}}{1 + q^{2^j p^{(j)}}} \). To this end observe that for \(j \not\equiv 0 \mod p\) we have

\[ V_p(j) = 1 \quad \text{and therefore} \quad \prod_{j=1}^{n} \frac{(-q; q)_n}{1 + q^{p^{(j)}}} = \prod_{2^j p^{(j)} \leq n} \frac{1 + q^{2^j p^{(j)}}}{1 + q^{2^j p^{(j)}}} \cdot \prod_{(2^j p^{(j)}) \leq n} \frac{1 + q^{(2^j p^{(j)}/p)}}{1 + q^{(2^j p^{(j)/p})}}. \]

For \(q \to 1\) the first product converges to 1.

Since \(\lim_{q \to 1} \frac{1 + q^{(2^j p^{(j))}/p}}{1 + q^{(2^j p^{(j)/p})}} = V_p((2^j + 1)p)\) and \(V_p \left( \left\lfloor \frac{n}{2} \right\rfloor \right) = V_p \left( \left\lfloor \frac{n}{2} \right\rfloor \right) \)

we get \(\lim_{q \to 1} \prod_{j=1}^{n} \frac{(-q; q)_n}{1 + q^{n p^{(j)}/p}} = p^{v_p(a^{(j)}/p)} \left( \left\lfloor \frac{n}{2} \right\rfloor \right) \).

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4. Conclusion

In the main part of this paper we have obtained some evidence that for non-negative integers \( r, m, k \) the polynomials \( \sum_{j=0}^{n} (-1)^j q^{j^2+ mj} \binom{n}{j} q^j \) are divisible by \( (q; q^2)_{n+1} \) in \( \mathbb{Z}[q] \) and that for odd primes \( p \) the polynomials \( \sum_{j=0}^{n} (-1)^j q^{j^2+ mj} \binom{n}{j} q^j \) have the factor \( \prod_{j=1}^{n+1} \left( 1 - q^{2j-1} \right) \) in \( \mathbb{Z}[q] \).

Further we were led to the conjecture that for odd prime numbers \( p \) and non-negative integers \( m \) the polynomials \( \sum_{j=0}^{n} q^{(2m+1)j} \binom{n}{j} q^j \) are not only divisible by \( \prod_{j=1}^{n+1} \left( 1 + q^{2j-1} \right) \) but even by \( \prod_{k=1}^{n} \left( 1 + q^{j^2+ mj} \right) \). Till now we could only provide proofs for some special cases.

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