Hankel determinants of Schröder-like numbers

Johann Cigler

Abstract

After a short survey about Schröder numbers and some generalizations which I call Schröder-like numbers I study some $q$– analogues which have simple Hankel determinants. Some special cases have already been considered in [2] and [3].

1. Schröder numbers

In the first paragraphs I state some results about Schröder numbers and Hankel determinants which are either well known or simple modifications of well-known results.

The (large) Schröder numbers $r_n$ can be defined by the generating function

$$F(z) = \sum_{n \geq 0} r_n z^n = \frac{1 - z - \sqrt{1 - 6z + z^2}}{2z}.$$  

The first terms of this sequence are (cf. [4] A006318)

$$(r_n)_{n \geq 0} = (1,2,6,22,90,\ldots).$$  

The generating function satisfies

$$F(z) = 1 + zF(z) + zF(z)^2.$$  

A closely related sequence (cf. [4], A001003) is the sequence (1,1,3,11,45,197,\ldots) of little Schröder numbers $s_n$, defined by $s_0 = 1$ and $s_n = \frac{r_n}{2}$ for $n > 0$.

Their generating function

$$f(z) = \frac{1 + F(z)}{2}$$

satisfies

$$f(z) = 1 - zf(z) + 2zf(z)^2.$$  

Historical remarks about these numbers can be found in [7].

It is well known that the Hankel determinants of these numbers are (cf. [4] A006318, A001003)

$$\det \begin{pmatrix} \frac{r_n}{2} \end{pmatrix}_{i,j=0}^{n-1} = 2^{\binom{n}{2}},$$  

$$\det \begin{pmatrix} \frac{r_{n+1}}{2} \end{pmatrix}_{i,j=0}^{n-1} = 2^{\binom{n+1}{2}},$$
\[
\det\left(s_{i+j}\right)_{i,j=0}^{n-1} = \det\left(s_{i+j+1}\right)_{i,j=0}^{n-1} = 2^n \quad (1.8)
\]

and
\[
\det\left(s_{i+j+2}\right)_{i,j=0}^{n-1} = 2^n \left(2^{n+1} - 1\right). \quad (1.9)
\]

There are many ways to prove such results. My favorite method uses orthogonal polynomials. In the sequel I consider only sequences of numbers \((a(n))\) with the property that \(a(0) = 1\) and \(\det(a(i+j))_{i,j=0}^{n-1} \neq 0\) for all \(n \geq 1\).

In this situation the polynomials
\[
p(n, x) = \frac{1}{\det(a(i+j))_{i,j=0}^{n-1}} \det \begin{pmatrix} a(0) & a(1) & \cdots & a(n-1) & 1 \\ a(1) & a(2) & \cdots & a(n) & x \\ a(2) & a(3) & \cdots & a(n+1) & x^2 \\ \vdots & \vdots & \ddots & \vdots \\ a(n) & a(n+1) & \cdots & a(2n-1) & x^n \end{pmatrix} \quad (1.10)
\]

are well defined and orthogonal with respect to the linear functional \(F\) defined by \(F(x^n) = a(n)\).

By Favard’s theorem they satisfy a recurrence of the form
\[
p(n, x) = (x - s(n-1))p(n-1, x) - t(n-2)p(n-2, x). \quad (1.11)
\]

Let \(a(n,k)\) be the uniquely determined coefficients in the expansion
\[
x^n = \sum_{k=0}^n a(n,k) p(k, x). \quad (1.12)
\]

They satisfy
\[
a(0,k) = [k = 0] \\
a(n,0) = s(0)a(n-1,0) + t(0)a(n-1,1) \\
a(n,k) = a(n-1,k-1) + s(k)a(n-1,k) + t(k)a(n-1,k + 1). \quad (1.13)
\]

Obviously
\[
a(n,0) = F(x^n) = a(n). \quad (1.14)
\]

We say that the sequences \((s(n))\) and \((t(n))\) are associated with the sequence \((a(n))\).

They contain the same information as the sequence \((a(n))\).

For the large Schröder numbers the associated sequences are (see (2.10))
\[
s(0) = 2, s(n) = 3 \\
t(n) = 2. \quad (1.15)
\]
This gives the Schröder triangle \([4] A133367\)

\[
\begin{array}{ccccccccc}
1 \\
2 & 1 \\
6 & 5 & 1 \\
22 & 23 & 8 & 1 \\
90 & 107 & 49 & 11 & 1
\end{array}
\]

For the little Schroeder numbers the generating function (1.5) implies (see (2.11))

\[
s(0) = 1, \quad s(n) = 3 \quad \text{(1.16)}
\]

and

\[
t(n) = 2. \quad \text{(1.17)}
\]

This gives the triangle

\[
\begin{array}{ccccccccc}
1 \\
1 & 1 \\
3 & 4 & 1 \\
11 & 17 & 7 & 1 \\
45 & 76 & 40 & 10 & 1
\end{array}
\]

The numbers \(a(n,k)\) have a well known combinatorial interpretation.
Consider lattice paths, so-called Motzkin paths, with upward steps \((n,k) \rightarrow (n+1,k+1)\), horizontal steps \((n,k) \rightarrow (n+1,k)\) and downward steps \((n,k+1) \rightarrow (n+1,k)\) which start in \((0,0)\) and never fall under the \(x-\)axis. To each path we associate a weight in the following way: Each upward step has weight 1, each horizontal step at height \(k\) has weight \(s(k)\) and each downward step which ends on height \(k\) has weight \(t(k)\). The weight of a path is the product of the weights of its steps. Then \(a(n,k)\) is the sum of the weights of all paths from \((0,0)\) to \((n,k)\).

Let now \(f_j(z)\) be the generating function of all paths which start and end at height \(j\) and never fall under this height. Then \(f_j(z)\) is the generating function

\[
f_j(z) = \sum_{n=0}^{\infty} a_j(n,0) z^n,
\]

where \(a_j(n,k)\) is given by (1.13) with \(s_j(n) = s(n+j)\) and \(t_j(n) = t(n+j)\).

Then a simple combinatorial argument gives

\[
f(z) = f_0(z) = 1 + s(0)zf(z) + t(0)z^2 f(z)f_j(z). \quad \text{(1.18)}
\]

This identity is equivalent with

\[
f(z) = \frac{1}{1 - s(0)z - t(0)z^2 f_j(z)}
\]

and therefore with the continued fraction

3
\[
\begin{align*}
  f(z) &= \frac{1}{1 - s(0)z - \frac{t(0)z^2}{1 - s(1)z - \frac{t(1)z^2}{1 - s(2)z - \cdots}}}.
\end{align*}
\] (1.19)

Let
\[
d(n,k) = \det (a(i + j + k))_{i,j=0}^{n-1}.
\] (1.20)

be the Hankel determinant of order \(k\) of \((a(n))\).

Then (cf. e.g. [6])
\[
d(n,0) = \prod_{i=1}^{n-1} \prod_{k=0}^{i-1} t(k) = t(0)^{n-1} t(1)^{n-2} \cdots t(n-3)^2 t(n-2)
\] (1.21)

and
\[
d(n,1) = (-1)^n p(n,0) d(n,0).
\] (1.22)

Sometimes the sequence \((d(n,0))\) is called the Hankel transform of the sequence \((a(n))\).

Under the stated requirements it would be better to call \((d(n,0))_{n \geq 1} \times (d(n,1))_{n \geq 1}\) the Hankel transform of \((a(n))\) since the sequence \((a(n))\) is uniquely determined by the sequences of Hankel determinants \((d(n,0))\) and \((d(n,1))\).

It is often convenient to consider a sequence \((a(n))\) which satisfies \(a(2n) = c(n)\) and \(a(2n+1) = 0\). In this case \(s(n) = 0\) for all \(n\). The corresponding lattice paths have no horizontal steps, i.e. they are so called Dyck paths.

Let
\[
(a_0(n)) = (a(2n,0)) = (c(n))
\] (1.23)

Then the associated sequences are
\[
s_0(0) = t(0), s_0(n) = t(2n-1) + t(2n), t_0(n) = t(2n)t(2n+1)
\] (1.24)

For
\[
(a_1(n)) = (a(2n+1,1))
\] (1.25)

we get
\[
s_1(0) = t(0) + t(1), s_1(n) = t(2n) + t(2n+1), t_1(n) = t(2n+1)t(2n+2).
\] (1.26)

As an example consider the sequence \((a(n))\) with \(a(2n) = r_n\) and \(a(2n+1) = 0\). In this case we get \(s(n) = 0, t(2n) = 2, t(2n+1) = 1\) (cf. (2.9)).
The corresponding triangle is

\[
\begin{array}{cccccccc}
1 \\
0 & 1 \\
2 & 0 & 1 \\
0 & 3 & 0 & 1 \\
6 & 0 & 5 & 0 & 1 \\
0 & 11 & 0 & 6 & 0 & 1 \\
22 & 0 & 23 & 0 & 8 & 0 & 1 \\
\end{array}
\]

Since \( a_i(n) = s_{n+1} \) are again little Schröder numbers, (1.26) gives a triangle for the little Schröder numbers \( s_{n+1} \). This is [4] A110440.

\[
1 \\
3 & 1 \\
11 & 6 & 1 \\
45 & 31 & 9 & 1 \\
197 & 156 & 60 & 12 & 1 \\
\]

The generating function for the sequence \((s_{n+1})\) is \( g(z) = \frac{f(z) - 1}{z} \) and satisfies \( g(z) = 1 + 3zg(z) + 2z^2g(z)^2 \).

### 2. Schröder-like numbers

We first want to study Schröder-like numbers \( A(n, x, y) \) defined by the generating function

\[
F(z) = \sum_{n=0}^{\infty} A(n, x, y)z^n \quad \text{which satisfies} \quad F(z) = 1 + xzF(z) + yzF(z)^2. \tag{2.1}
\]

A useful combinatorial interpretation of \( A(n, x, y) \) can be obtained in the following way: Consider lattice paths with upward and downward steps of length 1 and horizontal steps of length 2. If each upward step has weight 1, each downward step has weight \( xz \) and each horizontal step has weight \( yz \), then the weight of the set of all non-negative paths from \((0,0)\) to \((2n,0)\) is \( A(n, x, y)z^n \).

This implies that

\[
A(n, x, y) = \sum_{k=0}^{n} \binom{n+k}{2k} C_k x^{n-k} y^k = \sum_{k=0}^{n} \binom{2n-k}{k} C_k x^k y^{n-k}, \tag{2.2}
\]

where \( C_k = \frac{1}{k+1} \binom{2k}{k} \) is a Catalan number.
We first choose a \( k \)-subset \( 1 \leq a_1 < a_2 < \cdots < a_k < 2n \) of \( \{1, \cdots, 2n\} \) which contains no successive elements. This is equivalent with \( 1 \leq a_1 < a_2 - 1 < \cdots < a_k - k + 1 \leq 2n - k \).

Therefore there are \( \binom{2n-k}{k} \) possibilities to choose such a subset. Each \( a_j \) is the starting point of a horizontal step. On the \( 2n - 2k \) remaining points there are \( C_{n-k} \) non-negative Dyck paths.

By solving a quadratic equation we get the explicit formula

\[
F(z) = 1 - xz - \sqrt{1 - 2(x + 2y)z + x^2 z^2} \over 2yz.
\]

(2.3)

The numbers \( A(n,x,y) \) satisfy the simple recurrence relation

\[
A(n,x,y) = \frac{(2n-1)(x+y)A(n-1,x,y) - (n-2)x^2 A(n-2,x,y)}{n+1}
\]

(2.4)

with initial values

\[
A(0,x,y) = 1 \text{ and } A(1,x,y) = x + y.
\]

In order to show this we differentiate (2.1) and get

\[
F'(z) = xF(z) + xzF'(z) + yF(z)^2 + 2yzF(z)F'(z)
\]

Substituting (2.1) gives

\[
zF'(z)(1 - xz - 2yzF(z)) = yzF(z)^2 + xzF(z) = F(z) - 1
\]

Therefore

\[
zF'(z) = \frac{F(z) - 1}{(1 - xz - 2yzF(z))} = \frac{F(z) - 1}{\sqrt{1 - 2(x + 2y)z + x^2 z^2}} \over 1 - 2(x + 2y)z + x^2 z^2
\]

\[
= \frac{(F(z) - 1)(1 - xz - 2yzF(z))}{1 - 2(x + 2y)z + x^2 z^2} = \frac{(xz + 1 + (x + 2y)zF(z) - F(z))}{1 - 2(x + 2y)z + x^2 z^2}
\]

i.e.

\[
zF'(z)(1 - 2(x + 2y)z + x^2 z^2) = (xz + 1 + (x + 2y)zF(z) - F(z))
\]

Comparing coefficients we get

\[
nA(n,x,y) + 2(x + 2y)(n - 1)A(n-1,x,y) + x^2 (n-2)A(n-2,x,y)
\]

\[
= -A(n,x,y) + (x + 2y)A(n-1,x,y)
\]

and thus (2.4).

The simplest cases occur for \( x = 0 \) where \( A(n,0,y) = C_n y^n \) and for \( x + 2y = 0 \), where we get

\[
A(2n + 2,2,-1) = 0 \text{ and } A(2n + 1,2,-1) = (-1)^n C_n
\]
The equation \( F(z) = 1 + xzF(z) + yzF(z)^2 \) implies

\[
F(z) - yzF(z)^2 = 1 + xzF(z) = 1 - yzF(z) + (x + y)zF(z)
\]
or
\[
F(z) = 1 + (x + y)zF(z) f(z)
\]
with
\[
f(z) = \frac{1}{1 - yzF(z)}.
\]

Therefore (2.1) is equivalent with

\[
F(z) = 1 + (x + y)zF(z) f(z),
\]
\[
f(z) = 1 + yzF(z) f(z).
\]

This also gives

\[
f(z) = \frac{x + yF(z)}{x + y}.
\]

It is easily verified that

\[
f(z) = 1 - xzf(z) + (x + y)zf(z)^2.
\]

If we write \( f(z) = \sum_{n \geq 0} a(n, x, y) z^n \) then the numbers \( a(n, x, y) \) are a generalization of the little Schröder numbers.

Comparing with (1.18) we see that \( F(z^2) \) corresponds to

\[
s(n) = 0, \\
t(2n) = x + y, t(2n + 1) = y.
\]

Denote the corresponding \( a(n, k) \) by \( \alpha(n, k) \).

Since \( \alpha(2n, 0) = A(n, x, y) \) we get for the original sequence \((A(0, x, y), A(1, x, y), A(2, x, y), \ldots)\)

\[
s(0) = x + y, s(n) = x + 2y \\
t(n) = y(x + y).
\]

Observing that \( a(n, x, y) = A(n, -x, x + y) \) we get for the sequence \((a(n, x, y))\)

\[
s(0) = y, s(n) = x + 2y \\
t(n) = y(x + y).
\]

Let
\[ a_i(n) = \alpha (2n + 1, 1) = \frac{\alpha(2n + 2, 0)}{t(0)} = \frac{A(n + 1, x, y)}{x + y} = \frac{a(n + 1, x, y)}{y}. \]  \hspace{1cm} (2.12)

Its generating function is \( g(z) = \frac{f(z) - 1}{yz} \) and satisfies

\[ g(z) = 1 + (x + 2y)zg(z) + y(x + y)z^2g(z)^2. \]  \hspace{1cm} (2.13)

Therefore the corresponding values are

\[ s(n) = x + 2y, t(n) = y(x + y). \]  \hspace{1cm} (2.14)

From (2.1) we get the continued fraction

\[ F(z) = \frac{1}{1 - (x + y)z - \frac{y(x + y)z^2}{1 - (x + 2y)z - \frac{y(x + y)z^2}{1 - (x + 2y)z - \cdots}}}. \]  \hspace{1cm} (2.15)

By (2.6) and (1.18) we see that this can also be written as a so-called J-fraction

\[ F(z) = \frac{1}{1 - (x + y)z - \frac{y(x + y)z^2}{1 - (x + 2y)z - \frac{y(x + y)z^2}{1 - (x + 2y)z - \cdots}}}. \]  \hspace{1cm} (2.16)

It would be interesting if there is also a combinatorial proof, i.e. a bijection between the lattice paths defining these continued fractions.

The Hankel determinants are

\[ D(n, 0) = \det\left( A(i + j, x, y) \right)_{i, j=0}^{n-1} = \left( y(x + y) \right)^{\binom{n}{2}} \]  \hspace{1cm} (2.17)

and

\[ D(n, 1) = \det\left( A(i + j + 1, x, y) \right)_{i, j=0}^{n-1} = y^{\binom{n}{2}} \left( (x + y) \right)^{\binom{n+1}{2}}. \]  \hspace{1cm} (2.18)

The first result is obvious. The second follows from (1.22). Let \( r(n) = (-1)^n p(n, 0) \). Then

\[ r(n) = s(n - 1)r(n - 1) - t(n - 2)r(n - 2) = (x + 2y)r(n - 1) - y(x + y)r(n - 2) \]

with initial values \( r(0) = 1 \) and \( r(1) = (x + y) \).

This gives by induction \( r(n) = (x + y)^n \).
By changing $x \rightarrow -x, y \rightarrow x + y$ we get

$$d(n, 0) = \det \left( a(i + j, x, y) \right)_{i,j=0}^{n-1} = (y(x + y))^\binom{n}{2}$$

(2.19)

and

$$d(n, 1) = \det \left( a(i + j + 1, x, y) \right)_{i,j=0}^{n-1} = y^{\binom{n+1}{2}} (x + y)^{\binom{n}{2}}.$$  

(2.20)

The last identity also follows from (2.14). This can also be used to compute

$$d(n, 2) = \det \left( a(i + j + 2, x, y) \right)_{i,j=0}^{n-1} = y^{\binom{n}{2}} \det \left( a(i + j + 1) \right)_{i,j=0}^{n-1}.$$  

Using (1.22) we see that $d(n, 2) = r(n)d(n, 1)$, where  

$$r(n) = (x + 2y)r(n - 1) - y(x + y)r(n - 2)$$

with initial values $r(0) = 1$ and

$$r(1) = x + 2y = \frac{(x + y)^2 - y^2}{x}.$$  

This gives $r(n) = \frac{(x + y)^{n+1} - y^{n+1}}{x}$.

Therefore

$$d(n, 2) = y^{\binom{n+1}{2}} (x + y)^{\binom{n}{2}} \frac{(x + y)^{n+1} - y^{n+1}}{x}.$$  

(2.21)

**Remark**

If $F(z) = 1 + (x - y)zF(z) + yzF(z)^2$ then $G(z) = \frac{F(z) - 1}{xz}$ satisfies

$$G(z) = 1 + (x + y)zG(z) + xyz^2G(z)^2.$$  

(2.22)

Let $F(z) = \sum_{n \geq 0} A(n)z^n$ and $G(z) = \sum_{n \geq 0} B(n)z^n$.

Then $A(0) = 1$ and $A(n) = xB(n - 1)$ for $n \geq 1$.

There are many interesting sequences whose generating function satisfies (2.22).

E.g. for $(x, y) = (2, 1)$ we get $A(n) = r_n$ and $B(n) = s_{n+1}$,

for $(x, y) = (1, 1)$ we get the Catalan numbers and for $(x, y) = \left( \frac{1 + i \sqrt{3}}{2}, \frac{1 - i \sqrt{3}}{2} \right)$ we get the Motzkin numbers $B(n) = M_n$. 


3. q-analogues of Schröder-like numbers

Barcucci et al. [1] have introduced (large) $q$ – Schröder numbers by the generating function

$$F(z) = 1 + zF(z) + qzF(z)F(qz).$$  \hspace{1cm} (2.23)

The first terms are

$$(1,1+q,(1+q)(1+q+q^2),(1+q)(1+2q+3q^2+3q^3+q^4+q^5),\cdots)$$

We want to consider more generally the $q$ – Schröder-like numbers $A(n,x,y)$ with generating function

$$F(z) = 1 + xzF(z) + yzF(z)F(qz).$$  \hspace{1cm} (2.24)

For $(x,y) = (0,1)$ this reduces to $F(z) = 1 + zF(z)F(qz)$. Therefore $A(n,0,1) = C_n(q)$ are the $q$ – Catalan numbers of Carlitz.

**Theorem 1**

Let $F(z) = \sum_{n\geq 0} A(n,x,y)z^n$ satisfy the identity $F(z) = 1 + xzF(z) + yzF(z)F(qz)$.

Then

$$D(n,0) = \det (A(i+j,x,y))_{i,j=0}^{n-1} = q^{\frac{n(n-1)}{2}} y^{\binom{n}{2}} (x+y)^{n-1}(x+qy)^{n-2}\cdots(x+q^{n-2}y),$$ \hspace{1cm} (2.25)

$$D(n,1) = \det (A(i+j+1,x,y))_{i,j=0}^{n-1} = q^{\frac{n(n-1)}{2}} y^{\binom{n}{2}} (x+y)(x+qy)\cdots(x+q^{n-1}y)D(n,0)$$

$$= q^{\frac{n^2(n-1)}{2}} y^{\binom{n}{2}} (x+y)^n(x+qy)^{n-1}\cdots(x+q^{n-1}y)$$ \hspace{1cm} (2.26)

and

$$D(n,2) = q^{\frac{n(n-1)n(n+1)}{2}} y^{\binom{n}{2}} \prod_{j=0}^{n-1} (x+q^jy)^{n-j} \left( \prod_{j=1}^{n+1} (x+q^{j-1}y) - q^{\binom{n+1}{2}} y^{n+1} \right).$$ \hspace{1cm} (2.27)

An analogue of the little Schröder numbers is given by the generating function

$$f(z) = \sum_{n\geq 0} a(n,x,y)z^n = \frac{x + yF(z)}{x + y}.$$ \hspace{1cm} (2.28)

It is easily verified that it satisfies the equation

$$f(z) = 1 - xzf(qz) + (x + y)zf(z)f(qz)$$ \hspace{1cm} (2.29)

and that

$$F(z) = 1 + (x + y)ZF(z)f(qz).$$ \hspace{1cm} (2.30)
Theorem 2

Let \( f(z) = \sum_{n \geq 0} a(n, x, y) z^n \) satisfy the identity \( f(z) = 1 - xzf(qz) + (x + y)zf(z)f(qz). \)

Then
\[
d(n, 0) = \det \left( a(i + j, x, y) \right)_{i, j=0}^{n-1} = q^{\binom{n}{3}} y^{\binom{n}{2}} \prod_{j=1}^{n-1} (x + q^j y)^{n-j} \tag{2.31}
\]
and
\[
d(n, 1) = \det \left( a(i + j + 1, x, y) \right)_{i, j=0}^{n-1} = q^{\binom{n}{3}} y^{\binom{n+1}{2}} (x + qy)^{n-1} \cdots (x + q^{n-1} y). \tag{2.32}
\]

Whereas for \( q = 1 \) the formulae for the Hankel determinants for \( a(n, x, y) \) could be reduced to those of \( A(n, x, y) \) this is not true for the general case.

An analogue of (2.6) is
\[
F(z) = 1 + (x + y) F(z) f(qz), \quad F(z) = 1 + yF(z) f(qz). \tag{2.33}
\]

But in this form it seems to be of no use to find the continued fraction. Therefore I use an idea of the proof I have given in [2].

There is a uniquely determined series \( h(z) = h(z, y) = 1 + \sum_{n \geq 1} h_n z^n \) such that
\[
F(z) = F(z, y) = \frac{h(qz, y)}{h(z, y)}. \tag{2.34}
\]

From the defining equation for \( F(z) \) we get
\[
\frac{h(qz)}{h(z)} = 1 + xz \frac{h(qz)}{h(z)} + yz \frac{h(qz)}{h(z)} \frac{h(q^2 z)}{h(z)} \quad \text{and therefore}
\]
\[
h(qz) = h(z) + xzh(qz) + yzh(q^2 z). \]

Comparing coefficients we get
\[
(q^n - 1) h_n = q^{n-1} (x + q^{n-1} y) h_{n-1}. \]

This implies
\[
h(z) = \sum_{k \geq 0} q^{\binom{k}{2}} \frac{(x + y)(x + qy) \cdots (x + q^{k-1} y)}{(q - 1)(q^2 - 1) \cdots (q^k - 1)} z^k. \tag{2.35}
\]

On the other hand we have
\[
(x + y)h(z, qy) = xh(z, y) + yh(qz, y),
\]
i.e.
\[
\frac{h(z,qy)}{h(z,y)} = \frac{x+yF(z)}{x+y} = f(z) \tag{2.36}
\]

and

\[
h(qz,y) - h(z,y) = \sum_{k=0}^{qz-1} \frac{(x+y)(x+qy)\cdots(x+q^{k-1}y)}{(q-1)(q^2-1)\cdots(q^k-1)}(q^k-1)z^k
\]

\[
= (x+y)^2 \sum_{k=0}^{qz-1} \frac{(x+qy)\cdots(x+q^{k-1}y)}{(q-1)(q^2-1)\cdots(q^k-1)}(q^{k-1} = (x+y)zh(qz,qy)
\]

i.e.

\[
F(z) = 1 + (x+y)z \frac{h(qz,qy)}{h(z,y)} \tag{2.37}
\]

From

\[
F(z)f(qz) = \frac{h(qz,y)}{h(z,y)} \frac{h(qz,qy)}{h(z,qy)} = \frac{h(qz,qy)}{h(z,y)} \frac{h(qz,qy)}{h(z,qy)} = f(z)F(z,qy)
\]

we see that (2.33) can be written in the form

\[
F(z,y) = 1 + (x+y)zF(z,y)f(qz,y)
\]

\[
f(z,y) = 1 + yzf(z,y)F(z,qy). \tag{2.38}
\]

This gives the continued fraction

\[
F(z) = \frac{1}{1 - \frac{x+y}{(x+y)z}} \frac{1}{1 - \frac{qyz}{(x+qy)z}} \frac{1}{1 - \frac{q^2yz}{(q+q^2y)z}} \frac{1}{1 - \cdots} \tag{2.39}
\]

From this we can deduce the associated sequences \(s(n)\) and \(t(n)\) for \(F(z^2)\).

We get

\[
s(n) = 0, t(2n) = q^n(x+q^n y), t(2n+1) = q^{2n+1} y. \tag{2.40}
\]

This again implies the associated sequences for \(F(z)\) by (1.24). They are

\[
s(0) = x+y, s(n) = q^n(x+q^{n-1}y(1+q))
\]

\[
t(n) = q^{2n+1} y(x+q^n y). \tag{2.41}
\]

From this (2.25) follows immediately from (1.21).
In order to show (2.26) let \( r(n) = (-1)^n p(n,0) \). This gives
\[
r(n) = q^{-1} (x + q^{n-1} (1 + q) y) r(n-1) - q^{n-5} y (x + q^{n-2} y) r(n-2)
\]
with initial values \( r(0) = 1 \) and \( r(1) = x + y \).

It has to be shown that \( r(n) = q^{n\choose 2} (x + y) \cdots (x + q^{n-1} y) \),

i.e.
\[
q^{n\choose 2} (x + y) \cdots (x + q^{-1} y) = q^{n-1} (x + q^{n-2} (1 + q) y) q^{n\choose 2} (x + y) \cdots (x + q^{n-2} y) - q^{3n-5} y (x + q^{n-2} y) q^{n\choose 2} (x + y) \cdots (x + q^{n-3} y).
\]
But this is easily verified.

From (2.38) we get for \( f(z^2) \)
\[
s(n) = 0, t(2n) = q^{2n} y, t(2n+1) = q^n (x + q^{n+1} y).
\] \tag{2.42}
This implies that the associated sequences for \( f(z) \) are
\[
s(0) = y, s(n) = q^{n-1} (x + q^{n-1} (1 + q) y), t(n) = q^{3n} y (x + q^{n+1} y).
\] \tag{2.43}
This implies formulas (2.31) and (2.32).

Observing that \( a(0, x, y) = \frac{\frac{1}{2} \sum_{j=0}^{n-1} x + q^j y}{x + y} \), \( a(n, x, y) = \frac{y A(n, x, y)}{x + y} \) we get by expanding with respect to the first line
\[
d(n,0) = \frac{y^{n-1}}{(x + y)^n} (y D(n,0) + x D(n-1,2))
\] \tag{2.44}
or
\[
(x + y)^n q^{\frac{n\choose 3} 3} \prod_{j=1}^{n-1} (x + q^j y)^{n-j} - y^n q^{\frac{n(n-1)}{2} y^{n\choose 2}} (x + y)^{n-1} (x + qy)^{n-2} \cdots (x + q^{n-2} y)
\]
\[
= q^{\frac{n\choose 3} 3} \prod_{j=0}^{n-2} (x + q^j y)^{n-1-j} y^{n\choose 2} (x + y)(x + qy) \cdots (x + q^{n-1} y) - y^n q^{\frac{n(n-1)}{2}} = xy^{n-1} D(n-1,2)
\]
This implies
\[
D(n,2) = q^{\frac{n-1}(n+1)} y^{n\choose 2} \prod_{j=0}^{n-1} (x + q^j y)^{n-j} \left( \prod_{j=1}^{n+1} (x + q^j y) - q^{n+1\choose 2} y^{n+1} \right)
\] \tag{2.45}
Let now \( a_i(n) = \frac{A(n + 1, x, y)}{x + y} \).

Its generating function is \( g(z, y) = F(z)f(qz) = \frac{h(qz, y)h(qz, qy)}{h(z, y)h(qz, y)} = \frac{h(qz, qy)}{h(z, y)} \) by (2.38).

Substituting \( F(z, y) = 1 + (x + y)zg(z, y) \) into (2.24) we find that

\[
g(z, y) = 1 + (x + y)zg(z, y) + qyzg(qz, y) + qy(x + y)z^2g(z, y)g(qz, y).
\]  

(2.46)

Since

\[
\frac{g(qz, y)}{g(z, y)} = \frac{h(q^2z, qy)}{h(z, y)} \quad \frac{h(z, y)}{h(qz, qy)} = \frac{h(q^2z, qy)}{h(qz, qy)} = \frac{F(qz, qy)}{F(z, y)}
\]

we get

\[
g(z, y) = 1 + (x + y)zg(z, y) + qyzg(qz, y)(1 + (x + y)zg(z, y))
\]

\[
= 1 + (x + y)zg(z, y) + qyzg(qz, y)F(z) = 1 + (x + y)zg(z, y) + qyzg(z, y)F(qz, qy)
\]

\[
= 1 + (x + y)zg(z, y) + qyzg(z, y) + q^2yz^2(x + qy)g(z, y)g(qz, qy),
\]

i.e.

\[
g(z, y) = 1 + (x + (1 + q)y)zg(z, y) + q^2y(x + qy)z^2g(z, y)g(qz, qy).
\]  

(2.47)

This implies that the associated sequences are

\[
s(n) = q^n(x + q^n(1 + q)y) \quad \text{and} \quad t(n) = q^{3n + 2}y(x + q^{n + 1}y).
\]

We can now give another proof of (2.45).

By (1.22) we get \( D(n, 2) = D(n, 1)r(n) \),

where \( r(n) = s(n - 1)r(n - 1) - t(n - 2)r(n - 2) \)

with initial values \( r(0) = 1 \) and \( r(1) = s(0) = x + (1 + q)y = \frac{(x + y)(x + qy) - qy^2}{x} \).

We have to show that \( r(n) = q^{-\frac{n}{2}}\left(\prod_{j=1}^{n}(x + q^{j-1}y) - q^{n}\right) \) or

\[
\prod_{j=1}^{n}(x + q^{j-1}y) - q^{\frac{n+1}{2}}y^{n+1} = (x + q^{n-1}(1 + q)y)\left(\prod_{j=1}^{n}(x + q^{j-1}y) - q^{\frac{n}{2}}y^n\right)
\]

\[
- q^{n-1}y(x + q^{n-1}y)\left(\prod_{j=1}^{n}(x + q^{j-1}y) - q^{\frac{n-1}{2}}y^{n-1}\right).
\]

This is easily verified.
**Remark**

Let \( g(z) = g(z, x, y, q) \) satisfy

\[
g(z) = 1 + (x + y)zg(z) + qxyz^2g(z)g(qz) \tag{2.48}
\]

This gives \( q \)– analogues of several classical sequences.

The associated sequences are \( s(n) = q^n(x + y) \) and \( t(n) = q^{2n+1}xy \).

Define

\[
f(z) = f(z, x, y, q) = 1 + xzg(z). \tag{2.49}
\]

Then \( f(z) = \sum b(n, x, y, q)z^n \) satisfies the identity

\[
f(z) = 1 + xzf(z) - yzf(qz) + yzf(z) f(qz). \tag{2.50}
\]

This implies

\[
f(z, x, y, q) = \frac{1 - yzf(qz, x, y, q)}{1 - yzf(qz, x, y, q) - xz} = \frac{1}{1 - yzf(qz, x, y, q)}
\]

\[
= \frac{1}{1 - \frac{xz}{1 - \frac{yz}{1 - \frac{qxz}{1 - \frac{qyz}{1 - \frac{q^2xz}{1 - \cdots}}}}}}
\]

Therefore

\[
f(z, x, y, q) = \frac{1}{1 - xzf(z, y, q, z)}. \tag{2.51}
\]

We thus get

\[
f(z, x, y, q) = 1 + xzf(z, x, y, q)f(z, y, qx, q) \tag{2.51}
\]

or

\[
f(z, x, y, q) = 1 + xzf(z, x, y, q) + xyz^2f(z, x, y, q)g(z, y, qx, q). \tag{2.52}
\]

Therefore the associated sequences are

\[
s(0) = x, s(n) = q^{n-1}(qx + y), t(n) = q^{2n}y. \tag{2.53}
\]

It is easily verified that

\[
d(n, 0) = q^{\binom{n}{3}}(xy)^{\binom{n}{2}}, \tag{2.54}
\]

\[
d(n, 1) = x^n(xy)^{\binom{n}{2}} \frac{\sum x^k}{q^m}. \tag{2.55}
\]
and
\[ d(n,2) = x^n (qxy)^{\binom{n}{2}} \frac{x^{n+1} - y^{n+1}}{x - y} \sum_{k=0}^{\binom{n+1}{3}} q^{\binom{k}{2}} \frac{x^n (xy)^{\binom{k}{2}}}{x - y}. \] (2.56)

From (2.50) we get the recurrence relation
\[ b(n, x, y, q) = xb(n-1, x, y, q) + y \sum_{k=0}^{n-2} q^k b(k, x, y, q)b(n-1-k, x, y, q) \] (2.57)

with initial value \( b(0, x, y, q) = 1 \).

For \( x = 1 \) this reduces to a variant of the Pólya-Gessel \( q \)-Catalan numbers \( C_n(y; q, q^{-1}) \) (cf. [5],(5.5)). The well-known fact that \( C_n(q^2; q^2, q^{-2}) = C_n(q) \) can easily be seen by comparing the associated sequences (2.41) of \( C_n(q) \) and (2.53) of \( b(n, 1, q, q^2) \) which turn out to be \( s(0) = 1, s(n) = q^{2n-1}(1 + q) \) and \( t(n) = q^{4n+1} \).

References


