# Some results and conjectures about a class of q-polynomials with simple moments

*Johann Cigler*

johann.cigler@univie.ac.at

Fakultät für Mathematik, Universität Wien

Uni Wien Rossau, Oskar Morgenstern-Platz 1, A-1090 Wien

**Abstract**

*We collect some results and conjectures about $q-$analogues of a class of orthogonal polynomials with simple moments.*

In this note I collect some results and conjectures about $q-$analogues of orthogonal polynomials whose moments are $\mu_{2n}=\prod_{j=1}^{n}\frac{jr-1}{m+j}$ and $\mu_{2n+1}=0$ for some integers $r\geq 2$ and $m\geq 0.$ This is a supplement to my paper [7], where the case $r=2$ has been studied, where the corresponding orthogonal polynomials can be interpreted as a sort of interpolation between the $q-$Chebyshev polynomials for $m=0$ and $m=1$ and the discrete $q-$Hermite polynomials for $m=\infty.$ For $r>2$ I could not find concrete results in the literature, but due to the simple form of the moments it is highly probable that some of these results and conjectures are known. If so I would be very glad to receive references to the literature.

My aim has been to find sequences $\left(p_n(x)\right)_{n\geq 0}$ of monic polynomials with nice coefficients such that for the linear functional defined by $\Lambda\left(p_n\right)=[n=0]$ the moments $\Lambda\left(x^{2n}\right)$ are nice $q-$analogues of multiples of $\prod_{j=1}^{n}\frac{jr-1}{m+j}.$ In most cases proofs are straightforward and will be omitted.

## 0. Introduction

Let me first state some well-known results about orthogonal polynomials and Hankel determinants (cf. e.g. [4]).

Suppose that $\left(\mu_n\right)_{n\geq 0}$ is a sequence of real numbers such that $\mu_{2n+1}=0$ for all $n\in\mathbb{N}$ and that all Hankel determinants $\det\left(\mu_{i+j}\right)_{i,j=0}^{n-1}\neq 0.$

Then the polynomials

$$p_n(x) = \frac{1}{\det\left(\mu_{i+j}\right)_{i,j=0}^{n-1}} \det\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_n & x \\ \mu_2 & \mu_3 & \cdots & \mu_{n+1} & x^2 \\ \vdots & & & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} & x^n \end{pmatrix} \tag{0.1}$$

are orthogonal with respect to the linear functional $\Lambda$ on the polynomials defined by

$$\Lambda(x^n) = \mu_n. \tag{0.2}$$

This means that $\Lambda\left(p_n(x)p_m(x)\right) = 0$ if $m \neq n$ and $\Lambda\left(p_n(x)^2\right) \neq 0.$

In particular for $m = 0$ we get

$$\Lambda\left(p_n(x)\right) = [n = 0]. \tag{0.3}$$

These identities also characterize the linear functional $\Lambda.$

The polynomials $p_n(x)$ are of the form

$$p_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} a(n,k)x^k \tag{0.4}$$

with $a(n,k) = 0$ for $n + k \equiv 1 \bmod 2$ and satisfy a recurrence of the form

$$p_n(x) = xp_{n-1}(x) - \lambda_{n-2}p_{n-2}(x) \tag{0.5}$$

for some sequence $\left(\lambda_n\right)$ with initial values $p_0(x) = 1$ and $p_1(x) = x.$

As usual we call $\Lambda\left(x^{2n}\right) = \mu_{2n}$ the moments of $\Lambda.$

The corresponding Hankel determinants are given by

$$\det\left(\mu_{i+j}\right)_{i,j=0}^{n-1} = \prod_{i=1}^{n-1}\prod_{j=0}^{i-1}\lambda_j. \tag{0.6}$$

The simplest example of this situation occurs for $\lambda_n = 1$ for all $n \in \mathbb{N}.$ In this case $\mu_{2n} = \Lambda\left(x^{2n}\right) = \frac{1}{n+1}\binom{2n}{n} = C_n$ is a Catalan number and $p_n(x)$ a Fibonacci polynomial.

## 1. Polynomials with moments $\mu_{2n} = \prod_{j=1}^{n} (jr-1).$

### Theorem 1

*The polynomials*

$$p_n(x,r,s) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor}{k} \prod_{j=0}^{k-1} \left( \left\lfloor \frac{n+1-2j}{2} \right\rfloor r - 1 \right) s^k x^{n-2k} \qquad (1.1)$$

*satisfy the three-term recurrence*

$$p_n(x,r,s) = xp_{n-1}(x,r,s) + s\lambda_{n-2}(r) p_{n-2}(x,r,s) \qquad (1.2)$$

*with*

$$\begin{aligned} \lambda_{2n}(r) &= (n+1)r - 1, \\ \lambda_{2n+1}(r) &= (n+1)r \end{aligned} \qquad (1.3)$$

*and are therefore orthogonal with respect to the linear functional* $\Lambda$ *defined by* $\Lambda(p_n(x,r,s)) = [n=0].$

*Their moments are*

$$\mu_{2n}(r,s) = \Lambda\left(x^{2n}\right) = (-s)^n \prod_{j=1}^{n} (jr-1) \qquad (1.4)$$

*and* $\mu_{2n+1} = \Lambda\left(x^{2n+1}\right) = 0.$

From (1.3) we can obtain the Hankel determinants

$$\det\left(\mu_{i+j}(r,-1)\right)_{i,j=0}^{n-1} = r^{\left\lfloor \frac{(n-1)^2}{4} \right\rfloor} \prod_{i=0}^{n-1} \left\lfloor \frac{i}{2} \right\rfloor! \prod_{j=1}^{\lfloor \frac{n}{2} \rfloor} (jr-1)^{n+1-2j}, \qquad (1.5)$$

which for $r=2$ reduce to $\det\left(\mu_{i+j}(2,-1)\right)_{i,j=0}^{n-1} = \prod_{i=0}^{n-1} i!.$

Let $V$ be the linear operator on the polynomials defined by

$$Vx^n = \left( \left\lfloor \frac{n+1}{2} \right\rfloor r - 1 \right) \left\lfloor \frac{n}{2} \right\rfloor x^{n-2} \qquad (1.6)$$

for $n \geq 2$ and $Vx^n = 0$ for $n < 2$.

Then we can write

$$p_n(x,r,s)=\sum_{k\geq 0}\frac{s^k}{k!}V^k x^n = e^{sV}x^n. \tag{1.7}$$

This implies

$$p_n(x,r,s+t)=e^{sV}p_n(x,r,t)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{\left\lfloor \frac{n}{2}\right\rfloor}{k}\prod_{j=0}^{k-1}\left(\left\lfloor \frac{n+1-2j}{2}\right\rfloor r-1\right)t^k p_{n-2k}(x,r,s). \tag{1.8}$$

For $t=-s$ we get

$$\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{\left\lfloor \frac{n}{2}\right\rfloor}{k}\prod_{j=0}^{k-1}\left(\left\lfloor \frac{n+1-2j}{2}\right\rfloor r-1\right)(-s)^k p_{n-2k}(x,r,s)=x^n \tag{1.9}$$

and therefore (1.4).

If we choose $t=\frac{1-\lambda^2}{\lambda^2}s$ then we get

$$\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{\left\lfloor \frac{n}{2}\right\rfloor}{k}\prod_{j=0}^{k-1}\left(\left\lfloor \frac{n+1-2j}{2}\right\rfloor r-1\right)s^k\left(1-\lambda^2\right)^k\lambda^{n-2k}p_{n-2k}(x,r,s)=\lambda^n p_n\left(x,r,\frac{s}{\lambda^2}\right)=p_n\left(\lambda x,r,s\right).$$

Thus for $\lambda\in\mathbb{R}$ we have

$$p_n\left(\lambda x,r,s\right)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{\left\lfloor \frac{n}{2}\right\rfloor}{k}\prod_{j=0}^{k-1}\left(\left\lfloor \frac{n+1-2j}{2}\right\rfloor r-1\right)s^k\left(1-\lambda^2\right)^k\lambda^{n-2k}p_{n-2k}(x,r,s). \tag{1.10}$$

Let $p_n(x,r,s)=\sum_{k=0}^{n}a(n,k)x^k$, i.e.

$$\begin{aligned}a(2n,2k,r,s)&=\binom{n}{k}s^{n-k}\prod_{j=k+1}^{n}\left(jr-1\right),\\ a(2n+1,2k+1,r,s)&=\binom{n}{k}s^{n-k}\prod_{j=k+2}^{n+1}\left(jr-1\right)\end{aligned} \tag{1.11}$$

and let

$$A_n(r,s)=\left(a(i,j,r,s)\right)_{i,j=0}^{n-1}. \tag{1.12}$$

Then (1.8) implies

$$A_n(r,s+t)=A_n(r,s)A_n(r,t). \tag{1.13}$$

Therefore we get from (1.6)

$$A_n(r,s)=e^{B_n(r,s)} \tag{1.14}$$

with $b(i,k,r,s)=0$ for $i-k\neq 2$ and

$$\begin{aligned} &b(2i,2i-2,r,s)=i(ir-1),\\ &b(2i+1,2i-1,r,s)=i((i+1)r-1). \end{aligned} \tag{1.15}$$

Observe that for $r=2$

$$\binom{\left\lfloor \frac{n}{2}\right\rfloor}{k}\prod_{j=0}^{k-1}\left(\left(\left\lfloor \frac{n+1}{2}\right\rfloor-j\right)2-1\right)=\frac{n!}{k!(n-2k)!2^k}. \tag{1.16}$$

Therefore the polynomials $p_n(x,2,s)$ are the Hermite polynomials

$$p_n(x,2,s)=H_n(x,s)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}s^k\frac{n!}{k!(n-2k)!2^k}x^{n-2k}. \tag{1.17}$$

In this case we have $\lambda_n=n+1.$ The moments of $H_n(x,-1)$ are $(2n-1)!!.$

It is well known that the sequence $\left(H_n(a,s)\right)_{n\geq 0}$ is the moment sequence of the polynomials $P_n(x,a,2,s)=H_n(x-a,-s).$ It seems that an analogous result also holds in the general case.

**Conjecture 1**

*The sequence* $\left(p_n(a,r,s)\right.$ *is the sequence of moments of orthogonal polynomials* $P_n(x,a,r,s)$ *whose first terms are*

1,

$x-a,$

$x^2-\frac{ra}{r-1}x+\frac{a^2-(r-1)^2s}{r-1},$

$x^3-\frac{(r-2)a^3+(r-1)(2r-1)as}{(r-2)a^2+(r-1)^2s}x^2-\frac{(r-2)a^4+(r-3)(2r-1)a^2s}{(r-2)a^2+(r-1)^2s}x$

$+\frac{(r-2)a^5+(2r^2-6r+3)a^3s+(r-1)^2(2r-1)as^2}{(r-2)a^2+(r-1)^2s},\cdots.$

**2. Polynomials with moments** $\mu_{2n}=\prod_{j=1}^{n}\frac{jr-1}{m+j}.$

**Theorem 2**

*The orthogonal polynomials with moments* $\mu_{2n}=(-s)^n\prod_{j=1}^{n}\frac{jr-1}{m+j}$ *are*

$$f_n(x,m,r,s)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{\lfloor \frac{n}{2}\rfloor}{k}\prod_{j=0}^{k-1}\left(\left\lfloor \frac{n+1-2j}{2}\right\rfloor r-1\right)\frac{s^k}{\prod_{j=1}^{k}(m+n-j)}x^{n-2k}. \tag{2.1}$$

*They satisfy the three-term recurrence*

$$f_n(x,m,r,s)=xf_{n-1}(x,m,r,s)+s\lambda_{n-2}(m,r)f_{n-2}(x,m,r,s) \tag{2.2}$$

*with*

$$\begin{aligned}\lambda_{2n}(m,r)&=\frac{(m+n)\big((n+1)r-1\big)}{(m+2n)(m+2n+1)},\\ \lambda_{2n+1}(m,r)&=\frac{\big((m+n)r+1\big)(n+1)}{(m+2n+1)(m+2n+2)},\end{aligned} \tag{2.3}$$

*and with* $\lambda_0(m,r)=\frac{r-1}{m+1}$ *for all* $m\in\mathbb{N}.$

The first terms of this sequence are

$1,$

$x,$

$x^2+\frac{(r-1)s}{m+1},$

$x^3+\frac{(2r-1)s}{m+2}x,$

$x^4+\frac{2(2r-1)s}{m+3}x^2+\frac{(r-1)(2r-1)s^2}{(m+2)(m+3)},$

$x^5+\frac{2(3r-1)s}{m+4}x^3+\frac{(2r-1)(3r-1)s^2}{(m+3)(m+4)}x,$

$x^6+\frac{3(3r-1)s}{m+5}x^4+\frac{3(2r-1)(3r-1)s^2}{(m+4)(m+5)}x^2+\frac{(r-1)(2r-1)(3r-1)s^3}{(m+3)(m+4)(m+5)},$

We have

$$f_n(x,m,r,s+t)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{\lfloor \frac{n}{2}\rfloor}{k}\prod_{j=0}^{k-1}\left(\left\lfloor \frac{n+1-2j}{2}\right\rfloor r-1\right)a(n,n-2k,s,t)\,f_{n-2k}(x,m,r,s) \qquad (2.4)$$

with

$$a(n,n-2k,s,t)=\frac{t\sum_{j=0}^{k-1}\binom{k-1}{j}\binom{k}{j}j!\prod_{\ell=j}^{k-2}\left(m+n-2k+1+\ell\right)(-s)^j t^{k-1-j}}{\prod_{j=1}^{2k-1}\left(m+n-j\right)} \qquad (2.5)$$

for $k>0$ and

$$a(n,n,s,t)=1. \qquad (2.6)$$

For $t=-s$ we get from $\sum_{j=0}^{k-1}\binom{k-1}{j}\binom{k}{j}j!\prod_{\ell=j}^{k-2}\left(m+n-2k+1+\ell\right)=\prod_{j=1}^{k-1}(m+n-j)$

$$x^n=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{\lfloor \frac{n}{2}\rfloor}{k}\prod_{j=0}^{k-1}\left(\left\lfloor \frac{n+1-2j}{2}\right\rfloor r-1\right)\frac{(-s)^k}{\prod_{j=0}^{k-1}(m+n-k-j)}\,f_{n-2k}(x,m,r,s) \qquad (2.7)$$

Therefore the moments are $\mu_{2n}=(-s)^n\prod_{j=1}^{n}\frac{jr-1}{m+j}.$

For $r=2$ some aspects of these polynomials have been studied in [7]. We get from (1.16)

$$f_n(x,m,2,s)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{n!}{k!(n-2k)!2^k}\frac{s^k}{\prod_{j=1}^{k}\left(m+n-j\right)}x^{n-2k}. \qquad (2.8)$$

Note that $\lambda_n(m,2)=\lambda_n(m)=\frac{(n+1)(n+2m)}{(n+m)(n+m+1)}$ with $\lambda_0(m)=\frac{2}{m+1}$ for all $m\in\mathbb{N}.$

As a special case we get that $t_n(x)=f_n\left(x,0,2,-\frac{1}{2}\right)$ satisfies $t_n(x)=xt_{n-1}(x)-\lambda_{n-2}t_{n-2}(x)$ with $\lambda_0=2$ and $\lambda_n=1$ for $n>0.$

This implies that $T_n(x) = 2^{n-1} t_n(x)$ for $n > 0$ and $T_0(x) = 1,$ satisfies $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ with $T_0(x) = 1$ and $T_1(x) = x$. Thus the polynomials $T_n(x)$ are the Chebyshev polynomials of the first kind $T_n(x) = \frac{1}{2}\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n-k}{k} \frac{n}{n-k} (2x)^{n-2k}.$

In the same way we see that the polynomials

$$U_n(x) = 2^n f_n\left(x, 1, 2, -\frac{1}{2}\right) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k}$$

satisfy $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$ with $U_0(x) = 1$ and $U_1(x) = 2x$ and thus coincide with the Chebyshev polynomials of the second kind.

The well-known representations

$$T_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} x^{n-2k} \left(x^2 - 1\right)^k \tag{2.9}$$

and

$$U_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n+1}{2k+1} x^{n-2k} \left(x^2 - 1\right)^k \tag{2.10}$$

can be generalized to give

$$f_n(x, m, r, s) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{\left\lfloor \frac{n}{2} \right\rfloor}{k} \frac{\prod_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor - k - 1} \left((m+k+j)r + 1\right) \prod_{j=0}^{k-1} \left(\left(\left\lfloor \frac{n+1}{2} \right\rfloor - k + 1 + j\right) r - 1\right)}{\prod_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \left(\left(m + \left\lfloor \frac{n+1}{2} \right\rfloor + j\right) r\right)} x^{n-2k} \left(x^2 + rs\right)^k .$$

(2.11)

Polynomials with different $m$ are related by

$$f_n(x, m, r, s) = xf_{n-1}(x, m+1, r, s) + s\frac{\lambda_{n-2}(r)}{n+m-1} f_{n-2}(x, m+1, r, s) \tag{2.12}$$

where $\lambda_n(r)$ satisfies (1.3).

For $m = 0$ and $r = 2$ this reduces to $T_n(x) = xU_{n-1}(x) - U_{n-2}(x).$

**Conjecture 2**

*Let* $a(n,k,r,m)=[x^k]f_n(x,m,r,s)$ *and* $A_n(r,m)=\left(a(i,j,r,m)\right)_{i,j=0}^{n-1}$.

*Let* $B_n(r,m)=\left(b(i,j,r,m)\right)_{i,j=0}^{n-1}=\log A_n(r,m).$

*Then* $b(n+2k,n,r,m)=\gamma(k)\dfrac{a(n+2k,n,r,m)}{\prod_{j=1}^{k-1}(m+n+j)}$

*where* $\left(\gamma(k)\right)_{k\geq 0}=\left(0,1,-1,3,-14,80,-468,\cdots\right)$ *does not depend on* $n$ *and* $m$.

**Remark**

The sequence $\left(\gamma(k)\right)_{k\geq 0}$ also appears in OEIS [9], A027614 apparently in a similar setting.

In OEIS [9], A179320 it is stated without proof that the exponential generating function $R(x)=\sum_{n=1}^{\infty}\frac{\gamma(n)}{n!}x^n$ satisfies $R(x)=\frac{1-x}{1+x}R\left(\frac{x}{(1-x)^2}\right).$

## 3. Polynomials with moments $\mu_{2n}=\prod_{j=1}^{n}[jr-1]_q$.

**Theorem 3**

*The orthogonal polynomials with moments* $\mu_{2n}=\prod_{j=1}^{n}[jr-1]_q$ *are given by*

$$p_n(x,r,s,q)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}s^k q^{r\binom{k}{2}}\begin{bmatrix}\left\lfloor \frac{n}{2}\right\rfloor\\ k\end{bmatrix}_{q^r}\prod_{j=0}^{k-1}\left[\left(\left\lfloor \frac{n+1}{2}\right\rfloor-j\right)r-1\right]_q x^{n-2k}. \qquad (3.1)$$

*They satisfy*

$$p_n(x,r,s,q)=xp_{n-1}(x,r,s,q)+s\lambda_{n-2}(r,q)p_{n-2}(x,r,s,q) \qquad (3.2)$$

*with*

$$\begin{aligned}\lambda_{2n}(r,q)&=q^{nr}\left[(n+1)r-1\right]_q,\\ \lambda_{2n+1}(r,q)&=q^{(n+1)r-1}\left[(n+1)r\right]_q.\end{aligned} \qquad (3.3)$$

Some other recurrence relations are

$$p_{2n}(x,r,s,q) = xp_{2n-1}(x,r,q^r s,q) + s\left[nr-1\right]_q \; p_{2n-2}(x,r,q^r s,q),$$
$$p_{2n+1}(x,r,s,q) = xp_{2n}(x,r,q^r s,q) + s\left[nr\right]_q \; p_{2n-2}(x,r,q^r s,q), \tag{3.4}$$

and

$$p_{2n}(x,r,s,q) = q^{rn-1}xp_{2n-1}(x,r,s,q) + [rn-1]_q \left((1-q)x^2+s\right) p_{2n-2}(x,r,q^r s,q),$$
$$p_{2n+1}(x,r,s,q) = q^{rn}xp_{2n}(x,r,s,q) + [rn]_q \left((1-q)x^2+s\right) p_{2n-1}(x,r,q^r s,q). \tag{3.5}$$

Another formula for these polynomials is

$$p_n(x,r,s,q) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{\left\lfloor \frac{n-2k}{2} \right\rfloor \left(\left\lfloor \frac{n+1-2k}{2} \right\rfloor r-1\right)} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor \\ k \end{bmatrix}_{q^r} \prod_{j=0}^{k-1} \left[\left\lfloor \frac{n+1-2j}{2} \right\rfloor r-1\right]_q x^{n-2k} \prod_{j=0}^{k-1}\left(x^2(1-q)+q^{jr}s\right). \tag{3.6}$$

For $r=2$ we get the discrete $q-$Hermite polynomials

$$p_n(x,2,-1,q) = H_n(x,q) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k q^{2\binom{k}{2}} \frac{[n]_q!}{[k]_q![n-2k]_q!(-q;q)_k} x^{n-2k}. \tag{3.7}$$

In this case we get

$$\lambda_n(2,q) = q^n[n+1]. \tag{3.8}$$

Formula (3.6) reduces to

$$H_n(x,q) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{\binom{n-2k}{2}} \frac{[n]_q!}{[k]_q![n-2k]_q!(-q;q)_k} x^{n-2k} \prod_{j=0}^{k-1}\left((1-q)x^2 - q^{2j}\right). \tag{3.9}$$

Let $e(x,q) = \sum_n \frac{x^n}{[n]_q!}$ and $E(x,q) = \frac{1}{e(-x,q)} = \sum_{n\ge 0} q^{\binom{n}{2}} \frac{x^n}{[n]_q!}$ be the $q-$exponential series (cf. e.g. [1] or [2]) and define the linear operator $V_q$ on the polynomials by

$V_q x^n = \left[\left\lfloor \frac{n+1}{2} \right\rfloor r-1\right]_q \left\lfloor \frac{n}{2} \right\rfloor_{q^r} x^{n-2}$ and $V_q x = V_q 1 = 0.$

Then $p_n(x,r,s,q) = E\left(sV_q, q^r\right)x^n.$

From $\frac{e(az,q)}{e(bz,q)}=\sum_{k\geq 0}\frac{\prod_{j=0}^{k-1}\left(a-q^{j}b\right)}{[k]!}z^{k}$ (cf. [2]) we see that

$$p_n(x,r,s+t,q)=\frac{\sum_{k=0}^{\infty}\frac{(s+t)^k}{[k]_{q^r}!}q^{r\binom{k}{2}}V_q{}^k}{\sum_{k=0}^{\infty}\frac{s^k}{[k]_{q^r}!}q^{r\binom{k}{2}}V_q{}^k}\sum_{k=0}^{\infty}\frac{s^k}{[k]_{q^r}!}q^{r\binom{k}{2}}V_q{}^k x^n=\frac{E\left((s+t)V_q,q^r\right)}{E\left(sV_q,q^r\right)}p_n(x,r,s,q)$$

$$=\sum_{k\geq 0}\frac{\prod_{j=0}^{k-1}\left(-s+q^{jr}(s+t)\right)}{[k]_{q^r}!}V_q{}^k p_n(x,r,s,q).$$

Therefore we get

$$p_n(x,r,s+t,q)==\sum_{k\geq 0}\prod_{j=0}^{k-1}\left(-s+q^{jr}(s+t)\right)\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor\\ k\end{bmatrix}_{q^r}\prod_{j=0}^{k-1}\left[\left(\left\lfloor\frac{n+1}{2}\right\rfloor-j\right)r-1\right]_q p_{n-2k}(x,r,s,q).$$

(3.10)

For $t=-s$ this gives

$$\sum_{k\geq 0}(-s)^k\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor\\ k\end{bmatrix}_{q^r}\prod_{j=0}^{k-1}\left[\left(\left\lfloor\frac{n+1}{2}\right\rfloor-j\right)r-1\right]_q p_{n-2k}(x,r,s,q)=x^n \qquad (3.11)$$

and therefore $\mu_{2n}=(-s)^n\prod_{j=1}^{n}[jr-1]_q.$

If we choose $t=\frac{1-\lambda^2}{\lambda^2}s$ we get

$$p_n(\lambda x,r,s,q)=\sum_{k\geq 0}s^k\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor\\ k\end{bmatrix}_{q^r}\prod_{j=0}^{k-1}\left[\left(\left\lfloor\frac{n+1}{2}\right\rfloor-j\right)r-1\right]_q\lambda^{n-2k}\prod_{j=0}^{k-1}\left(q^{jr}-\lambda^2\right)p_{n-2k}(x,r,s,q).$$

(3.12)

For $r=2$ this gives

$$H_n(\lambda x,q)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^k\frac{[n]!}{[k]![n-2k]!\left(-q;q\right)_k}\lambda^{n-2k}\prod_{j=0}^{k-1}\left(q^{2j}-\lambda^2\right)H_{n-2k}(x,q). \qquad (3.13)$$

If $s$ and $t$ are not numbers but operators satisfying $ts = q^r st$ then $E\left((s+t)V, q^r\right) = E\left(tV, q^r\right)E\left(sV, q^r\right)$ (cf. e.g. [1] or [2]) and therefore

$$p_n(x,r,s+t,q) = E\left((s+t)V, q^r\right)x^n = E\left(tV, q^r\right)E\left(sV, q^r\right)x^n = E\left(tV, q^r\right)p_n(x,r,s,q).$$

Let $\varepsilon$ be the linear operator on the polynomials in $s$ defined by $\varepsilon p(s) = p\left(q^r s\right)$ and $\underline{s}$ the linear operator "multiplication by $s$" then $\varepsilon \underline{s} = q^r \underline{s}\varepsilon.$ Therefore $p_n(x,r,\underline{s}(1-\varepsilon),q)1 = x^n$ and we get another representation of $x^n$

$$x^n = E\left((-\underline{s}\varepsilon)V, q^r\right)p_n(x,r,\underline{s},q)1 = \sum_{k\geq 0}(-s)^k q^{2r\binom{k}{2}} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor \\ k \end{bmatrix}_{q^r} \prod_{j=0}^{k-1}\left[\left(\left\lfloor \frac{n+1}{2} \right\rfloor - j\right)r - 1\right]_q p_{n-2k}(x,r,q^{rk}s,q).$$

**Conjecture 3**

*Let* $A = \left(a(i,j,r)\right)$ *be defined by* $a(n,j,r) = [x^j]p_n(x,r,s,q).$

*Then the logarithm* $\log A$ *can be obtained from* $A$ *by multiplying the diagonals* $\left(i, i-2k\right)$

*with* $\gamma(k) = \dfrac{\prod_{j=1}^{k-1}\left(1-q^{-jr}\right)}{k}$ *and the principal diagonal with* $\gamma(0) = 0.$

For example $\log A_8$ looks like

$$\log A_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a(2,0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a(3,1) & 0 & 0 & 0 & 0 & 0 & 0 \\ a(4,0)\frac{1-q^{-r}}{2} & 0 & a(4,2) & 0 & 0 & 0 & 0 & 0 \\ 0 & a(5,1)\frac{1-q^{-r}}{2} & 0 & a(5,3) & 0 & 0 & 0 & 0 \\ a(6,0)\frac{\left(1-q^{-r}\right)\left(1-q^{-2r}\right)}{3} & 0 & a(6,2)\frac{1-q^{-r}}{2} & 0 & a(6,4) & 0 & 0 & 0 \\ 0 & a(7,1)\frac{\left(1-q^{-r}\right)\left(1-q^{-2r}\right)}{3} & 0 & a(7,3)\frac{1-q^{-r}}{2} & 0 & a(7,5) & 0 & 0 \end{pmatrix}$$

## 4. Orthogonal polynomials with moments $\prod_{j=1}^{n}\frac{[rj-1]_q}{[m+j]_{q^r}}$.

**Theorem 4**

*Let* $r\ge 2$ *and* $m\ge 0$ *be integers. Then the polynomials*

$$f_n(x,m,r,s,q)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} q^{r\binom{k}{2}}\begin{bmatrix}\lfloor \frac{n}{2}\rfloor\\ k\end{bmatrix}_{q^r}\prod_{j=0}^{k-1}\left[\left\lfloor\frac{n+1-2j}{2}\right\rfloor r-1\right]_q\frac{s^k}{\prod_{j=1}^{k}[m+n-j]_{q^r}}x^{n-2k} \qquad (4.1)$$

*are orthogonal with respect to the linear functional* $\Lambda$ *defined by* $\Lambda\left(f_n(x,m,r,s,q)\right)=[n=0]$

*and satisfy*

$$f_n(x,m,r,s,q)=xf_{n-1}(x,m,r,s,q)+s\lambda_{n-2}(m,r,q)f_{n-2}(x,m,r,s,q) \qquad (4.2)$$

*with*

$$\lambda_{2n}(m,r,q)=q^{nr}\frac{[m+n]_{q^r}\,[(n+1)r-1]_q}{[m+2n]_{q^r}\,[m+2n+1]_{q^r}},$$
$$\lambda_{2n+1}(m,r,q)=q^{(n+1)r-1}\frac{[n+1]_{q^r}\,[(n+m)r+1]_q}{[m+2n+1]_{q^r}\,[m+2n+2]_{q^r}}. \qquad (4.3)$$

*For* $m=0$ *this reduces to* $\lambda_0(0,r,q)=[r-1]_q$.

If we set

$$c(n,n-2k,r,q)=\begin{bmatrix}\lfloor \frac{n}{2}\rfloor\\ k\end{bmatrix}_{q^r}\prod_{j=0}^{k-1}\left[\left\lfloor\frac{n+1-2j}{2}\right\rfloor r-1\right]_q \qquad (4.4)$$

then

$$x^n=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}c(n,n-2k,r,q)\frac{(-s)^k}{\prod_{j=0}^{k-1}[m+n-k-j]_{q^r}}f_{n-2k}(x,r,m,s,q). \qquad (4.5)$$

This implies

$\Lambda\left(x^{2n+1}\right)=0$ and $\Lambda\left(x^{2n}\right)=(-s)^n\prod_{j=1}^{n}\frac{[rj-1]_q}{[m+j]_{q^r}}$.

Another representation is

$$f_{2n}(x,m,r,s,q)=\sum_{k=0}^{n} q^{(n-k)^2 r-(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^r} \frac{\left(q^{(m+k)r+1};q^r\right)_{n-k}\left(q^{(n-k+1)r-1};q^r\right)_k}{\left(q^{(m+n)r};q^r\right)_n} x^{2n-2k}\prod_{j=0}^{k-1}\left(x^2+q^{rj}[r]s\right),$$

$$f_{2n+1}(x,m,r,s,q)=\sum_{k=0}^{n} q^{2\binom{n-k+1}{2}r-(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^r} \frac{\left(q^{(m+k)r+1};q^r\right)_{n-k}\left(q^{(n-k+2)r-1};q^r\right)_k}{\left(q^{(m+n+1)r};q^r\right)_n} x^{2n+1-2k}\prod_{j=0}^{k-1}\left(x^2+q^{rj}[r]s\right).$$

(4.6)

There is an interesting relation between the polynomials with adjacent $m$.

$$f_{2n}(x,m,r,s,q)=xf_{2n-1}(x,m+1,r,s,q)+s\frac{q^{(n-1)r}\left[nr-1\right]_q}{[2n+m-1]_{q^r}} f_{2n-2}(x,m+1,r,s,q),$$
$$f_{2n+1}(x,m,r,s,q)=xf_{2n}(x,m+1,r,s,q)+s\frac{q^{nr-1}\left[rn\right]_q}{[2n+m]_{q^r}} f_{2n-1}(x,m+1,r,s,q), \quad (4.7)$$

For $m\to\infty$ (4.7) reduces to (3.2).

Another one is

$$f_{2n}(x,m,r,s,q)=xf_{2n-1}(x,m+1,r,q^r s,q)+s\frac{\left[nr-1\right]_q}{[2n+m-1]_{q^r}} f_{2n-2}(x,m+1,r,q^r s,q),$$
$$f_{2n+1}(x,m,r,s,q)=xf_{2n}(x,m+1,r,q^r s,q)+s\frac{\left[rn\right]_q}{[2n+m]_{q^r}} f_{2n-1}(x,m+1,r,q^r s,q), \quad (4.8)$$

For $m\to\infty$ it reduces to (3.4)

For $r=2$ the expressions can be simplified (cf. [5],[6],[7]).

We get

$$c(n,k,2,q)=\begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor \\ k \end{bmatrix}_{q^2} \prod_{j=0}^{k-1}\left[\left\lfloor \frac{n+1-2j}{2} \right\rfloor 2-1\right]_q = \frac{[n]_q!}{[k]_q![n-2k]_q!\left(-q;q\right)_k}. \quad (4.9)$$

Further we have

$$\lambda_{2n}(m,2,q)=q^{2n}\frac{\left[m+n\right]_{q^2}\left[2n+1\right]_q}{\left[m+2n\right]_{q^2}\left[m+2n+1\right]_{q^2}}=\frac{q^{2n}}{1+q}\frac{\left[2m+2n\right]_q\left[2n+1\right]_q}{\left[m+2n\right]_{q^2}\left[m+2n+1\right]_{q^2}}$$

$$\lambda_{2n+1}(m,2,q)=q^{2n+1}\frac{\left[n+1\right]_{q^2}\left[2(n+m)+1\right]_q}{\left[m+2n+1\right]_{q^2}\left[m+2n+2\right]_{q^2}}=\frac{q^{2n+1}}{1+q}\frac{\left[2m+2n+1\right]_q\left[2n+2\right]_q}{\left[m+2n\right]_{q^2}\left[m+2n+1\right]_{q^2}}$$

and thus

$$\lambda_n(m,2,q)=\frac{q^n}{1+q}\frac{\left[2m+n\right]_q\left[n+1\right]_q}{\left[m+n\right]_{q^2}\left[m+n+1\right]_{q^2}}=\frac{q^n}{(1+q)\left(1+q^{n+m}\right)\left(1+q^{n+m+1}\right)}\frac{\left[2m+n\right]_q\left[n+1\right]_q}{\left[m+n\right]_q\left[m+n+1\right]_q}.$$

Therefore we get

**Corollary 4.1**

*The polynomials*

$$f_n(x,m,2,s,q)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}q^{2\binom{k}{2}}\frac{[n]_q!}{[k]_q![n-2k]_q!(-q;q)_k}\frac{s^k}{\prod_{j=1}^{k}\left[m+n-j\right]_{q^2}}x^{n-2k} \tag{4.10}$$

*satisfy*

$$f_n(x,m,2,s,q)=xf_{n-1}(x,m,2,s,q)+s\lambda_{n-2}(m,2,q)f_{n-2}(x,m,2,s,q) \tag{4.11}$$

*with*

$$\lambda_n(m,2,q)=\frac{q^n}{1+q}\frac{\left[2m+n\right]_q\left[n+1\right]_q}{\left[m+n\right]_{q^2}\left[m+n+1\right]_{q^2}} \tag{4.12}$$

*and*

$$\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{[n]_q!}{[k]_q![n-2k]_q!(-q;q)_k}\frac{(-s)^k}{\prod_{j=1}^{k}\left[m+n-j\right]_{q^2}}f_{n-2k}(x,m,2,s,q)=x^n. \tag{4.13}$$

*Thus their moments are*

$$\Lambda\left(x^{2n}\right)=(-s)^n\prod_{j=1}^{n}\frac{[2j-1]_q}{[m+j]_{q^2}}. \tag{4.14}$$

**Remark**

These polynomials are related to the Big $q-$Jacobi polynomials ([8], 14.5)

$$P_n\left(x;q^{m-\frac{1}{2}},q^{m-\frac{1}{2}},-q^{m-\frac{1}{2}};q\right)={}_3\phi_2\left(\begin{matrix}q^{-n},q^{2m+n},x\\ q^{m+\frac{1}{2}},-q^{m+\frac{1}{2}}\end{matrix};q,q\right).$$

By [8], (14.5.4) the monic Big $q-$Jacobi polynomials $p_n(x,m)$ with these parameters satisfy the recurrence

$$p_{n+2}(x,m)=xp_{n+1}(x,m)+A_nC_{n+1}p_n(x,m)$$

with $A_n=\frac{[2m+n]_q}{[2m+2n]_q}$ and $C_n=q^{2m+n}\frac{[n]_q}{[2m+2n]_q}.$

This gives

$$A_nC_{n+1} == \frac{[2m+n]_q}{[2m+2n]_q} q^{2m+n+1} \frac{[n+1]_q}{[2m+2n+2]_q} = \frac{q^{2m+n+1}}{(1+q)^2} \frac{[2m+n]_q [n+1]_q}{[m+n]_{q^2} [m+n+1]_{q^2}} = \frac{q^{m+1}}{1+q} \lambda_n(m,2,q).$$

For $m=0$ we have

$$\lambda_n(0,2,q) = \frac{q^n}{1+q} \frac{[n]_q [n+1]_q}{[n]_{q^2} [n+1]_{q^2}} = \frac{q^n(1+q)}{(1+q^n)(1+q^{n+1})}$$ for $n>0$ and

$$\lambda_0(0,2,q) = \frac{1}{1+q} \frac{[2m]_q [1]_q}{[m]_{q^2} [m+1]_{q^2}}\bigg|_{m=0} = 1.$$

Comparing with [5] and [6] we see that the polynomials $f_n\left(x,0,2,\frac{qs}{1+q},q\right)$ are the monic $q-$Chebyshev polynomials of the first kind. For $n>0$ we have

$$f_n\left(x,0,2,\frac{qs}{1+q},q\right)(-q;q)_{n-1} = T_n(x,s,q).$$

In the same way we see that $\lambda_n(1,2,q) = \frac{q^n}{1+q} \frac{[2+n]_q [n+1]_q}{[1+n]_{q^2} [n+2]_{q^2}} = \frac{q^n(1+q)}{(1+q^{n+1})(1+q^{n+2})}.$

This implies that the polynomials $f_n\left(x,1,2,\frac{qs}{1+q},q\right)$ are the monic $q-$Chebyshev polynomials of the second kind.

In this case

$$f_n\left(x,1,2,\frac{qs}{1+q},q\right)(-q;q)_n = U_n(x,s,q).$$

Note that

$$f_n(x,0,2,s,q)\, f_n(x,0,2,qs,q) - (x^2+[2]s)\, f_{n-1}(x,1,2,qs,q)\, f_{n-1}(x,1,2,q^2s,q) = \frac{q^{\binom{n}{2}}(-[2]s)^n}{(-q;q)_{n-1}^2}$$

is a $q-$analogue of $T_n^2(x)-(x^2-1)U_{n-1}^2(x)=1.$

For $r=2$ identity (4.7) reduces to

$$f_n(x,m,2,s,q) = xf_{n-1}(x,m+1,2,s,q) + \frac{q^{n-2}[n-1]_q s}{[n+m-1]_{q^2}} f_{n-2}(x,m+1,2,s,q). \qquad (4.15)$$

Formula (4.6) reduces to

$$f_n(x,m,2,s,q)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor} q^{\binom{n-2k}{2}}\begin{bmatrix} n \\ 2k\end{bmatrix}[2k-1]!!\frac{\prod_{j=0}^{\left\lfloor \frac{n-2k-2}{2}\right\rfloor}\left[2m+2k+2j+1\right]}{\prod_{j=0}^{\left\lfloor \frac{n-2}{2}\right\rfloor}\left[2m+2\left\lfloor \frac{n+1}{2}\right\rfloor+2j\right]}x^{n-2k}\prod_{j=0}^{k-1}\left(x^2+q^{2j}(1+q)s\right)$$

$$=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor} q^{\binom{n-2k}{2}}\frac{[n]!}{[k]![n-2k]!\left(-q;q\right)_k}\frac{\left(q^{2m+2k+1};q^2\right)_{\left\lfloor \frac{n}{2}\right\rfloor-k}}{\left(q^{2m+2\left\lfloor \frac{n+1}{2}\right\rfloor};q^2\right)_{\left\lfloor \frac{n}{2}\right\rfloor}}(1-q)^k x^{n-2k}\prod_{j=0}^{k-1}\left(x^2+q^{2j}(1+q)s\right)$$

(4.16)

For $m\to\infty$ we get (3.9).

Special cases of (4.16) are (we set $(-q;q)_{-1}=1$ )

$$f_n(x,0,2,s,q)\left(-q;q\right)_{n-1}=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor} q^{\binom{n-2k}{2}}\begin{bmatrix} n \\ 2k\end{bmatrix}x^{n-2k}\prod_{j=0}^{k-1}\left(x^2+q^{2j}(1+q)s\right)$$

since

$$[2k-1]!!\frac{\prod_{j=0}^{\left\lfloor \frac{n-2k-2}{2}\right\rfloor}\left[2k+2j+1\right]}{\prod_{j=0}^{\left\lfloor \frac{n-2}{2}\right\rfloor}\left[2\left\lfloor \frac{n+1}{2}\right\rfloor+2j\right]}=\frac{1}{\left(-q;q\right)_{n-1}}$$

From $[2k-1]!!\frac{\prod_{j=0}^{\left\lfloor \frac{n-2k-2}{2}\right\rfloor}\left[2k+2j+3\right]}{\prod_{j=0}^{\left\lfloor \frac{n-2}{2}\right\rfloor}\left[2+2\left\lfloor \frac{n+1}{2}\right\rfloor+2j\right]}=\frac{[n+1]}{[2k+1}\frac{1}{\left(-q;q\right)_n}$

we get

$$f_n(x,1,2,s,q)\left(-q;q\right)_n=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor} q^{\binom{n-2k}{2}}\begin{bmatrix} n+1 \\ 2k+1\end{bmatrix}x^{n-2k}\prod_{j=0}^{k-1}\left(x^2+q^{2j}(1+q)s\right).$$

Another interesting formula is

$$\left[2m+2n\right]f_{n+1}(x,m,2,s,q)=q^n x[2m+n]f_n(x,m,2,s,q)+[n]\left(x^2+(1+q)s\right)f_{n-1}(x,m+1,2,q^2s,q).$$

For $m=0$ this reduces to

$$\left(1+q^n\right) f_{n+1}(x,0,2,s,q)=q^n x f_n(x,0,2,s,q)+\left(x^2+(1+q)s\right) f_{n-1}(x,1,2,q^2 s,q).$$

In general we have

$$\left[r(m+2n)\right] f_{2n+1}(x,m,r,s,q)=q^{nr}[(n+m)r]x f_{2n}(x,m,r,s,q)+[nr]\left(x^2+[r]s\right) f_{2n-1}(x,m+1,r,q^r s,q).$$
$$\left[r(m+2n)\right] f_{2n+2}(x,m,r,s,q)=q^{(n+1)r-1}[(n+m)r+1]x f_{2n+1}(x,m,r,s,q)$$
$$+[(n+1)r-1]\left(x^2+[r]s\right) f_{2n}(x,m+1,r,q^r s,q).$$

(4.17)

For $m\to\infty$ the polynomials $f_n(x,m,r,s,q)$ tend to $p_n(x,r,s,q)$. For $r=2$ they are a sort of interpolation between the $q-$Chebyshev and the discrete $q-$Hermite polynomials.

## 5. Other polynomials with related moments

### 5.1

There are also other choices with nice polynomials and moments but without orthogonality.

Let

$$d(n,n-2k,r,q)=\begin{bmatrix}\left\lfloor \frac{n}{2}\right\rfloor \\ k\end{bmatrix}_q \prod_{j=0}^{k-1}\left[\left(\left\lfloor \frac{n+1}{2}\right\rfloor-j\right)r-1\right]_q. \tag{5.1}$$

The polynomials

$$g_n(x,m,r,s,q)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor} s^k q^{\binom{k}{2}} d(n,n-2k,r,q)\frac{1}{\prod_{j=1}^{k}[m+n-j]_q} x^{n-2k} \tag{5.2}$$

satisfy

$$\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor} d(n,n-2k,r,q)\frac{(-s)^k}{\prod_{j=0}^{k-1}[m+n-k-j]_q} g_{n-2k}(x,m,r,q)=x^n. \tag{5.3}$$

Therefore the moments are

$$\mu_{2n}=(-s)^n\frac{\prod_{j=1}^{n}[jr-1]_q}{\prod_{j=1}^{n}[m+j]_q}. \tag{5.4}$$

**Remark**

For $m\in\mathbb{N}$ the Hankel determinants $\det\left(\mu_{i+j}\right)_{i,j=0}^{n-1}$ have no nice formula. But for $m=\infty$ we see that the polynomials $\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^k q^{\binom{k}{2}} d(n,n-2k,r,q)x^{n-2k}$ have the same moments $\mu_{2n}=\prod_{j=1}^{n}[jr-1]_q$ as the orthogonal polynomials $p_n(x,r,s)$. Thus in this case we have again nice Hankel determinants.

**5.2**

A slightly other choice is the following one:

Let $j(n,k,q)=\left(-q^{\left\lfloor \frac{n}{2}\right\rfloor+1-k};q\right)_k$.

The polynomials

$$l_n(x,m,r,s,q)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}s^k q^{\binom{k}{2}}c(n,n-2k,r,q)\frac{j(n,k,q)}{\prod_{j=1}^{k}[m+n-j]}x^{n-2k} \tag{5.5}$$

satisfy

$$\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}c(n,n-2k,r,q)(-s)^k\frac{j(n,k,q)}{\prod_{j=0}^{k-1}[m+n-k-j]}l_{n-2k}(x,m,r,q)=x^n. \tag{5.6}$$

Therefore the moments are

$$\Lambda\left(x^{2n}\right)=c(2n,0,r,q)\,j(2n,n,q)\frac{[m]!}{[m+n]!}=(-q;q)_n(-s)^n\frac{\prod_{j=1}^{n}[jr-1]_q}{\prod_{j=1}^{n}[m+j]_q}. \tag{5.7}$$

Since

$$c(n,n-2k,2,q)\,j(n,k,q)=\begin{bmatrix}\left\lfloor \frac{n}{2}\right\rfloor \\ k\end{bmatrix}\prod_{j=0}^{k-1}\left[2\left\lfloor \frac{n+1}{2}\right\rfloor-2j-1\right]\left(-q^{\left\lfloor \frac{n}{2}\right\rfloor+1-k};q\right)_k=\frac{[n]!}{[k]![n-2k]!}$$

(5.8)

we get

$$l_n(x,m,2,s,q)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}s^k q^{\binom{k}{2}}c(n,n-2k,2,q)\frac{j(n,k,q)}{\prod_{j=1}^{k}[m+n-j]}x^{n-2k}$$
$$=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}s^k q^{\binom{k}{2}}\frac{[n]!}{[k]![n-2k]!}\frac{1}{\prod_{j=1}^{k}[m+n-j]}x^{n-2k}. \qquad (5.9)$$

These polynomials have been introduced in [7] and satisfy the curious recurrence

$$l_n(x,m,2,q^m s,q)=\left(x-(1-q)sD_q\right)l_{n-1}(x,m,2,q^m s,q)+s\frac{[n-1]_q[n-2+2m]_q}{[n+m-2]_q[n+m-1]_q}l_{n-2}(x,m,2,q^m s,q),$$

(5.10)

where $D_q$ denotes the $q-$differentiation operator and $\lambda_n(m)=\frac{[n+1]_q[n+2m]_q}{[n+m]_q[n+m+1]_q}$ for $n>0$ and $\lambda_0(m)=1$ for $m>0$ and $\lambda_0(0)=2.$

For $m=0$ we get the $q-$Lucas polynomials

$$l_n(x,0,2,-1,q)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^k q^{\binom{k}{2}}\frac{[n]}{[n-k]}\begin{bmatrix}n-k\\ k\end{bmatrix}x^{n-2k} \qquad (5.11)$$

which satisfy

$$\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\begin{bmatrix}n\\ k\end{bmatrix}l_{n-2k}\left(x,0,2,-1,q\right)=x^n \qquad (5.12)$$

and whose moments are therefore the central $q-$binomial coefficients $\begin{bmatrix}2n\\ n\end{bmatrix}.$

For $m=1$ we get the $q-$Fibonacci polynomials

$$l_n(x,1,2,-1,q)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n-k\\ k\end{bmatrix}x^{n-2k} \qquad (5.13)$$

which satisfy

$$\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{q^k} \left( \begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix} \right) l_{n-2k}(x,1,2,-1,q) = x^n \tag{5.14}$$

and whose moments are the $q-$Catalan numbers $\frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}$.

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