

# Sum of cubes: Old proofs suggest new $q$ – analogues

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## Abstract

We prove a new  $q$  – analogue of Nicomachus’s theorem about the sum of cubes and some related results.

## 1. Introduction

In [1], [5] and [8] some  $q$  – analogues of the well-known formula

$$1^3 + 2^3 + \dots + n^3 = \binom{n+1}{2}^2 \quad (1)$$

have been found. Some information about this formula which is sometimes called Nicomachus’s theorem is given in [7].

In this note I propose another  $q$  – analogue which is inspired by an old result of C. Wheatstone [6].

He observed that the odd numbers can be grouped in such a way that the identities

$$\begin{aligned} 1 &= 1^3 \\ 3+5 &= 2^3 \\ 7+9+11 &= 3^3 \\ 13+15+17+19 &= 4^3 \\ &\dots \end{aligned} \quad (2)$$

hold, which implies that

$$1^3 + 2^3 + \dots + n^3 = 1 + 3 + \dots + \left( 2 \binom{n+1}{2} - 1 \right).$$

(1) follows from the well-known formula

$$1 + 3 + \dots + (2n-1) = n^2. \quad (3)$$

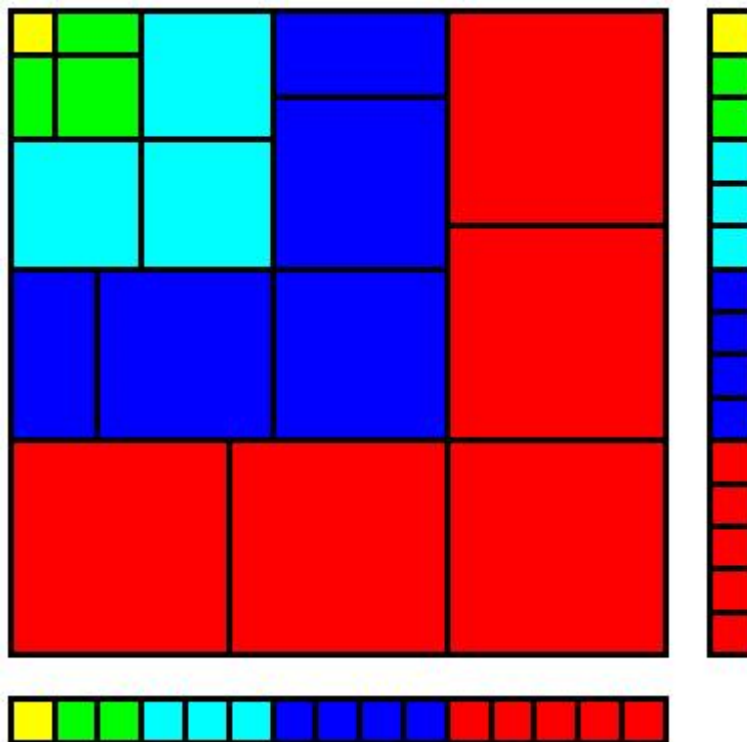
Identity (1) is usually stated in the form

$$1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2. \quad (4)$$

This is an immediate consequence of

$$1 + 2 + \dots + n = \binom{n+1}{2}. \quad (5)$$

A simple “proof without words” of (4) has been given in [4], which I reproduce here:



The simplest computational proof of (1) uses the trivial identity  $\binom{n+1}{2}^2 - \binom{n}{2}^2 = n^3$  which gives the telescoping sum

$$1^3 + 2^3 + \dots + n^3 = \binom{2}{2}^2 - \binom{1}{2}^2 + \binom{3}{2}^2 - \binom{2}{2}^2 + \dots + \binom{n+1}{2}^2 - \binom{n}{2}^2 = \binom{n+1}{2}^2 - \binom{1}{2}^2 = \binom{n+1}{2}^2.$$

## 2. $q$ – analogues.

As usual we let  $[n]_q = \frac{1-q^n}{1-q} = 1+q+\dots+q^{n-1}$  and  $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[1]_q[2]_q \cdots [k]_q}$  for  $0 < q < 1$ . It is clear that  $\lim_{q \rightarrow 1} [n]_q = n$  and  $\lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$ .

K.C. Garrett and K. Hummel [1] derived a combinatorial proof of the  $q$  – analogue

$$\sum_{k=1}^n q^{k-1} [k]_q^2 \frac{[k-1]_q + [k+1]_q}{[2]_q} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q, \quad (6)$$

S.O. Warnaar [5] proposed the identity

$$\sum_{k=1}^n q^{2n-2k} [k]_q^2 [k]_{q^2} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^2 \quad (7)$$

and G. Zhao and H. Feng [8] gave a combinatorial interpretation of

$$\sum_{k=0}^n q^{4(n-k)} [k]_q^2 \frac{1+q^2-2q^{k+1}}{1-q^2} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^2. \quad (8)$$

For a proof by induction observe that

$$\begin{aligned} \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^2 - q^{2a} \begin{bmatrix} n \\ 2 \end{bmatrix}_q^2 &= \frac{(1-q^n)^2}{(1-q)^2(1-q^2)^2} \left( (1-q^{n+1})^2 - (q^a - q^{n-1+a})^2 \right) \\ &= \frac{(1-q^n)^2}{(1-q)^2(1-q^2)^2} (1-q^{n+1} - (q^a - q^{n-1+a})) (1-q^{n+1} + (q^a - q^{n-1+a})) \end{aligned}$$

implies

$$\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^2 - \begin{bmatrix} n \\ 2 \end{bmatrix}_q^2 = q^{n-1} [n]_q^2 \frac{[n-1]_q + [n+1]_q}{[2]_q},$$

$$\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^2 - q^2 \begin{bmatrix} n \\ 2 \end{bmatrix}_q^2 = [n]_q^2 [n]_{q^2}$$

and

$$\left[ \begin{matrix} n+1 \\ 2 \end{matrix} \right]_q^2 - q^4 \left[ \begin{matrix} n \\ 2 \end{matrix} \right]_q^2 = [n]_q^2 \frac{1+q^2-2q^{n+1}}{1-q^2}.$$

A similar identity is

$$\left[ \left( \begin{matrix} n+1 \\ 2 \end{matrix} \right) \right]_q^2 - q^n \left[ \left( \begin{matrix} n \\ 2 \end{matrix} \right) \right]_q^2 = [n]_q [n^2]_q, \quad (9)$$

which follows from

$$\left( 1 - q^{\binom{n+1}{2}} \right)^2 - q^n \left( 1 - q^{\binom{n}{2}} \right)^2 = 1 - 2q^{\binom{n+1}{2}} + q^{2\binom{n+1}{2}} - q^n \left( 1 - 2q^{\binom{n}{2}} + q^{2\binom{n}{2}} \right) = (1 - q^n)(1 - q^{n^2}).$$

This identity gives the telescoping sum

$$\begin{aligned} \sum_{j=1}^n q^{\binom{n+1}{2} - \binom{j+1}{2}} [j]_q [j^2]_q &= q^{\binom{n+1}{2}} \sum_{j=1}^n q^{-\binom{j+1}{2}} \left( \left[ \left( \begin{matrix} j+1 \\ 2 \end{matrix} \right) \right]_q^2 - q^j \left[ \left( \begin{matrix} j \\ 2 \end{matrix} \right) \right]_q^2 \right) \\ &= q^{\binom{n+1}{2}} \left( q^{-\binom{2}{2}} \left[ \left( \begin{matrix} 2 \\ 2 \end{matrix} \right) \right]_q^2 - q^{-\binom{1}{2}} \left[ \left( \begin{matrix} 1 \\ 2 \end{matrix} \right) \right]_q^2 + q^{-\binom{3}{2}} \left[ \left( \begin{matrix} 3 \\ 2 \end{matrix} \right) \right]_q^2 - q^{-\binom{2}{2}} \left[ \left( \begin{matrix} 2 \\ 2 \end{matrix} \right) \right]_q^2 + \cdots + q^{-\binom{n+1}{2}} \left[ \left( \begin{matrix} n+1 \\ 2 \end{matrix} \right) \right]_q^2 - q^{-\binom{n}{2}} \left[ \left( \begin{matrix} n \\ 2 \end{matrix} \right) \right]_q^2 \right) \\ &= \left[ \left( \begin{matrix} n+1 \\ 2 \end{matrix} \right) \right]_q^2 - q^{\binom{n+1}{2}} q^{-\binom{1}{2}} \left[ \left( \begin{matrix} 1 \\ 2 \end{matrix} \right) \right]_q^2 = \left[ \left( \begin{matrix} n+1 \\ 2 \end{matrix} \right) \right]_q^2. \end{aligned}$$

Thus we have obtained our main result

### Theorem 1

$$\sum_{j=1}^n q^{\binom{n+1}{2} - \binom{j+1}{2}} [j]_q [j^2]_q = \left[ \left( \begin{matrix} n+1 \\ 2 \end{matrix} \right) \right]_q^2. \quad (10)$$

We now give two further proofs which generalize the beautiful proofs which we have sketched in the introduction and which originally led to this  $q$  – analogue.

We start with the following well-known (cf. [3])  $q$  – analogue of (3)

$$\sum_{k=1}^n q^{n-k} [2k-1]_q = [n]_q^2. \quad (11)$$

A computational proof uses the fact that

$$[n]_q^2 - q[n-1]_q^2 = \frac{q^{2n} - 2q^n + 1 - q^{2n-1} + 2q^n - q}{(q-1)^2} = \frac{q^{2n-1} - 1}{q-1} = [2n-1]_q.$$

But formula (11) has also a nice combinatorial interpretation.

Consider the squares  $S_n = \{(i, j)\}_{0 \leq i, j < n}$  and associate with each point  $(i, j) \in S_n$  the weight  $w(i, j) = q^i q^{n-1-j}$ . The weight of  $S_n$  is

$$w(S_n) = \sum_{i=0}^{n-1} q^i \sum_{j=0}^{n-1} q^{n-1-j} = [n]_q^2. \quad (12)$$

The square  $S_n$  is the union of the hooks

$$h_k = \{(k-1, 0), (k-1, 1), \dots, (k-1, k-1), (k-2, k-1), \dots, (0, k-1)\}, \quad 1 \leq k \leq n.$$

The weight of the hook  $h_k$  is

$$w(h_k) = q^n (q^{k-2} + q^{k-3} + \dots + q^0 + q^{-1} + \dots + q^{-k}) = q^{n-k} (1 + q + \dots + q^{2k-2}) = q^{n-k} [2k-1]_q.$$

The point  $(n, 0)$  will be called the base-point of the square.

As an example consider the case  $n = 6$  in matrix notation.

$$\begin{pmatrix} q^5 & q^6 & q^7 & q^8 & q^9 & q^{10} \\ q^4 & q^5 & q^6 & q^7 & q^8 & q^9 \\ q^3 & q^4 & q^5 & q^6 & q^7 & q^8 \\ q^2 & q^3 & q^4 & q^5 & q^6 & q^7 \\ q & q^2 & q^3 & q^4 & q^5 & q^6 \\ 1 & q & q^2 & q^3 & q^4 & q^5 \end{pmatrix} \quad (13)$$

Here we have  $w(h_1) = q^5$ ,  $w(h_2) = q^4 [3]_q$ ,  $w(h_3) = q^3 [5]_q, \dots$

## Second proof of Theorem 1

We first observe the nice  $q$  – analogues

$$\begin{aligned} 1 &= [1]_q [1^2]_q \\ q[3]_q + [5]_q &= [2]_q [2^2]_q \\ q^2[7]_q + q[9]_q + [11]_q &= [3]_q [3^2]_q \end{aligned}$$

and more generally

$$\sum_{j=1}^n q^{n-j} [n^2 - n + 2j - 1]_q = [n]_q [n^2]_q \quad (14)$$

of formulae (2).

They follow from the identity

$$\begin{aligned} \sum_{j=1}^n q^{n-j} [n^2 - n + 2j - 1]_q &= \frac{1}{1-q} \sum_{j=1}^n q^{n-j} (1 - q^{n^2 - n + 2j - 1}) = \frac{1}{1-q} \left( \sum_{j=0}^{n-1} q^{n-1-j} - \sum_{j=0}^{n-1} q^{n^2+j} \right) \\ &= \frac{[n]_q - q^{n^2} [n]_q}{1-q} = [n]_q [n^2]_q. \end{aligned}$$

Using (11) we get the desired result

$$\begin{aligned} \sum_{j=1}^n q^{\binom{n+1}{2} - \binom{j+1}{2}} [j]_q [j^2]_q &= \sum_{j=1}^n q^{\binom{n+1}{2} - \binom{j+1}{2}} \sum_{k=1}^j q^{j-k} [j^2 - j + 2k - 1]_q = \sum_{j=1}^n q^{\binom{n+1}{2}} \sum_{k=\binom{j}{2}+1}^{\binom{j+1}{2}} q^{-k} [2k-1]_q \\ &= \sum_{k=1}^{\binom{n+1}{2}} q^{\binom{n+1}{2} - k} [2k-1]_q = \left[ \binom{n+1}{2} \right]_q. \end{aligned}$$

## Third proof of Theorem 1

A combinatorial proof can also be given along the lines of the above “proof without words”.

For odd  $j$  the union  $R_j$  of the  $j$  hooks  $h_{\binom{j}{2}+1}, \dots, h_{\binom{j+1}{2}}$  is the union of  $j$  squares of side-

length  $j$  whose base points have weight  $q^{\binom{n+1}{2} - \binom{j+1}{2}} q^{mj}$ ,  $0 \leq m < j$ .

The weight of these squares is

$$\begin{aligned} q^{\binom{n+1}{2}-\binom{j+1}{2}} [j]_q^2 (1+q^j+\dots+q^{(j-1)j}) &= q^{\binom{n+1}{2}-\binom{j+1}{2}} [j]_q^2 [j]_{q^j} = q^{\binom{n+1}{2}-\binom{j+1}{2}} [j]_q \frac{1-q^j}{1-q} \frac{1-q^{j^2}}{1-q^j} \\ &= q^{\binom{n+1}{2}-\binom{j+1}{2}} [j]_q [j^2]_q. \end{aligned}$$

For  $j = 2\ell$  the union  $R_j$  of the hooks  $h_{\binom{j}{2}+1}, \dots, h_{\binom{j+1}{2}}$  is the union of  $j-1$  squares whose

base-points have weights  $q^{\binom{n+1}{2}-\binom{j+1}{2}} q^{\ell+2\ell m}$ ,  $0 \leq m \leq j-2$ , and of two rectangles with side lengths  $j$  and  $\ell$  as in the blue region in the above figure.

The weight of the uppermost rectangle is  $q^{\binom{n+1}{2}+\binom{j}{2}-\ell} [j]_q [\ell]_q$  and the weight of the leftmost rectangle is  $q^{\binom{n+1}{2}-\binom{j+1}{2}} [j]_q [\ell]_q$ .

Thus the total weight of this region is

$$\begin{aligned} & q^{\binom{n+1}{2}-\binom{2\ell+1}{2}} \sum_{m=0}^{2\ell-2} q^{\ell+2\ell m} [2\ell]_q^2 + q^{\binom{n+1}{2}+\binom{2\ell}{2}-\ell} [2\ell]_q [\ell]_q + q^{\binom{n+1}{2}-\binom{2\ell+1}{2}} [2\ell]_q [\ell]_q \\ &= q^{\binom{n+1}{2}-\binom{2\ell+1}{2}} [2\ell]_q [\ell]_q \left( \sum_{m=0}^{2\ell-2} q^{\ell+2\ell m} \frac{1-q^{2\ell}}{1-q^\ell} + q^{4\ell^2-\ell} + 1 \right) \\ &= q^{\binom{n+1}{2}-\binom{2\ell+1}{2}} [2\ell]_q [\ell]_q \left( \frac{q^\ell - q^{4\ell^2-\ell}}{1-q^\ell} + q^{4\ell^2-\ell} + 1 \right) = q^{\binom{n+1}{2}-\binom{2\ell+1}{2}} [2\ell]_q \frac{1-q^\ell}{1-q} \frac{(1-q^{4\ell^2})}{1-q^\ell} \\ &= q^{\binom{n+1}{2}-\binom{2\ell+1}{2}} [2\ell]_q [4\ell^2]_q. \end{aligned}$$

Thus we get  $\sum_{j=1}^n w(R_j) = w\left(S_{\binom{n+1}{2}}\right) = \left[ \binom{n+1}{2} \right]_q^2$  and thus again (10).

### 3. Related results

Let us note some related results. For any sequence of positive integers  $a(n)$  the sums

$$\sum_{j=1}^n q^{\sum_{i=1}^{j-1} a(i)} [a(j)]_q = \left[ \sum_{i=1}^n a(i) \right]_q \quad (15)$$

and

$$\sum_{j=1}^n q^{\sum_{i=j+1}^n a(i)} [a(j)]_q = \left[ \sum_{i=1}^n a(i) \right]_q \quad (16)$$

are  $q$ - analogues of  $\sum_{i=1}^n a(i)$ .

The proofs are obvious because

$$\left[ \sum_{i=1}^n a(i) \right]_q + q^{\sum_{i=1}^n a(i)} [a(n+1)]_q = \frac{1 - q^{\sum_{i=1}^n a(i)}}{1 - q} + \frac{q^{\sum_{i=1}^n a(i)} (1 - q^{a(n+1)})}{1 - q} = \frac{1 - q^{\sum_{i=1}^{n+1} a(i)}}{1 - q} = \left[ \sum_{i=1}^{n+1} a(i) \right]_q$$

and

$$q^{a(n+1)} \left[ \sum_{i=1}^n a(i) \right]_q + [a(n+1)]_q = \frac{q^{a(n+1)} - q^{\sum_{i=1}^{n+1} a(i)}}{1 - q} + \frac{1 - q^{a(n+1)}}{1 - q} = \frac{1 - q^{\sum_{i=1}^{n+1} a(i)}}{1 - q} = \left[ \sum_{i=1}^{n+1} a(i) \right]_q.$$

By choosing  $a(n) = n^3$  we get the following  $q$ - analogues of (1):

## Theorem 2

$$\sum_{j=1}^n q^{\binom{j}{2}} [j^3]_q = \left[ \binom{n+1}{2} \right]_q \quad (17)$$

and

$$\sum_{j=1}^n q^{\sum_{i=j+1}^n i^3} [j^3]_q = \left[ \binom{n+1}{2} \right]_q. \quad (18)$$



From the recurrence relations for the  $q$  – binomial coefficients

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$$

we get the well-known formulae

$$\sum_{j=1}^n q^{j-1} \begin{bmatrix} j \\ k \end{bmatrix}_q = q^{k-1} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \quad (19)$$

and

$$\sum_{j=1}^n q^{(k+1)(n-j)} \begin{bmatrix} j \\ k \end{bmatrix}_q = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q. \quad (20)$$

For  $q=1$  these sums can be used to compute  $\sum_{k=1}^n j^k$ . For example from  $n^3 = 6 \binom{n}{3} + 6 \binom{n}{2} + n$

$$\text{we get } \sum_{j=1}^n j^3 = 6 \binom{n+1}{4} + 6 \binom{n+1}{3} + \binom{n+1}{2} = \frac{n^2(n+1)^2}{4} = \binom{n+1}{2}^2.$$

Unfortunately in general the sums  $\sum_{k=1}^n q^{j-1} [j]^k$  don't have a simple expression.

For example from  $[n]_q^3 = q^3 [2]_q [3]_q \begin{bmatrix} n \\ 3 \end{bmatrix}_q + [2]_q [3]_q \begin{bmatrix} n \\ 2 \end{bmatrix}_q + q^{n-1} [n]_q$  we get

$$\sum_{j=1}^n q^{j-1} [j]_q^3 = q^5 [2]_q [3]_q \begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q + q [2]_q [3]_q \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_q + \sum_{j=1}^n q^{2(j-1)} [j]_q.$$

Now

$$\begin{aligned} \sum_{j=1}^n q^{2(j-1)} [j]_q &= \sum_{j=1}^n q^{(j-1)} [j]_q + \sum_{j=1}^n q^{(j-1)} (q^{(j-1)} - 1) [j]_q = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q - (1-q^2) \sum_{j=1}^n q^{(j-1)} \begin{bmatrix} j \\ 2 \end{bmatrix}_q \\ &= \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q - q(1-q^2) \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_q = \frac{[n+1]_q [n]_q}{[2]_q [3]_q} (1+q+q^2 - q(1+q)(1-q^{n-1})) = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q \frac{1+q^n+q^{n+1}}{1+q+q^2}. \end{aligned}$$

Thus we get

$$\sum_{j=1}^n q^{j-1} [j]_q^3 = q^5 [2]_q [3]_q \begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q + q [2]_q [3]_q \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_q + \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q \frac{1+q^n+q^{n+1}}{1+q+q^2}. \quad (21)$$

This formula cannot be simplified.

A curious generalization of (2) is due to P. Luthy [2].

He observed that

$$\begin{aligned} 1 &= 1^5 \\ 5 + 7 + 9 + 11 &= 2^5 \\ 19 + 21 + 23 + 25 + 27 + 29 + 31 + 33 + 35 &= 3^5 \\ \dots \end{aligned} \quad (22)$$

and more generally

$$\sum_{j=1}^{n^k} (n^{k+1} - n^k + 2j - 1) = n^{2k+1}. \quad (23)$$

To see this observe that  $\frac{(n^{k+1} - n^k + 2j - 1) + (n^{k+1} - n^k + 2(n^k - j) + 1)}{2} = n^{k+1}$  for  $1 \leq j \leq n^k$ .

Similar results also hold for the  $q$ - analogues of (23).

$$\sum_{j=1}^{n^k} q^{n^k-j} [n^{k+1} - n^k + 2j - 1]_q = [n^k]_q [n^{k+1}]_q. \quad (24)$$

For

$$\begin{aligned} \sum_{j=1}^{n^k} q^{n^k-j} [n^{k+1} - n^k + 2j - 1]_q &= \frac{1}{1-q} \sum_{j=1}^{n^k} q^{n^k-j} (1 - q^{n^{k+1} - n^k + 2j - 1}) = \frac{1}{1-q} \sum_{j=0}^{n^k-1} q^j - \frac{q^{n^{k+1}}}{1-q} \sum_{j=0}^{n^k-1} q^j \\ &= \frac{1}{(1-q)^2} (1 - q^{n^k}) (1 - q^{n^{k+1}}) = [n^k]_q [n^{k+1}]_q. \end{aligned}$$

#### 4. More $q$ – analogues of identities by C. Wheatstone [6].

##### 4.1

A  $q$  – analogue of  $\sum_{j=0}^{n-1} (n+1+2j) = 2n^2$  is

$$\sum_{j=0}^{n-1} [n+1+2j]_q q^{n-1-j} = [2]_q [n]_q [n]_{q^2}. \quad (25)$$

For

$$\sum_{j=0}^{n-1} [n+1+2j]_q q^{n-1-j} = \frac{1}{1-q} \left( \sum_{j=0}^{n-1} q^{n-1-j} - \sum_{j=0}^{n-1} q^{2n+j} \right) = \frac{(1-q^n)(1-q^{2n})}{(1-q)^2} = (1+q)[n]_q [n]_{q^2}.$$

##### 4.2

A similar identity is

$$\sum_{k=0}^n [mn+k]_{q^2} q^{n-k} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q [2m+1]_{q^n}. \quad (26)$$

The left-hand side is

$$\sum_{k=0}^n q^{n-k} \frac{1-q^{2mn+2k}}{1-q^2} = \frac{1}{1-q^2} \left( \sum_{k=0}^n q^{n-k} - q^{(2m+1)n+k} \right) = \frac{[n+1]_q}{1+q} \frac{1-q^{(2m+1)n}}{1-q} = \frac{[n+1]_q [n]_q}{1+q} [2m+1]_{q^n}.$$

##### 4.3

C. Wheatstone observed that

$$\sum_{j=\frac{3^n+1}{2}}^{\frac{3^{n+1}-1}{2}} j = 3^{2n} \quad (27)$$

implies

$$\sum_{j=1}^{\frac{3^{n+1}-1}{2}} j = \sum_{k=0}^n 3^{2k}. \quad (28)$$

A  $q$  – analogue is

**Theorem 3**

$$\sum_{j=\frac{3^n+1}{2}}^{\frac{3^{n+1}-1}{2}} [j]_{q^2} q^{\frac{3^{n+1}-1-2j}{2}} = [3^n]_q [3^n]_{q^2} \quad (29)$$

and therefore

$$\sum_{j=1}^{\frac{3^{n+1}-1}{2}} q^{\frac{3^{n+1}-2j-1}{2}} [j]_{q^2} = \sum_{k=0}^n q^{\frac{3^{n+1}-3^{k+1}}{2}} [3^k]_q [3^k]_{q^2} = \left[ \begin{matrix} 3^{n+1}+1 \\ 2 \\ 2 \end{matrix} \right]_q. \quad (30)$$

**Proof**

Observe that (cf. [3])

$$\sum_{k=0}^n [k]_{q^2} q^{n-k} = \left[ \begin{matrix} n+1 \\ 2 \end{matrix} \right]_q. \quad (31)$$

This is the special case  $m=0$  of (26).

For  $a(n) = \frac{3^n-1}{2}$  the left-hand side of (29) becomes

$$\begin{aligned} \left[ \begin{matrix} a(n+1)+1 \\ 2 \end{matrix} \right]_q - q^{a(n+1)-a(n)} \left[ \begin{matrix} a(n)+1 \\ 2 \end{matrix} \right]_q &= \frac{(1-q^{a(n+1)+1})(1-q^{a(n+1)}) - q^{a(n+1)-a(n)}(1-q^{a(n+1)})(1-q^{a(n)})}{(1-q)(1-q^2)} \\ &= \frac{(1-q^{a(n+1)+1})(1-q^{a(n+1)}) - q^{a(n+1)-a(n)}(1-q^{a(n+1)})(1-q^{a(n)})}{(1-q)(1-q^2)} = \frac{(1-q^{a(n+1)-a(n)})(1-q^{a(n+1)+a(n)+1})}{(1-q)(1-q^2)}. \end{aligned}$$

This gives

$$\sum_{j=\frac{3^n+1}{2}}^{\frac{3^{n+1}-1}{2}} [j]_{q^2} q^{\frac{3^{n+1}-1-2j}{2}} = \frac{(1-q^{3^n})(1-q^{2 \cdot 3^n})}{(1-q)(1-q^2)} = [3^n]_q [3^n]_{q^2} = \frac{(1-q^{3^n})^2(1+q^{3^n})}{(1-q)(1-q^2)} = [3^n]_q^2 \frac{1+q^{3^n}}{1+q}.$$

and therefore  $\sum_{j=1}^{\frac{3^{n+1}-1}{2}} q^{\frac{3^{n+1}-2j-1}{2}} [j]_{q^2} = \sum_{k=0}^n q^{\frac{3^{n+1}-3^{k+1}}{2}} [3^k]_q [3^k]_{q^2}$ , which by (31) implies (30).

#### 4.4

A  $q$  – analogue of  $\sum_{j=0}^{n-1} (2(n+1)j+1) = n^3$  is

$$\sum_{j=0}^{n-1} [2(n+1)j+1]_q q^{n^2-1-(n+1)j} = [n^2]_q [n]_{q^{n+1}}. \quad (32)$$

A  $q$  – analogue of  $\sum_{j=0}^{n-1} (2k+1)n = n^3$  is

$$\sum_{k=0}^{n-1} [(2k+1)n]_q q^{n(n-1)-nk} = [n^2]_q [n]_{q^n}. \quad (33)$$

More generally we get for  $m \in \mathbb{N}$

$$\sum_{k=0}^{n-1} [(2k+1)n^m]_q q^{n^m(n-1)-kn^m} = [n^{m+1}]_q [n]_{q^{n^m}}. \quad (34)$$

The proofs are straightforward and will be omitted.

#### 4.5

C. Wheatstone also observed that

$$\begin{aligned} 1^3 &= 1 \\ 2^3 &= 3 + 5 \\ 3^3 &= 6 + 9 + 12 \\ 4^3 &= 10 + 14 + 18 + 22 \\ 5^3 &= 15 + 20 + 25 + 30 + 35 \\ &\dots\dots\dots \end{aligned}$$

More generally this gives  $\sum_{j=0}^{n-1} \left( \binom{n+1}{2} + jn \right) = n^3$ .

We will now prove three different  $q$  – analogues of this identity.

**Theorem 4**

$$\sum_{j=0}^{n-1} \left[ \binom{n+1}{2} + nj \right]_q q^{\binom{n}{2} - \binom{j+1}{2}} = [n^2]_q \sum_{j=0}^{n-1} q^{\binom{n}{2} - \binom{j+1}{2}}, \quad (35)$$

$$\sum_{j=0}^{n-1} \left[ \binom{n+1}{2} + nj \right]_{q^2} q^{n(n-1)-nj} = [n^2]_{q^2} [n]_{q^n}, \quad (36)$$

and

$$\sum_{j=0}^{n-1} \left( q^{n-1-j} \left[ \binom{n+1}{2} \right]_q + q^{2n-j} [n]_q [j]_{q^2} \right) = [n]_q^2 [n]_{q^2}. \quad (37)$$

**Proof**

(35) follows from

$$\begin{aligned} \frac{1}{1-q} \sum_{j=0}^{n-1} \left( q^{\binom{n}{2} - \binom{j+1}{2}} \left( 1 - q^{\binom{n+1}{2} + nj} \right) \right) &= \frac{q^{\binom{n}{2}}}{1-q} \left( \sum_{j=0}^{n-1} q^{-\binom{j+1}{2}} - q^{\binom{n+1}{2}} \sum_{j=0}^{n-1} q^{nj - \binom{j+1}{2}} \right) \\ &= \frac{q^{\binom{n}{2}}}{1-q} \left( \sum_{j=0}^{n-1} q^{-\binom{j+1}{2}} - q^{\binom{n+1}{2}} \sum_{j=0}^{n-1} q^{\binom{n}{2} - \binom{n-j}{2}} \right) = \frac{1}{1-q} \sum_{j=0}^{n-1} q^{\binom{n}{2} - \binom{j+1}{2}} (1 - q^{n^2}). \end{aligned}$$

For the second identity the left-hand side is

$$\frac{1}{1-q^2} \sum_{j=0}^{n-1} \left( q^{n^2 - n - nj} \left( 1 - q^{2 \binom{n+1}{2} + 2nj} \right) \right) = \frac{1}{1-q^2} \left( \sum_{j=0}^{n-1} q^{nj} - q^{2n^2} \sum_{j=0}^{n-1} q^{nj} \right) = \frac{(1-q^{n^2})(1-q^{2n^2})}{(1-q^n)(1-q^2)},$$

which implies the right-hand side.

After dividing (37) by  $[n]_q$  and multiplying with  $(1-q)(1-q^2)$  the left-hand side is

$$\begin{aligned} (1-q) \sum_{j=0}^{n-1} \left( q^{n-1-j} (1-q^{n+1}) + q^{2n-j} (1-q^{2j}) \right) &= (1-q) \sum_{j=0}^{n-1} \left( q^{n-1-j} - q^{2n-j} + q^{2n-j} - q^{2n+j} \right) \\ &= (1-q) \left( \sum_{j=0}^{n-1} q^j - q^{n+1} \sum_{j=0}^{n-1} q^j + (q^{n+1} - q^{2n}) \sum_{j=0}^{n-1} q^j \right) = (1-q^n)(1-q^{2n}). \end{aligned}$$

This proves (37).

## 4.6

Finally we look for a  $q$  – analogue of  $\sum_{j=0}^{2n} ((2n-1)^2 + 8j) = (2n+1)^3$ .

### Theorem 5

$$\sum_{j=0}^{2n} [(2n-1)^2 + 8j]_{-q} q^{8n-4j} = [2n+1]_q [2n+1]_{q^4} [2n+1]_{q^{2n+1}}. \quad (38)$$

### Proof

$$\begin{aligned} \frac{1}{1-q} \sum_{j=0}^{2n} q^{8n-4j} (1 - q^{(2n-1)^2 + 8j}) &= \frac{1}{1-q} \sum_{j=0}^{2n} q^{4j} - \frac{q^{(2n-1)^2 + 8n}}{1-q} \sum_{j=0}^{2n} q^{4j} = \frac{(1 - q^{4(2n+1)})(1 - q^{(2n+1)^2})}{(1-q)(1-q^4)} \\ &= [2n+1]_q [2n+1]_{q^4} [2n+1]_{q^{2n+1}}. \end{aligned}$$

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