$q-$Catalan numbers and $q-$Narayana polynomials

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The $q-$Catalan numbers $1,1,1+q,1+2q+q^2+q^3,1+3q+3q^2+3q^3+2q^4+q^5+q^6,\cdots,$ which have been introduced by Carlitz and Riordan ([3] , cf. also[8] and [11]), are defined by $C_n(q)=\sum_{k=0}^{n-1}q^kC_k(q)C_{n-k-1}(q)$ with initial value $C_0(q)=1$. They are polynomials in q of degree $\binom{n}{2}$.

Let $f(z,q)=\sum_{k\geq 0}C_k(q)z^k$ be their generating function, which is uniquely determined by the functional equation $f(z,q)=1+zf(z,q)f(qz,q)$. This implies the well known fact that it can be represented in the form

$$f(z,q)=\frac{E_2(-qz)}{E_2(-z)}, \tag{1}$$

where $E_r(z)$ denotes the generalized $q-$exponential function

$$E_r(z)=\sum_{k\geq 0}q^{r\binom{k}{2}}\frac{z^k}{(1-q)(1-q^2)\cdots(1-q^k)}. \tag{2}$$

As shown in [8] the related polynomials $q^{\binom{n}{2}}C_n\left(\frac{1}{q}\right),$ which are given by $1,1,1+q,1+q+2q^2+q^3,1+q+2q^2+3q^3+3q^4+3q^5+q^6,\cdots$ have a nice combinatorial interpretation which implies that $q^{\binom{n}{2}}C_n\left(\frac{1}{q}\right)=1+p(1)q+\cdots+p(n-1)q^{n-1}+O(q^n),$ where $p(n)$ denotes the number of partitions of the number $n.$

This may also be seen from (1): If we compare the coefficients of z^n in $f(z,q)E_2(-z)=E_2(-qz)$, replace q by $\frac{1}{q}$ and multiply both sides with $q^{\binom{n}{2}}$ we get

$$\sum_{k=0}^{n}\frac{q^{\binom{n}{2}-\frac{k(k-3)}{2}}}{(1-q)\cdots(1-q^k)}C_{n-k}\left(\frac{1}{q}\right)=\frac{1}{(1-q)\cdots(1-q^n)}.$$

But the left-hand side is of the form

$q^{\binom{n}{2}}C_n\left(\frac{1}{q}\right)+O(q^n)$ and the right-hand side may be written as

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}+O(q^n)=\sum_{k=0}^{\infty}p(k)q^k+O(q^n).$$

In this note we want to sketch some extensions of (1).

a) First observe that

$$E_r(z)-E_r(qz)=zE_r(q^r z). \tag{3}$$

For $n\in\mathbb{N}$ define

$$G_r(z,n)=\sum_{k\ge 0}G(k,n,r)z^k:=\frac{E_r(-q^n z)}{E_r(-z)}. \tag{4}$$

Then we have

$$G_r(z,n+1)=G_r(z,n)+q^n zG_r(z,n+r). \tag{5}$$

Comparing coefficients we get

$$\frac{G(k,n+1,r)-G(k,n,r)}{q^n}=G(k-1,n+r,r) \tag{6}$$

with $G(k,0,r)=[k=0]$ and $G(0,n,r)=1.$

This implies

$$G_r(z,1)=1+zG_r(z,r). \tag{7}$$

These are the characteristic properties of the $q-$Gould polynomials (cf. [5]).

For $q=1$ they have the explicit formula $G(k,n,r)=\frac{n}{n+rk}\binom{n+rk}{k}$ (cf. e.g. [10]).

For general q special values are $G(k,n,0)=q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}$ and $G(k,n,1)=\begin{bmatrix}n+k-1\\k\end{bmatrix}$, where $\begin{bmatrix}n\\k\end{bmatrix}$ denotes a $q-$binomial coefficient. For $r>1$ no explicit formulas are known.

Note that $G_2(z,1)=1+zG_2(z,2)=1+zG_2(z,1)G_2(qz,1)=f(z,q)$ is the generating function of the $q-$Catalan numbers.

From $\frac{E_r(-q^n z)}{E_r(-z)}=\frac{E_r(-qz)}{E_r(-z)}\frac{E_r(-q^2 z)}{E_r(-qz)}\cdots\frac{E_r(-q^n z)}{E_r(-q^{n-1}z)}$
we get

$$G_r(z,n)=G_r(z,1)G_r(qz,1)\cdots G_r(q^{n-1}z,1) \tag{8}$$

and

$$G_r(z,m+n)=G_r(z,m)G_r(q^m z,n). \tag{9}$$

b) Let now $(a \dotplus b)^k := (a+b)(a+qb)\cdots(a+q^{k-1}b)$
and consider the modified exponential function

$$h(z,a,b,q) = \sum_{k\geq 0} q^{\binom{k}{2}} \frac{(a \dotplus b)^k}{(1 \dot{-} q)^k} (-z)^k, \tag{10}$$

which for $(a,b) = (0,1)$ reduces to $E_2(-z)$.
We want to study the series

$$f(z,a,b,q) = \frac{h(qz,a,b,q)}{h(z,a,b,q)}. \tag{11}$$

From

$$h(z,a,b,q) - h(qz,a,b,q) = \sum_{k\geq 0} q^{\binom{k}{2}} \frac{(a \dotplus b)^k}{(1 \dot{-} q)^k} (-z)^k (1-q^k) =$$

$$= -z \sum_{k\geq 0} q^{\binom{k-1}{2}} \frac{(a \dotplus b)^{k-1}(a+q^{k-1}b)}{(1 \dot{-} q)^{k-1}} (-qz)^{k-1}$$

we deduce

$$h(z,a,b,q) - h(qz,a,b,q) = \sum_{k\geq 0} q^{\binom{k}{2}} \frac{(a \dotplus b)^k}{(1 \dot{-} q)^k} (-z)^k (1-q^k) =$$

$$= -z \sum_{k\geq 0} q^{\binom{k-1}{2}} \frac{(a \dotplus b)^{k-1}(a+q^{k-1}b)}{(1 \dot{-} q)^{k-1}} (-qz)^{k-1} = -azh(qz,a,b,q) - bzh(q^2 z,a,b,q)$$

i.e.

$$f(z,a,b,q) = 1 + azf(z,a,b,q) + bzf(z,a,b,q)\, f(qz,a,b,q) \tag{12}$$

and

$$h(z,a,b,q) - h(qz,a,b,q) = -z \sum_{k\geq 0} q^{\binom{k-1}{2}} \frac{(a \dotplus b)^{k-1}(a+q^{k-1}b)}{(1 \dot{-} q)^{k-1}} (-qz)^{k-1} = -(a+b)zh(qz,a,qb,q).$$

Let now

$$g(z,a,b,q) = \frac{h(qz,a,qb,q)}{h(z,a,b,q)}. \tag{13}$$

Then

$$f(z,a,b,q) = 1 + (a+b)zg(z,a,b,q). \tag{14}$$

This implies

$$\frac{g(qz,a,b,q)}{g(z,a,b,q)} = \frac{h(q^2 z,a,qb,q)}{h(qz,a,b,q)} \frac{h(z,a,b,q)}{h(qz,a,qb,q)} = \frac{f(qz,a,qb,q)}{f(z,a,b,q)}. \tag{15}$$

If we write

$$f(z,a,b,q)=\sum_{n\geq 0} C_n(a,b,q)z^n, \qquad (16)$$

then $C_n(a,b,q)$ is a $q-$analogue of $C_n(a,b)=\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}b^{n-k}(a+b)^k$.

For it is easy to verify that the uniquely determined formal power series $f(z,a,b)$, which satisfies the functional equation $f(z,a,b)=1+azf(z,a,b)+bzf(z,a,b)^2$ has the series expansion

$$f(z,a,b)=1+\sum_{n\geq 1}\sum_{k=1}^{n}\frac{1}{n}\binom{n}{k}\binom{n}{k-1}b^{n-k}(a+b)^k z^n, \qquad (17)$$

The numbers $N(n,k)=\frac{1}{n}\binom{n}{k}\binom{n}{k-1}=\binom{n}{k}\binom{n-1}{k-1}-\binom{n}{k-1}\binom{n-1}{k}$ are called Narayana numbers. (cf. e.g. [11] or [2]). The first terms of this sequence are

1					
1	1				
1	3	1			
1	6	6	1		
1	10	20	10	1	
1	15	50	50	15	1

By choosing $a=s-1$ and $b=1$ and setting $F=f(z,s-1,1)-1$ we get the well-known fact (cf. [11]) that the generating function $F(z,s)=\sum_{n\geq 1}\sum_{k\geq 1}N(n,k)s^k z^n$ satisfies

$$zF(z,s)^2+(sz+z-1)F(z,s)+sz=0. \qquad (18)$$

It is easily verified that $G(z,s)=\frac{sz}{1-z}+\frac{s}{1-z}F(\frac{sz}{(1-z)^2},z)$ satisfies the same equation and therefore $G(z,s)=F(z,s)$. By comparing coefficients this implies that

$$\sum_{n\geq 1}N(n,k)z^n=\frac{\sum_{j=1}^{k-1}N(k-1,j)z^{k-1+j}}{(1-z)^{2k-1}}.$$

This is a refinement of the trivial fact that $\Delta^{2k-1}N(n,k)=(E-1)^{2k-1}N(n,k)=0$ if we denote by E the shift operator defined by $Ef(n)=f(n+1)$ and by $\Delta=E-1$ the difference operator.

It is clear that $N(n,k)=N(n,n+1-k)$. This can also be seen from the fact that $sF\left(sz,\frac{1}{s}\right)$ satisfies (18) too.

Of course $C_n(0,1)=\frac{1}{n+1}\binom{2n}{n}=C_n$ are the well known Catalan numbers.

We call $C_n(a,b,q)$ a $q-$Narayana polynomial.

$C_n(a,b,q)$ satisfies

$$C_n(a,b,q)=(a+b)C_{n-1}(a,b,q)+b\sum_{k=1}^{n-1}q^kC_k(a,b,q)C_{n-k-1}(a,b,q) \tag{19}$$

with initial value $C_0(a,b,q)=1$.
The first values are $1,a+b,a^2+(2+q)ab+(1+q)b^2,\cdots$.

From the definition is clear that $C_n(a,b,q)$ is a polynomial in a,b which is homogeneous of degree n. Therefore $C_n(a,b,q)$ has for $n\geq 1$ a unique representation in the form

$$C_n(a,b,q)=\sum_{k=1}^{n}N(n,k,q)(a+b)^kb^{n-k}. \tag{20}$$

In order to study the $q-$Narayana numbers $N(n,k,q)$ we may choose $a=s-1,b=1$.
Then we get

$$C_n(s-1,1,q)=\sum_{k=1}^{n}N(n,k,q)s^k \tag{21}$$

and the recurrence

$$C_n(s-1,1,q)=sC_{n-1}(s-1,1,q)+\sum_{k=1}^{n-1}q^kC_k(s-1,1,q)C_{n-k-1}(s-1,1,q). \tag{22}$$

The $q-$Narayana numbers are polynomials in q with integer coefficients.
The first values of these $q-$Narayana numbers are given in the following table:

1				
q	1			
q^3	$2q+q^2$	1		
q^6	$q^2+2q^3+2q^4+q^5$	$3q+2q^2+q^3$	1	
q^{10}	$2q^4+3q^6+2q^7+2q^8+q^9$	$3q^2+5q^3+4q^4+5q^5+2q^6+q^7$	$4q+3q^2+2q^3+q^4$	1

Some $q-$Narayana numbers can be explicitly given. E.g. $N(n,1,q)=q^{\binom{n}{2}}$ for $n\geq 1$.
Comparing the coefficients of s^{n-2} we get

$$N(2n+1,2,q)=q^{2n}N(2n,2,q)+q^{\binom{2n}{2}}+\sum_{k=1}^{2n-1}q^kq^{\binom{k}{2}}q^{\binom{2n-k}{2}}=q^{2n}N(2n,2,q)+2q^{n^2}\sum_{i=1}^{n}q^{i^2-i},$$

because $\binom{k+1}{2}+\binom{2n-k}{2}=n^2+2\binom{n-k}{2}$.
In the same way we get

$$N(2n,2,q)=q^{2n-1}N(2n-1,2,q)+q^{n^2-n}\left(1+2\sum_{i=1}^{n-1}q^{i^2}\right).$$

This implies that $N(n,2,q)$ satisfies the recurrence

$$(E^4-q^{n+2}(1+q)E^3+q^{n+2}(q^{n+2}-1)E^2+2q^{2n+3}E-q^{3n+3})N(n,2,q)=0.$$

Let now $M(n,k,q)=N(n,n+1-k,1).$ Then $M(n,1,q)=1$ and
$M(n,2,q)=q^{n-1}+2q^{n-2}+\cdots+(n-1)q$.
The last one follows from (22) by comparing coefficients of $s,$ which gives
$M(n,2,q)=M(n-1,2,q)+\sum_{k=1}^{n-1}q^k$. The first one by comparing the coefficients of $s^{n-1}.$
We get $(E-1)M(n,2,q)=q[n]$ and therefore $(E-1)(E-q)M(n,2,q)=q.$ This implies the homogeneous recurrence $(E-1)^2(E-q)M(n,2,q)=0.$
Computer experiments suggest the following facts:
1) For each k
$(E-1)^k(E-q)^{k-1}(E-q^2)^{k-2}\cdots(E-q^{k-1})M(n,k,q)=0$
and more generally
$(E-1)^{k-1}(E-q)^{k-1}(E-q^2)^{k-2}\cdots(E-q^{k-1})M(n,k,q)=q^{k-1}(1-q^2)^{k-2}(1-q^3)^{k-3}\cdots(1-q^{k-1}).$

2) $\deg(N(n,k,q)=\frac{(n-k)(n+k-1)}{2}$ for $n\geq k$ and
$q^{\frac{(n-k)(n+k-1)}{2}}N(n,k,\frac{1}{q})=\sum_{j=0}^{n-k}p(\binom{k}{2}+j,k-1)q^j+O(q^{n-k+1}),$
where $p(j,k-1)$ denotes the number of partitions of the number j with precisely $k-1$ different parts. Of course $p(j,2)=d(j)$ the number of divisors of $j.$ (Cf. The On-Line Encyclopedia of Integer Sequences OEIS A060177).
3) $\deg(N(n,n-k,q)=\frac{k(2n-k-1)}{2}=nk-\binom{k+1}{2}$ for $n\geq k+1$ and
$q^{\frac{k(2n-k-1)}{2}}N(n,n-k,\frac{1}{q})=\sum_{j=0}^{n-k}T(k^2+j,k)q^j+O(q^{n-k+1})$,
where $T(n,k)$ denotes the number of partitions of n with Durfee square of size k . (Cf. OEIS A115994).
Till now I have no proof of these results.

It is well known (cf. e.g. [11]) that the Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$ can be characterized by the values of their Hankel determinants. They satisfy $\det\left(C_{i+j}\right)_{i,j=0}^{n}=1$ and $\det\left(C_{i+j+1}\right)_{i,j=0}^{n}=1$.

We want to show that the $q-$Narayana polynomials can also be characterized by the values of their Hankel determinants. This shows that these polynomials are in some sense a natural generalization of the $q-$Catalan numbers.

Theorem
The polynomials $C_n(a,b,q)$ *are characterized by their Hankel determinants*

$$\det(C_{i+j}(a,b,q))\Big|_{i,j=0}^{n} = q^{\frac{n^2(n+1)}{2}} b^{\binom{n+1}{2}} (a+b)^n (a+qb)^{n-1}(a+q^2b)^{n-2}\cdots(a+q^{n-1}b) \tag{23}$$

and

$$\det(C_{i+j+1}(a,b,q))\Big|_{i,j=0}^{n} = q^{\frac{n(n+1)^2}{2}} b^{\binom{n+1}{2}} (a+b)^{n+1}(a+qb)^n(a+q^2b)^{n-1}\cdots(a+q^nb). \tag{24}$$

Remark
For the $q-$Catalan numbers this reduces to

$$\det(C_{i+j}(q))\Big|_{i,j=0}^{n} = q^{\frac{n(n+1)(4n-1)}{6}} \text{ and } \det(C_{i+j+1}(q))\Big|_{i,j=0}^{n} = q^{\frac{n(n+1)(4n+5)}{6}}.$$

These special cases have been proved by another method in [6].

In order to prove the theorem we need the following well-known (cf. e.g. [1] or [9])

Lemma
Let

$$\sum_{k\geq 0}\mu_k z^k = \cfrac{1}{1-s_0z-\cfrac{t_0z^2}{1-s_1z-\cfrac{t_1z^2}{1-\cdots}}} = \frac{1}{1-s_0z-}\frac{t_0z^2}{1-s_1z-}\frac{t_1z^2}{1-s_2z-}\cdots. \tag{25}$$

Then the Hankel determinants have the following values

$$\det(\mu_{i+j})_{i,j=0}^{n} = t_0^n t_1^{n-1}\cdots t_{n-1}$$

and

$$\det(\mu_{i+j+1})_{i,j=0}^{n} = d_{n+1} t_0^n t_1^{n-1}\cdots t_{n-1},$$

where $d_n = s_{n-1}d_{n-1} - t_{n-2}d_{n-2}, d_0 = 1, d_1 = s_0.$

Let F *be the linear functional defined by* $F(z^n) = \mu_n.$

Then the polynomials $p_n(z),$ *defined by*

$p_0(z) = 1, p_1(z) = z - s_0,$

and $p_k(z) = (z - s_{k-1})p_{k-1}(z) - t_{k-2}p_{k-2}(z)$

satisfy $F(p_np_m) = t_0\cdots t_{n-1}[n=m],$

i.e. are orthogonal with respect to the linear functional $F.$

Proof of the Theorem
Using (14) we can write (12) as

$$f(z,a,b,q) = 1 + (a+b)zf(z,a,b,q) + (a+b)bqz^2 f(z,a,b,q)g(qz,a,b,q), \tag{26}$$

where the series $g(z,a,b,q)$ satisfies

$$g(z,a,b,q) = 1 + (a+b)zg(z,a,b,q) + qbzg(qz,a,b,q) + qb(a+b)z^2 g(z,a,b,q)g(qz,a,b,q).$$

This is equivalent with

$$g(z,a,b,q) = 1 + (a+b+qb)zg(z,a,b,q) + q^2 b(a+qb)z^2 g(z,a,b,q)g(qz,a,qb,q). \qquad (27)$$

For (15) implies

$$qbzg(qz,a,b,q)(1+(a+b)zg(z,a,b,q)) = qbzg(qz,a,b,q)\,f(z,a,b,q)$$
$$= qbzg(z,a,b,q)\,f(qz,a,qb,q) = qbzg(z,a,b,q)(1+(a+qb)qzg(qz,a,qb,q)).$$

Now (27) is equivalent with

$$g(z,a,b,q) = \frac{1}{1 - s_1 z - t_1 z^2 g(qz,a,qb,q)}$$

with $s_1 = (a+b+qb)$ and $t_1 = q^2 b(a+qb)$.

This gives us a representation of $g(z,a,b,q)$ as a continued fraction of the form

$$\frac{1}{1-s_1 z -}\frac{t_1 z^2}{1-s_2 z -}\frac{t_2 z^2}{1-s_3 z -}\cdots$$

with

$s_n = q^{n-1}(a + q^{n-1}b + q^n b)$ and $t_n = q^{3n-1}b(q^n b + a)$.

From (26) we conclude that $f(z,a,b,q)$ has a representation as a continued fraction of the form (25) with the same s_n, t_n together with $s_0 = a+b$ and $t_0 = q(a+b)b.$

From this the theorem immediately follows.

It is clear that the sequence $\left(C_n(a,b,q)\right)$ is uniquely determined by these determinants.

Remark: It can be shown that the corresponding orthogonal polynomials are

$$p_n(z,a,b,q) = \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} z^k \sum_{j=k}^{n} q^{\binom{n+1}{2} - \binom{n+k+1-j}{2}} \begin{bmatrix} n+k-j \\ k \end{bmatrix} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix} b^{j-k} (a \underset{\cdot}{+} b)^{n-j}$$

and

$$p_n(z,0,1,q) = \sum_{k=0}^{n} (-1)^{n-k} q^{2\binom{n-k}{2}} \begin{bmatrix} n+k \\ 2k \end{bmatrix} z^k.$$

But we don't need this result.

c) Let now

$$h^*(z,a,b,q) := \sum_{k \geq 0} q^{\binom{k}{2}} \frac{r_k(a,b)}{(1 \underset{\cdot}{-} q)^k} (-z)^k, \qquad (28)$$

where $r_n(a,b,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a^k b^{n-k}$ is a Rogers-Szegö polynomial, which satisfies

$r_n(a,b,q) = (a+b)r_{n-1}(a,b,q) + ab(q^{n-1}-1)r_{n-2}(a,b,q)$ (cf. e.g. [4]).

Therefore we get

$$h^*(qz,a,b,q) - h^*(z,a,b,q) = -z(a+b)h^*(qz,a,b,q) - qabz^2 h^*(q^2 z,a,b,q).$$

If we define

$$f^*(z,a,b,q)=\frac{h^*(qz,a,b,q)}{h^*(z,a,b,q)}, \tag{29}$$

we see that $f^*(z,a,b,q)$ satisfies the functional equation

$$f^*(z,a,b,q)=1+(a+b)zf^*(z,a,b,q)+qabz^2 f^*(z,a,b,q)\,f^*(qz,a,b,q). \tag{30}$$

It is easy to see (compare (17) and (27)) that the series $f^*(z,a,b)=g(z,a-b,b,1)$ satisfying
$f^*(z,a,b)=1+(a+b)zf^*(z,a,b)+abz^2 f^*(z,a,b)^2$
has the expansion
$azf^*(z,a,b)=\sum_{n\geq 1}(\sum_{k=1}^{n}N(n,k)a^k b^{n-k})z^n$.
Therefore we write $af^*(z,a,b,q)=\sum_{n\geq 0}C^*_{n+1}(a,b,q)z^n$
with

$$C^*_n(a,b,q)=\sum_{k=1}^{n}N^*(n,k,q)a^k b^{n-k}. \tag{31}$$

This implies $N^*(n,k,0)=\binom{n-1}{k-1}$.
The first values of $(N^*(n,k,q))_{k=1}^{n}, n\geq 1,$ are

1				
1	1			
1	$2+q$	1		
1	$3+2q+q^2$	$3+2q+q^2$	1	
1	$4+3q+2q^2+q^3$	$6+6q+5q^2+2q^3+q^4$	$4+3q+2q^2+q^3$	1

Computer experiments suggest that $N^*(n,k,q)$ satisfies the minimal recurrence relation
$(E-1)^k(E-q)^{\left\lfloor \frac{k}{2}\right\rfloor}(E-q^2)^{\left\lfloor \frac{k}{3}\right\rfloor}(E-q^3)^{\left\lfloor \frac{k}{4}\right\rfloor}\cdots(E-q^{n-1})N^*(n,k,q)=0.$

Since $f^*(z,a,b)=f^*(z,b,a)$ we see that $N^*(n,k,q)=N^*(n,n-k+1,q).$

Let now $F(z,a,b,q)=1+azf^*(z,a,b,q).$ Then by (30)

$$F(z,a,b,q)=1+azF(z,a,b,q)-bzF(qz,a,b,q)+bzF(z,a,b,q)F(qz,a,b,q) \tag{32}$$

If we set $C^*_0(a,b,q)=1$, then (32) implies the recurrence

$$C^*_n(a,b,q)=aC^*_{n-1}(a,b,q)+b\sum_{k=0}^{n-2}q^k C^*_k(a,b,q)C^*_{n-1-k}(a,b,q). \tag{33}$$

Comparing with [8] (5.5) we see that $C^*_n(1,s,q)$ are the Pólya-Gessel $q-$Catalan numbers, the first values of which are

$1,1,1+s,1+2s+qs+s^2,1+3s+2qs+q^2s+3s^2+2qs^2+q^2s^2+s^3,\cdots$

By (30) the corresponding s_k, t_k are given by

$s_k = q^k(a+b)$ and $t_k = q^{2k+1}ab.$ Therefore their Hankel determinants are

$$\det(C^*_{i+j+1}(a,b,q))_{i,j=0}^n = (ab)^{\binom{n+1}{2}} q^{\sum_{i=0}^n i^2} = (ab)^{\binom{n+1}{2}} q^{\frac{n(n+1)(2n+1)}{6}} \tag{34}$$

and

$$\det(C^*_{i+j+2}(a,b,q))_{i,j=0}^n = (abq)^{\binom{n+1}{2}} q^{\frac{n(n+1)(2n+1)}{6}} \frac{a^{n+2}-b^{n+2}}{a-b}. \tag{35}$$

It is easy to verify that the corresponding orthogonal polynomials are

$$p_n(z,a,b) = \sum_{k=0}^n (-1)^{n-k} q^{\binom{n-k}{2}} z^k \sum_{j=k}^n \begin{bmatrix} n+k-j \\ k \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} a^{j-k} b^{n-j}.$$

In order to compute

$\det(C^*_{i+j}(a,b,q))_{i,j=0}^n$

we observe that from (32) we get

$$F(z,a,b,q) = \frac{1-bzF(qz,a,b,q)}{1-bzF(qz,a,b,q)-az} = \frac{1}{1-\dfrac{az}{1-bzF(qz,a,b,q)}} = \frac{1}{1-}\frac{az}{1-}\frac{bz}{1-}\frac{qaz}{1-}\cdots =$$

$$= \frac{1}{1-azF(z,b,qa,q)}$$

Therefore we have

$F(z,a,b,q) = 1+azF(z,a,b,q)F(z,b,qa,q)$

or

$F(z,a,b,q) = 1+azF(z,a,b,q)+abz^2F(z,a,b,q)f^*(z,b,qa,q)$

This gives us $t_k = q^{2k}ab$ for all $k \ge 0$ and therefore we have

$$\det(C^*_{i+j}(a,b,q))_{i,j=0}^n = (ab)^{\binom{n+1}{2}} q^{\frac{n(n+1)(n-1)}{3}}.$$

In this case the orthogonal polynomials are

$$p_n(z,a,b) = \sum_{k=0}^n (-1)^{n-k} q^{\binom{n-k}{2}} z^k \sum_{j=k}^n \begin{bmatrix} n+k-j \\ k \end{bmatrix} \begin{bmatrix} j-1 \\ j-k \end{bmatrix} b^{j-k} a^{n-j}.$$

For the special case $(a,b)=(1,s)$ these results have been proved by other methods in [7].

The generating function $f(z,q)$ of the $q-$Catalan numbers $C_n(q)$ has the continued fraction expansion

$$f(z,q) = \frac{1}{1-zf(qz,q)} = \frac{1}{1-\dfrac{z}{1-qzf(q^2z,q)}} = \frac{1-qzf(q^2z,q)}{1-qzf(q^2z,q)-z}.$$

Therefore we get

$f(z,q) = 1+zf(z,q)-qzf(q^2z,q)+qzf(z,q)f(q^2z,q).$

This implies $f(z,q) = F(z,1,q,q^2)$ and thus the well known (cf. [8]) result that $C_n(q) = C_n^*(1,q,q^2)$.

This can also directly be seen:

$$h^*(z,1,q,q^2) = \sum_{k=0}^{n} q^{2\binom{k}{2}} \left(\sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} q^k \right) \frac{(-z)^k}{(1-q^2)^k} = E_2(-z),$$

since

$$\sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} q^k = (1+q)^k \text{ (cf. [4]).}$$

From Gauss's formula $r_{2n+1}(1,-1) = 0, r_{2n}(1,-1 = (1-q)(1-q^3)\cdots(1-q^{2n-1})$ (cf e.g.[4]) we conclude in the same way that

$$C_{2n+1}^*(1,-1,q) = (-1)^n q^n C_n(q^2)$$

and $C_{2n+2}^*(1,-1,q) = 0.$

d) Another interesting special case is given by the $q-$Motzkin numbers $M_n(q)$ which have been considered in [6]. Their generating function $M(z) = \sum_{n \geq 0} M_n(q) z^n$ satisfies $M(z) = 1 + zM(z) + qz^2 M(z) M(qz).$ Therefore $M_n(q) = \frac{1-\sqrt{-3}}{2} C_n^*(\frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}, q)$.

The first values are

$1,1,1+q,1+2q+q^2,1+3q+3q^2+q^3+q^4,\cdots.$

The Hankel determinants are easily seen to be

$$\det(M_{i+j}(q))_{i,j=0}^n = q^{\frac{n(n+1)(2n+1)}{6}}$$

and

$$\det(M_{i+j+1}(q))_{i,j=0}^n = q^{\frac{n(n+1)(2n+1)}{6} + \binom{n+1}{2}} d_{n+1} = q^{2\binom{n+2}{3}} d_{n+1}.$$

Here $(d_n)_{n \geq 0} = (1,1,0,-1,-1,0,1,1,0,\cdots)$ is periodic with period 6.

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