Some remarks on q-Fibonacci polynomials and q-Catalan numbers

Johann Cigler

Abstract.

The moments of the Lucas polynomials are central binomial coefficients and the moments of the Fibonacci polynomials are Catalan numbers. This note gives simple proofs for special $q-$analogs of these results.

1. Some background material

The Fibonacci polynomials $F_n(x)=\sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^j\binom{n-j}{j}x^{n-2j}$ *are orthogonal with respect to the linear functional* Λ *defined by* $\Lambda(F_n)=[n=0].$ *They satisfy* $F_n(x)=xF_{n-1}(x)-F_{n-2}(x)$ *with initial values* $F_0(x)=1$ *and* $F_1(x)=x.$ *The corresponding moments are the Catalan numbers* $\Lambda\left(x^{2n}\right)=C_n=\binom{2n}{n}\frac{1}{n+1}$ *and* $\Lambda\left(x^{2n+1}\right)=0.$

The Lucas polynomials $L_n(x)$ *satisfy the same recurrence* $L_n(x)=xL_{n-1}(x)-L_{n-2}(x)$ *but with initial values* $L_0(x)=2$ *and* $L_1(x)=x.$ *For* $n\geq 2$ *we have*
$L_n(x)=\sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^j\binom{n-j}{j}\frac{n}{n-j}x^{n-2j}=F_n(x)-F_{n-2}(x).$ *The polynomials* $l_n(x)=L_n(x)$ *for* $n>0$ *and* $l_0(x)=1$ *are orthogonal with respect to the linear functional* M *defined by*

$\mathrm{M}\left(l_n\right)=[n=0]$ *with moments* $\mathrm{M}\left(x^{2n}\right)=\binom{2n}{n}$ *and* $\mathrm{M}\left(x^{2n+1}\right)=0.$

We want to give a simple approach to a special $q-$analog of these results.

In order to make the paper self-contained let me first recall some well-known definitions and facts about $q-$analogs of numbers and polynomials. Here q can be either an indeterminate or a real number with $|q|<1.$ We use the standard notations $[n]=[n]_q=1+q+\cdots+q^{n-1},$
$[n]!=[1][2]\cdots[n]$ and $\begin{bmatrix}n\\k\end{bmatrix}=\frac{[n]!}{[k]![n-k]!}$ for $0\leq k\leq n$ and $\begin{bmatrix}n\\k\end{bmatrix}=0$ else.

The $q-$binomial coefficients satisfy

$$\begin{bmatrix}n\\k\end{bmatrix}=q^k\begin{bmatrix}n-1\\k\end{bmatrix}+\begin{bmatrix}n-1\\k-1\end{bmatrix}=\begin{bmatrix}n-1\\k\end{bmatrix}+q^{n-k}\begin{bmatrix}n-1\\k-1\end{bmatrix}. \tag{1}$$

Let D be the $q-$differentiation operator on $\mathbb{C}(q)[x]$ defined by

$$Dp(x)=\frac{p(x)-p(qx)}{(1-q)x} \tag{2}$$

or equivalently by $Dx^n = [n]x^{n-1}$ for $n \in \mathbb{N}.$

Natural $q-$analogs of $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ are the Rogers-Szegö polynomials

$$r_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}. \tag{3}$$

They satisfy

$$Dr_n(x,y) = [n] r_{n-1}(x,y) \tag{4}$$

because

$$Dr_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} Dx^k y^{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} [k] x^{k-1} y^{n-k} = [n] \sum_{k=0}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^{k-1} y^{n-k} = [n] \sum_{k} \begin{bmatrix} n-1 \\ k \end{bmatrix} x^k y^{n-1-k}$$
$$= [n] r_{n-1}(x,y).$$

They also satisfy the recurrence

$$r_n(x,y) = \left(x + y\varepsilon\right)^n 1 \tag{5}$$

where ε denotes the linear operator on the polynomials defined by $\varepsilon p(x) = p(qx).$

This follows from

$$\left(x + y\varepsilon\right) \sum_{k} \begin{bmatrix} n-1 \\ k \end{bmatrix} x^k y^{n-1-k} = \sum_{k} \begin{bmatrix} n-1 \\ k \end{bmatrix} x^{k+1} y^{n-1-k} + \sum_{k} \begin{bmatrix} n-1 \\ k \end{bmatrix} q^k x^k y^{n-k}$$
$$= \sum_{k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^k y^{n-k} + \sum_{k} \begin{bmatrix} n-1 \\ k \end{bmatrix} q^k x^k y^{n-k} = \sum_{k} \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}.$$

Let us note that

$$\varepsilon = 1 + (q-1)xD \tag{6}$$

since $\left(1 + (q-1)xD\right) x^n = x^n + \left(q^n - 1\right) x^n = (qx)^n = \varepsilon x^n.$

Using (4), (5) and (6) we get the well-known recurrence

$$r_n(x,y) = \left(x + y\left(1 + (q-1)xD\right)\right) r_{n-1}(x,y) = (x+y) r_{n-1}(x,y) + \left(q^{n-1} - 1\right) xy r_{n-2}(x,y). \tag{7}$$

L. Carlitz [2] introduced the $q-$Fibonacci polynomials $f_n(x;q) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j q^{2\binom{j}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} x^{n-2j}$ which satisfy $f_n(x;q) = x f_{n-1}(x;q) - q^{n-2} f_{n-2}(x;q)$ with initial values $f_0(x;q) = 1$ and $f_1(x;q) = x.$

These polynomials are orthogonal with respect to the linear functional λ defined by $\lambda(f_n(x;q)) = [n=0]$ and their moments are the Carlitz $q-$Catalan numbers $c(n)$ which satisfy

$c(n)=\sum_{j=0}^{n-1} q^j c(j)c(n-1-j)$ with $c(0)=1$ (cf.[8]). Unfortunately, no closed formula for $c(n)$ is known.

A natural $q-$analog of C_n is $C_n(q)=\frac{1}{[n+1]}\begin{bmatrix}2n\\ n\end{bmatrix}$. But for the orthogonal polynomials whose moments are $C_n(q)$ neither closed formulas nor recursions are known.

A precise analog with closed formulas exists if we consider $q-$Chebyshev polynomials instead of $q-$Fibonacci polynomials (cf.[4]).

If we dispense with orthogonality there are also nice $q-$analogs for the Fibonacci and Lucas polynomials with closed formulas and recursions. In this note we give a new approach to these results with simplified proofs.

2. The main results

Let X be the multiplication operator with x and $X(q)=X+(1-q)D$ be the linear operator defined by $X(q)p(x)=xp(x)+(1-q)Dp(x)=xp(x)+\frac{p(x)-p(qx)}{x}$ on $\mathbb{C}(q)[x]$.

Theorem 1 ([1],[3],[4])

The $q-$Fibonacci polynomials

$$F_n(x;q)=\sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^j q^{\binom{j+1}{2}}\begin{bmatrix}n-j\\ j\end{bmatrix}x^{n-2j} \tag{8}$$

satisfy the recursion

$$F_n(x;q)=X(q)F_{n-1}(x;q)-F_{n-2}(x;q) \tag{9}$$

with initial values $F_0(x;q)=1$ and $F_1(x;q)=x.$

Let Λ be the linear functional defined by

$$\Lambda\left(F_n(x;q)\right)=[n=0]. \tag{10}$$

Then we get

$$\Lambda\left(x^{2n}\right)=q^n\frac{1}{[n+1]}\begin{bmatrix}2n\\ n\end{bmatrix} \tag{11}$$

and $\Lambda\left(x^{2n+1}\right)=0.$

Theorem 2 ([1],[3],[4])

The $q-$ *Lucas polynomials*

$$L_n(x;q)=\sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^j q^{\binom{j}{2}}\begin{bmatrix} n-j \\ j \end{bmatrix}\frac{[n]}{[n-j]}x^{n-2j} \tag{12}$$

satisfy the recursion

$$L_n(x;q)=X(q)L_{n-1}(x;q)-L_{n-2}(x;q) \tag{13}$$

with initial values $L_0(x;q)=2$ *and* $L_1(x;q)=x.$ *Let* $l_n(x;q)=L_n(x;q)$ *for* $n>0$ *and* $l_0(x;q)=1$ *and let* M *be the linear functional defined by*

$$\mathrm{M}\left(l_n(x;q)\right)=[n=0]. \tag{14}$$

Then we get

$$\mathrm{M}\left(x^{2n}\right)=\begin{bmatrix} 2n \\ n \end{bmatrix} \tag{15}$$

and $\mathrm{M}\left(x^{2n+1}\right)=0.$

3. Proofs

Identity (9) follows from

$$F_n(x;q)-XF_{n-1}(x;q)+F_{n-2}(x;q)=\sum_j(-1)^j q^{\binom{j+1}{2}}\begin{bmatrix} n-j \\ j \end{bmatrix}x^{n-2j}-\sum_j(-1)^j q^{\binom{j+1}{2}}\begin{bmatrix} n-1-j \\ j \end{bmatrix}x^{n-2j}$$
$$+\sum_j(-1)^j q^{\binom{j+1}{2}}\begin{bmatrix} n-2-j \\ j \end{bmatrix}x^{n-2-2j}=\sum_j(-1)^j q^{\binom{j+1}{2}}x^{n-2j}\left(\begin{bmatrix} n-j \\ j \end{bmatrix}-\begin{bmatrix} n-1-j \\ j \end{bmatrix}-q^{-j}\begin{bmatrix} n-1-j \\ j-1 \end{bmatrix}\right)$$
$$=\sum_j(-1)^j q^{\binom{j}{2}}x^{n-2j}\left(q^{n-j}-1\right)\begin{bmatrix} n-1-j \\ j-1 \end{bmatrix}=\sum_j(-1)^j q^{\binom{j}{2}}x^{n-2j}\left(q^{n+1-2j}-1\right)\begin{bmatrix} n-j \\ j-1 \end{bmatrix}$$
$$=\sum_j(-1)^j q^{\binom{j+1}{2}}x^{n-2-2j}\left(1-q^{n-1-2j}\right)\begin{bmatrix} n-1-j \\ j \end{bmatrix}=(1-q)D\sum_j(-1)^j q^{\binom{j+1}{2}}\begin{bmatrix} n-1-j \\ j \end{bmatrix}x^{n-1-2j}=(1-q)DF_{n-1}(x;q).$$

This implies that $F_n(x;q)=F_n\left(X(q)\right)1.$ More precisely let Φ be the linear operator defined by $\Phi\left(x^n\right)=X(q)^n1.$ Then $F_n(x;q)=\Phi\left(F_n(x)\right).$

This is true for $n=0$ and $n=1.$ By induction we get
$F_n(x;q)=X(q)F_{n-1}(X(q))1-F_{n-2}(X(q))1=\left(X(q)F_{n-1}(X(q))-F_{n-2}(X(q))\right)1=F_n(X(q))1.$

The corresponding $q-$Lucas polynomials satisfy

$$L_n(x;q)=F_n(x;q)-F_{n-2}(x;q) \tag{16}$$

for $n\geq 2$ because

$$F_n(x;q)-F_{n-2}(x;q)=\sum_j(-1)^j q^{\binom{j+1}{2}}\begin{bmatrix}n-j\\ j\end{bmatrix}x^{n-2j}+\sum_j(-1)^j q^{\binom{j}{2}}\begin{bmatrix}n-1-j\\ j-1\end{bmatrix}x^{n-2j}$$
$$=\sum_j(-1)^j q^{\binom{j}{2}}\frac{[n]}{[n-j]}\begin{bmatrix}n-j\\ j\end{bmatrix}x^{n-2j}=L_n(x;q).$$

By linearity it is clear that we also have

$$L_n(x;q)=X(q)L_{n-1}(x;q)-L_{n-2}(x;q). \tag{17}$$

Binet's formulas give $F_n(x)=\dfrac{\lambda(x)^{n+1}-\mu(x)^{n+1}}{\lambda(x)-\mu(x)}$ and $L_n(x)=\lambda(x)^n+\mu(x)^n$ with $\lambda(x)=\dfrac{x+\sqrt{x^2-4}}{2}$ and $\mu(x)=\dfrac{x-\sqrt{x^2-4}}{2}$.

Therefore we also have $F_n(x;q)=\Phi\left(\dfrac{\lambda(x)^{n+1}-\mu(x)^{n+1}}{\lambda(x)-\mu(x)}\right)$ and $L_n(x;q)=\Phi\left(\lambda(x)^n+\mu(x)^n\right)$.

Note that we need not define $\Phi\left(\lambda(x)\right)$ and $\Phi\left(\mu(x)\right)$.

Lemma 3

Let

$$\sum_{k=0}^{\left[\frac{n}{2}\right]}\begin{bmatrix}n\\ k\end{bmatrix}l_{n-2k}(x)=H_n(x;q). \tag{18}$$

Then

$$H_n(x;q)=xH_{n-1}(x;q)+\left(q^{n-1}-1\right)H_{n-2}(x;q) \tag{19}$$

with $H_0(x;q)=1$.

Proof

$$H_n(x;q)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\begin{bmatrix}n\\ k\end{bmatrix}l_{n-2k}(x)=\sum_k\left(\begin{bmatrix}n-1\\ k\end{bmatrix}+q^{n-k}\begin{bmatrix}n-1\\ k-1\end{bmatrix}\right)l_{n-2k}(x)$$
$$=\sum_k\begin{bmatrix}n-1\\ k\end{bmatrix}l_{n-2k}(x)+\sum_k\begin{bmatrix}n-1\\ k\end{bmatrix}q^{n-k-1}l_{n-2-2k}(x)$$

$$=\sum_k \begin{bmatrix} n-1 \\ k \end{bmatrix}\left(l_{n-2k}(x)+l_{n-2-2k}(x)\right)+\sum_k\left(q^{n-1}-1\right)\begin{bmatrix} n-2 \\ k \end{bmatrix} l_{n-2-2k}(x)$$
$$=x\sum_k \begin{bmatrix} n-1 \\ k \end{bmatrix} l_{n-1-2k}(x)+\left(q^{n-1}-1\right)\sum_k \begin{bmatrix} n-2 \\ k \end{bmatrix} l_{n-2-2k}(x)=xH_n(x;q)+\left(q^{n-1}-1\right)H_{n-2}(x;q).$$

Note that $H_n(x;q)$ is a variant of the continuous $q-$Hermite polynomials (cf. [6],[7]) and that $H_n(x,0)=F_n(x)$ and $H_n(x;1)=x^n.$

By (7) and (19) we get

Lemma 4

$$H_n(x;q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \lambda(x)^k \mu(x)^{n-k}. \tag{20}$$

Lemma 5

$$\Phi\left(H_n(x;q)\right)=H_n\left(X(q);q\right)1=x^n. \tag{21}$$

Proof

(21) is true for $n=0$ and $n=1.$ By induction we get

$$H_n\left(X(q);q\right)1=X(q)H_{n-1}\left(X(q);q\right)1+\left(q^{n-1}-1\right)H_{n-2}\left(X(q);q\right)1=X(q)x^{n-1}+\left(q^{n-1}-1\right)x^{n-2}$$
$$=\left(x+(1-q)D\right)x^{n-1}+\left(q^{n-1}-1\right)x^{n-2}=x^n+\left(1-q^{n-1}\right)x^{n-2}+\left(q^{n-1}-1\right)x^{n-2}=x^n.$$

Corollary 6

$$\sum_{k=0}^{\left[\frac{n}{2}\right]}\begin{bmatrix} n \\ k \end{bmatrix} l_{n-2k}(x;q)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\begin{bmatrix} n \\ k \end{bmatrix} l_{n-2k}(X(q))1=H_n(X(q);q)1=x^n. \tag{22}$$

For $q=1$ we get the well-known formula $\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{k} l_{n-2k}(x)=x^n.$

We also have

$$\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{k} l_{n-2k}\left(x;q\right)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{k} l_{n-2k}\left(X(q)\right)1=\left(\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{k} l_{n-2k}\left(X(q)\right)\right)1=X(q)^n 1. \tag{23}$$

Let M be the linear functional defined by $\mathrm{M}\left(l_n(x;q)\right)=[n=0].$ Then we get

$\mathrm{M}\left(x^{2n}\right)=\begin{bmatrix} 2n \\ n \end{bmatrix}$ and $\mathrm{M}\left(x^{2n+1}\right)=0.$

Corollary 7

$$\sum_k \left(\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix} \right) F_k(x;q) = x^n. \qquad (24)$$

Proof

$$x^n = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} l_{n-2k}(x;q) = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \left(F_{n-2k}(x;q) - F_{n-2-2k}(x;q) \right) = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} F_{n-2k}(x;q) - \sum_k \begin{bmatrix} n \\ k-1 \end{bmatrix} F_{n-2k}(x;q)$$
$$= \sum_k \left(\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix} \right) F_{n-2k}(x;q).$$

Corollary 8

Let Λ *be the linear functional defined by* $\Lambda\left(F_n(x;q)\right) = [n=0].$ *Then we get*

$\Lambda\left(x^{2n}\right) = q^n \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}$ *and* $\Lambda\left(x^{2n+1}\right) = 0.$

Remark

Formulas (22) and (24) can also be obtained with the inversion formulas by L. Carlitz [1].

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