Some elementary q-identities

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Abstract

We study $q-$analogues of two well-known polynomial identities. In some cases we get simple results which lead to another approach to Bressoud's polynomial version of the Rogers-Ramanujan identities.

1. Introduction

In the first part of this paper we collect some simple $q-$analogues of the identities

$$a(n,t)=\sum_{j\in\mathbb{Z}}(-1)^j\sum_{\ell\geq|j|}\binom{\left\lfloor \frac{n}{2}\right\rfloor - j}{\ell-j}\binom{\left\lfloor \frac{n+1}{2}\right\rfloor + j}{\ell+j}t^\ell=(1-t)^{\left\lfloor \frac{n+1}{2}\right\rfloor} \tag{1.1}$$

and

$$b(n,t)=\sum_{j\in\mathbb{Z}}(-1)^j\sum_{\ell\geq 0}\binom{\left\lfloor \frac{n}{2}\right\rfloor}{\ell-j}\binom{\left\lfloor \frac{n+1}{2}\right\rfloor}{\ell+j}t^\ell=(1+t)^{\left\lfloor \frac{n}{2}\right\rfloor}. \tag{1.2}$$

Most of them seem to be known, but I hope that the present approach gives some new insight.

Let

$$r(n,\ell)=\sum_{j=-\ell}^{\ell}(-1)^{\ell-j}\binom{\left\lfloor \frac{n}{2}\right\rfloor - j}{\ell-j}\binom{\left\lfloor \frac{n+1)}{2}\right\rfloor + j}{\ell+j} \tag{1.3}$$

be the coefficient of $(-1)^\ell t^\ell$ of $a(n,t)$ and

$$s(n,\ell)=\sum_{j=-\ell}^{\ell}(-1)^{j}\binom{\left\lfloor \frac{n}{2}\right\rfloor}{\ell-j}\binom{\left\lfloor \frac{n+1}{2}\right\rfloor}{\ell+j} \tag{1.4}$$

be the coefficient of t^ℓ of $b(n,t)$.

By comparing coefficients we get the well-known results

$$\begin{aligned} r(2n,\ell) &= r(2n-1,\ell) = \binom{n}{\ell}, \\ s(2n,\ell) &= s(2n+1,\ell) = \binom{n}{\ell}. \end{aligned} \tag{1.5}$$

Let $\begin{bmatrix} n \\ k \end{bmatrix}_q$ be a $q-$binomial coefficient. I have looked for exponents $a(j)$ and $b(j)$ such that the $q-$analogues

$$a(n,t,q) = \sum_{j\in\mathbb{Z}} (-1)^j q^{a(j)} \sum_{\ell=|j|}^{n} q^{(\ell-j)(\ell+j)} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor - j \\ \ell - j \end{bmatrix}_q \begin{bmatrix} \left\lfloor \frac{n+1}{2} \right\rfloor + j \\ \ell + j \end{bmatrix}_q t^{\ell} \tag{1.6}$$

and

$$b(n,t,q) = \sum_{j\in\mathbb{Z}} (-1)^j q^{b(j)} \sum_{\ell=|j|}^{n} q^{(\ell-j)(\ell+j)} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor \\ \ell - j \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n+1}{2} \right\rfloor \\ \ell + j \end{bmatrix} t^{\ell} \tag{1.7}$$

have nice evaluations. Previously P. Paule [8] found a large number of exponents $a(j)$ for which the special cases

$$a(n,1,q) = \sum_{j\in\mathbb{Z}} (-1)^j q^{a(j)} \begin{bmatrix} n \\ \left\lfloor \frac{n}{2} \right\rfloor + j \end{bmatrix} \tag{1.8}$$

have simple evaluations. Thus the present paper can be considered as an analogous search for arbitrary t.

As a by-product we get a natural framework for the polynomial versions of the Rogers-Ramanujan identities which have been found by David Bressoud [2] and simplified by R. Chapman [3] , G.E. Andrews and K. Eriksson [1] and in [6].

Finally we state some more general results for polynomials with exponents $a(j) = (r+1)j^2 + rj$ and $b(j) = (r+2)j^2 - rj.$

The exposition of the paper is elementary and does not need any knowledge of $q-$calculus.

We may assume that q is a real number with $|q|<1$ or alternatively an indeterminate. We use the following notations:

For $n\in\mathbb{N}$ let $(x;q)_n=(1-x)(1-qx)\cdots(1-q^{n-1}x)$ and $(x;q)_\infty=(1-x)(1-qx)(1-q^2x)\cdots.$

Let $\begin{bmatrix} x \\ k \end{bmatrix}=\begin{bmatrix} x \\ k \end{bmatrix}_q$ be a $q-$binomial coefficient defined by $\begin{bmatrix} x \\ k \end{bmatrix}=\prod_{j=0}^{k-1}\frac{1-q^{x-j}}{1-q^{k-j}}$ for $x\in\mathbb{R}$ and $k\in\mathbb{N}.$

We let $\begin{bmatrix} x \\ k \end{bmatrix}=0$ for $k<0.$ Note that $\begin{bmatrix} x \\ k \end{bmatrix}_q$ is a polynomial in $z=q^x$ of degree $k.$

As is well known and easily verified the $q-$binomial coefficients satisfy the recursions

$$\begin{bmatrix} x \\ k \end{bmatrix}=q^k\begin{bmatrix} x-1 \\ k \end{bmatrix}+\begin{bmatrix} x-1 \\ k-1 \end{bmatrix}=\begin{bmatrix} x-1 \\ k \end{bmatrix}+q^{x-k}\begin{bmatrix} x-1 \\ k-1 \end{bmatrix}. \tag{1.9}$$

We will make use of the simplest $q-$binomial theorem

$$(x;q)_n=\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}x^k \tag{1.10}$$

and the $q-$Chu-Vandermonde identity

$$\sum_k\begin{bmatrix} x \\ k \end{bmatrix}\begin{bmatrix} y \\ n-k \end{bmatrix}q^{k(y-n+k)}=\begin{bmatrix} x+y \\ n \end{bmatrix}, \tag{1.11}$$

which are easily proved by induction using (1.9).

To obtain the Rogers-Ramanujan identities we need to consider limits. For example for $|q|<1$ it is clear that $\lim_{n\to\infty}\begin{bmatrix} n \\ k \end{bmatrix}_q=\lim_{n\to\infty}\frac{(1-q^n)\cdots(1-q^{n-k+1})}{(1-q)\cdots(1-q^k)}=\frac{1}{(1-q)\cdots(1-q^k)}=\frac{1}{(q;q)_k}.$

If q is an indeterminate we consider limit relations in the ring of formal power series. This means that $\lim_{n\to\infty}\sum_{k\geq 0}a_{n,k}q^k=\sum_{k\geq 0}a_kq^k$ if for each K there exists N such that for $n\geq N$ $a_{n,k}=a_k$ for all k satisfying $0\leq k\leq K.$

For example $\lim_{n\to\infty}\begin{bmatrix} n \\ k \end{bmatrix}=\frac{1}{(q;q)_k}$ because $(q;q)_k\begin{bmatrix} n \\ k \end{bmatrix}=\sum_{j=0}^{k}(-1)^j\begin{bmatrix} k \\ j \end{bmatrix}q^{\frac{j(j+1-2k)}{2}}(q^n)^j=1+c_n(q)q^n$ for some formal power series $c_n(q).$

Let

$$r(n,\ell,q)=q^{\ell^2}\sum_{j=-\ell}^{\ell}(-1)^{\ell-j}q^{a(j)-j^2}\begin{bmatrix}\left\lfloor \frac{n}{2}\right\rfloor - j\\ \ell-j\end{bmatrix}\begin{bmatrix}\left\lfloor \frac{n+1}{2}\right\rfloor + j\\ \ell+j\end{bmatrix} \tag{1.12}$$

be the coefficient of $(-t)^{\ell}$.

For $t=1$ (2.1) gives

$$a(n,1,q)=\sum_{j\in\mathbb{Z}}(-1)^j q^{a(j)}\begin{bmatrix}n\\ \left\lfloor \frac{n}{2}\right\rfloor + j\end{bmatrix}. \tag{1.13}$$

This follows from the $q-$Chu-Vandermonde identity $\sum_k\begin{bmatrix}x\\ k\end{bmatrix}\begin{bmatrix}y\\ n-k\end{bmatrix}q^{k(y-n+k)}=\begin{bmatrix}x+y\\ n\end{bmatrix}$:

$$\sum_{\ell=|j|}^{n}q^{(\ell-j)(\ell+j)}\begin{bmatrix}\left\lfloor \frac{n}{2}\right\rfloor - kj\\ \ell-j\end{bmatrix}\begin{bmatrix}\left\lfloor \frac{n+1}{2}\right\rfloor + kj\\ \ell+j\end{bmatrix}=\sum_{\ell}q^{(\ell-j)(\ell+j)}\begin{bmatrix}\left\lfloor \frac{n}{2}\right\rfloor - kj\\ \ell-j\end{bmatrix}\begin{bmatrix}\left\lfloor \frac{n+1}{2}\right\rfloor + kj\\ \left\lfloor \frac{n+1}{2}\right\rfloor + kj-\ell-j\end{bmatrix}$$

$$=\sum_{i}q^{i(2j+i)}\begin{bmatrix}\left\lfloor \frac{n}{2}\right\rfloor - kj\\ i\end{bmatrix}\begin{bmatrix}\left\lfloor \frac{n+1}{2}\right\rfloor + kj\\ \left\lfloor \frac{n+1}{2}\right\rfloor + kj-2j-i\end{bmatrix}=\begin{bmatrix}n\\ \left\lfloor \frac{n+1}{2}\right\rfloor + kj-2j\end{bmatrix}=\begin{bmatrix}n\\ \left\lfloor \frac{n}{2}\right\rfloor + (2-k)j\end{bmatrix}.$$

If we let $n\to\infty$ in this identity we get the well-known identity

$$\sum_{\ell\ge|j|}\frac{q^{\ell^2-j^2}}{(q;q)_{\ell-j}(q;q)_{\ell+j}}=\frac{1}{(q;q)_\infty} \tag{1.14}$$

where both sides are interpreted as formal power series in q.

2. Product evaluations of $a(n,t,q)$.

In the first part of the paper we are looking for $q-$ analogues of the form

$$a(n,t,q)=\sum_{j\in\mathbb{Z}}(-1)^j q^{a(j)}\sum_{\ell=|j|}^{n} q^{(\ell-j)(\ell+j)}\begin{bmatrix}\left\lfloor \frac{n}{2}\right\rfloor - j\\ \ell-j\end{bmatrix}\begin{bmatrix}\left\lfloor \frac{n+1}{2}\right\rfloor + j\\ \ell+j\end{bmatrix}t^{\ell} \tag{2.1}$$

which are products of linear factors.

Note that $a(0)=0$ and $a(1,t,q)=1-q^{a(1)}t.$

Since $a(2,t,q)=1+qt-q^{a(-1)}t-q^{a(1)}t$ the only possible choices are $a(-1)=1$ or $a(1)=1.$

For $a(-1)=1$ we get $a(3,t,q)=1-t-qt+q^{a(2)}t^2.$

In order to get a product of the desired form we must choose $a(2)=1$ which gives $a(3,t,q)=(1-t)(1-qt).$

Then $a(4,t,q)=1-t-qt+qt^2-q^3t^2+q^{a(-2)}t^2$ which leads to $a(-2)=3$ and gives $a(4,0,t,q)=(1-t)(1-qt).$

Iterating this procedure leads to the conjecture that $a(j)=\binom{j}{2}$ gives a product representation of $a(n,0,t,q)$ for all $n\in\mathbb{N}.$

In the same way we are led from $a(-1)=1$ and $a(1)=1$ to $a(j)=j^2$ and from $a(-1)=1$ and $a(1)=2$ to $a(j)=\frac{j(3j+1)}{2}.$

For $a(1)=c>2$ we get $a(3,0,t,q)=1+q^2t-q^ct(1+q+q^2)+q^{a(2)}t^2$ which cannot be split into two factors of the form $1-q^bt.$

There remains the choice $a(1)=1.$ In this case $a(-1)=0$ leads to $a(j)=\binom{j+1}{2}=\binom{-j}{2}$ and $a(-1)=2$ to $\frac{j(3j-1)}{2}=\frac{-j(-3j+1)}{2}.$

We show all possibilities in the following table.

	$a(-2)$	$a(-1)$	$a(0)$	$a(1)$	$a(2)$
j^2	4	1	0	1	4
$\binom{j}{2}$	3	1	0	0	1
$\binom{j+1}{2}$	1	0	0	1	3
$\frac{j(3j+1)}{2}$	5	1	0	2	7
$\frac{j(3j-1)}{2}$	7	2	0	1	5

Lemma 2.1

If $a(j)=(r+1)j^2+rj$ *then* $a(2n,t,q)=a(2n-1,t,q)$. *Special cases are* $a(j)=\binom{j}{2}$, $a(j)=\frac{j(3j+1)}{2}$ *and* $a(j)=j^2$.

Proof

Comparing the coefficient of t^{ℓ} gives

$$\sum_j (-1)^j q^{a(j)+\ell^2-j^2}\begin{bmatrix} n-j \\ \ell-j \end{bmatrix}\begin{bmatrix} n+j \\ \ell+j \end{bmatrix}-\sum_j (-1)^j q^{a(j)+\ell^2-j^2}\begin{bmatrix} n-j-1 \\ \ell-j \end{bmatrix}\begin{bmatrix} n+j \\ \ell+j \end{bmatrix}$$
$$=\sum_j (-1)^j q^{a(j)+\ell^2-j^2+n-\ell}\begin{bmatrix} n-j-1 \\ \ell-j-1 \end{bmatrix}\begin{bmatrix} n+j \\ \ell+j \end{bmatrix}.$$

The map $j\to -j-1$ is a sign-reversing involution if $a(j)-j^2=a(-j-1)-(j+1)^2$. This is obviously true if $a(j)=(r+1)j^2+rj$.

These considerations lead us to the following identities:

Theorem 2.1

For $n \in \mathbb{N}$ *we get the identity*

$$a_1(n,t,q)=\sum_{j\in\mathbb{Z}}(-1)^j q^{\binom{j}{2}}\sum_{\ell\geq 0} q^{(\ell-j)(\ell+j)}\begin{bmatrix}\left\lfloor \frac{n}{2}\right\rfloor - j\\ \ell-j\end{bmatrix}\begin{bmatrix}\left\lfloor \frac{n+1}{2}\right\rfloor + j\\ \ell+j\end{bmatrix}t^\ell=\prod_{j=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\left(1-q^j t\right). \tag{2.2}$$

It satisfies $a_1(2n,t,q)=a_1(2n-1,t,q)$ *and is equivalent with*

$$\begin{aligned} r_1(2n,\ell,q)&=\sum_j(-1)^{\ell-j}q^{\ell^2-\binom{j+1}{2}}\begin{bmatrix}n-j\\ \ell-j\end{bmatrix}\begin{bmatrix}n+j\\ \ell+j\end{bmatrix}=q^{\binom{\ell}{2}}\begin{bmatrix}n\\ \ell\end{bmatrix},\\ r_1(2n-1,\ell,q)&=\sum_j(-1)^{\ell-j}q^{\ell^2-\binom{j+1}{2}}\begin{bmatrix}n-1-j\\ \ell-j\end{bmatrix}\begin{bmatrix}n+j\\ \ell+j\end{bmatrix}=q^{\binom{\ell}{2}}\begin{bmatrix}n\\ \ell\end{bmatrix}.\end{aligned} \tag{2.3}$$

For $t=1$ (2.2) *reduces to*

$$\sum_{j\in\mathbb{Z}}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}n\\ \left\lfloor \frac{n}{2}\right\rfloor + j\end{bmatrix}=[n=0]. \tag{2.4}$$

Theorem 2.2

For $n \in \mathbb{N}$ *we get the identity*

$$a_2(n,t,q)=\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{j(3j+1)}{2}}\sum_{\ell\geq 0} q^{(\ell-j)(\ell+j)}\begin{bmatrix}\left\lfloor \frac{n}{2}\right\rfloor - j\\ \ell-j\end{bmatrix}\begin{bmatrix}\left\lfloor \frac{n+1}{2}\right\rfloor + j\\ \ell+j\end{bmatrix}t^\ell=\prod_{j=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\left(1-q^{2\left\lfloor \frac{n+1}{2}\right\rfloor - j}t\right). \tag{2.5}$$

It satisfies $a_2(2n,t,q)=a_2(2n-1,t,q)$ *and is equivalent with*

$$\begin{aligned} r_2(2n,\ell,q)&=q^{\ell^2}\sum_j(-1)^{\ell-j}q^{\binom{j+1}{2}}\begin{bmatrix}n-j\\ \ell-j\end{bmatrix}\begin{bmatrix}n+j\\ \ell+j\end{bmatrix}=q^{\ell n+\binom{\ell+1}{2}}\begin{bmatrix}n\\ \ell\end{bmatrix},\\ r_2(2n-1,\ell,q)&=q^{\ell^2}\sum_j(-1)^{\ell-j}q^{\binom{j+1}{2}}\begin{bmatrix}n-1-j\\ \ell-j\end{bmatrix}\begin{bmatrix}n+j\\ \ell+j\end{bmatrix}=q^{\ell n+\binom{\ell+1}{2}}\begin{bmatrix}n\\ \ell\end{bmatrix}.\end{aligned} \tag{2.6}$$

For $t=1$(2.5) *reduces to*

$$\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{j(3j+1)}{2}}\begin{bmatrix}n\\ \left\lfloor \frac{n}{2}\right\rfloor + j\end{bmatrix}=\prod_{j=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\left(1-q^{2\left\lfloor \frac{n+1}{2}\right\rfloor - j}\right). \tag{2.7}$$

Theorem 2.3

For $n \in \mathbb{N}$ *we get the identity*

$$a_3(n,t,q) = \sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(3j-1)}{2}} \sum_{\ell\geq 0} q^{(\ell-j)(\ell+j)} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor - j \\ \ell - j \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n+1}{2} \right\rfloor + j \\ \ell + j \end{bmatrix} t^\ell = \prod_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(1 - q^{n-j} t\right). \quad (2.8)$$

It satisfies $a_3(2n,t,q) = a_3(2n-1,qt,q)$ *and is equivalent with*

$$\begin{aligned} r_3(2n,\ell,q) &= q^{\ell^2} \sum_j (-1)^{\ell-j} q^{\binom{j}{2}} \begin{bmatrix} n-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j \end{bmatrix} = q^{\ell n + \binom{\ell+1}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix}, \\ r_3(2n-1,\ell,q) &= q^{\ell^2} \sum_j (-1)^{\ell-j} q^{\binom{j}{2}} \begin{bmatrix} n-1-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j \end{bmatrix} = q^{\ell(n-1) + \binom{\ell+1}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix}. \end{aligned} \quad (2.9)$$

For $t=1$ (2.8) *reduces to the well-known identity* ([8], (10), (11))

$$\sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(3j-1)}{2}} \begin{bmatrix} n \\ \left\lfloor \frac{n}{2} \right\rfloor + j \end{bmatrix} = \prod_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(1 - q^{n-j}\right) = \frac{(q;q)_n}{(q;q)_{\left\lfloor \frac{n}{2} \right\rfloor}}. \quad (2.10)$$

Theorem 2.4

For $n \in \mathbb{N}$ *we get the identity*

$$a_4(n,t,q) = \sum_{j\in\mathbb{Z}} (-1)^j q^{j^2} \sum_{\ell\geq 0} q^{(\ell-j)(\ell+j)} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor - j \\ \ell - j \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n+1}{2} \right\rfloor + j \\ \ell + j \end{bmatrix} t^\ell = \prod_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(1 - q^{2j+1} t\right). \quad (2.11)$$

It satisfies $a_4(2n,t,q) = a_4(2n-1,t,q)$ *and is equivalent with*

$$\begin{aligned} q^{-\ell^2} r_4(2n,\ell,q) &= \sum_{j=-\ell}^{\ell} (-1)^{\ell-j} \begin{bmatrix} n-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j \end{bmatrix} = \begin{bmatrix} n \\ \ell \end{bmatrix}_{q^2}, \\ q^{-\ell^2} r_4(2n-1,\ell,q) &= \sum_{j=-\ell}^{\ell} (-1)^{\ell-j} \begin{bmatrix} n-1-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j \end{bmatrix} = \begin{bmatrix} n \\ \ell \end{bmatrix}_{q^2}. \end{aligned} \quad (2.12)$$

For $t=1$ (2.11) *reduces to*

$$\sum_{j\in\mathbb{Z}} (-1)^j q^{j^2} \begin{bmatrix} n \\ \left\lfloor \frac{n}{2} \right\rfloor + j \end{bmatrix} = \prod_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(1 - q^{2j+1}\right). \quad (2.13)$$

Theorem 2.5

For $n \in \mathbb{N}$ *we get the identity*

$$a_5(n,t,q) = \sum_{j\in\mathbb{Z}} (-1)^j q^{\binom{j+1}{2}} \sum_{\ell\geq 0} q^{(\ell-j)(\ell+j)} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor - j \\ \ell - j \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n+1}{2} \right\rfloor + j \\ \ell + j \end{bmatrix} t^\ell \tag{2.14}$$

which satisfies

$$\begin{aligned} a_5(2n,t,q) &= \prod_{j=0}^{n-1} \left(1 - q^j t\right), \\ a_5(2n+1,t,q) &= a_5(2n,t,q)\left(1 - q^{2n+1} t\right). \end{aligned} \tag{2.15}$$

It is equivalent with

$$\begin{aligned} r_5(2n,\ell,q) &= q^{\ell^2} \sum_{j=-\ell}^{\ell} (-1)^{\ell-j} q^{-\binom{j}{2}} \begin{bmatrix} n-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j \end{bmatrix} = q^{\binom{\ell}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix}, \\ r_5(2n-1,\ell,q) &= q^{\ell^2} \sum_{j=-\ell}^{\ell} (-1)^{\ell-j} q^{-\binom{j}{2}} \begin{bmatrix} n-1-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j \end{bmatrix} \\ &= q^{\binom{\ell}{2}} \begin{bmatrix} n-1 \\ \ell \end{bmatrix} + q^{2n+\frac{\ell(\ell-3)}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix} = q^{\binom{\ell}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix} \left(1 - q^{n-\ell} + q^n\right). \end{aligned} \tag{2.16}$$

Proofs

By Lemma 2.1 both formulae in (2.3), (2.6) and (2.12) are equivalent.

Lemma 2.2

If we change $q \to \frac{1}{q}$ *and take into account that* $\begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{q}} = q^{k^2-nk} \begin{bmatrix} n \\ k \end{bmatrix}$ *we get that*

$$\sum_j (-1)^{\ell-j} q^{c(j)+\ell^2} \begin{bmatrix} n-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j \end{bmatrix} = q^{b(\ell)} \begin{bmatrix} n \\ \ell \end{bmatrix} \tag{2.17}$$

is equivalent with

$$\sum_j (-1)^{\ell-j} q^{-c(j)+\ell^2} \begin{bmatrix} n-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j \end{bmatrix} = q^{\ell^2-b(\ell)+\ell n} \begin{bmatrix} n \\ \ell \end{bmatrix} \tag{2.18}$$

and that

$$\sum_j (-1)^{\ell-j} q^{\ell^2+c(j)} \begin{bmatrix} n-1-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j \end{bmatrix} = q^{b(\ell)} \begin{bmatrix} n \\ \ell \end{bmatrix} \tag{2.19}$$

is equivalent with

$$\sum_j (-1)^{\ell-j} q^{\ell^2-c(j)-j} \begin{bmatrix} n-1-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j \end{bmatrix} = q^{n\ell+\ell^2-\ell-b(\ell)} \begin{bmatrix} n \\ \ell \end{bmatrix}. \tag{2.20}$$

The equivalence of (2.17) and (2.18) implies that (2.3) and (2.6) are also equivalent.

Thus Theorem 2.1 and Theorem 2.2 are true if (2.2) holds.

If we denote the left-hand side of (2.2) by $a(n,t,q)$ then we have to show that $a(2n+1,t,q) = (1-q^n t)a(2n,t,q)$ or $r(2n+1,\ell,q) = r(2n,\ell,q) + q^n r(2n,\ell-1,q)$.

The last identity follows from

$$\begin{aligned}
&\sum_j (-1)^{\ell-j} q^{\ell^2-\binom{j+1}{2}} \begin{bmatrix} n-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+1+j \\ \ell+j \end{bmatrix} - \sum_j (-1)^{\ell-j} q^{\ell^2-\binom{j+1}{2}} \begin{bmatrix} n-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j \end{bmatrix} \\
&+q^n \sum_j (-1)^{\ell-j} q^{(\ell-1)^2-\binom{j+1}{2}} \begin{bmatrix} n-j \\ \ell-1-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell-1+j \end{bmatrix} \\
&= \sum_j (-1)^j q^{\ell^2-\ell-\binom{j+1}{2}+n+1} \begin{bmatrix} n-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j-1 \end{bmatrix} + q^n \sum_j (-1)^{\ell-j} q^{(\ell-1)^2-\binom{j}{2}} \begin{bmatrix} n-j \\ \ell-1-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell-1+j \end{bmatrix} \\
&= \sum_j (-1)^j q^{(\ell-1)^2-\binom{j}{2}+n} \begin{bmatrix} n+j \\ \ell+j-1 \end{bmatrix} \left(q^{\ell-j} \begin{bmatrix} n-j \\ \ell-j \end{bmatrix} + \begin{bmatrix} n-j \\ \ell-1-j \end{bmatrix} \right) \\
&= \sum_j (-1)^j q^{(\ell-1)^2-\binom{j}{2}+n} \begin{bmatrix} n+j \\ \ell+j-1 \end{bmatrix} \begin{bmatrix} n-j+1 \\ \ell-j \end{bmatrix} = 0.
\end{aligned}$$

To obtain the last line observe that $j \to -j+1$ gives a sign-reversing involution. Note that $\binom{j}{2} = \binom{-j+1}{2}$.

Proof of Theorem 2.3

In this case we have $a(2n,t,q)=a(2n-1,qt,q).$

The first identity in (2.9) is the same as the first identity in (2.6) by changing $j\to -j.$

By Lemma 2.2 we get from the second identity of (2.3)

$$\sum_j (-1)^{\ell-j} q^{\binom{j}{2}} \begin{bmatrix} n-1-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j \end{bmatrix} = q^{\ell n-\binom{\ell+1}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix}$$

which is equivalent with the second identity in (2.9).

Proof of Theorem 2.4

Since $\alpha(j)=j^2$ satisfies Lemma 2.1 both formulae (2.12) are equivalent.

Thus we have only to verify that the coefficient

$r_4(n,\ell,q)=\sum_{j=-\ell}^{\ell}(-1)^{\ell-j}q^{\ell^2}\begin{bmatrix} \left\lfloor \frac{n-2j}{2} \right\rfloor \\ \ell-j \end{bmatrix}\begin{bmatrix} \left\lfloor \frac{n+1+2j}{2} \right\rfloor \\ \ell+j \end{bmatrix}$ of $(-t)^\ell$ satisfies

$r_4(2n+1,\ell,q)-r_4(2n,\ell,q)-q^{2n+1}r_4(2n,\ell-1,q)=0.$

This follows from

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}(-1)^{\ell-j}q^{\ell^2}\begin{bmatrix} n-j \\ \ell-j \end{bmatrix}\begin{bmatrix} n+1+j \\ \ell+j \end{bmatrix}-\sum_{j\in\mathbb{Z}}(-1)^{\ell-j}q^{\ell^2}\begin{bmatrix} n-j \\ \ell-j \end{bmatrix}\begin{bmatrix} n+j \\ \ell+j \end{bmatrix}-q^{2n+1}\sum_{j\in\mathbb{Z}}(-1)^{\ell-1-j}q^{(\ell-1)^2}\begin{bmatrix} n-j \\ \ell-1-j \end{bmatrix}\begin{bmatrix} n+j \\ \ell-1+j \end{bmatrix}\\
&=\sum_{j\in\mathbb{Z}}(-1)^{\ell-j}q^{\ell^2+n-\ell+1}\begin{bmatrix} n-j \\ \ell-j \end{bmatrix}\begin{bmatrix} n+j \\ \ell+j-1 \end{bmatrix}+\sum_{j\in\mathbb{Z}}(-1)^{\ell-j}q^{\ell^2-2\ell+2n+2}\begin{bmatrix} n-j \\ \ell-1-j \end{bmatrix}\begin{bmatrix} n+j \\ \ell-1+j \end{bmatrix}\\
&=\sum_{j\in\mathbb{Z}}(-1)^{\ell-j}q^{\ell^2+n-\ell+1}\begin{bmatrix} n+j \\ \ell+j-1 \end{bmatrix}\left(\begin{bmatrix} n-j \\ \ell-j \end{bmatrix}+q^{n-\ell+1}\begin{bmatrix} n-j \\ \ell-1-j \end{bmatrix}\right)=\sum_{j\in\mathbb{Z}}(-1)^{\ell-j}q^{\ell^2+n-\ell+1}\begin{bmatrix} n+j \\ \ell+j-1 \end{bmatrix}\begin{bmatrix} n+1-j \\ \ell-j \end{bmatrix}=0.
\end{aligned}$$

Proof of Theorem 2.5

The first identity of (2.16) follows from (2.2) for even n by changing $j\to -j.$

Identity $a_5(2n+1,t,q)=a_5(2n,t,q)(1-q^{2n+1}t)$ or equivalently
$r_5(2n+1,\ell,q)=r_5(2n,\ell,q)+q^{2n+1}r_5(2n,\ell-1,q)$ follows from

$$r_5(2n+1,\ell,q)-r_5(2n,\ell,q)-q^{2n+1}r_5(2n,\ell-1,q)$$

$$=\sum_j(-1)^{\ell-j}q^{\ell^2-\binom{j}{2}}\begin{bmatrix}n-j\\ \ell-j\end{bmatrix}\begin{bmatrix}n+1+j\\ \ell+j\end{bmatrix}-\sum_j(-1)^{\ell-j}q^{\ell^2-\binom{j}{2}}\begin{bmatrix}n-j\\ \ell-j\end{bmatrix}\begin{bmatrix}n+j\\ \ell+j\end{bmatrix}$$

$$-q^{2n+1}\sum_j(-1)^{\ell-1-j}q^{(\ell-1)^2-\binom{j}{2}}\begin{bmatrix}n-j\\ \ell-1-j\end{bmatrix}\begin{bmatrix}n+j\\ \ell-1+j\end{bmatrix}$$

$$=q^{n-\ell+1}\sum_j(-1)^{\ell-j}q^{\ell^2-\binom{j}{2}}\begin{bmatrix}n-j\\ \ell-j\end{bmatrix}\begin{bmatrix}n+j\\ \ell+j-1\end{bmatrix}-q^{2n+1}\sum_j(-1)^{\ell-1-j}q^{(\ell-1)^2-\binom{j}{2}}\begin{bmatrix}n-j\\ \ell-1-j\end{bmatrix}\begin{bmatrix}n+j\\ \ell-1+j\end{bmatrix}$$

$$=q^{n-\ell+1}\sum_j(-1)^{\ell-j}q^{\ell^2-\binom{j}{2}}\begin{bmatrix}n-j\\ \ell-j\end{bmatrix}\begin{bmatrix}n+j\\ \ell-1+j\end{bmatrix}+q^{n-\ell+1}\sum_j(-1)^{\ell-j}q^{\ell^2-\binom{j}{2}}q^{n-\ell+1}\begin{bmatrix}n-j\\ \ell-1-j\end{bmatrix}\begin{bmatrix}n+j\\ \ell-1+j\end{bmatrix}$$

$$=q^{n+1-\ell}\sum_j(-1)^{\ell-j}q^{\ell^2-\binom{j}{2}}\begin{bmatrix}n-j+1\\ \ell-j\end{bmatrix}\begin{bmatrix}n+j\\ \ell-1+j\end{bmatrix}=0.$$

Remark

By letting $q\to\frac{1}{q}$ we see that the first line of (2.3) is equivalent with the first line of (2.6) and the first line of (2.9) with the first line of (2.16), whereas the first line of (2.12) remains unchanged.

In the same way the second line of (2.3) is equivalent with the second line of (2.9).

In the other cases we are led to $a(j)=2\binom{j}{2}$, $a(j)=3\binom{j}{2}$ and $a(j)=\frac{j(j-3)}{2}$.

We state the corresponding identities without proof.

For

$$a_6(n,t,q)=\sum_{j\in\mathbb{Z}}(-1)^j q^{j^2-j}\sum_{\ell\geq 0}q^{(\ell-j)(\ell+j)}\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor-j\\ \ell-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n+1}{2}\right\rfloor+j\\ \ell+j\end{bmatrix}t^\ell \tag{2.21}$$

we have

$$\begin{aligned}&a_6(2n+1,t,q)=\prod_{j=0}^{n}\left(1-q^{2j}t\right),\\ &a_6(2n,t,q)=a(2n+1,t,q)+q^n ta(2n-1,qt,q)=\sum_{\ell=0}^{n}(-1)^j q^{2\binom{\ell}{2}}\begin{bmatrix}n\\ \ell\end{bmatrix}_{q^2}\frac{1+q^{n+1+\ell}}{1+q^{n+1-\ell}}t^\ell.\end{aligned}\tag{2.22}$$

For

$$a_7(n,t,q)=\sum_{j\in\mathbb{Z}}(-1)^j q^{3\binom{j}{2}}\sum_{\ell\geq 0}q^{(\ell-j)(\ell+j)}\begin{bmatrix}\left\lfloor \frac{n}{2}\right\rfloor-j\\ \ell-j\end{bmatrix}\begin{bmatrix}\left\lfloor \frac{n+1}{2}\right\rfloor+j\\ \ell+j\end{bmatrix}t^\ell \tag{2.23}$$

we have

$$\begin{aligned}a_7(2n+1,t,q)&=(1-t)\left(q^{n+2}t;q\right)_n,\\ a_7(2n,t,q)&=\left(q^{n+2}t;q\right)_{n-1}\left(1-t+q^n t-q^{2n+1}t\right).\end{aligned} \tag{2.24}$$

For

$$a_8(n,t,q)=\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{j(j-3)}{2}}\sum_{\ell\geq 0}q^{(\ell-j)(\ell+j)}\begin{bmatrix}\left\lfloor \frac{n}{2}\right\rfloor-j\\ \ell-j\end{bmatrix}\begin{bmatrix}\left\lfloor \frac{n+1}{2}\right\rfloor+j\\ \ell+j\end{bmatrix}t^\ell \tag{2.25}$$

we get

$$\begin{aligned}a_8(2n-1,t,q)&=\left(\frac{t}{q};q\right)_n,\\ a_8(2n,t,q)&=\left(t;q\right)_{n-1}\frac{q-t\left(1-q^{n+1}+q^{2n+1}\right)}{q}.\end{aligned} \tag{2.26}$$

By Lemma 2.1 for $a(j)=\dfrac{j(5j+3)}{2}$ we also have $a_9(2n,t,q)=a_9(2n-1,t,q)$.

In this case we state the following

Conjecture 2.1

$$\begin{aligned}a_9(2n,t,q)&=\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{j(5j+3)}{2}}\sum_{\ell\geq 0}q^{(\ell-j)(\ell+j)}\begin{bmatrix}\left\lfloor \frac{n}{2}\right\rfloor-j\\ \ell-j\end{bmatrix}\begin{bmatrix}\left\lfloor \frac{n+1}{2}\right\rfloor+j\\ \ell+j\end{bmatrix}t^\ell\\ &=\sum_{k=0}^{n}q^{2\binom{k+1}{2}}\begin{bmatrix}n\\ k\end{bmatrix}\sum_{j=0}^{k}(-1)^j q^{j(2n-k)}q^{\binom{j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}\sum_{\ell=0}^{j}q^{\ell(\ell+1+k-j)}\begin{bmatrix}j\\ \ell\end{bmatrix}t^k.\end{aligned} \tag{2.27}$$

Note that $\sum_{j=0}^{k}(-1)^j q^{j(2n-k)}q^{\binom{j}{2}}\begin{bmatrix}k\\ j\end{bmatrix}\sum_{\ell=0}^{j}q^{\ell(\ell+1+k-j)}\begin{bmatrix}j\\ \ell\end{bmatrix}$ reduces to $(1-2)^k=(-1)^k$ for $q\to 1$.

3. Some nice $\mathbf{q}-$analogues of $b(n,t)$.

A search for simple product evaluations leads to $b(j)=\binom{2j}{2}$ and $b(j)=2j^2$.

As for $q=1$ there are close connections between $a(n,t)$ and $b(n,t)$.

This depends on

Lemma 3.2

Since $\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^k q^{kn-\binom{k}{2}} \begin{bmatrix} k-n-1 \\ k \end{bmatrix}$ *is a polynomial in* $x=q^n$ *of degree* k *we get from an identity of the form* $\sum_j (-1)^{\ell-j} q^{a(j)+\ell^2} \begin{bmatrix} n-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j \end{bmatrix} = q^{b(\ell)+c(\ell)n} \begin{bmatrix} n \\ \ell \end{bmatrix}$

the identity

$$\sum_{j=-\ell}^{\ell} (-1)^j q^{a(j)+j^2+\ell^2} \begin{bmatrix} n \\ \ell-j \end{bmatrix} \begin{bmatrix} n \\ \ell+j \end{bmatrix} = q^{b(\ell)+(c(\ell)-\ell)(\ell-n-1)+\binom{\ell}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix}, \qquad (3.1)$$

from

$$\sum_j (-1)^{\ell-j} q^{a(j)+\ell^2} \begin{bmatrix} n-1-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n+j \\ \ell+j \end{bmatrix} = q^{b(\ell)+nc(\ell)} \begin{bmatrix} n \\ \ell \end{bmatrix}$$

the identity

$$\sum_{j=-\ell}^{\ell} (-1)^j q^{a(j)+j^2+j+\ell^2} \begin{bmatrix} n+1 \\ \ell-j \end{bmatrix} \begin{bmatrix} n \\ \ell+j \end{bmatrix} = q^{b(\ell)+\binom{\ell+1}{2}+(c(\ell)-\ell)(\ell-n-1)} \begin{bmatrix} n \\ \ell \end{bmatrix} \qquad (3.2)$$

In the same way we get from

$$\sum_{j=-\ell}^{\ell} (-1)^j \begin{bmatrix} n \\ \ell-j \end{bmatrix} \begin{bmatrix} n \\ \ell+j \end{bmatrix} = \begin{bmatrix} n \\ \ell \end{bmatrix}_{q^2}$$

the identity

$$\sum_{j=-\ell}^{\ell} (-1)^j q^{j^2} \begin{bmatrix} n \\ \ell-j \end{bmatrix} \begin{bmatrix} n \\ \ell+j \end{bmatrix} = \begin{bmatrix} n \\ \ell \end{bmatrix}_{q^2}. \qquad (3.3)$$

-

From (2.3) we get

$$\sum_{j=-\ell}^{\ell}(-1)^j q^{\binom{j}{2}+\ell^2}\begin{bmatrix} n \\ \ell-j\end{bmatrix}\begin{bmatrix} n \\ \ell+j\end{bmatrix}=q^{n\ell}\begin{bmatrix} n \\ \ell\end{bmatrix}. \tag{3.4}$$

$$\sum_{j=-\ell}^{\ell}(-1)^j q^{\binom{j+1}{2}+\ell^2}\begin{bmatrix} n+1 \\ \ell-j\end{bmatrix}\begin{bmatrix} n \\ \ell+j\end{bmatrix}=q^{(n+1)\ell}\begin{bmatrix} n \\ \ell\end{bmatrix}. \tag{3.5}$$

From (2.6) we get

$$\sum_{j=-\ell}^{\ell}(-1)^j q^{\frac{j(3j+1)}{2}+\ell^2}\begin{bmatrix} n \\ \ell-j\end{bmatrix}\begin{bmatrix} n \\ \ell+j\end{bmatrix}=q^{\ell^2}\begin{bmatrix} n \\ \ell\end{bmatrix} \tag{3.6}$$

and

$$\sum_{j=-\ell}^{\ell}(-1)^j q^{3\binom{j+1}{2}+\ell^2}\begin{bmatrix} n+1 \\ \ell-j\end{bmatrix}\begin{bmatrix} n \\ \ell+j\end{bmatrix}=q^{\ell^2+\ell}\begin{bmatrix} n \\ \ell\end{bmatrix}. \tag{3.7}$$

From (2.9) we get

$$\sum_{j=-\ell}^{\ell}(-1)^j q^{\frac{j(3j-1)}{2}+\ell^2}\begin{bmatrix} n \\ \ell-j\end{bmatrix}\begin{bmatrix} n \\ \ell+j\end{bmatrix}=q^{\ell^2}\begin{bmatrix} n \\ \ell\end{bmatrix} \tag{3.8}$$

$$\sum_{j=-\ell}^{\ell}(-1)^j q^{\frac{j(3j+1)}{2}+\ell^2}\begin{bmatrix} n+1 \\ \ell-j\end{bmatrix}\begin{bmatrix} n \\ \ell+j\end{bmatrix}=q^{\ell^2}\begin{bmatrix} n \\ \ell\end{bmatrix}. \tag{3.9}$$

We are looking for polynomials of the form

$$b(n,t,q)=\sum_{j\in\mathbb{Z}}(-1)^j q^{b(j)}\sum_{\ell=|j|}^{n} q^{(\ell-j)(\ell+j)}\begin{bmatrix} \left\lfloor \frac{n}{2}\right\rfloor \\ \ell-j\end{bmatrix}\begin{bmatrix} \left\lfloor \frac{n+1}{2}\right\rfloor \\ \ell+j\end{bmatrix}t^{\ell}. \tag{3.10}$$

Let the coefficient of t^{ℓ} be

$$s(n,\ell,q)=q^{\ell^2}\sum_{j=-\ell}^{\ell}(-1)^j q^{b(j)-j^2}\begin{bmatrix} \left\lfloor \frac{n}{2}\right\rfloor \\ \ell-j\end{bmatrix}\begin{bmatrix} \left\lfloor \frac{n+1}{2}\right\rfloor \\ \ell+j\end{bmatrix}. \tag{3.11}$$

Let us first look for $q-$analogues which satisfy $b(2n+1,t,q)=b(2n,t,q)$.

This means

$$r(2n+1,\ell)-r(2n,\ell)=q^{\ell^2}\sum_{j=-\ell}^{\ell}(-1)^j q^{b(j)-j^2}\begin{bmatrix} n \\ \ell-j\end{bmatrix}\begin{bmatrix} n+1 \\ \ell+j\end{bmatrix}-q^{\ell^2}\sum_{j=-\ell}^{\ell}(-1)^j q^{b(j)-j^2}\begin{bmatrix} n \\ \ell-j\end{bmatrix}\begin{bmatrix} n \\ \ell+j\end{bmatrix}$$

$$=q^{\ell^2}\sum_{j=-\ell}^{\ell}(-1)^j q^{b(j)-j^2}\begin{bmatrix} n \\ \ell-j\end{bmatrix}q^{n+1-\ell-j}\begin{bmatrix} n \\ \ell+j-1\end{bmatrix}=0$$

and surely holds if $b(2j)-b(1-2j)=8j-2$.

Thus we get

Lemma 3.1

If $b(j)=(r+2)j^2-rj$ *for some* $r\in\mathbb{R}$ *then* $b(2n+1,t,q)=b(2n,t,q)$.

Special cases are $b(j)=2j^2$, $b(j)=\frac{j(3j+1)}{2}$ *and* $b(j)=\frac{j(5j-1)}{2}$.

This leads to the following Theorems:

Theorem 3.1

For $n\in\mathbb{N}$ *we get the identity*

$$b_1(n,t,q)=\sum_{j\in\mathbb{Z}}(-1)^j q^{2j^2}\sum_{\ell\geq 0}q^{\ell^2-j^2}\begin{bmatrix} \left\lfloor \frac{n}{2}\right\rfloor \\ \ell-j\end{bmatrix}\begin{bmatrix} \left\lfloor \frac{n+1}{2}\right\rfloor \\ \ell+j\end{bmatrix}t^\ell=\prod_{j=1}^{\left\lfloor \frac{n}{2}\right\rfloor}\left(1+q^{2j-1}t\right). \tag{3.12}$$

It satisfies $b_1(2n+1,t,q)=b_1(2n,t,q)$ *and is equivalent with*

$$s_1(n,\ell,q)=\sum_{j=-\ell}^{\ell}(-1)^j q^{\ell^2+j^2}\begin{bmatrix} \left\lfloor \frac{n}{2}\right\rfloor \\ \ell-j\end{bmatrix}\begin{bmatrix} \left\lfloor \frac{n+1}{2}\right\rfloor \\ \ell+j\end{bmatrix}=q^{\ell^2}\begin{bmatrix} \left\lfloor \frac{n}{2}\right\rfloor \\ \ell\end{bmatrix}_{q^2}. \tag{3.13}$$

For $t=1$(3.12) *reduces to*

$$\sum_{j\in\mathbb{Z}}(-1)^j q^{2j^2}\begin{bmatrix} n \\ \left\lfloor \frac{n}{2}\right\rfloor+2j\end{bmatrix}=\prod_{j=1}^{\left\lfloor \frac{n}{2}\right\rfloor}\left(1+q^{2j-1}\right). \tag{3.14}$$

The proof follows from (3.3).

Theorem 3.2

For $n\in\mathbb{N}$ *we get the identity*

$$b_2(n,t,q)=\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{j(3j+1)}{2}}\sum_{\ell\geq 0}q^{\ell^2-j^2}\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor\\ \ell-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n+1}{2}\right\rfloor\\ \ell+j\end{bmatrix}t^\ell=\sum_{\ell=0}^{\left\lfloor\frac{n}{2}\right\rfloor}q^{\ell\left\lfloor\frac{n}{2}\right\rfloor}\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor\\ \ell\end{bmatrix}t^\ell. \tag{3.15}$$

It satisfies $b_2(2n+1,t,q)=b_2(2n,t,q)$ *and is equivalent with*

$$s_2(n,\ell,q)=\sum_{j=-\ell}^{\ell}(-1)^j q^{\binom{j+1}{2}+\ell^2}\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor\\ \ell-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n+1}{2}\right\rfloor\\ \ell+j\end{bmatrix}=q^{\ell\left\lfloor\frac{n}{2}\right\rfloor}\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor\\ \ell\end{bmatrix}. \tag{3.16}$$

For $t=1$ (3.15) *reduces to*

$$\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{j(3j+1)}{2}}\begin{bmatrix}n\\ \left\lfloor\frac{n}{2}\right\rfloor+2j\end{bmatrix}=\sum_{\ell=0}^{\left\lfloor\frac{n}{2}\right\rfloor}q^{\ell\left\lfloor\frac{n}{2}\right\rfloor}\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor\\ \ell\end{bmatrix}. \tag{3.17}$$

The proof follows from (3.4) by changing $j\to -j.$

From (3.4) and (3.5) we also get

Theorem 3.3

For $n\in\mathbb{N}$ *we get the identity*

$$b_3(n,t,q)=\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{j(3j-1)}{2}}\sum_{\ell\geq 0}q^{\ell^2-j^2}\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor\\ \ell-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n+1}{2}\right\rfloor\\ \ell+j\end{bmatrix}t^\ell=\sum_{\ell=0}^{\left\lfloor\frac{n}{2}\right\rfloor}q^{\ell\left\lfloor\frac{n+1}{2}\right\rfloor}\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor\\ \ell\end{bmatrix}t^\ell. \tag{3.18}$$

It satisfies $b_3(2n+1,t,q)=b_3(2n,qt,q)$ *and is equivalent with*

$$s_3(n,\ell,q)=\sum_{j=-\ell}^{\ell}(-1)^j q^{\binom{j}{2}+\ell^2}\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor\\ \ell-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n+1}{2}\right\rfloor\\ \ell+j\end{bmatrix}=q^{\ell\left\lfloor\frac{n+1}{2}\right\rfloor}\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor\\ \ell\end{bmatrix}. \tag{3.19}$$

For $t=1$ (3.15) *reduces to*

$$\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix}n\\ \left\lfloor\frac{n+4j}{2}\right\rfloor\end{bmatrix}=\sum_{\ell=0}^{\left\lfloor\frac{n}{2}\right\rfloor}q^{\ell\left\lfloor\frac{n+1}{2}\right\rfloor}\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor\\ \ell\end{bmatrix}. \tag{3.20}$$

Theorem 3.4

For $n \in \mathbb{N}$ *we get the identity*

$$b_4(n,t,q) = \sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(5j-1)}{2}} \sum_{\ell \ge 0} q^{(\ell-j)(\ell+j)} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor \\ \ell - j \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n+1}{2} \right\rfloor \\ \ell + j \end{bmatrix} t^\ell = \sum_{\ell=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{\ell^2} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor \\ \ell \end{bmatrix} t^\ell. \qquad (3.21)$$

It satisfies $b_4(2n+1,t,q) = b_4(2n,t,q)$ *and is equivalent with*

$$\begin{aligned} q^{-\ell^2} s_4(2n,\ell,q) &= \sum_{j=-\ell}^{\ell} (-1)^j q^{\frac{j(3j-1)}{2}} \begin{bmatrix} n \\ \ell - j \end{bmatrix} \begin{bmatrix} n \\ \ell + j \end{bmatrix} = \begin{bmatrix} n \\ \ell \end{bmatrix} \\ q^{-\ell^2} s_4(2n+1,\ell,q) &= \sum_{j=-\ell}^{\ell} (-1)^j q^{\frac{j(3j-1)}{2}} \begin{bmatrix} n \\ \ell - j \end{bmatrix} \begin{bmatrix} n+1 \\ \ell + j \end{bmatrix} = \begin{bmatrix} n \\ \ell \end{bmatrix} \end{aligned} \qquad (3.22)$$

For $t = 1$ *we get*

$$\sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(5j-1)}{2}} \begin{bmatrix} n \\ \left\lfloor \frac{n+4j}{2} \right\rfloor \end{bmatrix} = \sum_{\ell=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{\ell^2} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor \\ \ell \end{bmatrix}. \qquad (3.23)$$

The proof follows from (3.6) by changing $j \to -j$.

(3.22) and more general results have been obtained in [9] as a special case of Rogers‘ $q-$ Dougall sum.

Theorem 3.5

For $n \in \mathbb{N}$ *we get the identity*

$$b_5(2n+1,t,q) = \sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(5j-3)}{2}} \sum_{\ell=|j|}^{n} q^{(\ell-j)(\ell+j)} \begin{bmatrix} n \\ \ell - j \end{bmatrix} \begin{bmatrix} n+1 \\ \ell + j \end{bmatrix} t^\ell = \sum_{\ell=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{\ell^2+\ell} \begin{bmatrix} n \\ \ell \end{bmatrix} t^\ell. \qquad (3.24)$$

It is equivalent with

$$\sum_{j=-\ell}^{\ell} (-1)^j q^{\frac{j(3j-3)}{2}+\ell^2} \begin{bmatrix} n \\ \ell - j \end{bmatrix} \begin{bmatrix} n+1 \\ \ell + j \end{bmatrix} = q^{\ell^2+\ell} \begin{bmatrix} n \\ \ell \end{bmatrix}. \qquad (3.25)$$

For $t = 1$ *this gives*

$$\sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(5j-3)}{2}} \begin{bmatrix} 2n+1 \\ n+1-2j \end{bmatrix} = \sum_{\ell=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{\ell^2+\ell} \begin{bmatrix} n \\ \ell \end{bmatrix}. \qquad (3.26)$$

Formula (3.25) follows from (3.7) by changing $j \to -j$.

Formula (3.23) for even n has first been found by Bressoud [2]. Other proofs have been given by Chapman [3], Andrews and Eriksson [1] and Cigler [6]. Formula (3.26) has first been proved in more general form in [9], Lemma 3.2.

Theorem 3.6

For $n \in \mathbb{N}$ we get the identity

$$b_6(n,t,q) = \sum_{j=-n}^{n} (-1)^j q^{\binom{2j}{2}} \sum_{\ell=|j|}^{n} q^{\ell^2-j^2} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor \\ \ell - j \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n+1}{2} \right\rfloor \\ \ell + j \end{bmatrix} t^\ell \tag{3.27}$$

with

$$\begin{aligned} b_6(2n+1,t,q) &= \prod_{j=1}^{n} \left(1+q^{2j}t\right), \\ b_6(2n,t,q) &= \left(1+q^n t\right) b(2n-1,t,q). \end{aligned} \tag{3.28}$$

It is equivalent with

$$s_6(2n+1,\ell,q) = \sum_{j=-\ell}^{\ell} (-1)^j q^{j^2-j+\ell^2} \begin{bmatrix} n \\ \ell - j \end{bmatrix} \begin{bmatrix} n+1 \\ \ell + j \end{bmatrix} = q^{\ell^2+\ell} \begin{bmatrix} n \\ \ell \end{bmatrix}_{q^2} \tag{3.29}$$

and

$$s_6(2n,\ell,q) = s_6(2n-1,\ell,q) + q^n s_6(2n,\ell-1,q) = q^{\ell^2+\ell} \begin{bmatrix} n \\ \ell \end{bmatrix}_{q^2} + q^{n+\ell^2-\ell} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}_{q^2}. \tag{3.30}$$

The first identity follows from

$$s_6(2n+1,\ell,q) = \sum_{j=-\ell}^{\ell} (-1)^j q^{j^2-j-\ell} \begin{bmatrix} n \\ \ell - j \end{bmatrix} \begin{bmatrix} n+1 \\ \ell + j \end{bmatrix} = \sum_{j=-\ell}^{\ell} (-1)^j q^{j^2-j-\ell} \begin{bmatrix} n \\ \ell - j \end{bmatrix} \left(q^{\ell+j} \begin{bmatrix} n \\ \ell + j \end{bmatrix} + \begin{bmatrix} n \\ \ell + j - 1 \end{bmatrix} \right)$$

$$= \sum_{j=-\ell}^{\ell} (-1)^j q^{j^2} \begin{bmatrix} n \\ \ell - j \end{bmatrix} \begin{bmatrix} n \\ \ell + j \end{bmatrix} + \sum_{j=-\ell}^{\ell} (-1)^j q^{j^2-j-\ell} \begin{bmatrix} n \\ \ell - j \end{bmatrix} \begin{bmatrix} n \\ \ell + j - 1 \end{bmatrix} = \begin{bmatrix} n \\ \ell \end{bmatrix}_{q^2} + 0 = \begin{bmatrix} n \\ \ell \end{bmatrix}_{q^2}$$

by using (3.13).

Since

$$s_6(2n,\ell,q)-s_6(2n-1,\ell,q)=\sum_{j=-\ell}^{\ell}(-1)^j q^{j^2-j+\ell^2}\begin{bmatrix} n \\ \ell-j\end{bmatrix}\begin{bmatrix} n \\ \ell+j\end{bmatrix}-\sum_{j=-\ell}^{\ell}(-1)^j q^{j^2-j+\ell^2}\begin{bmatrix} n-1 \\ \ell-j\end{bmatrix}\begin{bmatrix} n \\ \ell+j\end{bmatrix}$$

$$=\sum_{j=-\ell}^{\ell}(-1)^j q^{j^2+\ell^2-\ell+n}\begin{bmatrix} n-1 \\ \ell-j-1\end{bmatrix}\begin{bmatrix} n \\ \ell+j\end{bmatrix}$$

we must show that $\sum_{j=-\ell}^{\ell}(-1)^j q^{j^2}\begin{bmatrix} n-1 \\ \ell-j-1\end{bmatrix}\begin{bmatrix} n \\ \ell+j\end{bmatrix}=\begin{bmatrix} n-1 \\ \ell-1\end{bmatrix}_{q^2}$.

This follows from

$$\sum_{j=-\ell}^{\ell}(-1)^j q^{j^2}\begin{bmatrix} n-1 \\ \ell-j-1\end{bmatrix}\begin{bmatrix} n \\ \ell+j\end{bmatrix}=\sum_{j=-\ell}^{\ell}(-1)^j q^{j^2}\left(\begin{bmatrix} n \\ \ell-j\end{bmatrix}-q^{\ell-j}\begin{bmatrix} n-1 \\ \ell-j\end{bmatrix}\right)\begin{bmatrix} n \\ \ell+j\end{bmatrix}$$

$$=\sum_{j=-\ell}^{\ell}(-1)^j q^{j^2}\begin{bmatrix} n \\ \ell-j\end{bmatrix}\begin{bmatrix} n \\ \ell+j\end{bmatrix}-\sum_{j=-\ell}^{\ell}(-1)^j q^{j^2-j+\ell}\begin{bmatrix} n-1 \\ \ell-j\end{bmatrix}\begin{bmatrix} n \\ \ell+j\end{bmatrix}=\begin{bmatrix} n \\ \ell\end{bmatrix}_{q^2}-q^{2\ell}\begin{bmatrix} n-1 \\ \ell\end{bmatrix}_{q^2}=\begin{bmatrix} n-1 \\ \ell-1\end{bmatrix}_{q^2}.$$

4. The Rogers-Ramanujan identities

For $n\to\infty$ (3.23) gives the first Rogers-Ramanujan identity

$$\frac{1}{(q;q)_\infty}\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{j(5j-1)}{2}}=\sum_{\ell=0}^{\infty}\frac{q^{\ell^2}}{(q;q)_\ell}. \tag{4.1}$$

In the same way (3.26) gives the second Rogers-Ramanujan identity

$$\frac{1}{(q;q)_\infty}\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{j(5j-3)}{2}}=\sum_{\ell=0}^{\infty}\frac{q^{\ell^2+\ell}}{(q;q)_\ell}. \tag{4.2}$$

The usual product representation of the left-hand side follows by using Jacobi’s triple product identity.

Let me state some remarks about the history of this approach.

Bressoud‘s proof ([2]) uses an idea of Bailey which gives also other interesting $q-$identities such as (3.17) for even n.

He proves with a somewhat tricky method

$$\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{j(5j+1)}{2}}\begin{bmatrix}2n\\ n+2j\end{bmatrix}=\sum_{\ell=0}^{\left\lfloor\frac{n}{2}\right\rfloor}q^{\ell^2}\begin{bmatrix}n\\ \ell\end{bmatrix}.$$

For the second RR-identity he starts with

$$\frac{1}{1-q^{n+1}}\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{j(5j+3)}{2}}\begin{bmatrix}2n+2\\ n+2j+2\end{bmatrix}=\sum_{\ell=0}^{n}q^{\ell^2+\ell}\begin{bmatrix}n\\ \ell\end{bmatrix}.$$

R. Chapman [3] gave a simpler proof by finding recurrence relations which are satisfied by both sides.

More precisely let

$$S_a(n)=\sum_{j=0}^{n}q^{j^2+aj}\begin{bmatrix}n\\ j\end{bmatrix}. \tag{4.3}$$

Then

$$S_0(n)=\left(1+q^{2n-1}\right)S_0(n-1)+q^n\left(1-q^n\right)S_1(n-2), \tag{4.4}$$

$$S_1(n)=q^nS_0(n)+\left(1-q^n\right)S_1(n-1). \tag{4.5}$$

The same recurrence relations hold for the left-hand sides. Together with the initial values this proves the above identities.

G. Andrews and K. Eriksson [1] gave a further simplification:

Let $s_0(n)=\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{j(5j+1)}{2}}\begin{bmatrix}2n\\ n+2j\end{bmatrix}$ and $s_1(n)=\sum_{j\in\mathbb{Z}}(-1)^j q^{\frac{j(5j-3)}{2}}\begin{bmatrix}2n+1\\ n+2j\end{bmatrix}$.

They also show (4.5) and instead of (4.4)

$$S_0(n)=S_0(n-1)+q^nS_1(n-1) \tag{4.6}$$

which also uniquely determine $S_0(n)$ and $S_1(n)$.

Then they show that

$$s_0(n)-s_0(n-1)=q^ns_1(n-1)$$

and finally

$$s_1(n)-q^ns_0(n)=\left(1-q^n\right)s_1(n-1).$$

This gives $S_0(n)=s_0(n)$ and $S_1(n)=s_1(n)$.

In [6] I observed that by using $q-$Chu-Vandermonde's theorem

$\begin{bmatrix} 2n \\ n+2j \end{bmatrix} = \sum_{\ell \geq |j|} q^{(\ell-j)(\ell+j)} \begin{bmatrix} n \\ \ell-j \end{bmatrix} \begin{bmatrix} n \\ \ell+j \end{bmatrix}$ the identity $s_0(n) = S_0(n)$ can be written as

$$\sum_{\ell=0}^{n} q^{\ell^2} \begin{bmatrix} n \\ \ell \end{bmatrix} = \sum_{j \in \mathbb{Z}} (-1)^j q^{\frac{j(5j+1)}{2}} \begin{bmatrix} 2n \\ n+2j \end{bmatrix} = \sum_{j \in \mathbb{Z}} (-1)^j q^{\frac{j(5j+1)}{2}} \sum_{\ell \geq |j|} q^{(\ell-j)(\ell+j)} \begin{bmatrix} n \\ \ell-j \end{bmatrix} \begin{bmatrix} n \\ \ell+j \end{bmatrix}$$
$$= \sum_{\ell \geq 0} q^{\ell^2} \sum_{j=-\ell}^{\ell} (-1)^j q^{\frac{j(3j+1)}{2}} \begin{bmatrix} n \\ \ell-j \end{bmatrix} \begin{bmatrix} n \\ \ell+j \end{bmatrix}.$$

Therefore the identity holds if

$$\sum_{j=-\ell}^{\ell} (-1)^j q^{\frac{j(3j+1)}{2}} \begin{bmatrix} n \\ \ell-j \end{bmatrix} \begin{bmatrix} n \\ \ell+j \end{bmatrix} = \begin{bmatrix} n \\ \ell \end{bmatrix}.$$

In the same way I observed that $s_1(n) = S_1(n)$ can be reduced to

$$\sum_{j=-\ell}^{\ell} (-1)^j q^{3\binom{j+1}{2}} \begin{bmatrix} n+1 \\ \ell-j \end{bmatrix} \begin{bmatrix} n \\ \ell+j \end{bmatrix} = q^{\ell} \begin{bmatrix} n \\ \ell \end{bmatrix}.$$

This is essentially the same approach as in the present paper with the exception that in [6] I gave a direct proof of the last two identities.

5. Final observations

Finally let us consider the exponents

$$a(j) = (r+1)j^2 + rj \tag{4.7}$$

and

$$b(j) = (r+2)j^2 - rj \tag{4.8}$$

for some $r \in \mathbb{R}$.

By Lemma 2.1 and Lemma 3.1 we have $a(2n,t,r,q) = a(2n-1,t,r,q)$ and $b(2n,t,r,q) = b(2n+1,t,r,q)$ for the corresponding polynomials.

Let us write $a(2n,t,r,q) = \sum \begin{bmatrix} n \\ k \end{bmatrix} c(2n,k,r,q) t^k$ or equivalently

$$c(2n,k,r,q)=\sum_{j=-k}^{k}(-1)^j q^{rj^2+rj+k^2}\frac{\begin{bmatrix} n-j \\ k-j \end{bmatrix}\begin{bmatrix} n+j \\ k+j \end{bmatrix}}{\begin{bmatrix} n \\ k \end{bmatrix}}. \tag{4.9}$$

Then we get

Proposition 5.1

Let

$$\sum_{j=-k}^{k}(-1)^j q^{r(j^2+j)+k^2}\begin{bmatrix} n-j \\ k-j \end{bmatrix}\begin{bmatrix} n+j \\ k+j \end{bmatrix}=\begin{bmatrix} n \\ k \end{bmatrix}c(2n,k,r,q). \tag{4.10}$$

Then

$$c(2n,k,r,q)=\sum_{j=0}^{k}(-1)^j q^{2r\left(j^2+j\right)-j+k^2+k}\frac{[2j+1]}{[j+1]\begin{bmatrix} k+1+j \\ k \end{bmatrix}}\begin{bmatrix} n-1-j \\ k-j \end{bmatrix}\begin{bmatrix} n+j \\ j \end{bmatrix}. \tag{4.11}$$

For $q=1$ we get

$$\sum_{j=0}^{k}(-1)^{k-j}\frac{2j+1}{(j+1)\binom{k+1+j}{k}}\binom{n-1-j}{k-j}\binom{n+j}{n}=1 \tag{4.12}$$

and from Theorems 2.1, 2.2 and 2.4 we know that

$$c\left(2n,k,-\frac{1}{2},q\right)=(-1)^k q^{\binom{k}{2}}, \tag{4.13}$$

$$c\left(2n,k,\frac{1}{2},q\right)=(-1)^k q^{kn+\binom{k+1}{2}} \tag{4.14}$$

and

$$c(2n,k,0,q)=(-1)^k q^{k^2}\prod_{j=0}^{k-1}\frac{1+q^{n-j}}{1+q^{j+1}}. \tag{4.15}$$

To prove (4.11) observe that

$$\frac{\begin{bmatrix} n-j \\ k-j \end{bmatrix}\begin{bmatrix} n+j \\ k+j \end{bmatrix}-\begin{bmatrix} n-j-1 \\ k-j-1 \end{bmatrix}\begin{bmatrix} n+j+1 \\ k+j+1 \end{bmatrix}}{\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n-1-j \\ k-j \end{bmatrix}\begin{bmatrix} n+j \\ j \end{bmatrix}}=\frac{\begin{bmatrix} n-j-1 \\ k-j-1 \end{bmatrix}\begin{bmatrix} n+j \\ k+j \end{bmatrix}\left(\frac{[n-j]}{[k-j]}-\frac{[n+j+1]}{[k+j+1]}\right)}{\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n-1-j \\ k-j \end{bmatrix}\begin{bmatrix} n+j \\ j \end{bmatrix}}$$

$$=q^{k-j}\frac{\begin{bmatrix} n-j-1 \\ k-j-1 \end{bmatrix}\begin{bmatrix} n+j \\ k+j \end{bmatrix}\frac{[2j+1][n-k]}{[k-j][k+j+1]}}{\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n-1-j \\ k-j \end{bmatrix}\begin{bmatrix} n+j \\ j \end{bmatrix}}=q^{k-j}\frac{[2j+1]}{[j+1]\begin{bmatrix} k+1+j \\ k \end{bmatrix}}.$$

Therefore

$$\sum_{j=-k}^{k}(-1)^j q^{rj^2+rj+k^2}\begin{bmatrix} n-j \\ k-j \end{bmatrix}\begin{bmatrix} n+j \\ k+j \end{bmatrix}=\sum_{j=0}^{k}(-1)^j\left(q^{rj^2+rj+k^2}\begin{bmatrix} n-j \\ k-j \end{bmatrix}\begin{bmatrix} n+j \\ k+j \end{bmatrix}-q^{r(j+1)^2-r(j+1)+k^2}\begin{bmatrix} n-j-1 \\ k-j-1 \end{bmatrix}\begin{bmatrix} n+j+1 \\ k+j+1 \end{bmatrix}\right)$$

$$=\sum_{j=0}^{k}(-1)^j\left(q^{rj^2+rj+k^2}\left(\begin{bmatrix} n-j \\ k-j \end{bmatrix}\begin{bmatrix} n+j \\ k+j \end{bmatrix}-\begin{bmatrix} n-j-1 \\ k-j-1 \end{bmatrix}\begin{bmatrix} n+j+1 \\ k+j+1 \end{bmatrix}\right)\right)$$

$$=\sum_{j=0}^{k}(-1)^j q^{rj^2+rj+k^2}q^{k-j}\frac{[2j+1]}{[j+1]\begin{bmatrix} k+1+j \\ k \end{bmatrix}}\begin{bmatrix} n-1-j \\ k-j \end{bmatrix}\begin{bmatrix} n+j \\ j \end{bmatrix}.$$

In an analogous way let $b(j)=(r+2)j^2-rj.$

Proposition 5.2

Let

$$\sum_{j=-k}^{k}(-1)^j q^{(r+1)j^2+k^2}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n \\ k+j \end{bmatrix}=\begin{bmatrix} n \\ k \end{bmatrix}d(2n,k,r,q). \qquad (4.16)$$

Then

$$d(2n,k,r,q)=\sum_{j=0}^{k}(-1)^j q^{r(j^2+j)+j^2+k^2}\frac{[2j+1]}{[j+1]\begin{bmatrix} k+1+j \\ k \end{bmatrix}}\begin{bmatrix} n+1 \\ k-j \end{bmatrix}\begin{bmatrix} n-k \\ j \end{bmatrix}. \qquad (4.17)$$

We have

$$\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n \\ k+j \end{bmatrix} - q^{2j+1}\begin{bmatrix} n \\ k-j-1 \end{bmatrix}\begin{bmatrix} n \\ k+j+1 \end{bmatrix}$$
$$= \frac{[n]![n]!}{[k-j]![k+j]![n-k+j]![n-k-j]!}\left(1-q^{2j+1}\frac{[k-j][n-k-j]}{[k+j+1][n-k+j+1]}\right)$$
$$= \frac{[n]![n]!}{[k-j]![k+j]![n-k+j]![n-k-j]!}\frac{[2j+1][n+1]}{[k+j+1][n+j+1-k]}.$$

This implies

$$\frac{\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n \\ k+j \end{bmatrix} - q^{2j+1}\begin{bmatrix} n \\ k-j-1 \end{bmatrix}\begin{bmatrix} n \\ k+j+1 \end{bmatrix}}{\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n+1 \\ k-j \end{bmatrix}\begin{bmatrix} n-k \\ j \end{bmatrix}}$$
$$= \frac{[n]![n+1]![2j+1][k]![n-k]![k-j]![n+1-k+j]![j]![n-k-j]!}{[k-j]![k+j+1]![n-k+j+1]![n-k-j]![n]![n+1]![n-k]!}$$
$$= \frac{[2j+1][k]![j]!}{[k+j+1]!} = \frac{[2j+1]}{[j+1]\begin{bmatrix} k+j+1 \\ k \end{bmatrix}}.$$

Therefore we get

$$\sum_{j=-k}^{k}(-1)^j q^{(r+1)j^2-rj+k^2}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n \\ k+j \end{bmatrix}$$
$$= \sum_{j=0}^{k}(-1)^j\left(q^{(r+1)j^2+rj+k^2}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n \\ k+j \end{bmatrix} - q^{(r+1)(j+1)^2-r(j+1)+k^2}\begin{bmatrix} n \\ k-j-1 \end{bmatrix}\begin{bmatrix} n \\ k+j+1 \end{bmatrix}\right)$$
$$\sum_{j=0}^{k}(-1)^j q^{(r+1)j^2+rj+k^2}\left(\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n \\ k+j \end{bmatrix} - q^{2j+1}\begin{bmatrix} n \\ k-j-1 \end{bmatrix}\begin{bmatrix} n \\ k+j+1 \end{bmatrix}\right)$$
$$= \sum_{j=0}^{k}(-1)^j q^{(r+1)j^2+rj+k^2}\frac{[2j+1]}{[j+1]\begin{bmatrix} k+1+j \\ k \end{bmatrix}}\begin{bmatrix} n+1 \\ k-j \end{bmatrix}\begin{bmatrix} n-k \\ j \end{bmatrix}.$$

For $q=1$ this reduces to

$$\sum_{j=0}^{k}(-1)^j\frac{2j+1}{(j+1)\binom{k+1+j}{k}}\binom{n+1}{k-j}\binom{n-k}{j} = 1. \qquad (4.18)$$

Moreover Theorems 3.1, 3.2 and 3.4 give the special cases

$$d(2n,k,0,q)=q^{k^2}\prod_{j=0}^{k-1}\frac{1+q^{n-j}}{1+q^{j+1}}, \tag{4.19}$$

$$d\left(2n,k,-\frac{1}{2},q\right)=q^{kn} \tag{4.20}$$

and

$$d\left(2n,k,\frac{1}{2},q\right)=q^{k^2}. \tag{4.21}$$

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