1. Introduction

After recalling some properties of the $q$–Fibonacci and $q$–Lucas polynomials which I have introduced in [4] and [5] I apply these to derive some identities due to L. Carlitz [2] and H. Prodinger [8]. Finally I show that the $q$–Lucas polynomials allow an easy approach to L. Slater’s Bailey pairs $A(1)$–$A(8)$ and some related Rogers-Ramanujan type identities. I want to thank Andrew Sills for commenting on a previous version and pointing out to me the papers [10] and [11] of L. Slater.

Let

$$F_n(x, s) = \sum_{k=0}^{n-1} \binom{n-1-k}{k} s^k x^{n-2k}$$

and

$$L_n(x, s) = \sum_{k=0}^{n} \frac{n}{n-k} \binom{n-k}{k} s^k x^{n-k}$$

be the classical Fibonacci and Lucas polynomials. They satisfy the recurrence

$$F_n(x, s) = xF_{n-1}(x, s) + sF_{n-2}(x, s)$$

with initial values $F_0(x, s) = 0, F_1(x, s) = 1$ and

$$L_n(x, s) = xL_{n-1}(x, s) + sL_{n-2}(x, s)$$

with initial values $L_0(x, s) = 2$ and $L_1(x, s) = x$.

It will be convenient to define a variant $L'_n(x, s)$ by $L'_n(x, s) = L_n(x, s)$ for $n > 0$ and $L'_0(x, s) = 1$.

Let $\alpha = \frac{x + \sqrt{x^2 + 4s}}{2}$ and $\beta = \frac{x - \sqrt{x^2 + 4s}}{2}$ be the roots of the equation $z^2 - xz - s = 0.$
Then it is well-known and easily verified that \( F_n(x,s) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \) and \( L_n(x,s) = \alpha^n + \beta^n \).

This implies the well-known formula

\[
L_n(x,s) = F_{n+1}(x,s) + sF_{n-1}(x,s)
\]

(1.5)

for \( n > 0 \) because \( s = \alpha \beta \) and \( \alpha^{n+1} - \beta^{n+1} - \alpha \beta (\alpha^{n-1} - \beta^{n-1}) = \alpha^n (\alpha - \beta) + \beta^n (\alpha - \beta) \).

Another known formula is

\[
\sum_{k=0}^{\left\lfloor n/2 \right\rfloor} (-s)^k \binom{n}{k} L_{n-2k}(x,s) = x^n.
\]

(1.6)

For the proof it is convenient to consider this identity for odd \( n \) and even \( n \) separately.

For odd \( n \) the left-hand side is

\[
\sum_{k=0}^{\left\lfloor n/2 \right\rfloor} \binom{n}{k} (\alpha^{n-2k} + \beta^{n-2k}) = \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} \binom{n}{k} (\alpha^{n-k} + \beta^{n-k}) = \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} \binom{n}{k} \alpha^k \beta^{n-k} = (\alpha + \beta)^n = x^n.
\]

For \( n = 2m \) the same holds because for \( k = m \) the coefficient of \( (-s)^m \binom{2m}{m} \) is \( L'_0(x,s) = 1 \).

Using (1.5) we see that (1.6) is equivalent with

\[
\sum_{k=0}^{\left\lfloor n/2 \right\rfloor} \binom{n}{k} (\frac{n}{k} - \frac{n}{k-1}) (-s)^k F_{n+1-2k}(x,s) = x^n.
\]

(1.7)

Let us first review the simpler case of \( q - \) analogues of the binomial theorem

\[
(x + s)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} s^k = (x + s)(x + s)^{n-1}.
\]

It is well known that there are two important ones,

\[
p_n(x,s) = (x + s)(x + qs) \cdots (x + q^{n-1}s) = \sum_{k=0}^{n} \binom{n}{k} q^{\frac{k(k-1)}{2}} x^{n-k} s^k,
\]

(1.8)

which satisfy the recurrence relation \( p_n(x,s) = (x + q^{n-1}s)p_{n-1}(x,s) \) and the Rogers-Szegö polynomials

\[
r_n(x,s) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} s^k,
\]

(1.9)

which have no closed formula but satisfy the recursion
\[ r_n(x,s) = (x+s)r_{n-1}(x,s) + (q^{n-1} - 1)sx^{n-2}(x,s). \]  

(1.10)

Here \[ \binom{n}{k} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \] with \( (x;q)_n = \prod_{j=0}^{n-1} (1-q^jx) \) denotes the \( q \)–binomial coefficient.

We also use \[ [n] \] instead of \[ \begin{bmatrix} n \end{bmatrix} \].

A similar situation occurs with \( q \)–analogues of the Fibonacci polynomials. There are the polynomials studied by L. Carlitz

\[ f_n(x,s,q) = \sum_{k=0}^{n-1} \binom{n-1-k}{k} q^{k} x^{n-1-2k} s^k, \]  

(1.11)

which satisfy the recursion \( f_n(x,s,q) = xf_{n-1}(x,qs,q) + qsf_{n-2}(x,q^2s,q) \) and the polynomials \( F_n(x,s,q) \) with which I am concerned in this paper.

2. Definition and simple properties

Define the \( q \)–Fibonacci polynomials \( F_n(x,s,q) \) by

\[ F_n(x,s,q) = \sum_{k=0}^{n-1} q^{\binom{k+1}{2}} \binom{n-1-k}{k} s^k x^{n-1-2k} \]  

(2.1)

for \( n \geq 0 \).

The first polynomials are

\( 0,1,x,x^2 + qs,x^3 + (1+q)qsx,x^4 + qs[3]x^2 + q^3s^2, \cdots \)

Let us recall that these \( q \)–Fibonacci polynomials satisfy each of the recurrences

\[ F_n(x,s,q) = xf_{n-1}(x,qs,q) + qsF_{n-2}(x,qs,q), \]  

(2.2)

\[ F_n(x,s,q) = xf_{n-1}(x,s,q) + q^{-2}sF_{n-2}(x,\frac{s}{q},q) \]  

(2.3)

and

\[ F_n(x,s,q) = xf_{n-1}(x,s,q) + q^{-2}sF_{n-3}(x,s,q) + q^{-2}s^2F_{n-4}(x,s,q). \]  

(2.4)
The simple proofs follow by comparing coefficients and using the well-known recurrences for the \(q\)-binomial coefficients. We see that (2.2) is equivalent with
\[
\begin{bmatrix} n - 1 - k \end{bmatrix}_k = q \begin{bmatrix} n - 2 - k \end{bmatrix}_k + \begin{bmatrix} n - 2 - k \end{bmatrix}_{k - 1}
\]
and (2.3) with
\[
\begin{bmatrix} n - 1 - k \end{bmatrix}_k = \begin{bmatrix} n - 2 - k \end{bmatrix}_k + q^{n - 1 - 2k} \begin{bmatrix} n - 2 - k \end{bmatrix}_{k - 1}.
\]

Combining (2.2) and (2.3) we get (2.4).

As a consequence we get
\[
F_n(x, s, q) - x F_{n-1}(x, s, q) - s F_{n-2}(x, s, q)
\]
\[
= - s F_{n-2}(x, s, q) + q^{n-2} s F_{n-2}(x, s, q) - q^{n-2} s F_{n-1}(x, s, q) + q^{n-2} s x F_{n-1}(x, s, q) + q^{n-2} s^2 F_{n-4}(x, s, q)
\]
\[
= (q^{n-2} - 1) s F_{n-2}(x, s, q) - q^{n-2} s (F_{n-2}(x, s, q) - x F_{n-3}(x, s, q) - s F_{n-4}(x, s, q))
\]

Iterating this equation and observing that it holds for \(n = 2\) and \(n = 3\) gives
\[
F_n(x, s, q) - x F_{n-1}(x, s, q) - s F_{n-2}(x, s, q) = \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k q^{k(k-1)/2} \left(1 - q^{n-2k}\right) F_{n-2k}(x, s, q). \tag{2.5}
\]

**Remark 1**

From (2.2) we get the following combinatorial interpretation of the \(q\)-Fibonacci polynomials which is a \(q\)-analogue of the well-known Morse code model of the Fibonacci numbers. Consider words \(c = c_1 c_2 \cdots c_m\) of letters \(c_i \in \{a, b\}\) and associate with \(c\) the weight
\[
w(c) = w(c)(s) = q^{i_1 + i_2 + \cdots + i_k} s^k x^{m-k}, \quad \text{if} \quad c_i = \cdots = c_i = b, 1 \leq i_1 < \cdots < i_k \leq m, \quad \text{and all other} \quad c_i = a.
\]

The weight of the empty word \(\varepsilon\) is defined to be \(w(\varepsilon) = 1\).

We then have
\[
w(ac)(s) = xw(c)(qs),
\]
\[
w(bc)(s) = qsw(c)(qs),
\]
\[
w(ca)(s) = xw(c)(s),
\]
\[
w(cb)(s) = q^{m+1} sw(c)(s). \tag{2.6}
\]

Define the length \(l(c)\) of a word \(c\) consisting of \(k\) letters \(b\) and \(m-k\) letters \(a\) by
\[
l(c) = 2k + m - k = m + k.
\]

Let \(\Phi_n\) be the set of all words of length \(n-1\).

\(\Phi_n\) can also be identified with the set of all coverings of an \((n-1) \times 1\) rectangle with monominos (i.e. \(1 \times 1\) rectangles) and dominos (i.e. \(2 \times 1\) rectangles) or with Morse code sequences of length \(n-1\).

Let \(G_n(x, s) := \sum_{c \in \Phi_n} w(c)\) be the weight of \(\Phi_n\).
Then we get
\[ G_n(x,s) := \sum_{c \in \Phi_n} w(c) = F_n(x,s,q), \] (2.7)
i.e. \( F_n(x,s,q) \) is the weight of \( \Phi_n \).
For the proof observe that by considering the first letter of each word we see from (2.6) that
\[ G_n(x,s) = xG_{n-1}(x,qs) + qsG_{n-2}(x,qs). \]
Thus \( G_n(x,s) \) satisfies the same recurrence as \( F_n(x,s,q) \). Also the initial values coincide because \( G_0(x,s) = 0 \) and \( G_1(x,s) = 1 \).

For example \( \Phi_4 = \{aaa,ab,ba\} \) and
\[ G_4(x,s) = w(aaa) + w(ab) + w(ba) = x^3 + xq^2s + qsx = F_4(x,s,q). \]

This interpretation can also be used to obtain (2.3), which is equivalent with
\[ G_n(x,s) = xG_{n-1}(x,s) + q^{n-2}sG_{n-2} \left( x, \frac{s}{q} \right). \]
Here we consider the last letter of each word. From (2.6) we get \( w(ca)(s) = xw(c)(s) \), which gives the first term. To obtain the second term let us suppose that \( cb \in \Phi_n \) has \( k \) letters \( b \).

Then \( w(cb)(s) = q^msw(c)(s) = q^{n-1-k}s(q^{k}+\cdots+q^{k-1})x^{n-k-1} = q^{n-2}sw(c)\left( \frac{s}{q} \right) \). Since this expression is independent of \( k \), we get the second term.

Let \( D \) be the \( q \) – differentiation operator defined by
\[ Df(x) = \frac{f(x) - f(qx)}{(1-q)x}. \]
As has been shown in [5] these \( q \) – Fibonacci polynomials also satisfy
\[ F_n(x,s,q) = F_n(x+(q-1)sD,s)1. \] (2.8)
In order to show (2.8) we must verify that
\[ F_n(x,s,q) = xF_{n-1}(x,s,q) + (q-1)sDF_{n-1}(x,s,q) + sF_{n-2}(x,s,q). \] (2.9)
Comparing coefficients this amounts to
\[ q^k \begin{bmatrix} n-1-k \\ k \end{bmatrix} = q^k \begin{bmatrix} n-2-k \\ k \end{bmatrix} + \begin{bmatrix} n-2-k \\ k-1 \end{bmatrix} + q^{n-2k} \begin{bmatrix} n-1-k \\ k-1 \end{bmatrix}, \]
or
\[ (q^k-1) \begin{bmatrix} n-1-k \\ k \end{bmatrix} = (q^{n-2k}-1) \begin{bmatrix} n-1-k \\ k-1 \end{bmatrix}, \]
which is obviously true.
As in [5] we define the $q$–Lucas polynomials by

$$L_n(x, s, q) = L_n \left( x + (q-1)sD, s \right).$$

(2.10)

The first polynomials are

$$2, x, x^2 + (1 + q)s, x^3 + [3]sx, x^4 + [4]sx^2 + q(1 + q^2)s^2, \cdots$$

By applying the linear map

$$f(x) \rightarrow f(x + (q-1)sD),$$

(2.11)
to (1.5) we get

$$L_n(x, s, q) = F_{n+1}(x, s, q) + sF_{n-1}(x, s, q)$$

(2.12)

for $n > 0$.

This implies the explicit formula

$$L_n(x, s, q) = \sum_{k=0}^{n} q^k \binom{n}{k} \frac{[n]}{[n-k]} \left[ \frac{n-k}{k} \right] s^k x^{n-2k}$$

(2.13)

for $n > 0$, which is a very nice $q$–analogue of (1.2).

For the proof observe that

$$q^k \binom{n-k}{k} + \binom{n-k-1}{k-1} = q^k \frac{[n-k]}{[n-k]} \binom{n-k}{k} = \frac{[n]}{[n-k]} \left[ \frac{n-k}{k} \right].$$

Comparing coefficients we also get

$$L_n(x, qs, q) = F_{n+1}(x, s, q) + q^n sF_{n-1}(x, s, q).$$

(2.14)

This follows from

$$q^k \binom{n-k}{k} + q^{n-k} \binom{n-k-1}{k-1} = q^k \frac{[n-k]}{[n-k]} \binom{n-k}{k} = q^k \frac{[n-k]}{[n-k]} \left[ \frac{n-k}{k} \right].$$

**Remark 2**

(2.12) has the following combinatorial interpretation:

Consider a circle whose circumference has length $n$ and let monominos be arcs of length 1 and dominos be arcs of length 2 on the circle. Consider the set $\Lambda_n$ of all coverings with monominos and dominos and fix a point $P$ on the circumference of the circle. If $P$ is the initial point of a monomino or a domino of a covering then this covering can be identified
with a word \( c = c_1 \cdots c_m \). We define its weight in the same way as in the linear case.

Therefore the set of all those coverings has weight \( F_{n+1}(x,s,q) \).

If \( P \) is the midpoint of a domino we split \( b \) into \( b = b_0 b_1 \) and associate with this covering the word \( b_0 c_1 \cdots c_m b_0 \) with \( c_1 \cdots c_m \in \Phi_{n-1} \) and define its weight as \( sw(c_1 \cdots c_m) \).

Therefore \( w(\Lambda_n) = w(\Phi_{n+1}) + sw(\Phi_{n-1}) \).

E.g. \( \Lambda_4 = \{aaaa, aab, aba, baa, bb, b_a b_0, b_0 b_b, b_b b_b \} \). Thus

\[
w(\Lambda_4) = x^4 + x^2 q^3 s + x q s^2 x + q s q^2 s + s x^2 + q s x q = x^4 + [4] s x^2 + q(1 + q^2) s^2 = L_4(x,s,q).
\]

To give a combinatorial interpretation of (2.14) we consider all words of \( \Lambda_n \) with last letter \( a \) or the two last letters \( a b \). Their weight is \( F_n(x,s,q) + s F_{n-1}(x,s,q) = F_{n+1}\left( x, \frac{s}{q}, q \right) \).

There remains the set of all words in \( \Lambda_n \) with last letter \( b \). With the same argument as above we see that this is \( q^{n-1} s F_{n-1}\left( x, \frac{s}{q}, q \right) \). Therefore we have

\[
L_n(x,s,q) = F_{n+1}\left( x, \frac{s}{q}, q \right) + q^{n-1} s F_{n-1}\left( x, \frac{s}{q}, q \right),
\]

which is equivalent with (2.14).

We also need the polynomials \( L^*_n(x,s,q) \) which coincide with \( L_n(x,s,q) \) for \( n > 0 \), but have initial value \( L^*_0(x,s,q) = 1 \).

Comparing (2.10) with (1.4) we see that

\[
L_n(x,s,q) = x L_{n-1}(x,s,q) + (q-1)s DL_{n-1}(x,s,q) + sL_{n-2}(x,s,q).
\]  

(2.15)

This is a recurrence for the polynomials in \( x \) but not for individual numbers \( x \) and \( s \).

In order to find a recurrence for individual numbers I want to show first that for \( n > 2 \)

\[
L^*_n(x,s,q) - xL^*_{n-1}(x,s,q) - sL^*_{n-2}(x,s,q) = (1 - q^{n-1}) \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k q^{(k-1)(n-k-1)} s^k L^*_{n-2k}(x,s,q).
\]  

(2.16)

This reduces to (1.4) for \( q = 1 \).

It is easily verified that \( DL_n(x,s,q) = [n] F_n\left( x, \frac{s}{q}, q \right) \).

Therefore

\[
L_n(x,s,q) - xL_{n-1}(x,s,q) - sL_{n-2}(x,s,q) = (q-1)s DL_{n-1}(x,s,q) = \left( q^{n-1} - 1 \right) s F_{n-1}\left( x, \frac{s}{q}, q \right).
\]
By (2.14) we know that \( F_{n+1} \left( x, \frac{s}{q} \right) = L_{n-1}(x, s, q) - q^{n-3} s F_{n-3} \left( x, \frac{s}{q} \right) \).

Iteration gives (2.16).

From (2.16) we get

\[
L_n(x, s, q) - xL_{n-1}(x, s, q) - sL_{n-2}(x, s, q) = \left( q^{n-1} - 1 \right) sL_{n-2}(x, s, q)
\]

\[-q^{n-3} s \left[ \frac{n-1}{n-3} \right] \left( L_{n-2}(x, s, q) - xL_{n-3}(x, s, q) - sL_{n-4}(x, s, q) \right) \]

This can be written as

\[
L_n(x, s, q) = xL_n(x, s, q) - \frac{(1 + q)q^{n-3}}{n-3} sL_{n-2}(x, s, q) + \frac{[n-1]}{n-3} q^{n-3} s xL_{n-3}(x, s, q) + \frac{[n-1]}{n-3} q^{n-3} s L_{n-4}(x, s, q) (2.17)
\]

This recurrence holds for \( n \geq 4 \) if \( L_0(x, s, q) = 2 \).

3. Inversion theorems

L. Carlitz [2] has obtained two \( q \) – analogues of the Chebyshev inversion formulas. The first one ([2], Theorem 6) implies

**Theorem 3.1**

\[
\sum_{2k \leq n} \binom{n}{k} L_{n-2k}(x, s, q)(-s)^k = x^n \quad (3.1)
\]

and the second one ([2], Theorem 7) gives

**Theorem 3.2**

\[
\sum_{2k \leq n} \left( \binom{n}{k} - \binom{n}{k-1} \right) F_{n+1-2k}(x, s, q)(-s)^k = x^n. \quad (3.2)
\]

These are \( q \) – analogues of (1.6) and (1.7). We give another proof of these theorems:

Let

\[
A = x + (q-1)sD. \quad (3.3)
\]

Define

\[
\alpha(q) = \frac{A + \sqrt{A^2 + 4s}}{2} \quad (3.4)
\]

and
\[
\beta(q) = \frac{A - \sqrt{A^2 + 4s}}{2}. \tag{3.5}
\]

Since \(\alpha(q)^2 - A\alpha(q) - s = \beta(q)^2 - A\beta(q) - s = 0\) the sequences \((\alpha(q)^n)_{-\infty}^\infty\) and \((\beta(q)^n)_{-\infty}^\infty\) satisfy the recurrence

\[
\alpha(q)^n - A\alpha(q)^{n-1} - s\alpha(q)^{n-2} = \beta(q)^n - A\beta(q)^{n-1} - s\beta(q)^{n-2} = 0
\]

for all \(n \in \mathbb{Z}\).

Since the \(q\)–Fibonacci and the \(q\)–Lucas polynomials satisfy the same recurrence we get from the initial values

\[
L_n(x, s, q) = (\alpha(q)^n + \beta(q)^n)1 \tag{3.6}
\]

and

\[
F_n(x, s, q) = \frac{\alpha(q)^n - \beta(q)^n}{\alpha(q) - \beta(q)}1 \tag{3.7}
\]

for \(n \geq 0\). We can use these identities to extend these polynomials to negative \(n\).

We then get for \(n > 0\)

\[
L_{-n}(x, s, q) = (\alpha(q)^{-n} + \beta(q)^{-n})1 = (-1)^n \frac{\beta(q)^n + \alpha(q)^n}{s^n} = (-1)^n \frac{L_n(x, s, q)}{s^n} \tag{3.8}
\]

and

\[
F_{-n}(x, s, q) = \frac{\alpha(q)^{-n} - \beta(q)^{-n}}{\alpha(q) - \beta(q)}1 = (-1)^{n-1} \frac{F_n(x, s, q)}{s^n}. \tag{3.9}
\]

**Remark 3**

It is easily verified that the identities (2.2), (2.3), (2.4), (2.12) and (2.14) hold for all \(n \in \mathbb{Z}\).

We want to show that

\[
\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-s)^k \binom{n}{k} L_{n-2k}(x, s, q) = x^n. \tag{3.10}
\]

For odd \(n\) the left-hand side is
$$\sum_{k=0}^{n} \binom{n}{k} (\alpha(q)\beta(q))^k \binom{n}{k} (\alpha(q)^{n-k} + \beta(q)^{n-k}) = \sum_{k=0}^{n} \binom{n}{k} (\alpha(q)^{n-k} \beta(q)^k + \alpha(q)^k \beta(q)^{n-k}) \cdot \binom{n}{k} (\alpha(q)^k \beta(q)^{n-k})^1.$$ 

For \( n = 2m \) the same holds because for \( k = m \) the coefficient of \((-s)^{m}\) is

\[ L_0(x,s,q) = 1. \]

Therefore by (1.10) we see that \( R(n,x,s) := \sum_{k=0}^{n} (-s)^k \binom{n}{k} L_{n-2k}(x,s,q) \) satisfies

\[ R(n,x,s) = (\alpha_q + \beta_q) R(n-1,x,s) + \alpha_q \beta_q (q^{n-1} - 1) R(n-2,x,s) \]

\( = (x + (q-1)sD) R(n-1,x,s) - s (q^{n-1} - 1) R(n-2,x,s). \)

We have to show that \( R(n,x,s) = x^n. \)

This is obviously true for \( n = 0 \) and \( n = 1. \)

If it holds for \( m < n \) then

\[ R(n,x,s) = (x + (q-1)sD) x^{n-1} - s (q^{n-1} - 1) x^{n-2} = x^n \]

as asserted.

**Remark 4**

In [6] we have defined a new \( q \)–analogue of the Hermite polynomials \( H_n(x,s | q) = (x - sD)^n. \) By applying the linear map (2.11) to (1.6) and (1.7) these can be expressed as

\[ H_n(x,(q-1)s | q) = \sum_{k=0}^{n/2} \binom{n}{k} s^k L_{n-2k}^*(x,-s,q) = \sum_{k=0}^{(n+1)/2} \binom{n}{k} - \binom{n}{k-1} s^k F_{n+1-2k}(x,-s,q). \]

**Remark 5**

The polynomials \( F_{n+1}(x,s,q) \) are a basis of the polynomials in \( \mathbb{C}(s,q)[x]. \) Define a linear functional \( L \) on this vector space by \( L(F_{n+1}(x,s,q)) = [n = 0]. \)

Then from (3.2) we get

\[ L(x^{2n+1}) = 0 \] and \( L(x^{2n}) = (-qs)^n C_n(q), \) where \( C_n(q) = \frac{1}{(n+1) \binom{2n}{n}} \) is a \( q \)–analogue of the Catalan numbers. This is equivalent with

\[ \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \binom{2n-k}{k} C_{n-k}(q) = [n = 0]. \]
In the same way the linear functional $M$ defined by $M(L_n(x,s,q)) = [n = 0]$ gives

$M(x^{2n}) = \left[ \begin{array}{c} 2n \\ n \end{array} \right] (-s)^n$ and $M(x^{2n+1}) = 0$. This is equivalent with

$$\sum_{k=0}^{n} (-1)^k q^{\frac{k}{2}} \frac{[2n]}{[2n-k]} \left[ \begin{array}{c} 2n-k \\ k \end{array} \right] \left[ \begin{array}{c} 2n-2k \\ n-k \end{array} \right] = 0$$

for $n > 0$.

4. Some related identities

The classical Fibonacci and Lucas polynomials satisfy

$$L_n(x + y, -xy) = x^n + y^n$$

and

$$F_n(x + y, -xy) = \frac{x^n - y^n}{x - y}.$$ 

L. Carlitz [2] has given $q$–analogues of these theorems which are intimately connected with our $q$–analogues.

**Theorem 4.1 (L. Carlitz[2])**

Let $r_n(x, y) = \sum_{k=0}^{n} q^{\frac{n-k}{2}} \left[ \begin{array}{c} n-k \\ k \end{array} \right] x^k y^{n-k}$ be the Rogers-Szegö polynomials. Then

$$\frac{x^n - y^n}{x - y} = \sum_{k=0}^{n-1} q^{\frac{k}{2}} \left[ \begin{array}{c} n-k-1 \\ k \end{array} \right] (-xy)^k r_{n-1-2k}(x, y)$$

and

$$x^n + y^n = \sum_{k=0}^{n} q^{\frac{k}{2}} \left[ \begin{array}{c} n \\ n-k \end{array} \right] \left[ \begin{array}{c} n-k \\ k \end{array} \right] (-xy)^k r_{n-2k}(x, y).$$

These are polynomial identities which for $(x, y) \to (\alpha, \beta)$ immediately give the explicit formulae for the $q$–Fibonacci and $q$–Lucas-polynomials if we define them by (3.7) and (3.6).

To prove these theorems we use the identity

$$\sum_{j=0}^{k} (-1)^j q^{\frac{j}{2}} \left[ \begin{array}{c} k \\ j \end{array} \right] \left[ \begin{array}{c} n-j \\ k \end{array} \right] = 1$$
for $0 \leq k \leq n$ (cf. Carlitz[2]).

To show this identity let $n$ be fixed and let $U$ be the $\mathbb{C}(q)$–linear operator on the vector space of all sums $\sum_{k=0}^{n} c_k \left[\frac{x-k}{n-k}\right]$ with $c_k \in \mathbb{C}(q)$ defined by $U \left[\frac{x-k}{n-k}\right] = \left[\frac{x-k-1}{n-k}\right]$ for $0 \leq k \leq n$.

Since $(1-q^{n-k}U) \left[\frac{x-k}{n-k}\right] = \left[\frac{x-k-1}{n-k}\right] - q^{n-k} \left[\frac{x-k-1}{n-k}\right] = \left[\frac{x-k-1}{n-k}\right]

by using (1.8) we get the desired result

$$
\sum_{j=0}^{\infty} (-1)^{j} q^{\left(j+1\right)/2} \left[\frac{x}{n}\right] \left[\frac{x-j}{n}\right] = (1-qU) \cdots (1-q^{n-1}U) \left[\frac{x}{n}\right] = (1-qU) \cdots (1-q^{n-1}U) \left[\frac{x-1}{n-1}\right] = \cdots = \left[\frac{x-n}{0}\right].
$$

First we prove (4.3).

$$
\sum_{j=0}^{\infty} (-1)^{j} q^{\left(k+j\right)/2} \left[\frac{n-k-1}{k}\right] (-xy)^k \frac{r_{n-2k}(x,y)}{k!} = \sum_{j=0}^{\infty} \left[\frac{n-k-1}{k}\right] (-xy)^k \int_{j}^{n-2k} \left[\frac{n-1-2k}{i}\right] \frac{x^i}{y^{n-1-2k-j}}
$$

$$
= \sum_{i=0}^{n-1} x^{i} y^{n-1-i} \sum_{j=0}^{\infty} (-1)^{j} q^{\left(k+j\right)/2} \left[\frac{n-k-1}{k}\right] \left[\frac{n-1-2k}{i-k}\right] = \sum_{i=0}^{n-1} x^{i} y^{n-1-i} \sum_{j=0}^{\infty} (-1)^{j} q^{\left(k+j\right)/2} \left[\frac{n-1-k}{i}\right]
$$

$$
= \sum_{i=0}^{n-1} x^{i} y^{n-1-i} = \frac{x^n - y^n}{x - y}.
$$

For the proof of (4.4) observe that

$$
\frac{x^{n+1} - y^{n+1}}{x - y} - xy \frac{x^{n-1} - y^{n-1}}{x - y} = \frac{x^n (x-y) + y^n (x-y)}{x - y} = x^n + y^n.
$$

This implies

$$
x^n + y^n = \sum_{k} q^{\left(k+1\right)/2} \left[\frac{n-k}{k}\right] (-xy)^k \frac{r_{n-2k}(x,y)}{k!} - xy \sum_{k} q^{\left(k+1\right)/2} \left[\frac{n-k-2}{k}\right] (-xy)^k \frac{r_{n-2k}(x,y)}{k!}
$$

$$
= \sum_{k} q^{\left(k+1\right)/2} \left[\frac{n-k}{k}\right] (-xy)^k \frac{r_{n-2k}(x,y)}{k!} - \sum_{k} q^{\left(k+1\right)/2} \left[\frac{n-k-1}{k-1}\right] (-xy)^k \frac{r_{n-2k}(x,y)}{k!}
$$

$$
= \sum_{k} q^{\left(k+1\right)/2} \left[\frac{n-k}{k}\right] (-xy)^k \frac{r_{n-2k}(x,y)}{k!} - \sum_{k} q^{\left(k+1\right)/2} \left[\frac{n-k}{k}\right] (-xy)^k \frac{r_{n-2k}(x,y)}{k!}.
$$

For the classical Fibonacci polynomials the formula

$$
\sum_{j=0}^{\infty} \left(\frac{n-1-k}{i}\right) = \sum_{i=0}^{n-1} x^{i} y^{n-1-i} = \frac{x^n - y^n}{x - y}.
$$

For the classical Fibonacci polynomials the formula
\[
\sum_{k=0}^{n} (-1)^k x^k \binom{n}{k} F_{2n+m-k}(x,s) = s^n F_m(x,s)
\]  

(4.6)

holds for all \( m \in \mathbb{Z} \). This is an easy consequence of the Binet formula \( F_n(x,s) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \), where \( \alpha = \frac{x + \sqrt{x^2 + 4s}}{2} \) and \( \beta = \frac{x - \sqrt{x^2 + 4s}}{2} \). For (4.6) is equivalent with

\[
\frac{\alpha^{n+m}(\alpha - x)^n - \beta^{n+m}(\beta - x)^n}{\alpha - \beta} = s^n \frac{\alpha^m - \beta^m}{\alpha - \beta}.
\]

We now get

**Theorem 4.2**

\[
\sum_{k=0}^{n} (-1)^k q^k \binom{n}{k} x^k F_{2n+m-k}(x,s,q) = q^m s^n F_m \left( x, \frac{s}{q}, q \right).
\]

(4.7)

The case \( m = 0 \) gives

**Corollary 4.3 (H. Prodinger [8])**

\[
\sum_{k=0}^{n} (-1)^k q^k \binom{n}{k} x^k F_{2n-k}(x,s,q) = 0.
\]

(4.8)

**Proof**

For \( n = 0 \) this is trivially true for all \( m \in \mathbb{Z} \).

For \( n = 1 \) (4.7) reduces to

\[
F_{m+2}(x,s,q) - x F_{m+1}(x,s,q) = q^m s F_m \left( x, \frac{s}{q}, q \right),
\]

(4.9)

which also holds for \( m \in \mathbb{Z} \) by (2.3) and Remark 3.

Assume that (4.7) holds for \( i < n \) and all \( m \). Then we get
\[
\sum_{k} (-1)^k q^{\frac{k}{2}} \binom{n}{k} x^k F_{2n+m-k}(x,s,q) = \sum_{k} (-1)^k q^{\frac{k}{2}} \binom{n-1}{k} x^k F_{2n+m-k}(x,s,q)
\]
\[
+ \sum_{k} (-1)^k q^{\frac{k}{2}} \binom{n-1}{k-1} x^k F_{2n+m-k}(x,s,q)
\]
\[
= \sum_{k} (-1)^{k-1} q^{\frac{k}{2}} \binom{n-1}{k-1} x^{k-1} F_{2n+m-k+1}(x,s,q) + \sum_{k} (-1)^k q^{\frac{k}{2}} \binom{n-1}{k-1} x^k F_{2n+m-k}(x,s,q)
\]
\[
= \sum_{k} (-1)^{k-1} q^{\frac{k}{2}} \binom{n-1}{k-1} x^{k-1} F_{2n+m-k+1}(x,s,q) - x F_{2n+m-k}(x,s,q)
\]
\[
= \sum_{k} (-1)^{k} q^{\frac{k+1}{2}} \binom{n-1}{k} x^k q^{2n+m-k-2} F_{2n+m-k-2} \left( x, \frac{s}{q}, q \right)
\]
\[
= q^{2n+m-2} \sum_{k} (-1)^k q^{\frac{k}{2}} \binom{n-1}{k} x^k F_{2(n-1)+m-k} \left( x, \frac{s}{q}, q \right)
\]
\[
= q^{2n+m-2} \binom{n-1}{m(n-1)} s^{n(m-1)} F_m \left( x, \frac{s}{q^n}, q \right) = q^{\binom{n}{2}} s^n F_m \left( x, \frac{s}{q^n}, q \right).
\]

For
\[
h(n,m) = \sum_{k=0}^{n} (-1)^k q^{\frac{k}{2}} \binom{n}{k} x^k L_{2n+m-1-k}(x,s,q) \]
we get from
\[
L_n(x,s,q) = F_{n+1}(x,s,q) + s F_{n-1}(x,s,q)
\]
\[
h(n,m) = q^{\binom{n}{2}} s^n F_m \left( x, \frac{s}{q^n}, q \right) + s^{n+1} q^{\binom{n}{2} + mn - 2n} F_{m-2} \left( x, \frac{s}{q^n}, q \right).
\]

This implies
\[
\sum_{k=0}^{n} (-1)^k q^{\frac{k}{2}} \binom{n-1}{k} + \binom{n-1}{k} L_{2n+m-1-k}(x,s,q) = q^{n-1} h(n,m) + h(n-1,m+2).
\]

Because of (3.9) we get
\[
\sum_{k=0}^{n} (-1)^k q^{\frac{k}{2}} \binom{n-1}{k} + \binom{n-1}{k} L_{2n-1-k}(x,s,q) = q^{n-1} h(n,0) + h(n-1,2)
\]
\[
= q^{n-1} s^{n+1} q^{\binom{n}{2} - 2n} F_{-2} \left( x, \frac{s}{q^n}, q \right) + q^{\binom{n-1}{2} + 2(n-1)} s^{n-1} F_{2} \left( x, \frac{s}{q^{n-1}}, q \right)
\]
\[
= -s^{n+1} q^{\binom{n}{2} - n-1} \left( \frac{q^n}{s} \right) x + q^{\binom{n-1}{2} + 2n-2} s^{n-1} x = 0.
\]
This gives

**Theorem 4.4 (H. Prodinger [8])**

\[
\sum_{k=0}^{n} (-1)^k \binom{k}{2} q^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-1-k} (x, s, q) = 0. \tag{4.13}
\]

5. Some Rogers-Ramanujan type formulas

It is interesting that the \( q - \) Lucas polynomials give a simple approach to the Bailey pairs \( A(1) - A(8) \) of Slater’s paper [10].

Let us recall some definitions (cf. [1] or [7]) suitably modified for our purposes.

Two sequences \( a = (\alpha_n) \) and \( b = (\beta_n) \) are called a Bailey pair \((a, b)_m\), if

\[
\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q; q)_{n-k} (q; q)_{n+k+m}}
\]

for some \( m \in \{0, 1\} \). Note that \( (q; q)_n = (1 - q)(1 - q^2)\cdots(1 - q^n) \).

To obtain Bailey pairs we start with formula (3.1) and consider separately even and odd numbers \( n \). This gives

\[
\sum_{k=0}^{n} \binom{2n}{n-k} L_{2k}^s(x, -s(q), q) s(q)^{n-k} = x^{2n}
\]

and

\[
\sum_{k=0}^{n} \binom{2n+1}{n-k} L_{2k+1}^s(x, -s(q), q) s(q)^{n-k} = x^{2n+1}.
\]

Therefore

**Theorem 5.1**

\[
a = \left( L_{2n}^s(x, -s(q), q) s(q)^{-n} \right), b = \left( \frac{x^{2n}}{(q; q)_{2n}} \right)
\]

is a Bailey pair with \( m = 0 \)

and

\[
a = \left( L_{2n+1}^s(x, -s(q), q) s(q)^{-n} \right), b = \left( \frac{x^{2n+1}}{(q; q)_{2n+1}} \right)
\]

one with \( m = 1 \).
If we change $q \to \frac{1}{q}$ we get the Bailey pairs

$$a = \left( L_{n}^{2}(x, -s(q^{-1}), q^{-1})q^n s(q^{-1})^{-n} \right), b = \left( \frac{x^n q^n}{s(q^{-1})^n (q; q)_{2n}} \right)$$

(5.6)

with $m = 0$ and

$$a = \left( L_{n+1}^{2}(x, -s(q^{-1}), q^{-1})q^{n+1} s(q^{-1})^{-n} \right), b = \left( \frac{x^{n+1} q^{n+1}}{s(q^{-1})^n (q; q)_{2n+1}} \right)$$

(5.7)

with $m = 1$.

For each Bailey pair we consider the identity

$$\sum_{n \geq 0} q^{n^2 + mn} \beta_n = \sum_{n \geq 0} q^{n^2 + mn} \frac{\alpha_k}{\sum_{k=0}^{n} (q; q)_{n-k} (q; q)_{n+k+m}} = \sum_{n \geq 0} \frac{\alpha_k}{\sum_{k=0}^{n} (q; q)_{n-k} (q; q)_{n+k+m}} q^{n^2 + mn}.$$  

(5.8)

Here the inner sum $\sum_{n \geq k} (q; q)_{n-k} (q; q)_{n+k+m}$ can be easily computed:

For $k \in \mathbb{N}$ we have

$$\sum_{s \geq k} \frac{q^{s^2 + ks}}{(q; q)_{s-k}} = \frac{q^{s^2 + ki}}{(q; q)_{\infty}}$$

(5.9)

This is an easy consequence of the $q$–Vandermonde formula

$$\sum_{s=0}^{\infty} q^{(x-i)(s+i+k)} \left[ \frac{n+2i}{s+i+k} \right] \left[ \frac{n-2i}{s-i} \right] = \sum_{j=0}^{n+2i} q^{(2i+j+k)} \left[ \frac{n+2i}{n-j-k} \right] \left[ \frac{n-2i}{j} \right] = \left[ \frac{2n}{n-k} \right]$$

if we let $n \to \infty$.

Therefore we get

$$\sum_{n \geq 0} q^{n^2 + mn} \beta_n = \sum_{k \geq 0} \alpha_k \sum_{n \geq k} (q; q)_{n-k} (q; q)_{n+k+m} = \frac{1}{(q; q)_{\infty}} \sum_{k \geq 0} \alpha_k q^{k^2 + mk}.$$  

(5.10)

In the following formulas we set $x = 1$ and $m \in \{0, 1\}$.

For $s = 1$ we get from (5.10), (5.4) and (5.5)

$$\sum_{n \geq 0} q^{n^2 + mn} (q; q)_{2n+m} = \frac{1}{(q; q)_{\infty}} \sum_{j \geq 0} L_{2n+m}^{2}(1, -1, q)q^{j^2 + mj}.$$  

(5.11)
For $s = \frac{1}{q}$ we get

$$
\sum_{m=0}^{\infty} q^{n^2 + mn} (q; q)_{2n+m} = \frac{1}{(q; q)_\infty} \sum_{k=0}^{\infty} L^*_k (1, -q^{-1}, q) q^{s^2 + km}.
$$

(5.12)

In the same way we get from (5.6) and (5.7) for $s = 1$

$$
\sum_{m=0}^{\infty} q^{2n^2 + 2mn} (q; q)_{2n+m} = \sum_{m=0}^{\infty} q^{n^2 + mn} \sum_{i=0}^{\infty} L^*_i (1, -q^{-1}, q) q^2 \sum_{m=0}^{\infty} L^*_m (1, -q^{-1}, q) q^{2s^2 + 2mi}
$$

(5.13)

and for $s(q) = \frac{1}{q}$

$$
\sum_{m=0}^{\infty} q^{2n^2 - n + 2mn} (q; q)_{2n+m} = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} L^*_i (1, -q^{-1}, q) q^{2s^2 - i + mi} \sum_{m=0}^{\infty} L^*_m (1, -q^{-1}, q) q^{2s^2 - 2mi}.
$$

(5.14)

The main advantage of these formulas derives from the fact, that the $q$–Lucas polynomials have simple values for $x = 1$ and $s = -1$ or $s = -\frac{1}{q}$.

From (2.4) it is easily verified (cf. [5]) that

$$
F_{3n} \left(1, -\frac{1}{q}, q\right) = 0, \quad F_{3n+1} \left(1, -\frac{1}{q}, q\right) = (-1)^n q^{\frac{n(n-1)}{2}}, \quad F_{3n+2} \left(1, -\frac{1}{q}, q\right) = (-1)^n q^{\frac{n(n+1)}{2}}.
$$

(5.15)

Therefore by (2.14)

$$
L_{3n} (1, -1, q) = (-1)^n \left( q^{\frac{n(n-1)}{2}} + q^{\frac{n(n+1)}{2}} \right), \quad L_{3n+1} (1, -1, q) = (-1)^n q^{\frac{n(n+1)}{2}},
$$

(5.16)

$$
L_{3n+2} (1, -1, q) = (-1)^n q^{\frac{n(n-1)}{2}}.
$$

Of course in all formulas $L'_q (1, s, q) = 1$, although I shall not state this in each case explicitly.

(5.16) implies

$$
L'_{6n} (1, -1, q) = L'_{2(3n-1)} (1, -1, q) = -q^{6n^2 - 5n + 1}, \quad L'_{6n} (1, -1, q) = L'_{2(3n)} (1, -1, q)
$$

$$
= q^{6n^2 - n} + q^{6n^2 + n}, \quad L'_{6n+2} (1, -1, q) = L'_{2(3n+1)} (1, -1, q) = -q^{6n^2 + 5n + 1}
$$

(5.17)

and

$$
L'_{6n-1} (1, -1, q) = L'_{2(3n-1)+1} (1, -1, q) = q^{6n^2 - n}, \quad L'_{6n+3} (1, -1, q) = L'_{2(3n+1)+1} (1, -1, q)
$$

$$
= -q^{6n^2 + 5n + 1} + q^{6n^2 + 7n + 2}, \quad L'_{6n+1} (1, -1, q) = L'_{2(3n+1)} (1, -1, q) = q^{6n^2 + n}.
$$

(5.18)
The first terms of the sequence \( L_n'(1, -1, q) \) are therefore
\[1, -q, q^2, q^5 - q^7, -q^{12}, -q^{15}, q^{22} + q^{26}, \ldots\]
The sum of all these terms is Euler's pentagonal number series.
The same is true for the sequence \( L_{2n+1}'(1, -1, q) \), which begins with
\[1, -q - q^2, q^5, q^7, -q^{12}, -q^{15}, q^{22}, q^{26}, \ldots\]
This is an immediate consequence of (5.4) and (5.5) for \( x = s = 1 \), which reduce to
\[\sum_{k=0}^{n} L_{2k}'(1, -1, q) = \frac{1}{(q; q)_{2n}} \quad (5.19)\]
and
\[\sum_{k=0}^{n} L_{2k+1}'(1, -1, q) = \frac{1}{(q; q)_{2n+1}}. \quad (5.20)\]
If we let \( n \to \infty \) these formulas converge to
\[\sum_{k=0}^{\infty} L_{2k}'(1, -1, q) = (q; q)_\infty\]
and \( \sum_{k=0}^{\infty} L_{2k+1}'(1, -1, q) = (q; q)_\infty \) respectively.

By (2.12) we get
\[L_{3n}(1, -q^{-1}, q) = (-1)^n \begin{pmatrix} n(3n-1) \\ q \end{pmatrix} + q^{\frac{n(3n-5)}{2}} \quad \text{for } n > 0\]
\[L_{3n+1}(1, -q^{-1}, q) = (-1)^n q^{\frac{n(3n+1)}{2}}, \quad (5.21)\]
\[L_{3n+1}(1, -q^{-1}, q) = (-1)^n q^{\frac{(n-2)(3n-1)}{2}}. \]

This implies that
\[L_{6n-2}'(1, -\frac{1}{q}, q) = L_{2(3n-1)}'(1, -\frac{1}{q}, q) = -q^{6n^2-5n+1}, \quad L_{6n}'(1, -\frac{1}{q}, q) = L_{2(3n)}'(1, -\frac{1}{q}, q) \]
\[= q^{6n^2-n} + q^{6n^2-5n}, \quad L_{6n+2}'(1, -\frac{1}{q}, q) = L_{2(3n+1)}'(1, -\frac{1}{q}, q) = -q^{6n^2-n-1} \quad (5.22)\]
and
\[L_{6n-1}'(1, -\frac{1}{q}, q) = L_{2(3n-1)+1}'(1, -\frac{1}{q}, q) = q^{6n^2-7n+1}, \quad L_{6n+3}'(1, -\frac{1}{q}, q) = L_{2(3n+1)+1}'(1, -\frac{1}{q}, q) \]
\[= -q^{6n^2+5n+1} - q^{6n^2+n-1}, \quad L_{6n+1}'(1, -\frac{1}{q}, q) = L_{2(3n)+1}'(1, -\frac{1}{q}, q) = q^{6n^2+n} \quad (5.23)\]
Now it is time to harvest the Corollaries. We order them so that Corollary 5.1 corresponds to Slater's $A(i)$.

**Corollary 5.1** (cf. [9], A.79)

\[ \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_{2n}} = \frac{1}{(q; q)_\infty} \sum_{k \in \mathbb{Z}} (q^{15k^2 + k} - q^{15k^2 + 11k + 2}). \] 

(5.24)

**Proof.**

Choose $m = 0$ in (5.11) and observe that

\[ L_{6i} (1, -1, q) q^{(3i)k^2} = q^{15i^2-k} + q^{15i^2+k}, \quad L_{6i+2} (1, -1, q) q^{(3i+1)k^2} = q^{15i^2+11i+2} \]

and

\[ L_{6i-2} (1, -1, q) q^{(3i-1)k^2} = q^{15i^2-11i+2} \]

which implies

\[ \sum_{i \geq 0} L^*_{2i} (1, -1, q) q^{k^2} = \sum_{k \in \mathbb{Z}} (q^{15k^2+k} - q^{15k^2+11k+2}) \]

and thus (5.24).

**Corollary 5.2** (cf. [9], A.94)

\[ \sum_{n \geq 0} \frac{q^{n^2+n}}{(q; q)_{2n+1}} = \frac{1}{(q; q)_\infty} \sum_{k \in \mathbb{Z}} (q^{15k^2-4k} - q^{15k^2+14k+3}). \]

(5.25)

**Proof.**

We use formula (5.11) for $m = 1$ and compute

\[ L^*_{6n-1} (1, -1, q) q^{(3(n-1))k^2+3n-1} = q^{15(n-1)^2+4n+3}, \quad L^*_{6n+3} \left( 1, -\frac{1}{q}, q \right) q^{(3n+1)k^2+3n+1} = -q^{15n^2+14n+3} - q^{15n^2+16n+4}, \]

\[ L^*_{6n+1} \left( 1, -\frac{1}{q}, q \right) q^{(3n)k^2+3n} = q^{15n^2+4n}. \]

Since $15(n-1)^2 + 16(n-1) + 4 = 15n^2 - 14n + 3$ we get (5.25).

**Corollary 5.3** (cf. [9], A.99)

\[ \sum_{n \geq 0} \frac{q^{n^2+n}}{(q; q)_{2n}} = \frac{1}{(q; q)_\infty} \sum_{k \geq 0} L^*_{2k} (1, -q^{-1}, q) q^{k^2+k} = \frac{1}{(q; q)_\infty} \sum_{k \in \mathbb{Z}} (q^{15k^2+2k} - q^{15k^2+8k+1}). \]

(5.26)
This follows from (5.12) for $m = 0$ and
\[
L_{6n-2}^* \left( 1, -\frac{1}{q}, q \right) q^{(3n-1)^2 + 2(n-1)} = -q^{15n^2 - 7n + 1},
L_{6n}^* \left( 1, -\frac{1}{q}, q \right) q^{(3n)^2 + 3n} = q^{15n^2 + 2n} + q^{15n^2 - 2n},
L_{6n+2}^* \left( 1, -\frac{1}{q}, q \right) q^{(3n+1)^2 + 3n + 1} = -q^{15n^2 + 8n + 1}.
\]

**Corollary 5.4** (cf. [9], A.38)

\[
\sum_{n \geq 0} \frac{q^{n^2 + 2n}}{(q; q)_{2n+1}} = \frac{1}{(q; q)_5} \sum_{k \in \mathbb{Z}} \left( q^{15k^2 - 7k} - q^{15k^2 + 13k + 2} \right). \tag{5.27}
\]

**Proof.**

This follows from (5.12) for $m = 1$ and the computation
\[
L_{6n-3}^* \left( 1, -\frac{1}{q}, q \right) q^{(3n-1)^2 + 2(3n-1)} = -q^{15n^2 - 7n},
L_{6n+3}^* \left( 1, -\frac{1}{q}, q \right) q^{(3n)^2 + 2(3n+1)} = -q^{15n^2 + 17n + 4} - q^{15n^2 + 13n + 2},
L_{6n+1}^* \left( 1, -\frac{1}{q}, q \right) q^{(3n)^2 + 2(3n)} = q^{15n^2 + 7n}.
\]

Observe that $15(n - 1)^2 + 17(n - 1) + 4 = 15n^2 - 13n + 2$.

**Corollary 5.5** (cf. [9], A.39)

\[
\sum_{n \geq 0} \frac{q^{2n^2}}{(q; q)_{2n}} = \frac{1}{(q; q)_5} \sum_{k \in \mathbb{Z}} \left( q^{12k^2 + 4k} - q^{12k^2 + 7k + 1} \right). \tag{5.28}
\]

**Proof.**

Here we use (5.13) with $m = 0$.
\[
L_{6n-2}^* \left( 1, -\frac{1}{q}, q \right) q^{2(3n-1)^2} = -q^{12n^2 - 7n + 1},
L_{6n}^* \left( 1, -\frac{1}{q}, q \right) q^{2(3n)^2} = q^{12n^2 + n} + q^{12n^2 - n},
L_{6n+2}^* \left( 1, -\frac{1}{q}, q \right) q^{2(3n+1)^2} = -q^{12n^2 + 7n + 1}.
\]
Corollary 5.6 (cf. [9], A.84)

\[ \sum_{n \geq 0} \frac{q^{2n^2 + n}}{(q; q)_{2n+1}} = \frac{1}{(q; q)_\infty} \sum_{k \in \mathbb{Z}} (-1)^k q^{3k^2 - k} = (q^2; q^2)_\infty = (-q; q)_\infty. \]  

(5.29)

**Proof.**

We use (5.14) with \( m = 1 \).

\[ L^{*}_{6n-1} \left( 1, -q, \frac{1}{q} \right) q^{2(3n-1)^2 + 3n-1} = q^{3(2n)^2 - (2n)}, \]
\[ L^{*}_{6n+1} \left( 1, -q, \frac{1}{q} \right) q^{2(3n+1)^2 + 3n+1} = -q^{3(2n+1)^2 - (2n+1)} - q^{3(2n+1)^2 + (2n+1)}, \]
\[ L^{*}_{6n+3} \left( 1, -q, \frac{1}{q} \right) q^{2(3n)^2 + 3n} = q^{3(2n)^2 + (2n)}. \]

Therefore we get

\[ \sum_{i} L^{*}_{2i+1}(1, -q, q^{-1})q^{2i^2 + i} = \sum_{k \in \mathbb{Z}} (-1)^k q^{3k^2 - k}. \]

Corollary 5.7 (cf. [9], A.52)

\[ \sum_{n \geq 0} \frac{q^{2n^2 - n}}{(q; q)_{2n}} = \frac{1}{(q; q)_\infty} \sum_{k \in \mathbb{Z}} (-1)^k q^{3k^2 - k} = \frac{(q^2; q^2)_\infty}{(q; q)_\infty} = (-q; q)_\infty. \]  

(5.30)

**Proof.**

This follows from (5.14) with \( m = 0 \), because we get the same sums as in Corollary 5.6.

\[ L^{*}_{6n-2} \left( 1, -q, \frac{1}{q} \right) q^{2(3n-1)^2 + (3n-1)} = -q^{3(2n+1)^2 - (2n+1)}, \]
\[ L^{*}_{6n} \left( 1, -q, \frac{1}{q} \right) q^{2(3n)^2 - (3n)} = q^{3(2n)^2 - 2n} + q^{3(2n)^2 + 2n}, \]
\[ L^{*}_{6n+2} \left( 1, -q, \frac{1}{q} \right) q^{2(3n+1)^2 - (3n+1)} = -q^{3(2n+1)^2 - (2n+1)}. \]

The deeper reason for the simple results (5.29) and (5.30) are the formulas

\[ L^{*}_{2i+m} \left( 1, -q, \frac{1}{q} \right) q^{2j^2 + jm} = L^{*}_{2i+m} \left( 1, -1, q^2 \right) \]  

(5.31)

for \( m \in \{0,1\} \), which can easily be verified.
Corollary 5.8 (cf. [9], A.96)

\[
\sum_{n \geq 0} \frac{q^{2n^2+2n}}{(q; q)_{2n+1}} = \frac{1}{(q; q)_\infty} \sum_{k \geq 0} \left( q^{12k^2+5k} - q^{12k^2+13k+3} \right). 
\] (5.32)

Proof.
Here we use (5.13) with \( m = 1 \).
We get
\[
L_{6n+1}^* \left( 1, -1, \frac{1}{q} \right) q^{2(3n+1)^2+2(3n-1)} = q^{12n^2-5n}, \quad L_{6n+3}^* \left( 1, -1, \frac{1}{q} \right) q^{2(3n+1)^2+2(3n+1)} = -q^{12n^2+13n+3} - q^{12n^2+11n+2}, \]
\[
L_{6n+1}^* \left( 1, -1, \frac{1}{q} \right) q^{2(3n)^2+2(3n)} = q^{12n^2+5n}.
\]

We have only to verify that \( 12(n-1)^2 + 11(n-1) + 2 = 12n^2 - 13n + 3 \).

References


