Some determinants of Hessenberg matrices

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Abstract. We give short proofs of determinants of special Hessenberg matrices which have recently been published.

1. Introduction.

The papers [2]-[4] give representations of some numbers or polynomials as determinants of Hessenberg matrices. The purpose of this note is to give short and simple proofs of these results.

The main tool is the following Lemma which has already been used in [1] for analogous problems.

Lemma 1

Let $T=\left(t(i,j)\right)_{i,j\geq 0}$ *be a lower triangular matrix with* $t(i,i)=1$ *for all* i *and let* $T_{n,1}=\left(t(i+1,j)\right)_{i,j=0}^{n-1}$.

Then

$$\det\left(T_{n,1}\right)=\det\left(t(i+1,j)\right)_{i,j=0}^{n-1}=M_n, \qquad (1.1)$$

is equivalent with

$$\sum_{j=0}^{n}(-1)^{n-j}t(n,j)M_j=[n=0] \qquad (1.2)$$

for all n.

Proof

Let the first column of T^{-1} be $(1,-M_1,M_2,\cdots,(-1)^{n-1}M_{n-1})^T$.

By Cramer's rule we have $(-1)^k M_k=\det\left(t(i+1,j)\right)_{i,j=0}^{k-1}$ and by the definition of the inverse matrix we get (1.2).

2. Examples

1) The Bernoulli numbers can be defined by the symbolic formula $(B+1)^{n+1}-B^{n+1}=[n=0]$ which means $\sum_{j=0}^{n}\binom{n+1}{j}B_j=[n=0]$. If we write this in the form

$\sum_{j=0}^{n}(-1)^{n-j}\frac{1}{n+1}\binom{n+1}{j}(-1)^j B_j=[n=0]$ we get (1.2) with $t(n,j)=\frac{1}{n+1}\binom{n+1}{j}$ and $M_j=(-1)^j B_j$. Note that $t(i,i)=1$. Therefore we get (cf.[3])

$$\det\left(\frac{1}{i+2}\binom{i+2}{j}[j\le i+1]\right)_{i.j=0}^{n-1}=(-1)^n B_n. \tag{1.1}$$

2) The Euler numbers $\left(E_{2n}\right)_{n\ge 0}=\left(1,-1,5,-61,1385,\cdots\right)$ can be defined by

$$\sum_{k=0}^{n}\binom{2n}{2k}E_{2k}=[n=0]. \tag{1.2}$$

If we write (1.2) in the form

$$(2n)!\sum_{j=0}^{n}(-1)^{n-j}\frac{1}{(2n-2j)!}\frac{(-1)^j E_{2j}}{(2j)!}=[n=0] \tag{1.3}$$

we get the well-known determinant formula

$$(-1)^n\frac{E_{2n}}{(2n)!}=\det\begin{pmatrix} \frac{1}{2!} & 1 & & & & \\ \frac{1}{4!} & \frac{1}{2!} & 1 & & & \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & 1 & & \\ & & & & & \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & & & \frac{1}{4!} & \frac{1}{2!} \end{pmatrix}. \tag{1.4}$$

3) Identity (1.2) can also be written in the form

$$\sum_{j=0}^{n}(-1)^{n-j}\cos\left((n-j)\frac{\pi}{2}\right)\binom{n}{j}E_j(-1)^{\frac{j}{2}}=[n=0], \tag{1.5}$$

because $\sum_{j=0}^{n}(-1)^{n-j}\cos\left((n-j)\frac{\pi}{2}\right)\binom{n}{j}E_j(-1)^{\frac{j}{2}}=\sum_{j=0}^{n}(-1)^n\cos\left((n-2j)\frac{\pi}{2}\right)\binom{n}{2j}E_{2j}(-1)^j=0$

for odd n and $\sum_{j=0}^{2n}\cos\left((2n-2j)\frac{\pi}{2}\right)\binom{2n}{2j}E_{2j}(-1)^j=(-1)^n\sum_{j=0}^{2n}\binom{2n}{2j}E_{2j}=[n=0].$

By Lemma 1 identity (1.5) implies (cf. [4])

$$\det\left(\binom{i+1}{j}\cos\left((i-j+1)\frac{\pi}{2}\right)\right)_{i,j=0}^{n-1}=(-1)^{\frac{n}{2}}E_n. \tag{1.6}$$

4) The Euler polynomials $E_n(x)$ are defined by $E_n(x)+E_n(x+1)=2x^n$. If we write $E_n(x+1)=e^D E_n(x)$ where D denotes the differentiation operator then we get

$E_n(x)=\frac{2}{1+e^D}x^n$ and therefore $DE_n(x)=\frac{2}{1+e^D}Dx^n=n\frac{2}{1+e^D}x^{n-1}=nE_{n-1}(x).$

This implies $2x^n=\left(1+e^D\right)E_n(x)=E_n(x)+\sum_{k\geq 0}\frac{D^k}{k!}E_n(x)=2E_n(x)+\sum_{k\geq 1}\binom{n}{k}E_{n-k}(x).$

Lemma 1 then gives (cf. [2] and [4])

$$\det\left(t(i+1,j)\right)_{i,j=0}^{n-1}=(-1)^{n-1}E_{n-1}(x) \tag{1.7}$$

with $t(i+1,0)=x^i$, $t(i+1,j)=\binom{i}{j-1}\frac{1}{2}$ for $0<j<i+1$, $t(i+1,i+1)=1$, and $t(i,j)=0$ for $j>i+1$.

5) Define the Fibonacci polynomials $F_n(s)$ by the recursion $F_n(s)=sF_{n-1}(s)+F_{n-2}(s)$ with initial values $F_0(s)=0$ and $F_1(s)=1$.

Then it is well known that

$$F_{n+1}(s)=\det\left(t(i+1,j)\right)_{i,j=0}^{n-1} \tag{1.8}$$

with $t(i,i)=s$, $t(i,i-1)=-1$, $t(i,i+1)=1$ and $t(i,j)=0$ else.

By Lemma 1 this is equivalent with $-F_{n-1}(s)-sF_n(s)+F_{n+1}(s)$ for $n\geq 2$ and $F_2(s)-sF_1(s)=0$ and $F_1(s)=1$.

The main result of [4] is equivalent with

$$\det\left(t(i+1,j)\right)_{i,j=0}^{n-1}=(-1)^n(n-1)!F_{n-1}(s), \tag{1.9}$$

if $t(n,n)=1$, $t(n,n-1)=-(n-1)s$, $t(n,n-2)=-(n-1)(n-2)$ for $n>2$, $t(2,0)=-1$ and $t(n,j)=0$ for all other values j.

By Lemma 1 formula (1.9) is equivalent with

$$\sum_{j=0}^{n}(-1)^{n-j}t(n,j)M_j=[n=0] \tag{1.10}$$

with $M_0=1$ and $M_j=(-1)^j(j-1)!F_{j-1}(s)$ for $j>0$.

For $n\geq 2$ formula (1.10) reduces to

$$t(n,n-2)(n-3)!F_{n-3}(s)+t(n,n-1)(n-2)!F_{n-2}(s)-t(n,n)(n-1)!F_{n-1}(s)$$
$$=-(n-1)!\left(F_{n-3}(s)+sF_{n-2}(s)-F_{n-1}(s)\right)=0.$$

For $n=0$ we get $M_0=1$ and for $n=1$ we also have $-t(1,0)M_0+t(1,1)M_1=0.$

References

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