

Some operator identities related to q -Hermite polynomials

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Abstract

Some q -analogues of the normal ordering of the operator $(X + sD)^n$ on the polynomials are derived.

1. Introduction

Consider the multiplication operator X on the polynomials in x defined by $Xf(x) = xf(x)$ and the differentiation operator D which satisfies $Df(x) = f'(x)$.

Define polynomials $H_n(x, s)$ by

$$H_n(x, s) = xH_{n-1}(x, s) + (n-1)sH_{n-2}(x, s) \quad (1.1)$$

with initial values $H_0(x, s) = 1, H_1(x, s) = x$. These are a variant of the Hermite polynomials.

Then the following operator identity holds.

Theorem1 (J.L. Burchnell [1])

$$(X + sD)^n = \sum_{k=0}^n \binom{n}{k} H_{n-k}(X, s) s^k D^k. \quad (1.2)$$

As far as I know formula (1.2) is (in an equivalent form) due to J. L. Burchnell [1]. See also Gian-Carlo Rota [7], p.45. Their proof uses the fact that

$$(X + sD) = e^{-\frac{X^2}{2s}} (sD) e^{\frac{X^2}{2s}}, \quad (1.3)$$

which follows from the operator form

$$Df(X) = f(X)D + f'(X) \quad (1.4)$$

of the differentiation rule for products $D(f(x)g(x)) = f(x)D(g(x)) + f'(x)g(x)$.

I shall give another proof which will be generalized in the sequel.

Let $G_n(x, s) = (X + sD)^n 1$.

This means that

$$G_n(x, s) = xG_{n-1}(x, s) + sG'_{n-1}(x, s). \quad (1.5)$$

Then $G_n(x, s)$ is of the form $G_n(x, s) = \sum_{j=0}^n c(n, j) s^j x^{n-2j}$. This gives

$$c(0, j) = [j=0], \quad c(n, 0) = 1 \text{ and}$$

$$c(n, j) = c(n-1, j) + (n+1-2j)c(n-1, j-1).$$

From

$$\frac{n!}{2^j j!(n-2j)!} - \frac{(n-1)!}{2^j j!(n-1-2j)!} - \frac{(n-1)!(n+1-2j)}{2^{j-1}(j-1)!(n+1-2j)!} = \frac{(n-1)!}{2^j j!(n-2j)!} (n - (n-2j) - 2j) = 0$$

we conclude that $c(n, j) = \frac{n!}{2^j j!(n-2j)!}$.

Thus

$$G_n(x, s) = \sum_j s^j \frac{n!}{2^j j!(n-2j)!} x^{n-2j} \quad (1.6)$$

and

$$G'_n(x, s) = nG_{n-1}(x, s). \quad (1.7)$$

Thus

$G_n(x, s) = xG_{n-1}(x, s) + (n-1)sG_{n-2}(x, s)$. Together with the initial values $G_0(x, s) = 1$ and $G_1(x, s) = x$ this implies $G_n(x, s) = H_n(x, s)$.

Therefore we get the following well-known properties

$$(X + sD)^n 1 = H_n(x, s) = \sum_j s^j \frac{n!}{2^j j!(n-2j)!} x^{n-2j} = e^{\frac{sD^2}{2}} x^n \quad (1.8)$$

and

$$H'_n(x, s) = nH_{n-1}(x, s). \quad (1.9)$$

The multiplication operators $H_n(X, s)$ defined by $H_n(X, s)f(x) = H_n(x, s)f(x)$ satisfy

$$DH_n(X, s) = H_n(X, s)D + nH_{n-1}(X, s). \quad (1.10)$$

Now it is easy to derive (1.2). For

$$\begin{aligned} (X + sD)^{n+1} &= (X + sD) \sum_k \binom{n}{k} H_{n-k}(X, s) (sD)^k = \sum_k \binom{n}{k} XH_{n-k}(X, s) (sD)^k \\ &+ \sum_k \binom{n}{k} H_{n-k}(X, s) (sD)^{k+1} + \sum_k \binom{n}{k} (n-k) H_{n-k-1}(X, s) (sD)^k \\ &= \sum_k \binom{n}{k} (XH_{n-k}(X, s) + (n-k)H_{n-k-1}(X, s)) (sD)^k + \sum_k \binom{n}{k-1} H_{n-k+1}(X, s) (sD)^k \\ &= \sum_k \binom{n}{k} H_{n-k+1}(X, s) (sD)^k + \sum_k \binom{n}{k-1} H_{n-k+1}(X, s) (sD)^k = \sum_{k=0}^n \binom{n+1}{k} H_{n+1-k}(X, s) (sD)^k. \end{aligned}$$

Changing $k \rightarrow n-k$ in (1.2) and setting $k = m+j$ we get

Corollary 1

$$(X + sD)^n = \sum_{m=0}^n \sum_{j=0}^{\text{Min}(m, n-m)} \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_j s^{n-m} X^{m-j} D^{n-m-j}. \quad (1.11)$$

Here the so called Weyl binomial coefficients $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_j$ are given by

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_j = \frac{n!}{2^j j! (m-j)! (n-m-j)!}. \quad (1.12)$$

In this form (1.2) has been rediscovered several times (cf. e.g. A. Varvak [8]) as normal ordering of $(X + sD)^n$, i.e. as a representation in terms of operators of the form $X^k D^\ell$.

Note that

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_j = \left\{ \begin{matrix} n \\ n-m \end{matrix} \right\}_j = \binom{n-2j}{m-j} \left\{ \begin{matrix} n \\ j \end{matrix} \right\}_j \quad (1.13)$$

and

$$H_n(x, s) = \sum_j \frac{n!}{2^j j! (n-2j)!} s^j x^{n-2j} = \sum_{2j \leq n} \left\{ \begin{matrix} n \\ j \end{matrix} \right\}_j s^j x^{n-2j}. \quad (1.14)$$

2. Normal ordering of $(X + q^{n-1}sD_q)(X + q^{n-2}sD_q)\cdots(X + sD_q)$.

I have found two q -analogues connected with a variant of the discrete q -Hermite polynomials which give simple formulas.

Let

$$h_n(x, s) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{j^2} s^j \frac{[n]!}{(1+q)(1+q^2)\cdots(1+q^j)[j]![n-2j]!} x^{n-2j} \quad (2.1)$$

be this variant and let D_q be the q -differentiation operator defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}. \quad (2.2)$$

Then it is well known (cf. e.g. [2]) that

$$h_n(x, s) = (X + q^{n-1}sD_q)(X + q^{n-2}sD_q)\cdots(X + sD_q)1. \quad (2.3)$$

For a direct proof of this assertion let

$$h_n(x, s) = (X + q^{n-1}sD)(X + q^{n-2}sD)\cdots(X + sD)1 = \sum_{j=0}^n c(n, j) s^j x^{n-2j}. \quad (2.4)$$

This implies $c(0, j) = [j=0]$ and

$$c(n, j) = c(n-1, j) + [n+1-2j]q^{n-1}c(n-1, j-1).$$

It is now easily verified that $c(n, 0) = 1$ and

$$c(n, j) = q^{j^2} \begin{bmatrix} n \\ 2j \end{bmatrix} [2j-1]!! = q^{j^2} \frac{[n]!}{(1+q)(1+q^2)\cdots(1+q^j)[j]![n-2j]}.$$

For this is true for $n = 0$. With induction we get that these values satisfy

$$\begin{aligned} & c(n, j) - c(n-1, j) - [n+1-2j]q^{n-1}c(n-1, j-1) \\ &= q^{j^2} \frac{[n]!}{(1+q)(1+q^2)\cdots(1+q^j)[j]![n-2j]!} - q^{j^2} \frac{[n-1]!}{(1+q)(1+q^2)\cdots(1+q^j)[j]![n-1-2j]!} \\ & - q^{n-1}[n+1-2j]q^{(j-1)^2} \frac{[n-1]!}{(1+q)(1+q^2)\cdots(1+q^{j-1})[j-1]![n+1-2j]!} \\ &= q^{j^2} \frac{[n-1]!}{(1+q)(1+q^2)\cdots(1+q^j)[j]![n-2j]!} ([n] - [n-2j] - q^{n-2j}(1+q^j)[j]) = 0. \end{aligned}$$

Using the q -analogue

$$E_q(z) = \sum_{k \geq 0} \frac{q^{\binom{k}{2}} z^k}{[k]_q!} \quad (2.5)$$

of the exponential series this can be expressed in the form

$$h_n(x, s) = E_{q^2} \left(\frac{qsD_q^2}{[2]_q} \right) x^n. \quad (2.6)$$

For

$$E_{q^2} \left(\frac{qsD_q^2}{[2]_q} \right) x^n = \sum_j \frac{q^{j^2} s^j}{(1+q)^j [j]_{q^2}!} D_q^{2j} x^n = \sum_j \frac{q^{j^2} s^j}{\prod_{i=0}^j (1+q^i) [i]_q} D_q^{2j} x^n = \sum_j \frac{q^{j^2} s^j [n]_q!}{\prod_{i=0}^j (1+q^i) [j]_q! [n-2j]_q!} x^{n-2j}.$$

This implies

$$D_q h_n(x, s) = [n] h_{n-1}(x, s). \quad (2.7)$$

By (2.4) we get the well-known recurrence relation (cf. e.g. [2])

$$h_n(x, s) = x h_{n-1}(x, s) + q^{n-1} s [n-1] h_{n-2}(x, s). \quad (2.8)$$

For the operator

$$F(n) = (X + q^{n-1} s D_q) (X + q^{n-2} s D_q) \cdots (X + s D_q) \quad (2.9)$$

we get

Theorem 2

$$(X + q^{n-1} s D_q) (X + q^{n-2} s D_q) \cdots (X + s D_q) = \sum_{k=0}^n g_n(k, X, s) s^k D_q^k \quad (2.10)$$

with

$$g_n(k, x, s) = \begin{bmatrix} n \\ k \end{bmatrix} \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} s^j q^{j^2 + kj + \binom{k}{2}} \begin{bmatrix} n-k \\ 2j \end{bmatrix} [2j-1]!! \prod_{i=0}^{k-1} \frac{1+q^{n-j-i}}{1+q^{j+1+i}} x^{n-k-2j}. \quad (2.11)$$

For $k=0$ this reduces to $g_n(0, x, s) = h_n(x, s)$.

Proof

$$F(n) = \sum_{k=0}^n g_n(k, X, s) s^k D_q^k = (X + q^{n-1} s D) \sum_{k=0}^{n-1} g_{n-1}(k, X, s) s^k D_q^k$$

implies

$$g_n(k, x, s) = x g_{n-1}(k, x, s) + q^{n-1} g_{n-1}(k-1, qx, s) + q^{n-1} s (D_q g_{n-1}(k, x, s)).$$

Let now

$$g_n(k, x, s) = \sum_j c(n, k, j) s^j x^{n-k-2j}. \quad (2.12)$$

Then

$$c(n, k, j) = c(n-1, k, j) + q^{2n-1-k-2j} c(n-1, k-1, j) + q^{n-1} [n+1-k-2j] c(n-1, k, j-1) \quad (2.13)$$

It now suffices to show that

$$\begin{aligned} c(n, k, j) &= \begin{bmatrix} n \\ k \end{bmatrix} q^{j^2+kj+\binom{k}{2}} \begin{bmatrix} n-k \\ 2j \end{bmatrix} [2j-1]!! \frac{(1+q^{n-j})(1+q^{n-j-1}) \cdots (1+q^{n-j-k+1})}{(1+q^{j+1})(1+q^{j+2}) \cdots (1+q^{j+k})} \\ &= \frac{[n]! q^{j^2+kj+\binom{k}{2}} (1+q^{n-j})(1+q^{n-j-1}) \cdots (1+q^{n-j-k+1})}{[k]! [n-k-2j]! [j]! (1+q)(1+q^2) \cdots (1+q^{j+k})}. \end{aligned}$$

This is verified by the following computation:

$$\begin{aligned} &c(n, k, j) - c(n-1, k, j) - q^{2n-1-k-2j} c(n-1, k-1, j) - q^{n-1} [n+1-k-2j] c(n-1, k, j-1) \\ &= \frac{[n-1]! q^{j^2+kj} (1+q^{n-j-1}) \cdots (1+q^{n-j-k+1})}{[k]! [n-k-2j]! [j]! (1+q)(1+q^2) \cdots (1+q^{j+k})} \\ &\left([n] q^{\binom{k}{2}} (1+q^{n-j}) - q^{\binom{k}{2}} (1+q^{n-j-k}) [n-k-2j] - q^{2n-3-3j+\binom{k-2}{2}} [k] (1+q^{j+k}) - q^{n-2j+\binom{k-1}{2}} (1+q^{n-j}) [j] (1+q^{j+k}) \right) = 0. \end{aligned}$$

Corollary 2

$$\begin{aligned} &(X + q^{n-1} s D_q) (X + q^{n-2} s D_q) \cdots (X + s D_q) \\ &= \sum_{m=0}^n \sum_{j=0}^{\text{Min}(m, n-m)} q^{\binom{j+1}{2} + \binom{n-m}{2}} \frac{(1+q^{m+1})(1+q^{m+2}) \cdots (1+q^{n-j}) [n]! s^{n-m}}{(1+q)(1+q^2) \cdots (1+q^{n-m}) [j]! [m-j]! [n-m-j]!} X^{m-j} D_q^{n-m-j}. \end{aligned} \quad (2.14)$$

3. Normal ordering of $(X + qsD_q)(X + q^3sD_q)\cdots(X + q^{2n-1}sD_q)$.

Another more natural q – analogue of Theorem 1 is

Theorem 3

$$(X + qsD_q)(X + q^3sD_q)\cdots(X + q^{2n-1}sD_q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{kn} h_{n-k}(X, s)(sD_q)^k. \quad (3.1)$$

Proof

Consider

$$G(n) = (X + qsD_q)(X + q^3sD_q)\cdots(X + q^{2n-1}sD_q).$$

We show first that $G(n)1 = h_n(x, s)$.

Let

$$f_n(x, s) = (X + sD_q)(X + q^3sD_q)\cdots(X + q^{2n-1}sD_q)1 = \sum_{j=0}^n c(n, j)s^j x^{n-2j}. \quad (3.2)$$

Then $f_n(x, s) = xf_{n-1}(x, q^2s) + qsD_q f_n(x, q^2s)$.

This implies $c(0, j) = [j = 0]$ and

$$c(n, j) = q^{2j}c(n-1, j) + [n+1-2j]q^{2j-1}c(n-1, j-1).$$

This gives $c(n, 0) = 1$ and $c(n, j) = q^{j^2} \begin{bmatrix} n \\ 2j \end{bmatrix} [2j-1]!! = q^{j^2} \frac{[n]!}{(1+q)(1+q^2)\cdots(1+q^j)[j]![n-2j]!}$.

This is true for $n = 0$. With induction we get that these values satisfy

$$\begin{aligned} & c(n, j) - q^{2j}c(n-1, j) - [n+1-2j]q^{2j-1}c(n-1, j-1) \\ &= q^{j^2} \frac{[n]!}{(1+q)(1+q^2)\cdots(1+q^j)[j]![n-2j]!} - q^{j^2+2j} \frac{[n-1]!}{(1+q)(1+q^2)\cdots(1+q^j)[j]![n-1-2j]!} \\ & \quad - q^{2j-1}[n+1-2j]q^{(j-1)^2} \frac{[n-1]!}{(1+q)(1+q^2)\cdots(1+q^{j-1})[j-1]![n+1-2j]!} \\ &= q^{j^2} \frac{[n-1]!}{(1+q)(1+q^2)\cdots(1+q^j)[j]![n-2j]!} \left([n] - q^{2j}[n-2j] - (1+q^j)[j] \right) = 0. \end{aligned}$$

Therefore we get $f_n(x, s) = h_n(x, s)$.

This implies that $h_n(x, s)$ also satisfies the recurrence

$$h_n(x, s) = xh_{n-1}(x, q^2s) + qsD_q h_n(x, q^2s). \quad (3.3)$$

From (2.1) it is clear that $h_n(qx, q^2s) = q^n h_n(x, s)$.

The proof of (3.1) follows from

$$\begin{aligned} & (X + qsD_q) \sum_k \binom{2}{k} q^{kn} (q^2s)^k h_{2-k}(X, q^2s) D_q^k = \sum_k \binom{n}{k} q^{kn} (q^2s)^k X h_{n-k}(X, q^2s) D_q^k \\ & + qs \sum_k \binom{n}{k} q^{kn} (q^2s)^k h_{n-k}(qX, q^2s) D_q^{k+1} + qs \sum_k \binom{n}{k} q^{kn} (q^2s)^k (D_q h_{n-k}(X, q^2s)) D_q^k \\ & = \sum_k \binom{n}{k} q^{kn} (q^2s)^k (X h_{n-k}(X, q^2s) + qs (D_q h_{n-k}(X, q^2s))) D_q^k + \sum_k \binom{n}{k} q^{(k+1)(n+1)} s^k h_{n-k}(X, s) D_q^{k+1} \\ & = \sum_k q^k \binom{n}{k} q^{k(n+1)} s^k h_{n+1-k}(X, s) D_q^k + \sum_k \binom{n}{k-1} q^{k(n+1)} s^k h_{n-k+1}(X, s) D_q^k \\ & = \sum_{k=0}^n \binom{n+1}{k} q^{k(n+1)} s^k h_{n+1-k}(X, s) D_q^k. \end{aligned}$$

Corollary 3

$$\begin{aligned} & (X + qsD_q)(X + q^3sD_q) \cdots (X + q^{2n-1}sD_q) \\ & = \sum_{m=0}^n \sum_{j=0}^{\text{Min}(m, n-m)} \frac{q^{n^2+j^2-(m+j)n} [n]! s^{n-m}}{(1+q)(1+q^2) \cdots (1+q^j) [j]! [m-j]! [n-m-j]!} X^{m-j} D_q^{n-m-j}. \end{aligned}$$

4. Normal ordering of $(X + sD_q)^n$.

The first terms of $(X + sD_q)^n$ in the ordering of Corollary 1 are

$$\begin{aligned} & sD_q + X, \\ & s^2D_q^2 + (1+q)sXD + s + X^2, \\ & s^3D_q^3 + (1+q+q^2)s^2XD_q^2 + (2+q)s^2D_q + (1+q+q^2)sX^2D_q + (2+q)sX + X^3, \\ & s^4D_q^4 + (1+q+q^2+q^3)s^3XD_q^3 + (3+2q+q^2)s^3D_q^2 + (1+q+2q^2+q^3+q^4)s^2X^2D_q^2 \\ & + (3+5q+3q^2+q^3)s^2XD_q + (2+q)s^2 + (1+q+q^2+q^3)sX^3D_q + (3+2q+q^2)sX^2 + X^4. \end{aligned}$$

Consider the variant of the q -Hermite polynomials introduced in [7] by

$$H_n(x, s | q) = (X + sD_q)^n 1. \quad (4.1)$$

They are related to the q -Lucas polynomials, which have been studied in [3] and [4].

We define these q -Lucas polynomials by

$$L_n(x, s) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{k}{2}} \frac{[n]}{[n-k]} \begin{bmatrix} n-k \\ k \end{bmatrix} s^k x^{n-2k} \quad (4.2)$$

for $n > 0$ with initial value

$$L_0(x, s) = 1. \quad (4.3)$$

Observe that the choice of initial value is different from the one used in [3].

The first values are $1, x, x^2 + (1+q)s, x^3 + (1+q+q^2)sx, x^4 + (1+q+q^2+q^3)sx^2 + (q+q^3)s^2, \dots$

It is easily verified that they satisfy (cf. [3])

$$(X + (1-q)sD_q)L_n(x, -s) = L_{n+1}(x, -s) + sL_{n-1}(x, -s) \quad (4.4)$$

for $n \geq 2$,

$$(X + (1-q)sD_q)L_1(x, -s) = x^2 + (1-q)s = L_2(x, -s) + sL_0(x, -s) + s \quad (4.5)$$

and

$$(X + (1-q)sD_q)L_0(x, -s) = x = L_1(x, -s). \quad (4.6)$$

As shown in [5]

$$H_n(x, (1-q)s | q) = (X + (1-q)sD_q)^n 1 = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{j} s^j L_{n-2j}(x, -s). \quad (4.7)$$

In order to make this paper self-contained we give a new proof:

(4.7) is obviously true for $n = 0$ and $n = 1$. In the general case we get

$$\begin{aligned}
H_{2n+1}(x, (1-q)s | q) &= (X + (1-q)sD_q)H_{2n}(x, s | q) = (X + (1-q)sD_q) \sum_{j=0}^n \binom{2n}{j} s^j L_{2n-2j}(x, -s) \\
&= \sum_{j=0}^n \binom{2n}{j} s^j (X + (1-q)sD_q) L_{2n-2j}(x, -s) = \sum_{j=0}^{n-1} \binom{2n}{j} s^j (L_{2n+1-2j}(x, -s) + sL_{2n-1-2j}(x, -s)) + \binom{2n}{n} s^n x \\
&= \sum_{j=0}^{n-1} \binom{2n}{j} s^j L_{2n+1-2j}(x, -s) + \sum_{j=1}^{n-1} \binom{2n}{j-1} s^j L_{2n+1-2j}(x, -s) + \binom{2n}{n-1} s^n x + \binom{2n}{n} s^n x = \sum_{j=0}^n \binom{2n+1}{j} s^j L_{2n+1-2j}(x, -s)
\end{aligned}$$

and

$$\begin{aligned}
H_{2n}(x, (1-q)s | q) &= (X + (1-q)sD_q)H_{2n-1}(x, s | q) = (X + (1-q)sD_q) \sum_{j=0}^{n-1} \binom{2n-1}{j} s^j L_{2n-1-2j}(x, -s) \\
&= \sum_{j=0}^{n-1} \binom{2n-1}{j} s^j (X + (1-q)sD_q) L_{2n-1-2j}(x, -s) = \sum_{j=0}^{n-2} \binom{2n-1}{j} s^j (L_{2n-2j}(x, -s) + sL_{2n-2-2j}(x, -s)) \\
&\quad + \binom{2n-1}{n-1} s^{n-1} (x^2 + (1-q)s) \\
&= \sum_{j=0}^{n-2} \binom{2n-1}{j} s^j L_{2n-2j}(x, -s) + \sum_{j=1}^{n-1} \binom{2n-1}{j-1} s^j L_{2n-2j}(x, -s) + \binom{2n-1}{n-1} s^{n-1} (x^2 + (1-q)s) \\
&= \sum_{j=0}^{n-2} \binom{2n}{j} s^j L_{2n-2j}(x, -s) + \binom{2n-1}{n-2} s^{n-1} L_2(x, -s) + \binom{2n-1}{n} s^{n-1} (x^2 + (1-q)s) \\
&= \sum_{j=0}^{n-2} \binom{2n}{j} s^j L_{2n-2j}(x, -s) + \binom{2n}{n-1} s^{n-1} L_2(x, -s) + \binom{2n}{n} s^n = \sum_{j=0}^n \binom{2n}{j} s^j L_{2n-2j}(x, -s).
\end{aligned}$$

For the next theorem we need a generalization $L_n^{(k)}(x, s)$ of the Lucas polynomials. We define them by

$$L_n^{(k)}(x, s) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n+k]}{[n+k-j]} \begin{bmatrix} n+k-j \\ k \end{bmatrix} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \quad (4.8)$$

with $L_0^{(k)}(x, s) = 1$. It is clear that $L_n^{(0)}(x, s) = L_n(x, s)$.

Computer calculations led to

Theorem 4

$$\left(X + (1-q)sD_q\right)^n = \sum_{k=0}^n A(n, k, X)(1-q)^k s^k D_q^k \quad (4.9)$$

with

$$A(n, k, x) = \sum_{i=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{n}{i} s^i L_{n-2i-k}^{(k)}(x, -s). \quad (4.10)$$

Before proving this theorem let us make some remarks.

$\sum_{k=0}^n A(n, k, X)D_q^k$ is a linear combination of terms of the form $x^{n-2i-k-2j}D_q^k$.

Let $m = i + k + j, \ell = i + j$ be the uniquely determined integers such that $2i + k + 2j = m + \ell$ and $k = m - \ell$. Then we get

$$\begin{aligned} A(n, k, x) &= \sum_{i=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{n}{i} s^i \sum_{j=0}^{\lfloor \frac{n-k-2i}{2} \rfloor} q^{\binom{j}{2}} \frac{[n-2i]}{[n-2i-j]} \begin{bmatrix} n-2i-j \\ k \end{bmatrix} \begin{bmatrix} n-k-2i-j \\ j \end{bmatrix} (-s)^j x^{n-k-2i-2j} \\ &= \sum_{i, m, \ell} \binom{n}{i} (-1)^{\ell-i} s^\ell q^{\binom{\ell-i}{2}} \frac{[n-2i]}{[n-i-\ell]} \begin{bmatrix} n-i-\ell \\ m-\ell \end{bmatrix} \begin{bmatrix} n-m-i \\ \ell-i \end{bmatrix} x^{n-m-\ell} \end{aligned} \quad (4.11)$$

Comparing coefficients we see that (4.9) is equivalent with

$$\left(X + sD_q\right)^n = \sum_{m, \ell} \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{\ell, q} X^{n-m-\ell} s^m D_q^{m-\ell}, \quad (4.12)$$

where the q -Weyl binomial coefficients are given by

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{\ell, q} = \frac{1}{(1-q)^\ell} \sum_i \binom{n}{i} (-1)^{\ell-i} q^{\binom{\ell-i}{2}} \frac{[n-2i]}{[n-i-\ell]} \begin{bmatrix} n-i-\ell \\ m-\ell \end{bmatrix} \begin{bmatrix} n-m-i \\ \ell-i \end{bmatrix}. \quad (4.13)$$

For the special case $m = \ell$ we get

$$H_n(x, s | q) = \sum_{\ell} \left\{ \begin{matrix} n \\ \ell \end{matrix} \right\}_{\ell, q} x^{n-2\ell} s^{\ell} \quad (4.14)$$

with

$$\left\{ \begin{matrix} n \\ \ell \end{matrix} \right\}_{\ell, q} = \frac{1}{(1-q)^{\ell}} \sum_i \binom{n}{i} (-1)^{\ell-i} q^{\binom{\ell-i}{2}} \frac{[n-2i]}{[n-i-\ell]} \begin{bmatrix} n-\ell-i \\ \ell-i \end{bmatrix}. \quad (4.15)$$

Comparing (4.13) with (4.15) we see that

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{\ell, q} = \begin{bmatrix} n-2\ell \\ m-\ell \end{bmatrix} \left\{ \begin{matrix} n \\ \ell \end{matrix} \right\}_{\ell, q}. \quad (4.16)$$

My original proof of (4.9) has been rather clumsy. But as has been observed by J. Zeng [9] Theorem 4 follows immediately from (4.15), which has been proved in [5] and [6], and (4.16) which has been proved by A. Varvak [8], Theorem 6.4.

The same idea can be used to give a direct proof of Theorem 4:

Define coefficients $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{\ell, q}$ by

$$(X + sD_q)^n = \sum_{m, \ell} \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{\ell, q} X^{m-\ell} s^{n-m} D_q^{n-m-\ell}.$$

Then

$$\sum_{m, \ell} \left\{ \begin{matrix} n+1 \\ m \end{matrix} \right\}_{\ell, q} X^{m-\ell} s^{n+1-m} D_q^{n+1-m-\ell} = (X + sD_q)^{n+1} = (X + sD_q) \sum_{m, \ell} \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{\ell, q} X^{m-\ell} s^{n-m} D_q^{n-m-\ell}.$$

Since $D_q f(X) = D_q(f(X)) + f(qX)D_q$ we get

$$\left\{ \begin{matrix} n+1 \\ m \end{matrix} \right\}_{\ell, q} = \left\{ \begin{matrix} n \\ m-1 \end{matrix} \right\}_{\ell} + [m+1-\ell] \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{\ell-1, q} + q^{m-\ell} \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{\ell, q}. \quad (4.17)$$

This recurrence together with the initial values $\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}_{0,q} = 1$ and all other $\begin{Bmatrix} 0 \\ m \end{Bmatrix}_{\ell,q} = 0$ determines $\begin{Bmatrix} n \\ m \end{Bmatrix}_{\ell,q}$ uniquely if we set $\begin{Bmatrix} n \\ m \end{Bmatrix}_{\ell,q} = 0$ for $m < 0$ or $\ell < 0$.

It now suffices to show that

$$\begin{Bmatrix} n \\ m \end{Bmatrix}_{\ell,q} := \begin{bmatrix} n-2\ell \\ m-\ell \end{bmatrix} \begin{Bmatrix} n \\ \ell \end{Bmatrix}_{\ell,q} \quad (4.18)$$

satisfies (4.17) and the initial values.

First observe that (4.15) is an easy consequence of (4.7):

By (4.14) and (4.7) we get

$$\begin{aligned} H_n(x, (1-q)s | q) &= \sum_{\ell} \begin{Bmatrix} n \\ \ell \end{Bmatrix}_{\ell,q} x^{n-2\ell} (1-q)^{\ell} s^{\ell} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{j} s^j L_{n-2j}(x, -s) \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{i} s^i \sum_{k=0}^{\lfloor \frac{n-2i}{2} \rfloor} q^{\binom{k}{2}} \frac{[n-2i]}{[n-2i-k]} \begin{bmatrix} n-2i-k \\ k \end{bmatrix} (-s)^k x^{n-2i-2k}. \end{aligned}$$

Setting $i+k = \ell$ and comparing coefficients of $x^{n-2\ell}$ we get

$$\begin{Bmatrix} n \\ \ell \end{Bmatrix}_{\ell,q} (1-q)^{\ell} s^{\ell} = \sum \binom{n}{i} (-1)^{\ell-i} q^{\binom{\ell-i}{2}} \frac{[n-2i]}{[n-i-\ell]} \begin{bmatrix} n-i-\ell \\ \ell-i \end{bmatrix} s^{\ell},$$

which gives (4.15).

Now we must verify (4.17). This becomes

$$\begin{aligned} \begin{bmatrix} n+1-2\ell \\ m-\ell \end{bmatrix} \begin{Bmatrix} n+1 \\ \ell \end{Bmatrix}_{\ell,q} &= \begin{bmatrix} n-2\ell \\ m-1-\ell \end{bmatrix} \begin{Bmatrix} n \\ \ell \end{Bmatrix}_{\ell,q} + [m+1-\ell] \begin{bmatrix} n-2\ell+2 \\ m-\ell+1 \end{bmatrix} \begin{Bmatrix} n \\ \ell-1 \end{Bmatrix}_{\ell-1,q} + q^{m-\ell} \begin{bmatrix} n-2\ell \\ m-\ell \end{bmatrix} \begin{Bmatrix} n \\ \ell \end{Bmatrix}_{\ell,q} \\ &= \left(\begin{bmatrix} n-2\ell \\ m-1-\ell \end{bmatrix} + q^{m-\ell} \begin{bmatrix} n-2\ell \\ m-\ell \end{bmatrix} \right) \begin{Bmatrix} n \\ \ell \end{Bmatrix}_{\ell,q} + [m+1-\ell] \begin{bmatrix} n-2\ell+2 \\ m-\ell+1 \end{bmatrix} \begin{Bmatrix} n \\ \ell-1 \end{Bmatrix}_{\ell-1,q} \\ &= \begin{bmatrix} n+1-2\ell \\ m-\ell \end{bmatrix} \begin{Bmatrix} n \\ \ell \end{Bmatrix}_{\ell,q} + [m+1-\ell] \begin{bmatrix} n-2\ell+2 \\ m-\ell+1 \end{bmatrix} \begin{Bmatrix} n \\ \ell-1 \end{Bmatrix}_{\ell-1,q} \end{aligned}$$

and reduces to

$$\left\{ \begin{matrix} n+1 \\ \ell \end{matrix} \right\}_{\ell,q} = \left\{ \begin{matrix} n \\ \ell \end{matrix} \right\}_{\ell,q} + [n+2-2\ell] \left\{ \begin{matrix} n \\ \ell-1 \end{matrix} \right\}_{\ell-1,q}$$

which is clear by (4.14) and (4.1).

Thus we get again Theorem 4.

References

- [1] J. L. Burchall, A note on the polynomials of Hermite, *Quart. J. Math.* 12(1941), 9-11
- [2] J. Cigler, Elementare q -Identitäten, *Sém. Lotharingien Comb.*, B05a (1981), 29 pp.
- [3] J. Cigler, A new class of q -Fibonacci polynomials, *Electr. J. Comb.* 10 (2003), #R 19
- [4] J. Cigler, q -Lucas polynomials and associated Rogers-Ramanujan type identities, arXiv:0907.0165
- [5] J. Cigler and J. Zeng, A curious q -analogue of Hermite polynomials, arXiv:0905.0228
- [6] M. Josuat-Vergès, Rook placements in Young diagrams and permutation enumeration, arXiv: 0811.0524
- [7] G.-C. Rota, *Finite Operator Calculus*, Academic Press 1975
- [8] A. Varvak, Rook numbers and the normal ordering problem, *J. Comb. Th. A* 112 (2005), 292-307
- [9] J. Zeng, A remark on the q -Weyl binomial coefficients, Personal communication, October 11, 2010