Some remarks and conjectures about Rogers-Szegö polynomials

Johann Cigler

Fakultät für Mathematik

Universität Wien

johann.cigler@univie.ac.at

Abstract

This note is a supplement to my paper "Some elementary results and conjectures about $q-$ Newton binomials". It studies some non-trivial $q-$ analogues of $(1-1)^n = [n=0]$ and of $(1+1)^n = 2^n$ and motivates some conjectures about Hankel determinants of "normalized" Rogers-Szegö polynomials.

0. Introduction

This note presents some observations about the Rogers-Szegö polynomials

$$r(n,s,q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q s^j. \tag{1}$$

By replacing the binomial coefficients $\binom{n}{j}$ with the $q-$ binomial coefficients $\begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n \\ j \end{bmatrix}_q = \prod_{i=0}^{j-1} \frac{1-q^{n-i}}{1-q^{j-i}}$ they seem to be the simplest $q-$ analogue of the binomial theorem $\sum_{j=0}^{n} \binom{n}{j} s^j = (1+s)^n.$ Whereas the related $q-$ binomials $\sum_{j=0}^{n} q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix} s^j$ admit the closed formula

$$\sum_{j=0}^{n} q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix} s^j = (-s;q)_n = (1+s)(1+qs)\cdots\left(1+q^{n-1}s\right), \tag{2}$$

for the Rogers-Szegö polynomials no such formulae exist. They satisfy instead the recurrence

$$r(n,s,q) = (1+s)r(n-1,s,q) + \left(q^{n-1}-1\right) sr(n-2,s,q). \tag{3}$$

Despite that, nevertheless there are closed formulae for some $q-$ analogues of the special cases $(1-1)^n = [n=0]$ and $\left(1+1\right)^n = 2^n$.

By a famous theorem of Gauss we have

$r(2n,-1,q)=\left(q;q^2\right)_n=(1-q)\left(1-q^3\right)\cdots\left(1-q^{2n-1}\right)$

and $r(2n+1,-1,q)=0.$ A related result is $r(n,-q,q)=\left(q;q^2\right)_{\left\lfloor\frac{n+1}{2}\right\rfloor}.$

On the other hand we have $r\left(n,q,q^2\right)=\sum_{j=0}^{n}q^j\begin{bmatrix}n\\j\end{bmatrix}_{q^2}=(-q;q)_n=(1+q)\left(1+q^2\right)\cdots\left(1+q^n\right).$

It turns out that these results lead to some interesting extensions.

1. Some results and conjectures

As a first extension of these closed formulae it is shown in [3] and [6] that the alternating sums $\sum_{j=0}^{n}(-1)^j\begin{bmatrix}n\\j\end{bmatrix}_q q^{rj}$ are divisible by $\left(q;q^2\right)_{\left\lfloor\frac{n+1}{2}\right\rfloor}$ in $\mathbb{Z}\left[q,q^{-1}\right]$ for all $r\in\mathbb{Z},$ whereas the non-alternating sums $\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_{q^2}q^{rj}$ are divisible by $(-q;q)_n$ for all odd integers $r.$

Let us recall a proof of these facts. If not otherwise stated we freely use the notations and results introduced in [3]. Since $\sum_{j=0}^{n}q^{-rj}\begin{bmatrix}n\\j\end{bmatrix}=\sum_{j=0}^{n}q^{-r(n-j)}\begin{bmatrix}n\\j\end{bmatrix}=q^{-rn}\sum_{j=0}^{n}q^{rj}\begin{bmatrix}n\\j\end{bmatrix}$ we can confine ourselves to consider q^r with $r\in\mathbb{N}.$

Let us introduce the "normalized" Rogers-Szegö polynomials

$$f(n,s,q)=\frac{\sum_{j=0}^{n}s^j\begin{bmatrix}n\\j\end{bmatrix}_{q^2}}{(-q;q)_n} \tag{4}$$

and

$$F(n,s,q)=\frac{\sum_{j=0}^{n}(-s)^j\begin{bmatrix}n\\j\end{bmatrix}_q}{\left(q;q^2\right)_{\left\lfloor\frac{n+1}{2}\right\rfloor}}. \tag{5}$$

Theorem 1

The normalized Rogers-Szegö polynomials have the following expansions:

$$f(n,s,q)=\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}\frac{\prod_{i=0}^{j-1}\left(s-q^{2i+1}\right)}{(-q;q)_j}, \tag{6}$$

$$F(2n,s,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q^2}\frac{\prod_{j=0}^{2k-1}\left(q^j-s\right)}{\left(q;q^2\right)_k} \tag{7}$$

and

$$F(2n+1,s,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q^2}\frac{\prod_{j=0}^{2k}\left(q^j-s\right)}{\left(q;q^2\right)_{k+1}}. \tag{8}$$

Proof

If we expand $f(n,s,q)$ with respect to the polynomials $\left(\frac{s}{q^{2j-1}};q^2\right)_j$ we get

$$f(n,s,q)=\frac{\sum_{j=0}^{n}s^j\begin{bmatrix} n \\ j \end{bmatrix}_{q^2}}{(-q;q)_n}=\sum_{j=0}^{n}(-1)^j q^{j^2}\left(q^{n+1-j};q\right)_j\frac{\left(\frac{s}{q^{2j-1}};q^2\right)_j}{\left(q^2;q^2\right)_j}=\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}\frac{\prod_{i=0}^{j-1}\left(s-q^{2i+1}\right)}{(-q;q)_j}.$$

To prove this observe that

$$\sum_{n\geq 0}\sum_{j=0}^{n}s^j\begin{bmatrix} n \\ j \end{bmatrix}_{q^2}\frac{z^n}{\left(q^2;q^2\right)_n}=e_{q^2}(sz)e_{q^2}(z)=\frac{1}{\left(sz;q^2\right)_\infty\left(z;q^2\right)_\infty}=\frac{\left(qz;q^2\right)_\infty}{\left(sz;q^2\right)_\infty}\frac{1}{\left(z;q^2\right)_\infty\left(qz;q^2\right)_\infty}$$

$$=\sum_k\frac{\left(\frac{q}{s};q^2\right)_k}{\left(q^2;q^2\right)_k}(sz)^k\sum_\ell\frac{z^\ell}{(q;q)_\ell}=\sum_k(-1)^k q^{k^2}\frac{\left(\frac{s}{q^{2k-1}};q^2\right)_k}{\left(q^2;q^2\right)_k}\sum_\ell\frac{z^\ell}{(q;q)_\ell}$$

implies

$$\sum_{j=0}^{n}s^j\begin{bmatrix} n \\ j \end{bmatrix}_{q^2}=\left(q^2;q^2\right)_n\sum_{j=0}^{n}(-1)^j q^{j^2}\left(\frac{s}{q^{2j-1}};q^2\right)_j\frac{1}{\left(q^2;q^2\right)_j(q;q)_{n-j}}$$

$$=(-q;q)_n\sum_{j=0}^{n}(-1)^j q^{j^2}\left(\frac{s}{q^{2j-1}};q^2\right)_j\frac{(q;q)_n}{\left(q^2;q^2\right)_j(q;q)_{n-j}}.$$

Comparing coefficients in

$$\sum_n \sum_{j=0}^{n} (-s)^j \begin{bmatrix} n \\ j \end{bmatrix} \frac{z^n}{(q;q)_n} = e_q(-sz) e_q(z) = \frac{e_q(-sz)}{e_q(-z)} e_q(-z) e_q(z) = \sum_{k \geq 0} q^{\binom{k}{2}} \left(\frac{s}{q^{k-1}}; q \right)_k \frac{z^k}{(q;q)_k} \sum_{\ell \geq 0} \frac{z^{2\ell}}{\left(q^2;q^2\right)_\ell}$$

gives

$$\sum_{j=0}^{n} (-s)^j \begin{bmatrix} n \\ j \end{bmatrix} = \sum_{j+2\ell=n} q^{\binom{j}{2}} \left(\frac{s}{q^{j-1}}; q \right)_j \frac{(q;q)_n}{(q;q)_j \left(q^2;q^2\right)_\ell}.$$

This implies

$$F(2n,s,q) = \sum_{\ell=0}^{n} q^{\binom{2n-2\ell}{2}} \left(\frac{s}{q^{2n-2j-1}}; q \right)_{2n-2j} \frac{\left(q^2;q^2\right)_n}{(q;q)_{2n-2j} \left(q^2;q^2\right)_j},$$

$$F(2n+1,s,q) = \sum_{\ell=0}^{n} q^{\binom{2n-2\ell+1}{2}} \left(\frac{s}{q^{2n-2j}}; q \right)_{2n-2j+1} \frac{\left(q^2;q^2\right)_n}{(q;q)_{2n-2j+1} \left(q^2;q^2\right)_j}.$$

This can be simplified to give (7) and (8).

Corollary 1

For $r \in \mathbb{N}$ *we get*

$$f\left(n, q^{2r+1}, q\right) = \frac{\sum_{j=0}^{n} q^{(2r+1)j} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2}}{(-q;q)_n} = \sum_{j=0}^{r} (-1)^j q^{j^2} \begin{bmatrix} r \\ j \end{bmatrix}_{q^2} \left(q^{n-j+1};q\right)_j \in \mathbb{Z}[q],$$

$$F(2n,q^r,q) = \frac{\sum_{j=0}^{2n} (-1)^j q^{rj} \begin{bmatrix} 2n \\ j \end{bmatrix}_q}{\left(q;q^2\right)_n} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \frac{\prod_{j=0}^{2k-1} \left(q^j - q^r\right)}{\left(q;q^2\right)_k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^{\binom{2k}{2}} \begin{bmatrix} r \\ 2k \end{bmatrix} \left(q^2;q^2\right)_k$$

$$= \sum_{k=0}^{\left\lfloor \frac{r}{2} \right\rfloor} q^{\binom{2k}{2}} \begin{bmatrix} r \\ 2k \end{bmatrix} \left(q^{2n-2k+2};q^2\right)_k \in \mathbb{Z}[q]$$

and

$$F(2n+1,q^r,q) = \sum_{k=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} q^{\binom{2k+1}{2}} \begin{bmatrix} r \\ 2k+1 \end{bmatrix} \left(q^{2n-2k+2};q^2\right)_k \in \mathbb{Z}[q].$$

Corollary 2

For each $r\in\mathbb{Z}$ *the sequences* $\left(f\left(n,q^{2r+1},q\right)\right)_{n\geq 0}$, $\left(F\left(2n,q^{r},q\right)\right)_{n\geq 0}$, $\left(F\left(2n+1,q^{r},q\right)\right)_{n\geq 0}$ *and* $\left(F\left(n,q^{r},q\right)\right)_{n\geq 0}$ *satisfy homogeneous linear recurrences with constant coefficients.*

Proof

If a sequence $\left(x(n)\right)$ satisfies the recurrence $\sum_{k=0}^{r}a(k)x(n-k)=0$ then the sequence $\left(\lambda^{n}x(n)\right)$ satisfies $\sum_{k=0}^{r}a(k)\lambda^{k}\left(\lambda^{n-k}x(n-k)\right)=0.$ Therefore we can limit ourselves to $r\in\mathbb{N}.$

By (2) we see that for $m\in\mathbb{N}$ the sequence $\left(q^{mn}\right)_{n\geq 0}$ satisfies the recurrences

$$\sum_{j=0}^{r}(-1)^{j}q^{\binom{j}{2}}\begin{bmatrix}r\\ j\end{bmatrix}q^{m(n-j)}=0 \text{ for } r>m \text{ because } \sum_{j=0}^{r}(-1)^{j}q^{\binom{j}{2}}\begin{bmatrix}r\\ j\end{bmatrix}q^{mn-mj}=q^{mn}\prod_{j=0}^{r-1}\left(1-q^{j}\frac{1}{q^{m}}\right)=0.$$

By Corollary 1 $f\left(n,q^{2r-1},q\right)$ is a linear sum of terms q^{mn} with $0\leq m<r$ and therefore satisfies the recurrence

$$\sum_{j=0}^{r}(-1)^{j}q^{\binom{j}{2}}\begin{bmatrix}r\\ j\end{bmatrix}f\left(n-j,q^{2r-1},q\right)=0.$$

In the same way we see that $F(2n,q^{r},q)$ and $F(2n+1,q^{r},q)$ are linear sums of q^{2mn} for $0\leq m\leq\left\lfloor\frac{r}{2}\right\rfloor$ and satisfy therefore recurrences of the form

$$\sum_{j=0}^{N}(-1)^{j}q^{2\binom{j}{2}}\begin{bmatrix}N\\ j\end{bmatrix}_{q^{2}}F(n-2j,q^{r},q)=0 \quad \text{for } N>\left\lfloor\frac{r}{2}\right\rfloor.$$

If a sequence $\left(x(n)\right)_{n\geq 0}$ satisfies a linear homogeneous recurrence of order r with constant coefficients then the rows and columns of the Hankel matrix $\left(x(i+j)\right)_{i,j=0}^{n}$ are linear dependent for $n\geq r$ and therefore their determinants vanish.

Thus we get

Corollary 3

For each $r\in\mathbb{Z}$ *the Hankel determinants* $\det\left(f(i+j,q^{2r+1},q)\right)_{i,j=0}^{n}$, $\det\left(F(i+j,q^{r},q)\right)_{i,j=0}^{n}$, $\det\left(F(2i+2j,q^{r},q)\right)_{i,j=0}^{n}$ *and* $\det\left(F(2i+2j+1,q^{r},q)\right)_{i,j=0}^{n}$ *vanish for all sufficiently large* $n\in\mathbb{N}.$

It turns out that the corresponding Hankel determinants for general s have closed formulae. Since the verification of all details seems to be rather complicated I did not try it and state these results as conjectures. The reader is invited to fill in the missing steps.

Some conjectured Hankel Determinants

$$d(n,s,q)=\det\left(f(i+j,s,q)\right)_{i,j=0}^{n}=d(n,0,q)\prod_{j=0}^{n}\left(\frac{s}{q^{2j-1}};q^{2}\right)_{2j}, \tag{9}$$

$$\begin{aligned}D_0(n,s,q)&=\det\left(F(2i+2j,s,q)\right)_{i,j=0}^{n}\\&=D_0(n,0,q)\prod_{j=0}^{n}\left(\left(s-q^{2j}\right)\left(s-q^{2j+1}\right)\left(s-\frac{1}{q^{2j}}\right)\left(s-\frac{1}{q^{2j+1}}\right)\right)^{n-j},\end{aligned} \tag{10}$$

$$\begin{aligned}D_1(n,s,q)&=\det\left(F(2i+2j+1,s,q)\right)_{i,j=0}^{n}\\&=D_1(n,0,q)(1-s)^{n+1}\prod_{j=0}^{n}\left(\left(s-q^{2j+2}\right)\left(s-q^{2j+1}\right)\left(s-\frac{1}{q^{2j+2}}\right)\left(s-\frac{1}{q^{2j+1}}\right)\right)^{n-j}\end{aligned} \tag{11}$$

and

$$\begin{aligned}D(n,s,q)&=\det\left(F(i+j,s,q)\right)_{i,j=0}^{n}\\&=D(n,0,q)\frac{\prod_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\left(\left(1-q^{2j}s\right)\left(1-q^{2j+1}s\right)\left(1-\frac{s}{q^{2j}}\right)\left(1-\frac{s}{q^{2j+1}}\right)\right)^{n-2j}}{(1-s)^{2\left\lfloor\frac{n+1}{2}\right\rfloor}}.\end{aligned} \tag{12}$$

We now provide some evidence that these conjectures are true. For a proof it would suffice to verify the identities (14) or (16). The corresponding determinants for $s=0$ can be explicitly computed.

All of these Hankel determinants are polynomials in $\mathbb{Q}(q)[s]$. The roots of $D(n,s,q)$, $D_0(n,s,q)$ and $D_1(n,s,q)$ are of the form $q^{\pm r}$ and the roots of $d(n,s,q)$ are of the form $q^{\pm(2r+1)}$ for integers $r\in\mathbb{Z}$. It follows again that for each integer r the Hankel determinants $D\left(n,q^{r},q\right),D_0\left(n,q^{r},q\right),D_1\left(n,q^{r},q\right)$ and $d\left(n,q^{2r+1},q\right)$ vanish for all large $n\in\mathbb{N}$.

2. Orthogonal polynomials and Hankel determinants

In order to make the paper intelligible for readers who are not familiar with orthogonal polynomials and Hankel determinants I first recall some well-known facts (cf. e.g. [1]). If not otherwise stated we use the notation introduced in [3].

A sequence of real-valued monic polynomials $\left(p_n(x)\right)_{n\geq 0}$ with $\deg p_n = n$ is called orthogonal with respect to a linear functional F if $F\left(p_n(x)p_m(x)\right)=0$ for $m \neq n$ and $F\left(p_n(x)^2\right) \neq 0.$

In particular for $m=0$ we get

$$F\left(p_n(x)\right)=[n=0]. \tag{13}$$

Since $\deg p_n = n$ the linear functional F is uniquely determined by (13).

On the other hand F is also uniquely determined by the moments $a(n)=F\left(x^n\right)$ for $n \in \mathbb{N}.$

By Favard's theorem there exist numbers $\sigma(n), \tau(n)$ such that

$$p_n(x)=(x-\sigma(n-1))p_{n-1}(x)-\tau(n-2)p_{n-2}(x). \tag{14}$$

On the other hand for given sequences $\sigma(n)$ and $\tau(n) \neq 0$ formula (14) together with the initial values $p_0(x)=1$ and $p_{-1}(x)=0$ defines a sequence of polynomials which are orthogonal with respect to the linear functional F defined by (13). If $F\left(x^n\right)=a(n)$ then

$$p_n(x)=\frac{1}{\det(a(i+j))_{i,j=0}^{n-1}}\det\begin{pmatrix} a(0) & a(1) & \cdots & a(n-1) & 1 \\ a(1) & a(2) & \cdots & a(n) & x \\ a(2) & a(3) & \cdots & a(n+1) & x^2 \\ \vdots & & & & \vdots \\ a(n) & a(n+1) & \cdots & a(2n-1) & x^n \end{pmatrix}. \tag{15}$$

If we define $a(n,j)$ by

$$\begin{aligned} &a(0,j)=[j=0] \\ &a(n,0)=\sigma(0)a(n-1,0)+\tau(0)a(n-1,1) \\ &a(n,j)=a(n-1,j-1)+\sigma(j)a(n-1,j)+\tau(j)a(n-1,j+1), \end{aligned} \tag{16}$$

then $a(n,0)=a(n)$ and the Hankel determinant $\det\left(a(i+j)\right)_{i,j=0}^{n}$ is given by

$$\det\left(a(i+j)\right)_{i,j=0}^{n}=\prod_{i=1}^{n}\prod_{j=0}^{i-1}\tau(j). \tag{17}$$

Thus if all $\tau(j)$ have a closed expression so does the Hankel determinant.

Under general conditions a sequence $\left(a(n)\right)_{n\geq 0}$ can be characterized by the sequences of the Hankel determinants $\det\left(a(i+j)\right)_{i,j=0}^{n}$ and $\det\left(a(i+j+1)\right)_{i,j=0}^{n}$. For from $\det\left(a(0)\right)$, $\det\left(a(1)\right)$, $\det\begin{pmatrix} a(0) & a(1) \\ a(1) & a(2) \end{pmatrix}$, $\det\begin{pmatrix} a(1) & a(2) \\ a(2) & a(3) \end{pmatrix}, \cdots$ we can compute $a(0), a(1), a(2), a(3), \cdots$.

The Hankel determinant $\det\left(a(i+j+1)\right)_{i,j=0}^{n}$ can be easily computed by observing (15). We get

$$\det\left(a(i+j+1)\right)_{i,j=0}^{n} = (-1)^{n} \det\left(a(i+j)\right)_{i,j=0}^{n} p_{n+1}(0). \tag{18}$$

This scenario can be used to guess and in some cases also to prove the Hankel determinants of a given sequence $\left(a(n)\right)$.

Let me demonstrate this method in the known case (cf. [4]) of the sequence of Rogers-Szegö polynomials $\left(r_n(s,q)\right)_{n\geq 0} = \left(\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q s^j \right)_{n\geq 0}$. Using (15) we compute the polynomials $p_n(x,s,q)$ for small values of n and determine $\sigma(n,s,q)$ and $\tau(n,s,q)$ using (14). Then we guess that

$$\sigma(n,s,q) = q^n \left(1+s\right) \tag{19}$$

and

$$\tau(n,s,q) = q^n s\left(q^{n+1}-1\right). \tag{20}$$

By (17) the Hankel determinants become

$$d(n,s,q) = \det\left(r_{i+j}(s,q)\right)_{i,j=0}^{n} = q^{\binom{n+1}{3}} (-s)^{\binom{n+1}{2}} \prod_{j=1}^{n} \left(q;q\right)_j. \tag{21}$$

Till now (21) is only a guess. In order to prove it we compute $a(n,j) = a(n,j,s,q)$ by (16).

Again we guess that

$$a(n,j,s,q) = r_{n-j}(s,q) \begin{bmatrix} n \\ j \end{bmatrix}_q . \tag{22}$$

Then it suffices to verify (16) and we are done.

The same procedure can be applied to obtain

$$p_n(x,s,q) = \sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix} r_j\left(s,q^{-1}\right) x^{n-j} \tag{23}$$

by using Favard's theorem (14).

Remark

Comparing with [5], 14.24, we see that the polynomials $p_n(x,s,q)$ are the

Al-Salam-Carlitz I polynomials $p_n(x,s,q)=U_n^{(s)}(x;q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}} s^k \prod_{j=0}^{n-k-1}\left(x-q^j\right).$

The above result is therefore equivalent with the fact that the Rogers-Szegö polynomials $r_n(s,q)=\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q s^j$ are the moments of the Al-Salam-Carlitz I polynomials with respect to the linear functional F.

The second Hankel determinants of the Rogers-Szegö polynomials are given by

$$\det\left(r(i+j+1,s,q)\right)_{i,j=0}^{n}=(-1)^n d(n,s,q)\,p_{n+1}(0)=d(n,s,q)\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}+\binom{n-k}{2}} s^k.$$

Note that $\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}+\binom{n-k}{2}} s^k = q^{\binom{n}{2}} r\left(n,s,\frac{1}{q}\right).$

3. Hankel determinants of $f(n,s,q)=\dfrac{\sum_{j=0}^{n} s^j \begin{bmatrix} n \\ j \end{bmatrix}_{q^2}}{(-q;q)_n}.$

By (6) $f(n,s,q)=\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}\dfrac{\prod_{i=0}^{j-1}\left(s-q^{2i+1}\right)}{(-q;q)_j}.$

Let us consider more generally

$$h(n,s,z,q)=\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q \frac{\prod_{i=0}^{j-1}\left(s-q^{2i+1}z\right)}{(-qz;q)_j}. \tag{24}$$

Note that besides $h(n,s,1,q)=f(n,s,q)$ we also have $h(n,s,0,q)=r(n,s,q)=\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q s^j.$

The sequence $\left(h(n,s,z,q)\right)$ satisfies the recurrence

$$h(n,s,z,q)=\frac{(1+s)h(n-1,s,z,q)+\left(q^{n-1}-1\right)sh(n-2,s,z,q)}{1+q^n z}$$

with initial values $h(0,s,z,q)=1$ and $h(1,s,z,q)=\dfrac{1+s}{1+qz}.$

We also get

$$h(n,s,z,q)=\frac{\sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^j q^{j^2}\left(q;q^2\right)_j \begin{bmatrix} n \\ 2j \end{bmatrix}(sz)^j r(n-2j,s,q)}{\left(-qz;q\right)_n}.$$

It is easy to guess that

$$\sigma(n,s,z)=q^n(1+s)\frac{1+q^{n-1}(1+q)z-q^{2n}z}{\left(1+q^{2n-1}z\right)\left(1+q^{2n+1}z\right)} \tag{25}$$

and

$$\tau(n,s,z)=-\frac{q^n\left(1+q^n z\right)\left(1-q^{n+1}\right)}{\left(1+q^{2n}z\right)\left(1+q^{2n+1}z\right)^2\left(1+q^{2n+2}z\right)}\left(s-q^{2n+1}z\right)\left(1-q^{2n+1}sz\right). \tag{26}$$

This implies

$$d(n,s,q)=q^{\frac{n^2(n+1)}{2}}\frac{\prod_{j=0}^{n}\left(\frac{s}{q^{2j-1}};q^2\right)_{2j}\prod_{j=0}^{n}(q;q)_j}{\prod_{j=0}^{n}(-q;q)_{j+n}}. \tag{27}$$

Finally we also guess that

$$\begin{aligned} p(n,x,s,z,q)&=\sum_{j=0}^{n}(-1)^j q^{\binom{j}{2}}\begin{bmatrix} n \\ j \end{bmatrix} h\left(j,s,q^{2n}z,\frac{1}{q}\right)x^{n-j} \\ &=\sum_{j=0}^{n} q^{\binom{n-j}{2}}\begin{bmatrix} n \\ j \end{bmatrix}\frac{\prod_{i=0}^{n-j-1}\left(q^{2j+2i+1}z-s\right)}{\left(-q^{n+j}z;q\right)_{n-j}}\prod_{i=0}^{j-1}\left(x-q^i\right) \end{aligned} \tag{28}$$

and

$$a(n,k,s,z)=\begin{bmatrix} n \\ k \end{bmatrix} h\left(n-k,s,q^{2k}z,q\right). \tag{29}$$

To give a formal proof of (27) we should verify (16) for these polynomials which I have not done.

4. Hankel determinants of $F(n,s,q)$.

Let us consider

$$H(n,s,z,q)=\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \frac{\prod_{i=0}^{2j-1}\left(q^i z-s\right)}{\left(qz^2;q^2\right)_j} \tag{30}$$

which satisfies $H(n,s,1,q)=F(2n,s,q)$, $H(n,s,q,q)=\frac{1-q}{1-s}F(2n+1,s,q)$ and $H(n,s,0,q)=\sum_{j=0}^{n} s^{2j}\begin{bmatrix} n \\ j \end{bmatrix}_{q^2}$.

The sequence $\left(H(n,s,z,q)\right)$ satisfies the recurrence

$$H(n,s,z,q)=\frac{\left(1+s^2-q^{2n-2}(1+q)sz\right)H(n-1,s,z,q)+\left(1-q^{2n-2}\right)s^2H(n-2,s,z,q)}{1-q^{2n-1}z^2}$$

with initial values $H(0,s,z,q)=1$ and $H(1,s,z,q)=\frac{1+(1+q)s+s^2}{1-qz^2}$.

We also get

$$H(n,s,z,q)=\frac{\sum_{j=0}^{n}(-1)^j\left(-q^{n+1-j};q\right)_j q^{\binom{j}{2}}\begin{bmatrix} n \\ j \end{bmatrix} s^j z^j H(n-j,s,0,q)}{\left(qz^2;q^2\right)_n}.$$

Here we get

$$\sigma(n,s,z)=q^{2n-2}\frac{(1+s^2)q^2\left(1-q^{2n-1}z^2-q^{2n-3}z^2+q^{4n-1}z^2\right)+sz(1+q)\left(1-q^{2n}-q^{2n+2}+q^{4n-1}z^2\right)}{\left(1-q^{4n+1}z^2\right)\left(1-q^{4n-3}z^2\right)} \tag{31}$$

and

$$\tau(n,s,z)=\tau(n,0,z)\left(s-q^{2n}z\right)\left(s-q^{2n+1}z\right)\left(1-q^{2n}sz\right)\left(1-q^{2n+1}sz\right) \tag{32}$$

with

$$\tau(n,0,z)=-q^{2n}\frac{(1-q^{2n-1}z^2)(1-q^{2n+2})}{(1-q^{4n-1}z^2)(1-q^{4n+1}z^2)^2(1-q^{4n+3}z^2)}. \tag{33}$$

This implies (10) and (11).

In this case we get

$$p_n(x,s,z)=\sum_{j=0}^{n}(-1)^j q^{2\binom{j}{2}}\begin{bmatrix} n\\ j\end{bmatrix}_{q^2} H\left(j,s,q^{2n-1}z,\frac{1}{q}\right)$$
$$=\sum_{j=0}^{n}\begin{bmatrix} n\\ j\end{bmatrix}_{q^2} q^{2\binom{n-j}{2}}\frac{\prod_{i=0}^{2n-2j-1}\left(s-q^{2j+i}z\right)}{\left(q^{2n-1+2j}z^2;q^2\right)_{n-j}}\prod_{i=0}^{j-1}\left(x-q^{2i}\right). \tag{34}$$

and

$$a(n,k,s,z)=\begin{bmatrix} n\\ k\end{bmatrix}_{q^2} H\left(n-k,s,q^{2k}z,q\right). \tag{35}$$

Finally we want to determine the Hankel determinants $D(n,s,q)=\det\left(F(i+j,s,q)\right)_{i,j=0}^{n}$.

Here we get

$$\sigma(n,s)=(-1)^n\frac{\left(1+s^2\right)u(n)+sv(n)}{(1-s)\left(1-q^{2n-1}\right)\left(1-q^{2n+1}\right)} \tag{36}$$

with

$$u(2n)=q^{2n}\left(1-q^{2n-1}+q^{2n+1}-q^{4n+1}\right),$$
$$u(2n+1)=q^{2n+2}\left(1-q^{4n+1}+q^{2n-1}-q^{2n+1}\right) \tag{37}$$

and

$$v(2n)=1-3q^{2n}-q^{2n+1}+q^{4n-1}-q^{4n+1}+q^{6n-1}+q^{6n}+2q^{6n+1}-q^{8n},$$
$$v(2n+1)=1-q^{2n}-q^{2n+1}-2q^{2n+2}-q^{4n+1}+q^{4n+3}+3q^{6n+3}+q^{6n+4}-q^{8n+4}.$$

The corresponding $\tau(n,s)$ are

$$\tau(2n,s)=\frac{q^{4n+1}\left(1-\dfrac{s}{q^{2n}}\right)\left(1-\dfrac{s}{q^{2n+1}}\right)\left(1-q^{2n}s\right)\left(1-q^{2n+1}s\right)}{(1-s)^2\left(1-q^{4n+1}\right)^2}, \tag{38}$$

$$\tau(2n+1,s)=-(1-s)^2q^{2n+2}\frac{\left(1-q^{2n+1}\right)\left(1-q^{2n+2}\right)}{\left(1-q^{4n+3}\right)^2}. \tag{39}$$

By (17) we get (12).

For $s=0$ we get

$$D(n,0,q)=(-1)^{\left\lfloor\frac{(n+1)^2}{4}\right\rfloor}q^{n^2+n\sum_{j=1}^{n-1}\left\lfloor\frac{j}{2}\right\rfloor}\prod_{j=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\left(\frac{1}{\left(1-q^{4j+1}\right)}\right)^{2n-2j}\left(\frac{\left(1-q^{2j+1}\right)\left(1-q^{2j+2}\right)}{\left(1-q^{4j+3}\right)^2}\right)^{n-1-2j}. \tag{40}$$

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