Some notes on q-Gould polynomials

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Abstract

In a recent paper F. Chapoton and J. Zeng studied polynomials which are related to the $q-$ ballot numbers of Carlitz and Riordan and rediscovered some results of my 1996 paper on $q-$Gould polynomials. Since those results apparently are unknown I recall some of them from a slightly different point of view.

0. Introduction

F. Chapoton and J. Zeng [1] studied polynomials which are related to the $q-$ballot numbers of Carlitz and Riordan. Similar polynomials had also been introduced in [3] and [5], but it seems that nobody has taken note of them. Perhaps this is due to the fact that they were written in German and appeared in a journal with low dissemination. This note gives a survey of some of my old results in English from a slightly different point of view. In order to save space I shall freely use notations and results of my recent paper [7].

1. Background material

Let me first sketch some background material.

Consider the special class of Fibonacci polynomials $f_n(x)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{n-k}{k}(-1)^k x^{n-2k}$ and the coefficients $c(n,k)$ in the representation $x^n=\sum_{k=0}^{n}c(n,k)f_k(x).$

They satisfy $c(n,k)=c(n-1,k-1)+c(n-1,k+1)$ and are explicitly given by $c(2n+k-1,k-1)=\binom{2n+k-1}{n}-\binom{2n+k-1}{n-1}=\frac{k}{2n+k}\binom{2n+k}{n}$ and $c(2n+k,k-1)=0.$ Note that $c(2n,0)=C_n=\frac{1}{n+1}\binom{2n}{n}$ is a Catalan number.

The matrix $\left(c(n,k)\right)_{n,k=0}^{\infty}$ is often called Catalan triangle (cf. OEIS [9], A053121).

Recall the well-known combinatorial interpretation of the numbers $c(n,k)$ as the number of non-negative lattice paths (Dyck paths) in $\mathbb{Z}^2$ which start in $(0,0)$ and end in (n,k) with up-steps $(1,1)$ and down-steps $(1,-1).$

Let $G_n(x)$ be the number of such paths with n down-steps which end at height $x-1$ for some $x \geq 1$. Then

$$G_n(x) = c(n+x-1, x-1) = \frac{x}{2n+x}\binom{2n+x}{n} = \binom{2n+x-1}{n} - \binom{2n+x-1}{n-1}. \qquad (1.1)$$

These and related polynomials have been extensively studied by Henry W. Gould (cf. e.g. [8]) and have therefore been called Gould polynomials by Gian-Carlo Rota [10].

These Gould polynomials are uniquely determined by the recurrence

$$\Delta G_n(x) = G_n(x+1) - G_n(x) = G_{n-1}(x+2) = E^2 G_{n-1}(x) \qquad (1.2)$$

and the initial values $G_n(0) = [n=0]$. Here Δ denotes the difference operator and E the shift operator defined by $Ef(x) = f(x+1)$.

The polynomials $f_n(x)$ are the special case $m=2$ of the polynomials (cf. [7])

$f_n^{(m)}(x) = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \binom{n-(m-1)k}{k}(-1)^k x^{n-mk}$ for $m \geq 1$, which satisfy the recursion $f_n^{(m)}(x) = x f_{n-1}^{(m)}(x) - f_{n-m}^{(m)}(x)$. This implies that the coefficients $c(n,k,m)$ of the expansion $x^n = \sum_{k=0}^{n} c(n,k,m) f_k^{(m)}(x)$ satisfy $c(n,k,m) = c(n-1,k-1,m) + c(n-1,k+m-1,m)$ and

can be interpreted as the numbers of all non-negative lattice paths with up-steps $(1,1)$ and down-steps $(1,1-m)$ from $(0,0)$ to (n,k).

Let $G_n(x,m)$ be the number of such paths with n down-steps which end at height $x-1$ for some $x \geq 1$. Then

$$G_n(x,m) = c(mn+x-1, x-1, m) = \frac{x}{mn+x}\binom{mn+x}{n} = \binom{mn+x-1}{n} - (m-1)\binom{mn+x-1}{n-1}. \qquad (1.3)$$

For $x=1$ we get the generalized Catalan numbers

$$C_n^{(m)} = \frac{1}{1+(m-1)n}\binom{mn}{n} = \frac{1}{1+mn}\binom{mn+1}{n} = G_n(1,m). \qquad (1.4)$$

Note that $C_n^{(m)} = \Lambda\left(x^{mn}\right)$, if we define the linear functional Λ on the polynomials by $\Lambda\left(f_n^{(m)}\right) = [n=0]$.

The Gould polynomials $G_n(x,m)$ could independently of the above interpretation also be defined as the uniquely determined polynomials satisfying $E^{-m}\Delta G_n(x,m) = G_{n-1}(x,m)$ with initial values $G_n(0,m) = [n=0]$. With this definition also $m=0$ is possible and gives $G_n(x,0) = \binom{x}{n}$.

This follows from

$$\Delta G_n(x,m) = \binom{mn+x-1}{n-1} - (m-1)\binom{mn+x-1}{n-2}$$
$$= \binom{m(n-1)+x+m-1}{n-1} - (m-1)\binom{m(n-1)+x+m-1}{n-2} = G_{n-1}\left(x+m,m\right) = E^m G_{n-1}(x,m)$$

with initial values $G_n(0,m) = [n=0]$.

For $m=1$ we get $G_n(x,1) = \frac{x}{n+x}\binom{n+x}{n} = \binom{n+x-1}{n}$.

Let us note the well-known Rothe-Hagen identities

$$\frac{x+y}{x+y+mn}\binom{x+y+mn}{n} = G_n(x+y,m) = \sum_{k=0}^{n} G_k(x,m)G_{n-k}(y,m)$$
$$= \sum_{k=0}^{n} \frac{x}{x+mk}\binom{x+mk}{k}\frac{y}{y+m(n-k)}\binom{y+m(n-k)}{n-k}, \tag{1.5}$$

$$\binom{x+y+mn}{n} = \sum_{k=0}^{n} G_k(x,m)\binom{y+m(n-k)}{n-k} \tag{1.6}$$

and the generating function

$$G(x,m,z) = \sum_{n\geq 0} G_n(x,m)z^n = G(1,m,z)^x. \tag{1.7}$$

The generating function of the generalized Catalan numbers

$C(z,m) = \sum_{n\geq 0} C_n^{(m)} z^n = G(1,m,z)$ satisfies

$$C(z,m) = 1 + zC(z,m)^m. \tag{1.8}$$

2. q-Gould polynomials

Some $q-$ analogues of these Gould polynomials have been found in [3] and [5].

As already stated the notations are the same as in [7]. Moreover I use the symbols

$$\begin{bmatrix} x \\ k \end{bmatrix} = \frac{[x][x-1]\cdots[x-k+1]}{[k]!} \text{ and } \binom{x}{k}_q = \frac{x(x-[1])\cdots(x-[k-1])}{[k]!}.$$

Since $q^j[x-j]=[x]-[j]$ these are related by $q^{\binom{k}{2}}\begin{bmatrix} x \\ k \end{bmatrix} = \binom{[x]}{k}_q$.

Let E_q be the $q-$shift operator which satisfies $E_q f(x) = f(qx+1)$ or equivalently $E_q f([x]) = f([x+1])$ and Δ_q the $q-$difference operator $\Delta_q f(x) = \frac{E_q f(x) - f(x)}{E_q x - x}$, which satisfies $\Delta_q \binom{x}{n}_q = \binom{x}{n-1}_q$.

These operators are $q-$commuting

$$\Delta_q E_q = q E_q \Delta_q, \tag{2.1}$$

$$E_q \Delta_q f(x) = \frac{E_q^2 f(x) - E_q f(x)}{E_q(E_q x - x)} = \frac{E_q^2 f(x) - E_q f(x)}{q(E_q x - x)}.$$

Let us also recall the $q-$binomial theorem $(a+b)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a^k b^{n-k}$ for elements a,b with $ba = qab$ (cf. e.g. [2]).

Consider the $q-$ analogues $f_n^{(m)}(x,q) = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} q^{m\binom{k}{2}} \begin{bmatrix} n-(m-1)k \\ k \end{bmatrix} (-1)^k x^{n-mk}$ of the polynomials $f_n^{(m)}(x)$.

These polynomials satisfy (cf. [7])

$f_n^{(m)}(x,q) = x f_{n-1}^{(m)}(x,q) - q^{n-m} f_{n-m}^{(m)}(x,q)$. Therefore the coefficients $c(n,k,m,q)$ in

$$x^n = \sum_{k=0}^{n} c(n,k,m,q) f_k^{(m)}(x,q) \tag{2.2}$$

satisfy $c(n,k,m,q) = c(n-1,k-1,m,q) + q^k c(n-1,k+m-1,m,q)$.

If we define the linear functional Λ by $\Lambda\left(f_n^{(m)}(x,q)\right)=[n=0]$ we get

$$\Lambda\left(x^{mn}\right)=C_n^{(m)}(q)=c(mn,0,m,q). \tag{2.3}$$

The numbers $c(n,k,m,q)$ have the following combinatorial interpretation.

Consider paths in $\mathbb{Z}^2$ which start at $(0,0)$ with up-steps $(1,1)$ and down-steps $(1,1-m)$ for some $m\geq 1$. To each path we associate a weight w such that each up-step has weight 1 and each down-step with endpoint on height $k\in\mathbb{Z}$ has weight q^k. To each path of length n we associate the word $y_{i_1}y_{i_2}\cdots y_{i_n}$ where y_0 corresponds to an up-step and y_1 to a down-step. Now suppose that $y_0y_1=qy_1y_0$ holds. Then each word v has a representation $v=y_{i_1}y_{i_2}\cdots y_{i_n}=\alpha(v)y_1^d y_0^u$ where d denotes the number of down-steps y_1 and u denotes the number of up-steps y_0. For example let $m=3$ and consider the path $(0,0)\to(1,-2)\to(2,-1)\to(3,0)\to(4,1)\to(5,2)\to(6,3)\to(7,1)$ with weight $q^{-2}q=q^{-1}.$

The corresponding word is $y_1y_0y_0y_0y_0y_0y_1$ which can be reduced to $y_1y_0^5y_1=q^5y_1^2y_0^5.$

We show that

$$\alpha(v)=q^{(m-1)\binom{d+1}{2}}w(v). \tag{2.4}$$

In our example we have $w(v)=q^{-1}$ and $\alpha(v)=q^5.$ Thus $\alpha(v)=q^{(3-1)\binom{3}{2}}w(v)=q^6q^{-1}=q^5.$

Identity (2.4) holds for words of length 1 since $\alpha(y_0)=w(y_0)=1$ and $\alpha(y_1)=1=q^{(m-1)\binom{2}{2}}w(y_1).$ Now we use induction on the length of the word.

(2.4) holds for vy_0 since $\alpha(vy_0)=\alpha(v)=q^{(m-1)\binom{d+1}{2}}w(v)=q^{(m-1)\binom{d+1}{2}}w(vy_0).$

For vy_1 we have $w(vy_1)=q^{u-(m-1)(d+1)}w(v)$ and $\alpha(vy_1)=q^u\alpha(v)$ because $vy_1=\alpha(v)y_1^d y_0^u y_1=q^u\alpha(v)y_1^{d+1}y_0^u.$

Thus $\alpha(vy_1)=q^u\alpha(v)=q^{u+(m-1)\binom{d+1}{2}}w(v)=q^{(m-1)\binom{d+1}{2}+(m-1)(d+1)}w(vy_1)=q^{(m-1)\binom{d+2}{2}}w(vy_1).$

Let now $c(n,k,m,q)$ be the weight of all **non-negative** paths from $(0,0)$ to (n,k). Then we get $c(n,0,m,q)=c(n-1,m-1,m,q)$ and
$c(n,k,m,q)=c(n-1,k-1,m,q)+q^k c(n-1,k+m-1,m,q).$

In general there is no closed formula for $c(n,k,m,q)$ and for the $q-$ analogues of the Gould polynomials. Therefore we must characterize them by other properties.

Consider for some $x\geq 1$ the weight $c(mn+x-1,x-1,m,q)$ of all non-negative paths from $(0,0)$ to $(mn+x-1,x-1)$ with precisely n down-steps. Each such path contains a point of the form $(mn-1,mk-1)$ for some $k>0.$ Each path to $(mn-1,mk-1)$ has $n-k$ down-steps and $(m-1)n+k-1$ up-steps. The remaining path from $(mn-1,mk-1)$ to $(mn+x-1,x-1)$ has length x and k down-steps. Each word $y_{i_1}y_{i_2}\cdots y_{i_x}$ corresponds to such a path since it can never go beneath the $x-$axis. Since $y_0y_1=qy_1y_0$ the $q-$binomial theorem gives
$\left(y_1+y_0\right)^x=\sum_{k=0}^{x}\begin{bmatrix}x\\k\end{bmatrix}y_1^k y_0^{x-k}.$

Therefore to each non-negative path from $(0,0)$ to $(mn+x-1,x-1)$ corresponds a code of the form $v\begin{bmatrix}x\\k\end{bmatrix}y_1^k y_0^{x-k}$ for some $k\geq 1.$ We know that $\alpha(v)=q^{(m-1)\binom{n-k+1}{2}}w(v)$

Summing over all v gives

$$\sum_v v\begin{bmatrix}x\\k\end{bmatrix}y_1^k y_0^{x-k}=\begin{bmatrix}x\\k\end{bmatrix}\sum_v \alpha(v)y_1^{n-k}y_0^{mn-1-n+k}y_1^k y_0^{x-k}=\begin{bmatrix}x\\k\end{bmatrix}q^{k(mn-n+k-1)}\sum_v \alpha(v)y_1^n y_0^{(m-1)n+x-1}$$
$$=\begin{bmatrix}x\\k\end{bmatrix}q^{k(mn-n+k-1)}\sum_v q^{(m-1)\binom{n-k+1}{2}}w(v)y_1^n y_0^{(m-1)n+x-1}$$
$$=c(nm-1,mk-1,m,q)\begin{bmatrix}x\\k\end{bmatrix}q^{(m+1)\binom{k}{2}+(m-1)\binom{n+1}{2}}y_1^n y_0^{(m-1)n+x-1}$$

Thus by (2.4) the weight of these paths is for each positive integer x

$$c(mn+x-1,x-1,m,q)=\sum_{k=1}^{n}c(nm-1,mk-1,m,q)q^{(m+1)\binom{k}{2}}\begin{bmatrix}x\\k\end{bmatrix}. \tag{2.5}$$

Let us now introduce the polynomials

$$G_n(x,m,q)=\sum_{k=1}^{n}c(mn-1,mk-1,m,q)q^{m\binom{k}{2}}\binom{x}{k}_q. \tag{2.6}$$

These polynomials have been called $q-$Gould polynomials in [3].

The first terms are $1,\ x,\ [m]x+q^m\binom{x}{2}_q$.

Then (2.5) reduces to

$$G_n\left([x],m,q\right)=c(mn+x-1,x-1,m,q) \tag{2.7}$$

For $x=1$ we get a $q-$ analogue of the generalized Catalan numbers

$$G_n(1,m,q)=c(mn,0,m,q)=C_n^{(m)}(q). \tag{2.8}$$

The identity (2.6) gives

$$G_n(x,m,q)=\sum_{k=1}^{n}G_{n-k}\left([mk],q\right)q^{m\binom{k}{2}}\binom{x}{k}_q. \tag{2.9}$$

The identity

$$c(mn+x-1,x-1,m,q)=c(mn+x-2,x-2,m,q)+q^{x-1}c(mn+x-2,x+m-2,m,q)$$

gives

$$G_n\left([x],m,q\right)=G_n\left([x-1],m,q\right)+q^{x-1}G_{n-1}\left([x+m-1],m,q\right). \tag{2.10}$$

Thus if we consider $x+1$ instead of x in (2.10) we get

$$\Delta_q G_n\left([x],m,q\right)=G_{n-1}\left([x+m],m,q\right)=E_q^m G_{n-1}\left([x],m,q\right).$$

Since $G_n(x,m,q)$ is a polynomial we get the following

Theorem 1

Define the Gould polynomials $G_n(x,m,q)$ by

$$G_n(x,m,q)=\sum_{k=1}^{n}c(mn-1,mk-1,m,q)q^{m\binom{k}{2}}\binom{x}{k}_q. \tag{2.11}$$

Then $G_n(x,m,q)$ is the uniquely determined polynomial which satisfies

$$\Delta_q G_n\left(x,m,q\right)=E_q^m G_{n-1}\left(x,m,q\right) \tag{2.12}$$

and $G_n(0,m,q)=[n=0]$.

Moreover

$$G_n\left([x],m,q\right)=c(mn+x-1,x-1,m,q). \tag{2.13}$$

Corollary 1

For $m=1$ *we get*

$$G_n([x],1,q)=\begin{bmatrix} x+n-1 \\ n \end{bmatrix} \tag{2.14}$$

and

$$G_n(x,1,q)=\frac{x\left(Ex\right)\cdots\left(E^{n-1}x\right)}{[n]!}=\frac{x\left(qx+[1]\right)\left(q^2x+[2]\right)\cdots\left(q^{n-1}x+[n-1]\right)}{[n]!}. \tag{2.15}$$

This follows from

$$\Delta_q G_n\left(x,1,q\right)=\frac{\left(Ex\right)\cdots\left(E^{n-1}x\right)}{[n]!}\frac{E^n x-x}{1+(q-1)x}=\frac{\left(Ex\right)\cdots\left(E^{n-1}x\right)}{[n-1]!}=EG_{n-1}(x,1,q).$$

Since $\Delta_q \binom{x}{n}_q = \binom{x}{n-1}_q$ we could define $G_n(x,0,q)=\binom{x}{n}_q$.

Now we prove a $q-$ analogue of (1.5):

Theorem 2

$$G_n\left([x+y],m,q\right)=\sum_{k=0}^{n} G_k\left([x],m,q\right)q^{ky}G_{n-k}\left([y],m,q\right). \tag{2.16}$$

Proof

$G_n\left([x+y],m,q\right)=c(mn+x+y-1,x+y-1,q)$ is the weight of all non-negative paths from $(0,0)$ to $(mn+x+y-1,x+y-1)$ with n down-steps. Consider the largest path starting from $(0,0)$ which ends at height $y-1$. Let $n-k$ be the number of down-steps of this path. The next step is an up-step. The remaining path is a non-negative path with k down-steps whose weight is the same as the weight of a path from $(0,0)$ to $(*,x-1)$ where each down-step which ends at height ℓ has weight $q^{y+\ell}$. This gives (2.16).

Note that this argument could also be applied for $m=0$ and gives then a version of the $q-$ Vandermonde identity.

Let us give also another proof . Let $G_n([x+y],m,q)=\sum_{k=0}^{n} a(n,k)G_k([x],m,q).$

Then

$$a(n,k)=L\left(E_q^{-m}\Delta_q\right)^k G_n([x+y],m,q)=L\left(E_q^{-m}\Delta_q\right)^k E_q^y G_n([x],m,q)$$
$$=q^{ky}L\left(E_q^y\left(E_q^{-m}\Delta_q\right)^k G_n([x],m,q)\right)=q^{ky}G_{n-k}([y],m,q).$$

The last method of proof gives also a $q-$ analogue of (1.6)

Theorem 3

$$\begin{bmatrix} x+y+mn \\ n \end{bmatrix}=\sum_{k=0}^{n} q^{-m\binom{k}{2}+ky+kmn+\binom{n-k}{2}-\binom{n}{2}}\begin{bmatrix} y+m(n-k) \\ n-k \end{bmatrix}G_k([x],m,q). \tag{2.17}$$

Proof

Let $\begin{bmatrix} x+y+mn \\ n \end{bmatrix}=\sum_{k=0}^{n} a(n,k)G_k([x],m,q).$ Then

$$a(n,k)=L\left(E_q^{-m}\Delta_q\right)^k\begin{bmatrix} x+y+mn \\ n \end{bmatrix}=L\left(E_q^{-m}\Delta_q\right)^k E^{y+mn}q^{-\binom{n}{2}}\binom{[x]}{n}_q$$
$$=Lq^{-m\binom{k}{2}-\binom{n}{2}}E_q^{-km}\Delta_q^k E^{y+mn}\binom{[x]}{n}_q=Lq^{-m\binom{k}{2}-\binom{n}{2}+k(y+mn)}E_q^{y+m(n-k)}\binom{[x]}{n-k}_q$$
$$=\sum_{k=0}^{n} q^{-m\binom{k}{2}+ky+kmn+\binom{n-k}{2}-\binom{n}{2}}\begin{bmatrix} y+m(n-k) \\ n-k \end{bmatrix}.$$

For $m\geq 2$ no simple closed formulae are known. But a simple inverse of (2.11) follows by setting $y=-mn$ in (2.17).

Theorem 4

$$\binom{x}{n}_q=\sum_{k=0}^{n} q^{-m\binom{k}{2}}\binom{[-mk]}{n-k}_q G_k(x,m,q). \tag{2.18}$$

Since $G_n(x,1,q)$ has a closed formula it is also interesting to expand $G_n(x,m,q)$ in terms of $G_n(x,1,q)$. We get

Theorem 5

$$G_n(x,m,q)=\sum_{k=0}^{n} q^{(m-1)\binom{k}{2}} G_{n-k}\left([(m-1)k],m,q\right)G_1(x,1,q). \tag{2.19}$$

Proof

Write for the moment

$$G_n(x,m,q)=\sum_{k=0}^{n} a(n,k)G_1(x,1,q).$$

Then

$$a(n,k)=L\left(E_q^{-1}\Delta_q\right)^k G_n(x,m,q)=q^{(m-1)\binom{k}{2}}E_q^{k(m-1)}\left(E_q^{-m}\Delta_q\right)^k G_n(x,m,q)$$
$$=q^{(m-1)\binom{k}{2}}G_{n-k}([(m-1)k],m,q).$$

Theorem 2 implies

Theorem 6

For positive integers x,y *the generating function*

$$G([x],m,z,q)=\sum_{n\ge 0} G_n\left([x],m,q\right)z^n \tag{2.20}$$

satisfies

$$G([x+y],m,z,q)=G([y],m,z,q)G([x],m,q^y z,q). \tag{2.21}$$

Let $C(z,m,q)$ denote the generating function of the $(m,q)-$ Catalan numbers $C_n^{(m)}(q)=G_n(1,m,q)$. Then we get as $q-$analogue of (1.7)

Corollary 2

$$G([x],m,z,q)=C(z,m,q)C(qz,m,q)\cdots C(q^{x-1}z,m,q).$$

A $q-$analogue of (1.8) is

$$C(z,m,q)=1+zC(z,m,q)C(qz,m,q)\cdots C(q^{m-1}z,m,q). \tag{2.22}$$

This follows from

$G_n([m],m,q) = c(mn+m-1,m-1,m,q) = c(mn+m,0,m,q) = C_{n+1}^{(m)}(q),$ because

$$C(z,m,q) = 1+z\sum_n C_{n+1}^{(m)}(q)z^n = 1+z\sum_n G_n([m],m,q)z^n = 1+zG([m],m,z,q)$$
$$= 1+zC(z,m,q)C(qz,m,q)\cdots C(q^{m-1}z,m,q).$$

We have also (cf. e.g. [6])

$$C(z,m,q) = \frac{E^{(m)}(-qz)}{E^{(m)}(-z)} \tag{2.23}$$

with

$$E^{(m)}(z) = \sum_{n\geq 0} \frac{q^{m\binom{n}{2}}}{(q;q)_n} z^n. \tag{2.24}$$

For

$$E^{(m)}(z) - E^{(m)}(qz) = \sum_{n\geq 0} \frac{q^{m\binom{n}{2}}}{(q;q)_n}(1-q^n)z^n = z\sum_{n\geq 0} \frac{q^{m\binom{n}{2}}}{(q;q)_{n-1}} z^{n-1} = zE^{(m)}(q^m z)$$

implies

$$\frac{E^{(m)}(-qz)}{E^{(m)}(-z)} = 1+\frac{E^{(m)}(-q^m z)}{E^{(m)}(-z)} = 1+\frac{E^{(m)}(-qz)}{E^{(m)}(-z)}\frac{E^{(m)}(-q^2 z)}{E^{(m)}(-qz)}\cdots\frac{E^{(m)}(-q^m z)}{E^{(m)}(-q^{m-1}z)}.$$

In [4] we proved

Theorem 7

$$G_n([k],m,q) = \sum_{(c_1,c_2,\cdots,c_n)} q^{c_1+c_2+\cdots+c_n} \tag{2.25}$$

where $(c_1,c_2,\cdots,c_n)$ *is the set of all* $n-$ *tuples with* $0\leq c_1<k$ *and* $0\leq c_{i+1}\leq c_i+m-1.$

Proof

Let c_1 be the height of the endpoint of last down-step and if c_i is the height of the down-step d_i then let c_{i+1} be the height of the endpoint of the last down-step before d_i. Then clearly $0\leq c_1<k$ and $0\leq c_{i+1}\leq c_i+m-1.$ The path is uniquely determined by these numbers.

3. Remarks

As already mentioned the paper [1] also studies the case $m = 2$ with somewhat different concepts.

They define polynomials $C_n(x \mid q)$ which satisfy $C_1(x \mid q) = 1$ and

$$\Delta_q C_{n+1}(x \mid q) = qE_q^2 C_n\left(x \mid q\right) \text{ with } C_n\left(-\frac{1}{q} \mid q\right) = 0.$$

Since $G_n(qx+1,2,q) = E_q G_n(x,2,q)$ satisfies

$$\Delta_q G_n(qx+1,2,q) = \Delta_q E_q G_n(x,2,q) = qE_q \Delta_q G_n(x,2,q) = qE_q E_q^2 G_{n-1}(x,2,q) = qE_q^2 G_{n-1}(qx+1,2,q)$$

and $G_{n+1}(qx+1,2,q) = 0$ for $x = -\frac{1}{q}$ this implies that

$$C_{n+1}\left(x \mid q\right) = G_n\left(qx+1,2,q\right) \tag{3.1}$$

or equivalently

$$G_n\left([x+1],2,q\right) = C_{n+1}\left([x] \mid q\right). \tag{3.2}$$

They also study expansions analogous to (2.11) where the coefficients are expressed in terms of the $q-$ballot numbers $f(n,k \mid q)$ introduced by Carlitz and Riordan. Comparing with formula [1], (2.4) we see that these numbers are connected with the numbers $c(n,k,2,q)$ by

$$f(n,k \mid q) = c\left(n+k, n-k, 2, \frac{1}{q}\right) q^{kn-\binom{k}{2}}. \tag{3.3}$$

Finally I want to mention the analogues of the Gould polynomials for the monic $q-$ Chebyshev polynomials of the second kind.

The monic $q-$ Chebyshev polynomials of the second kind (cf. [7])

$$u_n(x,q) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \begin{bmatrix} n-k \\ k \end{bmatrix} q^{k^2} (-1)^k \frac{x^{n-2k}}{\left(-q;q\right)_k \left(-q^{n+1-k};q\right)_k} \tag{3.4}$$

satisfy the recurrence relation

$$u_n(x,q) = xu_{n-1}(x,q) - \frac{q^{n-1}}{\left(1+q^{n-1}\right)\left(1+q^n\right)} u_{n-2}(x,q) \tag{3.5}$$

with initial values $u_0(x,q) = 1$ and $u_1(x,q) = x.$

This leads (cf. [5]) to paths with up-step $(1,1)$ and down-step $(1,-1)$ where the weight of the down-steps with endpoint k is $\lambda_k(q)=\frac{q^{k+1}}{(1+q^{k+1})(1+q^{k+2})}$.

Let $c(n,k,q)=c(n-1,k-1,q)+\lambda_k(q)c(n-1,k+1,q)$

Then

$c(2n+x,x,q)=c(2n+x-1,x-1,q)+\lambda_x(q)c(2n+x-1,x+1,q)$

Let the analogue of the Gould polynomials be

$$g_n([x],q)=c(2n+x-1,x-1,q). \tag{3.6}$$

This means that

$$g_n([x+1],q)=g_n([x],q)+\frac{q^{x+1}}{(1+q^{x+1})(1+q^{x+2})}g_{n-1}([x+2],q) \tag{3.7}$$

The identity (cf. [7])

$$\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{\begin{bmatrix}n\\k\end{bmatrix}-\begin{bmatrix}n\\k-1\end{bmatrix}}{(-q;q)_k(-q^{n+2-2k};q)_k}u_{n-2k}(x,q)=x^n$$

implies the closed formula

$$g_n([x],q)=\frac{[x]}{[2n+x]}\begin{bmatrix}2n+x\\n\end{bmatrix}\frac{q^n}{(-q;q)_n(-q^{x+1};q)_n}. \tag{3.8}$$

Note that these functions are no longer polynomials.

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