Some divisibility properties of q-Fibonacci numbers

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Abstract

We give a survey of some known and some new results about factors of different sorts of $q-$ Fibonacci numbers.

0. Introduction

Let $(F_n)_{n\ge 0}=(0,1,1,2,3,5,8,\cdots)$ be the sequence of Fibonacci numbers and let $v_p(n)$ be the $p-$adic valuation of $n,$ i.e. the highest power of the prime number p which divides $n.$ The Fibonacci numbers satisfy (cf. [7]) $v_5(F_n)=v_5(n),$ $v_2(F_{3n})=1$ for odd n and $v_2(F_{6n})=v_2(n)+3.$ If p is a prime different from 2 and 5 then either F_{p-1} or F_{p+1} is divisible by $p.$

For $q\in\mathbb{C}$ let $[n]=[n]_q=\frac{1-q^n}{1-q}=1+q+\cdots+q^{n-1}$ and let $\begin{bmatrix} n \\ k \end{bmatrix}=\frac{[n]\cdots[n-k+1]}{[1][2]\cdots[k]}$ be a $q-$ binomial coefficient.

The Schur-Carlitz $q-$Fibonacci numbers $F_n(q)=\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} q^{k^2}\begin{bmatrix} n-1-k \\ k \end{bmatrix}$ and $G_n(q)=\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} q^{k^2+k}\begin{bmatrix} n-1-k \\ k \end{bmatrix}$ (cf. [9],[2]) which have been introduced by I. Schur in his proof of the Rogers-Ramanujan identities inherit some of the properties for odd primes and the $q-$ Fibonacci numbers $f(n,q)=\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} q^{\binom{k}{2}}\begin{bmatrix} n-1-k \\ k \end{bmatrix}$ introduced in [4] inherit divisibility properties by $2.$

1. Divisibility properties for odd primes $p\ne 5.$

1.1. The $q-$Fibonacci numbers $F_n(q)$ satisfy the recurrence

$$F_n(q)=F_{n-1}(q)+q^{n-2}F_{n-2}(q) \tag{1.1}$$

with initial values $F_0(q)=0$ and $F_1(q)=1.$

The first terms are

$0,1,1,1+q,1+q+q^2,1+q+q^2+q^3+q^4,1+q+q^2+q^3+2q^4+q^5+q^6,\cdots.$

It is clear that $F_n(1)=F_n.$

Theorem 1.1 (George E. Andrews , Leonard Carlitz [1])

If p is an odd prime with $p\equiv\pm 2 \mod 5$ then $F_{p+1}(q)\equiv 0 \mod [p]_q.$

For $q=1$ this can be proved (cf. [6], Theorem 180) using Binet's formula

$F_n=\frac{1}{2^n\sqrt{5}}\left(\left(1+\sqrt{5}\right)^n-\left(1-\sqrt{5}\right)^n\right).$

This gives

$$2^pF_{p+1}=\binom{p+1}{1}+\binom{p+1}{3}5+\cdots+\binom{p+1}{p}5^{\frac{p-1}{2}}.$$

Here all binomial coefficients are divisible by p except the first and last one. Therefore

$2^pF_{p+1}\equiv 1+5^{\frac{p-1}{2}} \mod p.$ Hence $F_{p+1}\equiv 0 \mod p$ if $5^{\frac{p-1}{2}}\equiv\left(\frac{5}{p}\right)=-1 \mod p.$ By the quadratic reciprocity law $\left(\frac{p}{5}\right)\left(\frac{5}{p}\right)=1.$ This implies $\left(\frac{p}{5}\right)=-1$ and thus $p\equiv\pm 2 \mod 5.$

Since there is no analogue of Binet's formula for $q-$Fibonacci numbers L. Carlitz used the polynomial version of the first Rogers-Ramanujan identity (cf. [9], [5])

$$F_{n+1}(q)=\sum_{k=-\left\lfloor\frac{n+2}{5}\right\rfloor}^{\left\lfloor\frac{n+2}{5}\right\rfloor}(-1)^kq^{\frac{k(5k-1)}{2}}\begin{bmatrix} n \\ \left\lfloor\frac{n+5k}{2}\right\rfloor\end{bmatrix}. \tag{1.2}$$

He showed more generally

Lemma 1.1

Let $\Phi_n(q)$ be the $n-th$ cyclotomic polynomial. Then $F_{n+1}(q)$ is divisible by $\Phi_n(q)$ if and only if $n\equiv\pm 2 \mod 5,$ where n is an arbitrary positive integer.

For a prime $n=p$ the cyclotomic polynomial reduces to $1+q+\cdots+q^{p-1}=[p]_q$ and therefore implies Theorem 1.1.

Since $\begin{bmatrix} n \\ k \end{bmatrix}$ is divisible by $\Phi_n(q)$ for $1 \le k \le n-1$ we get by (1.2)

$$F_{n+1}(q) \equiv (-1)^r q^{\frac{r(5r+1)}{2}} \begin{bmatrix} n \\ \left\lfloor \frac{n-5r}{2} \right\rfloor \end{bmatrix} + (-1)^r q^{\frac{r(5r-1)}{2}} \begin{bmatrix} n \\ \left\lfloor \frac{n+5r}{2} \right\rfloor \end{bmatrix} \bmod \Phi_n(q) \tag{1.3}$$

with $r = \left\lfloor \frac{n+2}{5} \right\rfloor$.

It suffices to verify that $F_{n+1}(q) \equiv 0 \mod \Phi_n(q)$ for $n = 10m+2,\ 10m+3,\ 10m+7,\ 10m+8.$

This is shown in the following table:

n	$r = \left\lfloor \frac{n+2}{5} \right\rfloor$	$e(r) = \left\lfloor \frac{n+5r}{2} \right\rfloor$	$e(-r) = \left\lfloor \frac{n-5r}{2} \right\rfloor$	$\begin{bmatrix} n \\ e(r) \end{bmatrix}$	$\begin{bmatrix} n \\ e(-r) \end{bmatrix}$
$10m+2$	$2m$	$10m+1$	1	$[n]$	$[n]$
$10m+3$	$2m+1$	$10m+4$	-1	0	0
$10m+7$	$2m+1$	$10m+6$	1	$[n]$	$[n]$
$10m+8$	$2m+2$	$10m+9$	-1	0	0

In each case both terms of (1.3) vanish modulo $\Phi_n(q)$.

Also observe that $f(q) \equiv 0 \bmod \Phi_n(q)$ for a polynomial $f(q)$ is equivalent with $f(\zeta_n) = 0$ for a primitive $n-$th root of unity ζ_n.

1.2. The $q-$Fibonacci numbers $G_n(q)$ satisfy the recurrence

$$G_n(q) = G_{n-1}(q) + q^{n-1} G_{n-2}(q) \tag{1.4}$$

with initial values $G_0(q) = 0$ and $G_1(q) = 1$. The first terms are

$0,1,1,1+q^2,1+q^2+q^3,1+q^2+q^3+q^4+q^6,1+q^2+q^3+q^4+q^5+q^6+q^7+q^8,\cdots.$

The polynomial version of the second Rogers-Ramanujan identity (cf.[9] ,[5]) gives

$$G_n(q) = \sum_{k=-\left\lfloor \frac{n+2}{5} \right\rfloor}^{\left\lfloor \frac{n+2}{5} \right\rfloor} (-1)^k q^{\frac{k(5k-3)}{2}} \begin{bmatrix} n \\ \left\lfloor \frac{n-1+5k}{2} \right\rfloor \end{bmatrix}. \tag{1.5}$$

For $n=5m$ this implies

$$G_{5m}(q)=\sum_{k=-m}^{m}(-1)^k q^{\frac{k(5k-3)}{2}}\begin{bmatrix}5m\\ \left\lfloor \frac{5(m+k)-1}{2}\right\rfloor\end{bmatrix}_q \equiv 0 \bmod \Phi_{5m}(q) \tag{1.6}$$

since no $q-$binomial coefficient reduces to 1.

As has been observed by H. Pan [8] for $n\not\equiv 0 \bmod 5$ there remain modulo $\Phi_n(q)$ only the terms with $k=r(n),$ where $r(n)=\left\lfloor \frac{n+2}{5}\right\rfloor$ if $n\equiv 3 \bmod 5$ and $n\equiv 4 \bmod 5$ and $r(n)=-\left\lfloor \frac{n+2}{5}\right\rfloor$ if $n\equiv 1 \bmod 5$ or $n\equiv 2 \bmod 5.$

This leads to the following table where the congruences are modulo $\Phi_n(q).$

n	$r(n)$	$G_n(q)$
$5m$	0	0
$5m+1$	$-m$	$(-1)^m q^{\frac{m(5m+3)}{2}} \equiv q^m$
$5m+2$	$-m$	$(-1)^m q^{\frac{m(5m+3)}{2}} \equiv -q^{3m+1}$
$5m+3$	$m+1$	$(-1)^{m+1} q^{\frac{(m+1)(5m+2)}{2}} \equiv -q^{2m+1}$
$5m+4$	$m+1$	$(-1)^{m+1} q^{\frac{(m+1)(5m+2)}{2}} \equiv q^{4m+3}$

The congruences in the right column are easily verified. For example we have for $n=5m+2$ and even m

$$(-1)^m q^{\frac{m(5m+3)}{2}} \equiv q^{\frac{m}{2}(5m+2)} q^{\frac{m}{2}} \equiv -q^{\frac{m}{2}+\frac{5m+2}{2}} = -q^{3m+1}$$

and for odd m

$$(-1)^{m+1} q^{\frac{m(5m+3)}{2}} \equiv q^{\frac{m(5m+3)}{2}-\frac{(5m+2)(m-1)}{2}} \equiv q^{3m+1}.$$

Theorem 1.2 (H. Pan [8])

If p *is a prime with* $p\equiv \pm 1 \bmod 5$ *then* $G_{p-1}(q)\equiv 0 \bmod [p]_q.$

For example

$$G_{10}(q)=[11]_q[5]_{q^2}\left(1-q+q^3-q^4+q^6\right).$$

Let me sketch H. Pan's proof.

By (1.5) we get

$$G_{n-1}(q)=\sum_{k=-\left\lfloor\frac{n+1}{5}\right\rfloor}^{\left\lfloor\frac{n+1}{5}\right\rfloor}(-1)^k q^{\frac{k(5k-3)}{2}}\begin{bmatrix} n-1 \\ \left\lfloor\frac{n-2+5k}{2}\right\rfloor\end{bmatrix}.$$

For $n=5m+1$ this reduces to

$$G_{5m}(q)=\sum_{k=-m+1}^{m}(-1)^k q^{\frac{k(5k-3)}{2}}\begin{bmatrix} 5m \\ \left\lfloor\frac{5(m+k)-1}{2}\right\rfloor\end{bmatrix}=\sum_{k=-m+1}^{m}(-1)^k q^{\frac{k(5k-3)}{2}}\prod_{j=1}^{\left\lfloor\frac{5(m+k)-1}{2}\right\rfloor}\frac{[5m+1-j]_q}{[j]_q}$$

$$=\sum_{k=-m+1}^{m}(-1)^k q^{\frac{k(5k-3)}{2}}\prod_{j=1}^{\left\lfloor\frac{5(m+k)-1}{2}\right\rfloor}\frac{[5m+1]_q-[j]_q}{q^j[j]_q}\equiv\sum_{k=-m+1}^{m}(-1)^{k+\left\lfloor\frac{5(m+k)-1}{2}\right\rfloor}q^{\frac{k(5k-3)}{2}-\binom{\left\lfloor\frac{5(m+k)+1}{2}\right\rfloor}{2}}\bmod\Phi_n(q).$$

Now observe that

$$\ell(m,k)=\frac{k(5k-3)}{2}-\binom{\left\lfloor\frac{5(m+k)+1}{2}\right\rfloor}{2}\text{ satisfies }\ell(m,2k-1)-\ell(m,2k)=5m+1=n$$

if $m\equiv 0\bmod 2$ and $\ell(m,2k+1)-\ell(m,2k)=-5m-1=-n$ if $m\equiv 1\bmod 2.$

Therefore each pair of adjacent terms in $G_m(q)=\sum_{k=-m+1}^{m}(-1)^{k+\left\lfloor\frac{5(m+k)-1}{2}\right\rfloor}q^{\ell(m,k)}\bmod\Phi_n(q)$

satisfies $\pm q^{\ell(m,2k-1)}\mp q^{\ell(m,2k)}=0\bmod\Phi_n(q)$ if m is even and
$\pm q^{\ell(m,2k)}\mp q^{\ell(m,2k+1)}=0\bmod\Phi_n(q)$ if m is odd.

For $n=5m+4$ and

$$G_{5m+3}(q)=\sum_{k=-m}^{m+1}(-1)^k q^{\frac{k(5k-3)}{2}}\begin{bmatrix} 5m+3 \\ \left\lfloor\frac{5(m+k)+2}{2}\right\rfloor\end{bmatrix}\bmod\Phi_n(q)$$

the situation is analogous.

With the same arguments H. Pan has shown that

$$F_{5n}(q)\equiv 0\bmod\Phi_{5n}(q). \tag{1.7}$$

By (1.2) we get

$$F_{5n}(q)=\sum_{k=-n+1}^{n}(-1)^k q^{\frac{k(5k-1)}{2}}\begin{bmatrix} 5n-1 \\ \frac{5n+5k-1}{2}\end{bmatrix}_q$$

and as above each pair of adjacent elements sums to $0.$

1.3. Let $A(x)=\begin{pmatrix}1 & x\\ 1 & 0\end{pmatrix}.$ Then it is easily verified (cf. [3]) that

$$A\left(q^{n-1}\right)A\left(q^{n-2}\right)\cdots A\left(q\right)A(1)=\begin{pmatrix}F_{n+1}(q) & G_n(q)\\ F_n(q) & G_{n-1}(q)\end{pmatrix}. \tag{1.8}$$

If we take the determinant of (1.8) we get the $q-$Cassini formula

$$F_{n+1}(q)G_{n-1}(q)-F_n(q)G_n(q)=(-1)^n q^{\binom{n}{2}}. \tag{1.9}$$

If q is a primitive $n-$th root of unity then $q^{\binom{n}{2}}=\left(q^{\frac{n}{2}}\right)^{n-1}=-1$ if $n\equiv 0 \bmod 2$ and $q^{\binom{n}{2}}=\left(q^n\right)^{\frac{n-1}{2}}=1$ if $n\equiv 1 \bmod 2.$ Therefore we get

$$(-1)^n q^{\binom{n}{2}}\equiv -1 \bmod \Phi_n(q). \tag{1.10}$$

The above results and Cassini's formula give

Corollary 1.1

If $n\equiv 0 \bmod 5$ then $F_n(q)G_n(q)\equiv 0 \bmod \Phi_n(q)$ and if $n\not\equiv 0 \bmod 5$ then

$$F_n(q)G_n(q)\equiv 1 \bmod \Phi_n(q). \tag{1.11}$$

More generally we get

Corollary 1.2

Let ζ_k be a primitive $k-th$ root of unity. Then

$$\begin{aligned}F_{kn}\left(\zeta_k\right)&=F_nF_k\left(\zeta_k\right),\\ G_{kn}\left(\zeta_k\right)&=F_nG_k\left(\zeta_k\right),\end{aligned} \tag{1.12}$$

and therefore

$$F_{kn}\left(\zeta_k\right)G_{kn}\left(\zeta_k\right)=\begin{cases}0 & \text{if } k\equiv 0 \bmod 5\\ F_n^2 & \text{if } k\not\equiv 0 \bmod 5.\end{cases} \tag{1.13}$$

Proof

Let $j = mk + \ell$ with $0 \le \ell < k.$ Then for $km + \ell \le k(n-m) - \ell - 1$

$$\begin{bmatrix} k(n-m)-\ell-1 \\ km+\ell \end{bmatrix}_q = \prod_{i=1}^{m} \frac{1-q^{k(n-m-i)}}{1-q^{ki}} * \frac{\left(1-q^{k(n-m)-\ell-1}\right)\cdots\left(1-q^{k(n-m)-k+1}\right)\left(1-q^{k(n-m)-k-1}\right)\cdots\left(1-q^{k(n-m)-k-\ell}\right)}{(1-q)\cdots\left(1-q^{k-1}\right)}$$

$$*\cdots*\frac{\left(1-q^{k-\ell-1}\right)\cdots\left(1-q^{k-2\ell-1}\right)}{(1-q)\cdots\left(1-q^{\ell}\right)}$$

If we let $q \to \zeta_k$ then the first term converges to $\binom{n-m-1}{m},$ the middle terms give 1 because the factors of the numerator are a permutation of the factors of the denominator, and the last term converges to $\begin{bmatrix} k-\ell-1 \\ \ell \end{bmatrix}_{\zeta_k}$.

Therefore we get

$$F_{kn}(\zeta_k) = \sum_j \begin{bmatrix} kn-1-j \\ j \end{bmatrix}_{\zeta_k} \zeta_k^{j^2} = \sum_m \sum_\ell \begin{bmatrix} k(n-m)-1-\ell \\ km+\ell \end{bmatrix}_{\zeta_k} \zeta_k^{(km+\ell)^2}$$

$$= \sum_m \sum_\ell \binom{n-m-1}{m} \begin{bmatrix} k-\ell-1 \\ \ell \end{bmatrix}_{\zeta_k} \zeta_k^{\ell^2} = \sum_\ell \begin{bmatrix} k-\ell-1 \\ \ell \end{bmatrix}_{\zeta_k} \zeta_k^{\ell^2} \sum_m \binom{n-m-1}{m} = F_n F_k(\zeta_k).$$

The proof for $G_n(q)$ is essentially the same.

2. The main result for $F_{5n}(q)$ and $G_{5n}(q)$.

Theorem 2.1

Let $n = 5^k m$ with $k \ge 1$ and $m \not\equiv 0 \bmod 5.$ Then

$$F_{5^k m}(q) \text{ and } G_{5^k m}(q) \text{ are divisible by } \left[5^k\right]_{q^m}. \tag{2.1}$$

For example

$F_5(q) = [5]_q,$

$F_{10}(q) = [5]_{q^2}\left(1+q+q^4[9]_q\right) = [5]_q\left(1-q+q^2-q^3+q^4\right)\left(1+q+q^4[9]_q\right),$

$G_5(q) = [5]_q(1-q+q^2),$

$G_{10}(q) = [5]_{q^2}[11]_q\left(1-q+q^3-q^4+q^6\right).$

Let us first recall how to prove that $v_5\left(F_n\right)=v_5\left(n\right).$ By Binet's formula we get

$$F_n=\frac{1}{2^n\sqrt{5}}\left(\left(1+\sqrt{5}\right)^n-\left(1-\sqrt{5}\right)^n\right)=\frac{1}{2^{n-1}}\sum_{k=0}^{n}\binom{n}{2k+1}5^k=\frac{1}{2^{n-1}}\sum_{k=0}^{n}\frac{n}{2k+1}\binom{n-1}{2k}5^k.$$

For each $k>0$ we have $v_5\left(\frac{5^k n}{2k+1}\right)>v_5(n)$ and for $k=0$ we have $v_5\left(\frac{n}{1}\binom{n-1}{0}\right)=v_5(n).$

This implies $v_5(F_n)=v_5(n).$

It is rather trivial that $F_{5n}(q)$ and $G_{5n}(q)$ are divisible by $[5]_q.$

To show this observe that $q^n\equiv q^{n(\mathrm{mod}\,5)}(\mathrm{mod}[5]_q).$ Therefore $F_{5n}(q)\equiv 0\left(\mathrm{mod}[5]_q\right)$ by (1.1) implies $F_{5n+2}(q)\equiv F_{5n+1}(q),$ $F_{5n+3}(q)\equiv F_{5n+2}(q)+qF_{5n+1}(q)\equiv F_{5n+1}(q)\left(1+q\right),$
$F_{5n+4}(q)\equiv F_{5n+3}(q)+q^2F_{5n+2}(q)\equiv\left(1+q+q^2\right)F_{5n+1}(q)$ and finally
$F_{5n+5}(q)\equiv F_{5n+4}(q)+q^3F_{5n+3}(q)\equiv\left(1+q+q^2+q^3+q^4\right)F_{5n+1}(q)\equiv 0\left(\mathrm{mod}[5]_q\right).$

Analogously $G_{5n}(q)\equiv 0\left(\mathrm{mod}[5]_q\right)$ by (1.1) implies $G_{5n+2}(q)\equiv G_{5n+1}(q),$
$G_{5n+3}(q)\equiv G_{5n+2}(q)+q^2G_{5n+1}(q)\equiv G_{5n+1}(q)\left(1+q^2\right),$
$G_{5n+4}(q)\equiv G_{5n+3}(q)+q^3G_{5n+2}(q)\equiv\left(1+q^2+q^3\right)G_{5n+1}(q)$ and finally
$G_{5n+5}(q)\equiv G_{5n+4}(q)+q^4G_{5n+3}(q)\equiv\left(1+q^2+q^3+q^4+q^6\right)G_{5n+1}(q)=[5]_q\left(1-q+q^2\right)G_{5n+1}(q)$
$\equiv 0\left(\mathrm{mod}[5]_q\right).$

For the general case observe that by (1.7) and (1.6) $F_{5^\ell r}(q)\equiv 0\,\mathrm{mod}\,\Phi_{5^\ell r}(q)$ and $G_{5^\ell r}(q)\equiv 0\,\mathrm{mod}\,\Phi_{5^\ell r}(q)$ for each factor $5^\ell r$ of $5^k m$ with $\ell\geq 1$ and that all $\Phi_{5^\ell r}(q)$ are irreducible. Therefore the product of all these cyclotomic polynomials divides $F_{5^k m}(q)$ and $G_{5^k m}(q).$ But this product coincides with $\left[5^k\right]_{q^m}$ because $\left[5^k\right]_{q^m}=\frac{1-q^{5^k m}}{1-q^m}=\frac{\prod_{d|5^k m}\Phi_d(q)}{\prod_{d|m}\Phi_d(q)}.$

For example we see that $F_{10}(q)$ is divisible by $\Phi_5(q)\Phi_{10}(q),$ $F_{15}(q)$ is divisible by $\Phi_5(q)\Phi_{15}(q),$ or $F_{20}(q)$ is divisible by $\Phi_5(q)\Phi_{10}(q)\Phi_{20}(q).$

3. The Fibonacci numbers $f_r(n,q)$.

Let for some $r\in\mathbb{Z}$

$$f_r(n,q)=\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}q^{\binom{k}{2}+2rk}\begin{bmatrix}n-1-k\\k\end{bmatrix}. \tag{3.1}$$

These polynomials satisfy the recurrence (cf. [4])

$$f_r(n,q) = f_r(n-1,q) + q^{n-3+2r} f_r(n-3,q) + q^{n-4+4r} f_r(n-4,q) \tag{3.2}$$

with initial values $f_r(0,q)=0,\ \ f_r(1,q)=1,\ \ f_r(2,q)=1,\ \ f_r(3,q)=1+q^{2r}$ and $f_r(4,q)=1+q^{2r}+q^{1+2r}.$

Of special interest is $f(n,q)=f_0(n,q).$ The first terms of $f(n,q)$ are

$0,1,1,2,2+q,2+2q+q^2,2(1+q)\left(1+q^2\right),2+2q+2q^2+4q^3+2q^4+q^5,\cdots.$

Conjecture 3.1

Let $n=2^k(2m+1)$ *with* $k\geq 0.$ *Then*

$$f(6n,q)=f\left(6\cdot(2m+1)\cdot 2^k,q\right) \text{ is divisible by } 2\left[2^{k+2}\right]_{q^{2m+1}}. \tag{3.3}$$

For example $f(12,q)=2[8]_q\left(1+q^3+q^5+q^6+q^7+2q^8+q^9+q^{11}\right)$ and $f(18,q)$ is divisible by $2[4]_{q^3}$.

Let me prove some trivial facts:

The Fibonacci numbers F_n satisfy $F_{6n}\equiv 0 \mod 8.$ For $\left(F_n \bmod 8\right)_{n\geq 0}=\left(0,1,1,2,3,5,0,1,1,2,3,5,0,\cdots\right).$

Now observe that $f(3n,q)$ is even because $\sum_{k=0}^{\left\lfloor\frac{3n-1}{2}\right\rfloor}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}3n-1-k\\ k\end{bmatrix}=0.$

Cf. [4],Theorem 3.2 and the literature cited there.

It is also easy to show by induction that

$f(3n,q)\bmod 2=0,\ \ f(3n+1,q)\bmod 2=q^{\frac{n(3n-1)}{2}},\ \ f(3n+2,q)\bmod 2=q^{\frac{n(3n+1)}{2}}.$

Observe that $f(6,q)=2\left(1+q+q^2+q^3\right)=2[4]_q.$

The sequence $\left(q^n \bmod [4]\right)=\left(1,q,q^2,-1-q-q^2,\cdots\right)$ is periodic with period 4.

This implies that the sequence $f(n+24,q)\bmod[4]_q$ satisfies the same recurrence. It is easily verified that it also has the same initial values. Therefore the sequence $f(n,q)\bmod[4]_q$

has period $24.$ Since it satisfies $f(6n,q)\equiv 0 \bmod [4]_q$ we finally get that $f(6n,q)$ is divisible by $2(1+q)\left(1+q^2\right).$

For general r we get

Conjecture 3.2

Let $n=2^k(2m+1)$ *with* $k\geq 0.$ *Then*

$f_r(6n,q)=f\left(6\cdot(2m+1)\cdot 2^k,q\right)$ *is divisible by* $\left[2^{k+2}\right]_{q^{2m+1}}.$

For example $f_r\left(6,q\right)=\left(1+q^{2r}\right)\left(1+q^{2r+1}+q^{2r+2}+q^{2r+3}\right)$ is a multiple of $(1+q)\left(1+q^2\right).$ If r is even then i and -1 are roots of the second factor, if r is odd then i is a root of the first factor and -1 is a root of the second factor of $f_r\left(6,q\right).$

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