Finite q-identities related to well-known theorems of Euler and Gauss

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Abstract
We give generalizations of a finite version of Euler's pentagonal number theorem and of a $q-$identity of Gauss by introducing a new parameter.

0. Introduction

Let $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q^{n-k+1})\cdots(1-q^n)}{(1-q)\cdots(1-q^k)}$ be a $q-$binomial coefficient.

A. Berkovich and F. G. Garvan [1] and with another method S.O. Warnaar [4] have proved that

$$\sum_{j=-L}^{2L} (-1)^j q^{\frac{j(3j+1)}{2}} \begin{bmatrix} 2L-j \\ L+j \end{bmatrix} = 1$$

for all $L \in \mathbb{N}$. This is a finite version of Euler's pentagonal number theorem, because for $L \to \infty$ it reduces to the pentagonal number theorem

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(3j+1)}{2}}.$$

Let $r_n(x,a) = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k a^{n-k}$. A famous theorem of Gauss states that $r_{2n+1}(1,-1) = 0$ and

$$r_{2n}(1,-1) = \frac{(q;q)_{2n}}{(q^2;q^2)_n} = (1-q)(1-q^3)\cdots(1-q^{2n-1}).$$

Generalizing these results we give explicit evaluations of the sums

$$h(L,k) = \sum_{j=-L}^{2L} (-1)^j q^{\frac{j(3j+1)}{2}+kj} \begin{bmatrix} 2L-j \\ L+j \end{bmatrix}$$

and

$$r_n\left(1,-q^k\right) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j q^{kj}.$$

1. Variations of a finite version of Euler's pentagonal number theorem

Generalizing the formula of Berkovich and Garvan we give explicit evaluations of the sums

$$h(L,k)=\sum_{j=-L}^{2L}(-1)^j q^{\frac{j(3j+1)}{2}+kj}\begin{bmatrix}2L-j\\ L+j\end{bmatrix}. \tag{1.1}$$

For small values of $k\in\mathbb{N}$ we get the following formulas for $h(L,k)$:

$$h(L,0)=1$$
$$h(L,1)=q^L\begin{bmatrix}1\\1\end{bmatrix}$$
$$h(L,2)=-q^{-1}\begin{bmatrix}2\\0\end{bmatrix}+q^{L-1}\begin{bmatrix}2\\1\end{bmatrix}$$
$$h(L,3)=-q^{-2}\begin{bmatrix}3\\0\end{bmatrix}+q^{2L-2}\begin{bmatrix}3\\2\end{bmatrix}-q^{3L-1}\begin{bmatrix}3\\3\end{bmatrix}$$
$$h(L,4)=-q^{L-4}\begin{bmatrix}4\\1\end{bmatrix}+q^{2L-4}\begin{bmatrix}4\\2\end{bmatrix}-q^{4L-2}\begin{bmatrix}4\\4\end{bmatrix}$$

For $q=1$ the left hand side of each equation reduces to 1. The right hand side reduces to a difference of sums of the form $A_{k,i}=\sum_{j\equiv i\,(\mathrm{mod}\,3)}\binom{k}{j}$. E.g.
$A_{7,1}-A_{7,2}=\left(\binom{7}{1}+\binom{7}{4}+\binom{7}{7}\right)-\left(\binom{7}{2}+\binom{7}{5}\right)=(7+35+1)-(21+21)=1.$ The existence of such a representation becomes obvious from

$(1+\rho)^n=A_{n,0}+A_{n,1}\rho+A_{n,2}\rho^2=(-\rho^2)^n,$
where ρ denotes a third root of unity.
E.g. if $n=3m+1$ we have $(-\rho^2)^{3m+1}=(-1)^{m+1}\rho^2=(-1)^m(1+\rho).$ Therefore
$A_{n,1}-A_{n,2}=(-1)^m.$

In the general case we define a sequence $w(n)$ for $n\in\mathbb{Z}$ by $w(n)=0$ if $n\equiv 2(\mathrm{mod}\,3)$ and $w(n)=(-1)^{\left\lfloor\frac{n}{3}\right\rfloor}q^{-\frac{n(n-1)}{6}}$ else. This means

$$\begin{aligned}w(3k)&=(-1)^k q^{-\frac{k(3k-1)}{2}}\\ w(3k+1)&=(-1)^k q^{-\frac{k(3k+1)}{2}}\\ w(3k+2)&=0.\end{aligned} \tag{1.2}$$

From the definition follows that

$$w(-n) = w(n+1). \tag{1.3}$$

Then we have

Theorem 1

For each $L \in \mathbb{N}$ *and* $k \in \mathbb{N}$ *the sum* $h(L,k) = \sum_{j=-L}^{2L} (-1)^j q^{\frac{j(3j+1)}{2}+kj} \begin{bmatrix} 2L-j \\ L+j \end{bmatrix}$ *has the value*

$$h(L,k) = \sum_{j=0}^{k} q^{\binom{j+1}{2}} q^{jL} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j w(-k-j) = \sum_{j=0}^{k} q^{\binom{j+1}{2}} q^{jL} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j w(k+j+1). \tag{1.4}$$

Since $w(-k-j) = 0$ if $-k-j \equiv 2 \pmod 3$ there remain only $q-$binomial coefficients $\begin{bmatrix} k \\ j \end{bmatrix}$ with $j \equiv -k \pmod 3$ or $j \equiv -k+2 \pmod 3$. The sign of the coefficient of q^{jL} is given by $(-1)^{j+\left\lfloor -\frac{k+j}{3} \right\rfloor} = (-1)^{j+3+\left\lfloor -\frac{k+j+3}{3} \right\rfloor}$. Therefore all terms with j in the same residue class $\bmod 3$ have the same sign.

E.g. $h(L,7) = A_{7,1}(L,q) - A_{7,2}(L,q)$ with $A_{7,1}(L,q) = q^{L-11} \begin{bmatrix} 7 \\ 1 \end{bmatrix} + q^{4L-12} \begin{bmatrix} 7 \\ 4 \end{bmatrix} + q^{7L-7} \begin{bmatrix} 7 \\ 7 \end{bmatrix}$ and $A_{7,2}(L,q) = q^{2L-12} \begin{bmatrix} 7 \\ 2 \end{bmatrix} + q^{5L-11} \begin{bmatrix} 7 \\ 5 \end{bmatrix}$.

The expansion of $h(L,n)$ has a vanishing constant term if and only if $n \equiv 1 \pmod 3$. Therefore we get $h(3k+1) = \lim_{L\to\infty} h(L,3k+1) = 0$.

For $n = 3k$ the constant term is $w(-3k) = (-1)^k q^{-\frac{k(3k+1)}{2}}$ and therefore $h(3k) = (-1)^k q^{-\frac{k(3k+1)}{2}}$.

Analogously we get $h(3k-1) = (-1)^k q^{-\frac{k(3k-1)}{2}}$.

From the definition of $h(L,n)$ we see that

$$h(3k+i)\prod_{n=1}^{\infty}(1-q^n) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(3j+1)}{2}+(3k+i)j}.$$

This can also be verified by using Jacobi's triple product identity. The case $h(0) = 1$ gives Euler's pentagonal number theorem.

For the triple product implies

$$\sum_{j\in\mathbb{Z}} (-1)^j q^{aj^2+bj} = \prod_{n=0}^{\infty} (1-q^{2an+a+b})(1-q^{2an+a-b})(1-q^{2an+2a}).$$

Therefore

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(3j+1)}{2}+(3k+i)j} = \prod_{n\ge 0} (1-q^{3n+2+3k+i})(1-q^{3n+1-3k-i})(1-q^{3n+3}).$$

For $i=3l+1$ the product vanishes because one of the middle factors vanishes.
For $i=3l$ we get

$$\prod_{n\geq 0}(1-q^{3n+2+3k+i})(1-q^{3n+1-3k-i})(1-q^{3n+3})=q^{-\frac{(k+l)(3(k+l)+1)}{2}}\prod_{n\geq 1}(1-q^n)=h(3k+i)\prod_{n\geq 1}(1-q^n)$$

because to each term $1-q^{3n+2}, 0\leq n<k+l,$ which does not occur in the first factors of the product there corresponds precisely one term $1-q^{-(3n+2)}$ in the middle factors and $2+5+\cdots+(3(k+l)-1)=\frac{(k+l)(3(k+l)+1)}{2}$. The case $i=3l-1$ can be treated in the same way.

Proof of Theorem 1

In order to prove this theorem we use some results about the $q-$Fibonacci polynomials $F_n(x,s)=\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\begin{bmatrix}n-k-1\\k\end{bmatrix}q^{\binom{k+1}{2}}x^{n-1-2k}s^k$ from our previous paper [2]. These are $q-$analogues of the Fibonacci polynomials $f_n(x,s)=\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-1-k}{k}x^{n+2k-1}s^k$, which are characterized by the recurrence relation $f_n(x,s)=xf_{n-1}(x,s)+sf_{n-2}(x,s)$ and the initial values $f_0(x,s)=0$ and $f_1(x,s)=1$.

Let

$$G(L,i,s)=\sum_{j=-L}^{2L+i}s^jq^{\frac{j(3j-1)}{2}-ij}\begin{bmatrix}2L+i-j\\L+j\end{bmatrix}\tag{1.5}$$

and

$$f_n(s)=\sum_{k=0}^{n-1}\begin{bmatrix}n-1-k\\k\end{bmatrix}_{\frac{1}{q}}q^{-\binom{k+1}{2}}s^k.\tag{1.6}$$

Then $f_n(s)=F_n(1,s)\big|_{q\to\frac{1}{q}}$ and therefore

$$f_n(qs)=f_{n-1}(s)+sf_{n-2}(s)\tag{1.7}$$

by [2] (2.2).

Using the easily verified formula $q^{-\binom{k}{2}}\begin{bmatrix} n-k \\ k \end{bmatrix}_{\frac{1}{q}} = q^{-\binom{n}{2}} q^{k^2+\binom{n-k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}$

we get

$$\sum_{j=-L}^{2L+i} s^{j+L} q^{\frac{j(3j-1)}{2}-ij} \begin{bmatrix} 2L+i-j \\ L+j \end{bmatrix} = q^{\frac{L(3L+1)}{2}+iL} \sum_{k=0}^{3L+i} q^{-\binom{k+1}{2}} \begin{bmatrix} 3L+i-k \\ k \end{bmatrix}_{\frac{1}{q}} s^k. \tag{1.8}$$

Then (1.8) can be written as

$$G(L,i,s) = q^{\frac{L(3L+1)}{2}+iL} s^{-L} f_{3L+i+1}(s). \tag{1.9}$$

Combining (1.7) and (1.9) we get

$$G(L,i,qs) = G(L,i-1,s) + q^L s G(L,i-2,s). \tag{1.10}$$

By [2] (3.2)

$$f_{3n}(-q) = 0,\ f_{3n+1}(-q) = (-1)^n q^{-\frac{n(3n-1)}{2}},\ f_{3n+2}(-q) = (-1)^n q^{-\frac{n(3n+1)}{2}}. \tag{1.11}$$

The formula $F_{-n}(1,s) = (-1)^{n-1}\dfrac{F_n(1,s)}{s^n}$ (cf. [2] (2.7))

gives

$F_{-n}(1,-\frac{1}{q}) = -F_n(1,-\frac{1}{q})q^n.$

Therefore we get

$$F_{-3k+1}(1,-\frac{1}{q}) = (-1)^k q^{\frac{k(3k+1)}{2}}, F_{-3k+2}(1,-\frac{1}{q}) = (-1)^k q^{\frac{k(3k-1)}{2}}, F_{-3k}(1,-\frac{1}{q}) = 0.$$

This implies that (1.11) holds for all $n \in \mathbb{Z}$.

We use (1.9) to extend $G(L,i,s)$ to values with $3L+i+1<0$.
Then it is easy to verify that

$$G(L,n,-q) = w(n) \tag{1.12}$$

for all $n \in \mathbb{Z}$. Note that the right-hand side does not depend on $L \in \mathbb{N}$.

Formula (1.10) gives immediately

$$G(L,i,q^k s)=\sum_{j=0}^{k} q^{\binom{j}{2}} q^{jL} s^j \begin{bmatrix} k \\ j \end{bmatrix} G(L,i-k-j,s). \tag{1.13}$$

This follows by induction from

$$\begin{aligned}
&G(L,i,q^{k+1}s)=G(L,i-1,q^k s)+q^{L+k}sG(L,i-2,q^k s)\\
&=\sum_{j=0}^{k} q^{\binom{j}{2}} q^{jL} s^j \begin{bmatrix} k \\ j \end{bmatrix} G(L,i-k-j-1,s)+\sum_{j=0}^{k} q^{\binom{j}{2}} q^{jL+L} q^k s^j \begin{bmatrix} k \\ j \end{bmatrix} G(L,i-k-j-2,s)\\
&=\sum_{j\geq 0} q^{\binom{j}{2}} q^{jL} s^j \begin{bmatrix} k \\ j \end{bmatrix} G(L,i-k-j-1,s)+\sum_{j\geq 0} q^{\binom{j}{2}} q^{jL} s^j q^k \begin{bmatrix} k \\ j \end{bmatrix} G(L,i-k-j-1,s)\\
&=\sum_{j=0}^{k+1} q^{\binom{j}{2}} q^{jL} s^j \begin{bmatrix} k+1 \\ j \end{bmatrix} G(L,i-k-1-j,s).
\end{aligned}$$

The theorem follows if in (1.13) we set $s=-q$ and $i=0$.

2. Variations of a q-identity of Gauss

Consider now the Rogers-Szegö polynomials

$$r_n(x,a)=\sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k a^{n-k}. \tag{2.1}$$

Gauss's theorem states that
$r_{2n+1}(1,-1)=0$ and

$$r_{2n}(1,-1)=\frac{(q;q)_{2n}}{(q^2;q^2)_n}=(1-q)(1-q^3)\cdots(1-q^{2n-1}). \tag{2.2}$$

A very simple proof uses Euler's $q-$exponential series

$$e(x)=\frac{1}{(x;q)_\infty}=\frac{1}{(1-x)(1-qx)(1-q^2x)\cdots}=\sum_{k\geq 0}\frac{x^k}{(q;q)_k}. \tag{2.3}$$

The generating function of the Rogers-Szegö polynomials is given by

$$\sum_n \frac{r_n(x,a)}{(q;q)_n} z^n = e(xz)e(az). \tag{2.4}$$

This follows from

$$\begin{aligned}
e(xz)e(az)&=\sum_k \frac{x^k z^k}{(q;q)_k}\sum_\ell \frac{a^\ell z^\ell}{(q;q)_\ell}=\sum_{k,\ell}\frac{z^{k+\ell}}{(q;q)_{k+\ell}}\frac{(q;q)_{k+\ell}}{(q;q)_k(q;q)_\ell}x^k a^\ell=\sum_n \frac{z^n}{(q;q)_n}\sum_{k+\ell=n}\begin{bmatrix} k+\ell \\ k \end{bmatrix} x^k a^\ell\\
&=\sum_n \frac{r_n(x,a)}{(q;q)_n} z^n.
\end{aligned}$$

As a special case we get

$$\sum_n \frac{r_n(1,-1)}{(q;q)_n} z^n = e(-z)e(z) = \frac{1}{(1+z)(1+qz)(1+q^2z)\cdots}\frac{1}{(1-z)(1-qz)(1-q^2z)\cdots}$$
$$= \frac{1}{(1-z^2)(1-q^2z^2)(1-q^4z^2)\cdots} = \sum_n \frac{(z^2)^n}{(q^2;q^2)_n} = \sum_n \frac{z^{2n}}{(q;q)_{2n}}(1-q)(1-q^3)\cdots(1-q^{2n-1}), \tag{2.5}$$

which is equivalent with Gauss's theorem by comparing coefficients.

B.A Kupershmidt [3] has given a formula for $r_n(1,-x) = (-1)^n r_n(x,-1)$ which can be easily deduced from the generating function

$$\sum_n \frac{r_n(x,-1)}{(q;q)_n} z^n = e(xz)e(-z) = \left(\frac{e(xz)}{e(z)}\right)\left(e(z)e(-z)\right).$$

He raised the problem of explicitly evaluating $r_n(1,-q^k)$. In the following we give two such formulas.

For the first one we generalize the method we used to prove Gauss's theorem:

From $e(qx) = \dfrac{1}{(1-qx)(1-q^2x)\cdots} = (1-x)e(x)$

we get

$$\sum_n \frac{r_n(1,-q)}{(q;q)_n} z^n = e(-qz)e(z) = \frac{1+z}{(1+z)(1+qz)(1+q^2z)\cdots}\frac{1}{(1-z)(1-qz)(1-q^2z)\cdots}$$
$$= \frac{1+z}{(1-z^2)(1-q^2z^2)(1-q^4z^2)\cdots} = (1+z)\sum_n \frac{(z^2)^n}{(q^2;q^2)_n}.$$

This implies $r_{2n}(1,-q) = r_{2n}(1,-1)$ and $r_{2n-1}(1,-q) = r_{2n}(1,-1)$.

More generally for $k \geq 1$

$$\sum_n \frac{r_n(1,-q^k)}{(q;q)_n} z^n = e(-q^kz)e(z) = \frac{(1+z)\cdots(1+q^{k-1}z)}{(1+z)(1+qz)(1+q^2z)\cdots}\frac{1}{(1-z)(1-qz)(1-q^2z)\cdots}$$
$$= (1+z)\cdots(1+q^{k-1}z)\sum_n \frac{r_n(1,-1)}{(q;q)_n} z^n = \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix} z^j \sum_n \frac{z^{2n}}{(q^2;q^2)_n}.$$

Comparing coefficients we get

$$r_n(1,-q^k) = (q;q)_n \sum_{j+2\ell=n} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix} \frac{1}{(q^2;q^2)_\ell}. \tag{2.6}$$

This may be written in the form

$$r_{2n-1}(1,-q^k) = r_{2n}(1,-1) \sum_{0 \leq \ell < n} q^{\binom{2n-2\ell-1}{2}} \begin{bmatrix} k \\ 2n-2\ell-1 \end{bmatrix} \frac{(q^2;q^2)_{n-1}}{(q^2;q^2)_\ell}$$
$$= r_{2n}(1,-1) \sum_{j \geq 0} q^{\binom{2j+1}{2}} \begin{bmatrix} k \\ 2j+1 \end{bmatrix} \prod_{i=n-j}^{n-1} (1-q^{2i})$$

and

$$r_{2n}(1,-q^k)=r_{2n}(1,-1)\sum_{j\geq 0} q^{\binom{2j}{2}}\begin{bmatrix} k \\ 2j \end{bmatrix}\prod_{i=n-j+1}^{n}(1-q^{2i})\,.$$

The first values are

$$\frac{r_{2n-1}(1,-q)}{r_{2n}(1,-1)}=1$$

$$\frac{r_{2n-1}(1,-q^2)}{r_{2n}(1,-1)}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\frac{r_{2n-1}(1,-q^3)}{r_{2n}(1,-1)}=\begin{bmatrix} 3 \\ 1 \end{bmatrix}+q^3(1-q^{2n-2})$$

$$\frac{r_{2n-1}(1,-q^4)}{r_{2n}(1,-1)}=\begin{bmatrix} 4 \\ 1 \end{bmatrix}+q^3\begin{bmatrix} 4 \\ 3 \end{bmatrix}(1-q^{2n-2})$$

$$\frac{r_{2n-1}(1,-q^5)}{r_{2n}(1,-1)}=\begin{bmatrix} 5 \\ 1 \end{bmatrix}+\begin{bmatrix} 5 \\ 3 \end{bmatrix}q^3(1-q^{2n-2})+\begin{bmatrix} 5 \\ 5 \end{bmatrix}q^{10}(1-q^{2n-2})(1-q^{2n-4})$$

and

$$\frac{r_{2n}(1,-q)}{r_{2n}(1,-1)}=1$$

$$\frac{r_{2n}(1,-q^2)}{r_{2n}(1,-1)}=1+q\left(1-q^{2n}\right)$$

$$\frac{r_{2n}(1,-q^3)}{r_{2n}(1,-1)}=1+q(1-q^{2n})\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\frac{r_{2n}(1,-q^4)}{r_{2n}(1,-1)}=1+q(1-q^{2n})\begin{bmatrix} 4 \\ 2 \end{bmatrix}+q^6(1-q^{2n})(1-q^{2n-2})\begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

These are polynomials in q^{2n}. Now we want to compute the coefficients of these polynomials.
In order to do this we start from the formula

$$r_n(q^2x,a)-(1-\frac{qx}{a})r_n(qx,a)-q^{n+1}\frac{x}{a}r_n(x,a)=0, \tag{2.7}$$

which is easily verified by comparing coefficients:

$$q^{2k}\begin{bmatrix} n \\ k \end{bmatrix}-q^k\begin{bmatrix} n \\ k \end{bmatrix}+q^k\begin{bmatrix} n \\ k-1 \end{bmatrix}-q^{n+1}\begin{bmatrix} n \\ k-1 \end{bmatrix}=q^k\left((q^k-1)\begin{bmatrix} n \\ k \end{bmatrix}+(1-q^{n+1-k})\begin{bmatrix} n \\ k-1 \end{bmatrix}\right)=0.$$

Let now

$$b(n,k) = -\frac{r_{2n-1}(q^k,-1)}{r_{2n}(1,-1)}. \tag{2.8}$$

Then we get

$$b(n,k+2) - (1+q^{k+1})b(n,k+1) + q^{2n+k}b(n,k) = 0. \tag{2.9}$$

Define now polynomials $f(k,s)$ by the recurrence

$$f(k,s) = (1+q^{k-1})f(k-1,s) - q^{k-2}sf(k-2,s) \tag{2.10}$$

and initial values $f(0,s)=0$ and $f(1,s)=1$.
Then

$$b(n,k) = f(k,q^{2n}). \tag{2.11}$$

The polynomial $f(k,s)$ is a $q-$analogue of the Fibonacci polynomial

$$f_k(2,-s) = \sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} (-1)^j s^j \binom{k-1-j}{j} 2^{k-1-2j}.$$

It is easily verified that this formula has the direct $q-$analogue

$$f(k,s) = \sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} (-1)^j q^{j^2} s^j \begin{bmatrix} k-j-1 \\ j \end{bmatrix}_{q^2} \prod_{i=1}^{k-1-2j} (1+q^i). \tag{2.12}$$

For by comparing coefficients the recursion (2.10) is equivalent with the identity

$$\begin{bmatrix} k-j-1 \\ j \end{bmatrix}_{q^2} (1+q^{k-1-2j}) - (1+q^{k-1}) \begin{bmatrix} k-j-2 \\ j \end{bmatrix}_{q^2} - q^{k-1-2j} \begin{bmatrix} k-j-2 \\ j-1 \end{bmatrix}_{q^2} (1+q^{k-1-2j}) = 0.$$

This is trivial, because we get

$$\begin{bmatrix} k-j-1 \\ j \end{bmatrix}_{q^2} - \begin{bmatrix} k-j-2 \\ j \end{bmatrix}_{q^2} - q^{2k-2-4j} \begin{bmatrix} k-j-2 \\ j-1 \end{bmatrix}_{q^2}$$

$$+q^{k-1-2j} \left(\begin{bmatrix} k-j-1 \\ j \end{bmatrix}_{q^2} - q^{2j} \begin{bmatrix} k-j-2 \\ j \end{bmatrix}_{q^2} - \begin{bmatrix} k-j-2 \\ j-1 \end{bmatrix}_{q^2} \right) = 0$$

by using both recurrences for the $q-$binomial coefficients.

From

$$\frac{r_{2n+1}(1,-q^{k+1})}{r_{2n}(1,-1)} - \frac{r_{2n+1}(1,-q^k)}{r_{2n}(1,-1)} = q^k(1-q^{2n+1}) \frac{r_{2n}(1,-q^k)}{r_{2n}(1,-1)}.$$

we see that

$$c(n,k) = \frac{r_{2n}(1,-q^k)}{r_{2n}(1,-1)} = q^{-k}(b(n+1,k+1) - b(n+1,k)). \tag{2.13}$$

This gives

$$c(n,k)=\sum_{j=0}^{\left\lfloor \frac{k}{2}\right\rfloor}(-1)^j q^{j^2} q^{2jn}\frac{[k]}{[2k-2j]}\begin{bmatrix}k-j\\ j\end{bmatrix}_{q^2}\prod_{i=1}^{k-2j}(1+q^i)\,.$$

Therefore we have proved

Theorem 2

For each fixed $k\geq 0$ the following identities hold:

$$\frac{r_{2n-1}(1,-q^k)}{r_{2n}(1,-1)}=\sum_{j=0}^{\left\lfloor \frac{k-1}{2}\right\rfloor}(-1)^j q^{j^2} q^{2jn}\begin{bmatrix}k-j-1\\ j\end{bmatrix}_{q^2}\prod_{i=1}^{k-1-2j}(1+q^i) \tag{2.14}$$

and

$$\frac{r_{2n}(1,-q^k)}{r_{2n}(1,-1)}=\sum_{j=0}^{\left\lfloor \frac{k}{2}\right\rfloor}(-1)^j q^{j^2} q^{2jn}\frac{[k]}{[2k-2j]}\begin{bmatrix}k-j\\ j\end{bmatrix}_{q^2}\prod_{i=1}^{k-2j}(1+q^i). \tag{2.15}$$

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