Some determinants associated with Catalan numbers

and a conjecture about Hankel determinants

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Abstract

Starting with some matrices whose determinants are Catalan numbers we obtain some generalizations and $q-$analogues and a curious conjecture about Hankel determinants.

1. Introduction

This paper has an unusual genesis. In an internet posting an amateur mathematician, Tony Foster, looked for interesting properties of Pascal's triangle and observed that the elements of those "diagonals" $\{1,1\}$, $\{1,3,1\}$, $\{1,6,5,1\},\cdots,$ whose sums are Fibonacci numbers, yield matrices whose determinants are Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ such as

$$\det\begin{pmatrix}1\end{pmatrix} = 1 = C_1,\ \det\begin{pmatrix}1 & 1\\ 1 & 3\end{pmatrix} = 2 = C_2,\ \det\begin{pmatrix}1 & 1 & 0\\ 1 & 3 & 1\\ 1 & 6 & 5\end{pmatrix} = 5 = C_3,$$

$$\det\begin{pmatrix}1 & 1 & 0 & 0\\ 1 & 3 & 1 & 0\\ 1 & 6 & 5 & 1\\ 1 & 10 & 15 & 7\end{pmatrix} = 14 = C_4$$

or more generally

$$\det\left(\binom{i+j+1}{2j}\right)_{i,j=0}^{n-1} = \det\left(\binom{i+j+1}{i-j+1}\right)_{i,j=0}^{n-1} = C_n. \tag{1}$$

This caught my attention and led me to some proofs of this fact. Elementary column operations give the equivalent identity

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n+j}{n-j}C_j = 0. \tag{2}$$

Different interpretations of this identity lead to different proofs of (1):

We can interpret the left-hand side of (2) as $\Lambda\left(F_{2n}(x)\right)$, where $F_n(x) = \sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^j\binom{n-j}{j}x^{n-2j}$ is a Fibonacci polynomial and Λ the linear functional on the polynomials defined by

$\Lambda\left(F_n(x)\right) = [n=0]$. The moments of the Fibonacci polynomials, which are orthogonal with respect to Λ, are $\Lambda\left(x^{2n}\right) = C_n$ which gives (2).

Other methods which lead to (2) are expansions of the polynomials $\binom{x-1}{n}$ with respect to Gould polynomials $G_n(x) = \frac{x}{2n+x}\binom{2n+x}{n}$ or hypergeometric summation formulae.

These methods also lead to

$$\det\left(\binom{i+j+k}{i-j+1}\right)_{i,j=0}^{n-1} = \det\left(\binom{j+k}{i-j+1}\right)_{i,j=0}^{n-1} = C_n^{(k)} \tag{3}$$

where the numbers $C_n^{(k)} = \frac{k}{n+k}\binom{2n+k-1}{n} = \frac{k}{2n+k}\binom{2n+k}{n}$ are the coefficients of the $k-$th power of the generating function of the Catalan numbers, $\sum_{n\geq 0} C_n^{(k)} z^n = \left(\sum_{n\geq 0} C_n z^n\right)^k$.

Another way to prove (1) uses a result by Christian Krattenthaler (cf. Lemma 3) about determinants of binomial coefficients and establishes a surprising relation with Hankel determinants:

$$\det\left(\binom{i+j+m}{i-j+m}\right)_{i,j=0}^{n-1} = \det\left(C_{n+i+j}\right)_{i,j=0}^{m-1}. \tag{4}$$

The appearance of the Hankel determinants $\det\left(C_{n+i+j}\right)_{i,j=0}^{m-1}$ seemed at first glance to be a lucky coincidence, but some experimentation led to the conjecture that it depends on the orthogonality of the Fibonacci polynomials. It seems that for arbitrary monic orthogonal polynomials $r_n(x) = \sum_{j=0}^{n}(-1)^{n-j} r(n,j) x^j$ with moments M_n the identity

$$\det\left(r(i+m,j)\right)_{i,j=0}^{n-1} = \frac{\det\left(M_{n+i+j}\right)_{i,j=0}^{m-1}}{\det\left(M_{i+j}\right)_{i,j=0}^{m-1}} \tag{5}$$

holds. For $n=1$ this reduces to the well-known identity $\det\left(M_{i+j+1}\right)_{i,j=0}^{m-1} = (-1)^m r_m(0) \det\left(M_{i+j}\right)_{i,j=0}^{m-1}$, but for $n>1$ this result seems to be new.

It is also natural to ask for $q-$analogues. Here we obtained as analogues of (3)

$$\det\left(q^{\binom{i-j}{2}}\begin{bmatrix} i+j+k \\ i-j+1 \end{bmatrix}\right)_{i,j=0}^{n-1} = C_n^{(k)}(q) \tag{6}$$

and

$$\det\left(q^{2\binom{i-j+1}{2}}\frac{\left(-q^{j+k};q\right)_{i-j+1}}{\left(-q;q\right)_{i-j+1}}\begin{bmatrix} j+k \\ i-j+1 \end{bmatrix}\right)_{i,j=0}^{n-1} = \frac{\left(-q^{n+1};q\right)_{k-1}}{\left(-q;q\right)_{k-1}}C_n^{(k)}(q) \tag{7}$$

with $C_n^{(k)}(q) = \dfrac{[k]}{[2n+k]}\begin{bmatrix} 2n+k \\ n \end{bmatrix}$.

A common generalization of the above results for which we do not have a $q-$analogue is

$$\det\left(\binom{i+j+k+m}{i-j+m}\right)_{i,j=0}^{n-1} = \det\left(\binom{j+k+m}{i-j+m}\right)_{i,j=0}^{n-1} = \det\left(C_{n-i+j}^{(2i+k+1)}\right)_{i,j=0}^{m-1}, \tag{8}$$

The proofs of these results also gave other interesting results.

I want to thank Christian Krattenthaler and Michael Schlosser for helpful informations.

2. The case q=1

The matrices $\left(\binom{i+j+1}{i-j+1}\right)_{i,j=0}^{n-1}$ are "almost triangular" in the sense that all entries $a(i,j)$ with $j > i+1$ vanish. Our first proof uses elementary matrix operations.

Lemma 1

Let $T = \left(t(i,j)\right)_{i,j\geq 0}$ *be a triangular matrix with* $t(i,i) = 1$ *for all* i *and let* $T_{n,1} = \left(t(i+1,j)\right)_{i,j=0}^{n-1}$. *If there are numbers* $M_n \neq 0$ *such that*

$$\sum_{j=0}^{n}(-1)^{n-j}t(n,j)M_j = [n=0] \tag{9}$$

then

$$\det\left(T_{n,1}\right) = \det\left(t(i+1,j)\right)_{i,j=0}^{n-1} = M_n. \tag{10}$$

Proof

Let t_j be column j of $T_{n,1} = \left(t(i+1,j)\right)_{i,j=0}^{n-1}$ and let S_n be the matrix with columns $s_j = \left(s(i,j)\right)_{i=0}^{n-1} = t_j + \frac{1}{M_j}\sum_{k=1}^{j}(-1)^k M_{j-k} t_{j-k}$.

Then we get $M_j s_j = \sum_{k=0}^{j}(-1)^{j-k} M_k t_k = M_j t_j - M_{j-1} s_{j-1}$.

For each i this implies

$$s(i,j)M_j = \sum_{k=0}^{j}(-1)^k M_{j-k} t(i+1,j-k) = 0$$

for $j = i+1$ by (9) and $s(i,j)M_j = M_j t(i,j) - M_{j-1} s(i,j-1) = 0$ for $j > i+1$ by induction. Thus S_n is triangular.

Furthermore we get

$$s(i,i)M_i = -\sum_{k=0}^{i+1}(-1)^k M_{i+1-k} t(i+1,i+1-k) + M_{i+1} = M_{i+1}.$$

Thus $s(i,i) = \frac{M_{i+1}}{M_i}$.

Since S_n is derived from $T_{n,1}$ by elementary column operations we have

$$\det T_{n,1} = \det S_n = \prod_{i=0}^{n-1}\frac{M_{i+1}}{M_i} = M_n.$$

For the matrices (1) we choose

$$T = A = \left(a(i,j)\right)_{i,j\geq 0} = \left(\binom{i+j}{i-j}\right)_{i,j\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 6 & 5 & 1 & 0 & \cdots \\ 1 & 10 & 15 & 7 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

In this case identity (9) reduces to the identity

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n+j}{n-j} C_j = 0 \qquad (11)$$

which implies (1).

Remark

Let $\left(b(i,j)\right)_{i,j\geq 0}$ be the inverse matrix of $\left(\binom{i+j}{i-j}\right)_{i,j\geq 0}$. Then (11) is equivalent with

$b(n,0)=(-1)^n C_n$. Note that by Cramer's rule $b(n,0)=(-1)^n \det\left(\binom{i+j+1}{i-j+1}\right)_{i,j=0}^{n-1}$.

2.1. Some proofs of identity (11).

2.1.1. Using orthogonal polynomials

As already mentioned identity (11) occurs in a natural way in the computation of the Hankel determinants $\det\left(C_{i+j}\right)_{i,j=0}^{n-1}$ of the Catalan numbers with the method of orthogonal polynomials.

Let me recall some relevant facts.

Lemma 2.

Let $(a(n))$ *be a sequence of real numbers with* $a(0)=1.$

If all Hankel determinants $\det(a(i+j))_{i,j=0}^{n-1}\neq 0$ *then the polynomials*

$$p_n(x)=\frac{1}{\det(a(i+j))_{i,j=0}^{n-1}}\det\begin{pmatrix} a(0) & a(1) & \cdots & a(n-1) & 1\\ a(1) & a(2) & \cdots & a(n) & x\\ a(2) & a(3) & \cdots & a(n+1) & x^2\\ \vdots & & & & \vdots\\ a(n) & a(n+1) & \cdots & a(2n-1) & x^n \end{pmatrix} \qquad (12)$$

are orthogonal with respect to the linear functional Λ *defined by*

$$\Lambda(x^n)=a(n). \qquad (13)$$

This means that $\Lambda\left(p_n(x)p_m(x)\right)=0$ *if* $m\neq n$ *and* $\Lambda\left(p_n^2(x)\right)\neq 0.$

In particular for $m=0$ *we get*

$$\Lambda\left(p_n(x)\right)=[n=0]. \qquad (14)$$

By Favard's theorem there exist numbers $s(n),t(n)$ *such that*

$$p_n(x)=(x-s(n-1))p_{n-1}(x)-t(n-2)p_{n-2}(x). \qquad (15)$$

If we define $c(n,j)$ *by*

$$
\begin{aligned}
&c(0,j) = [j=0] \\
&c(n,0) = s(0)c(n-1,0) + t(0)c(n-1,1) \\
&c(n,j) = c(n-1,j-1) + s(j)c(n-1,j) + t(j)c(n-1,j+1)
\end{aligned}
\tag{16}
$$

then $c(n,0) = a(n)$ *and*

$$\sum_{k=0}^{n} c(n,k) p_k(x) = x^n. \tag{17}$$

This implies

$$c(n,k) = \frac{\Lambda\left(x^n p_k(x)\right)}{\Lambda\left(p_k(x)^2\right)}. \tag{18}$$

For $a(2n) = C_n$ and $a(2n+1) = 0$ the orthogonal polynomials are the special Fibonacci polynomials

$$F_n(x) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j \binom{n-j}{j} x^{n-2j} \tag{19}$$

which satisfy $F_n(x) = xF_{n-1}(x) - F_{n-2}(x)$ with initial values $F_0(x) = 1$ and $F_1(x) = x$ and the linear functional Λ is given by $\Lambda\left(F_n(x)\right) = 0$ for $n > 0$ and $\Lambda\left(F_0(x)\right) = \Lambda(1) = 1.$ These polynomials are orthogonal with respect to Λ with moments $\Lambda\left(x^{2n}\right) = C_n$ and $\Lambda\left(x^{2n+1}\right) = 0.$ Moreover we have (cf. e.g. [5])

$$c(2n+k,k) = \frac{k+1}{2n+k+1}\binom{2n+k+1}{n}. \tag{20}$$

Applying Λ to $F_{2n}(x) = \sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} x^{2n-2j} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n+j}{n-j} x^{2j}$ we get (11).

Since $F_{2n+k}(x) = \sum_{-k \le 2j \le 2n} (-1)^{n-j} \binom{n+k+j}{n-j} x^{2j+k}$

we get $\dfrac{\Lambda\left(F_{2n+k}(x)F_k(x)\right)}{\Lambda\left(F_k(x)^2\right)} = \sum_{-k \le 2j \le 2n} (-1)^{n-j} \binom{n+k+j}{n-j} \dfrac{\Lambda\left(x^{2j+k}F_k(x)\right)}{\Lambda\left(F_k(x)^2\right)}.$

Therefore we get by (18) and $\Lambda\left(x^{2j+k}F_k(x)\right) = 0$ for $0 \le 2j+k < k$

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n+k+j}{n-j}C_j^{(k+1)}=[n=0] \tag{21}$$

which implies

$$\det\left(\binom{i+j+k}{i-j+1}\right)_{i,j=0}^{n-1}=C_n^{(k)}. \tag{22}$$

From

$$x^{n-1}F_{n+k+1}(x)=\sum_{0\le 2j\le n+k+1}(-1)^j\binom{n+k+1-j}{j}x^{2n+k-2j}$$
$$=\sum_j(-1)^{n-j}\binom{k+1+j}{n-j}x^{k+2j}$$

we get

$\sum_j(-1)^{n-j}\binom{k+1+j}{n-j}C_j^{(k+1)}=\frac{\Lambda\left(x^{n-1}F_{n+k+1}(x)F_k(x)\right)}{\Lambda\left(F_k(x)^2\right)}=[n=0]$ and therefore

$$\det\left(\binom{j+k}{i-j+1}\right)_{i,j=0}^{n-1}=C_n^{(k)}. \tag{23}$$

This method has wide applications. I will only mention the analogous results for Lucas polynomials and central binomial coefficients which are a sort of companion for the Fibonacci polynomials and Catalan numbers.

We consider in fact a variant of the Lucas polynomials defined by

$$L_n(x)=\sum_{j=0}^{n}(-1)^j\frac{n}{n-j}\binom{n-j}{j}x^{n-2j} \tag{24}$$

for $n>0.$ and $L_0(x)=1.$ These polynomials are monic and orthogonal with $s(n)=0,$ $t(0)=2$ and $t(n)=1$ for $n>0.$ Their moments are $\Lambda\left(x^{2n}\right)=\binom{2n}{n}$ and $\Lambda\left(x^{2n+1}\right)=0.$

More generally the numbers $c(n,k)$ are given by

$$c(2n+k,k)=\binom{2n+k}{n} \tag{25}$$

and $c(n,j)=0$ else.

From $\Lambda\left(F_{2n}(x)\right)=\sum_{j=0}^{n}(-1)^{n-j}\frac{2n}{n+j}\binom{n+j}{n-j}\binom{2j}{j}=[n=0]$ we get

$$\det\left(\frac{2i+2}{i+j+1}\binom{i+j+1}{i-j+1}\right)_{i,j=0}^{n-1}=\binom{2n}{n}. \tag{26}$$

$$L_{2n+k}(x)=\sum_{-k\le 2j\le 2n}(-1)^{n-j}\frac{2n+k}{n+k+j}\binom{n+k+j}{n-j}x^{2j+k}$$

implies $\frac{\Lambda\left(L_{2n+k}(x)L_k(x)\right)}{\Lambda\left(L_k(x)^2\right)}=\sum_{-k\le 2j\le 2n}(-1)^{n-j}\frac{2n+k}{n+k+j}\binom{n+k+j}{n-j}\frac{\Lambda\left(x^{2j+k}L_k(x)\right)}{\Lambda\left(L_k(x)^2\right)}.$

Therefore we get by (18) and $\Lambda\left(x^{2j+k}L_k(x)\right)=0$ for $j<0$

$$\sum_{j=0}^{n}(-1)^{n-j}\frac{2n+k}{n+k+j}\binom{n+k+j}{n-j}\binom{2j+k}{j}=[n=0] \tag{27}$$

which implies

$$\det\left(\frac{2i+k+1}{i+j+k}\binom{i+j+k}{i-j+1}\right)_{i,j=0}^{n-1}=\binom{2n+k-1}{n}. \tag{28}$$

From

$$x^{n-1}L_{n+k+1}(x)=\sum_{0\le 2j\le n+k+1}(-1)^{j}\frac{n+k+1}{n+k+1-j}\binom{n+k+1-j}{j}x^{2n+k-2j}$$
$$=\sum_{j}(-1)^{n-j}\frac{n+k+1}{k+1+j}\binom{k+1+j}{n-j}x^{k+2j}$$

we get

$\sum_{j}(-1)^{n-j}\frac{n+k+1}{k+1+j}\binom{k+1+j}{n-j}\binom{2j+k}{j}=\frac{\Lambda\left(x^{n-1}L_{n+k+1}(x)L_k(x)\right)}{\Lambda\left(L_k(x)^2\right)}=[n=0]$ and therefore

$$\det\left(\frac{i+k+1}{j+k}\binom{j+k}{i-j+1}\right)_{i,j=0}^{n-1}=\binom{2n+k-1}{n}. \tag{29}$$

2.1.2. Expansion with respect to Gould polynomials

The generating function $c(z)=\sum_{n\geq 0} C_n z^n$ of the Catalan numbers satisfies $c(z)=1+zc(z)^2$ and $c(z)^k=\sum_{n\geq 0} C_n^{(k)} z^n$ with $C_n^{(k)}=\frac{k}{n+k}\binom{2n+k-1}{n}=\frac{k}{2n+k}\binom{2n+k}{n}$.

From $c(z)^k=c(z)^{k-1}+z^2c(z)^{k+1}$ we get

$$C_n^{(k)}=C_n^{(k-1)}+C_{n-1}^{(k+1)}. \tag{30}$$

Consider the special Gould polynomials $G_n(x)=G_n(x,2)=\frac{x}{2n+x}\binom{2n+x}{n}$ for $n>0$ and $G_0(x)=1.$ Let $Ef(x)=f(x+1)$ be the shift operator and $\Delta=E-1$ the difference operator on the polynomials. These Gould polynomials satisfy

$$E^{-2}\Delta G_n(x)=G_n(x-1)-G_n(x-2)=G_{n-1}(x) \tag{31}$$

because $G_n(x)$ is a polynomial of degree n and (30) shows that (31) is true for infinitely many non-negative integers $x=k$.

Let us expand the polynomial $\binom{x-1}{n}$ as linear combinations of Gould polynomials,

$$\binom{x-1}{n}=\sum_{j=0}^{n} b(n,j)G_j(x).$$

Since $\left(E^{-2}\Delta\right)^m G_n(x)\Big|_{x=0}=G_{n-m}\left(x\right)\Big|_{x=0}=[n=m]$ we get

$$b(n,j)=\left(E^{-2}\Delta\right)^j\binom{x-1}{n}\Big|_{x=0}=\binom{x-2j-1}{n-j}\Big|_{x=0}=\binom{-2j-1}{n-j}=(-1)^{n-j}\binom{n+j}{n-j}$$

Thus we have

$$\binom{x-1}{n}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n+j}{n-j}G_j(x).$$

For $x=1$ this gives (11) and thus also (1).

If we expand $\binom{x-k}{n}=\sum_{j=0}^{n} b(n,j)G_j(x)$ we get in the same way

$$\binom{x-k}{n}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n+j+k-1}{n-j}G_j(x).$$

For $x=k$ this gives $\binom{0}{n}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n+j+k-1}{n-j}C_j^{(k)}$,

which implies

$$\det\left(\binom{i+j+k}{i-j+1}\right)_{i,j=0}^{n-1}=C_n^{(k)}. \tag{32}$$

Expanding the polynomials $\binom{x+n-1}{n}$ we get

$\binom{x+n-1-k}{n}=\sum_{j=0}^{n}(-1)^{n-j}\binom{j+k}{n-j}G_j(x)$ which for $x=k$ reduces to

$\sum_{j=0}^{n}(-1)^{n-j}\binom{j+k}{n-j}C_j^{(k)}=\binom{n-1}{n}=[n=0].$ This implies

$$\det\left(\binom{j+k}{i-j+1}\right)_{i,j=0}^{n-1}=C_n^{(k)}. \tag{33}$$

Remarks

Formulae (32) and (33) are equivalent by elementary row operations using Vandermonde's identity

$$\sum_{\ell=0}^{i}\binom{i}{i-\ell}\binom{j+k}{\ell-j}=\binom{i+j+k}{i-j}.$$

The general Gould polynomials $G_n(x,r)=\frac{x}{rn+x}\binom{rn+x}{n}$ satisfy

$E^{-r}\Delta G_n(x,r)=G_{n-1}(x,r)$ (cf. [14], p.55). In the same way as above we get

$$\binom{x-k}{n}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n+(r-1)j+k-1}{n-j}G_j(x,r)$$

which for $x=k$ reduces to

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n+(r-1)j+k-1}{n-j}\frac{k}{k+rj}\binom{k+rj}{j}=[n=0]. \tag{34}$$

This implies

$$\det\left(\binom{i+(r-1)j+k}{i-j+1}\right)_{i,j=0}^{n-1}=\frac{k}{rn+k}\binom{rn+k}{n}. \tag{35}$$

Similarly we also get (cf. [9])

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{(r-1)j+k-1}{n-j}\frac{k}{k+rj}\binom{k+rj}{j}=[n=0] \tag{36}$$

and

$$\det\left(\binom{(r-1)j+k}{i-j+1}\right)_{i,j=0}^{n-1}=\frac{k}{rn+k}\binom{rn+k}{n}. \tag{37}$$

2.1.3. Using hypergeometric identities

Using the notation $(x)_n = x(x+1)\cdots(x+n-1)$ and Chu-Vandermonde's formula (cf. [1], 2.2.3)

$$\sum_{j=0}^{n}\frac{(a)_j(-n)_j}{j!(c)_j}={}_2F_1\left(\begin{matrix}a,-n\\ c\end{matrix};1\right)=\frac{(a-c)_n}{(c)_n} \tag{38}$$

we get another proof of (11).

After changing $j\to n-j$ we get

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n+j}{n-j}C_j=\sum_{j=0}^{n}(-1)^{j}\binom{2n-j}{j}\frac{1}{(n-j+1)}\binom{2n-2j}{n-j}$$
$$=\sum_{j=0}^{n}(-1)^{j}\frac{(2n-j)!}{j!(2n-2j)!}\frac{(2n-2j)!}{(n-j)!(n+1-j)!}=\sum_{j=0}^{n}\frac{(2n)!}{j!(-2n)_j}\frac{(-n)_j}{n!}\frac{\left(-n-1\right)_j}{(n+1)!}$$
$$=C_n\sum_{j=0}^{n}\frac{(-n)_j\left(-n-1\right)_j}{j!(-2n)_j}=C_n\,{}_2F_1\left[\begin{matrix}-n-1,-n\\ -2n\end{matrix};1\right]=C_n\,\frac{\left(1-n\right)_n}{\left(-2n\right)_n}=[n=0].$$

2.2. A different method to prove (1) uses

Lemma 3 (C. Krattenthaler [12],Th. 27)

With the usual notations of $q-$*calculus we have*

$$\det\left(q^{jL_i}\begin{bmatrix} L_i + A - j \\ L_i + j \end{bmatrix}\right) = q^{\sum_{i=1}^{n} iL_i}\prod_{i=1}^{n}\frac{\left[L_i + A - n\right]!}{\left[L_i + n\right]!\left[A-2i\right]!}\prod_{j=1}^{n}\prod_{i=1}^{j-1}\left[L_i - L_j\right]\left[L_i + L_j + A + 1\right]. \quad (39)$$

For $q=1$ this reduces to

$$\det\left(\binom{L_i + A - j}{L_i + j}\right)_{i,j=1}^{n} = \prod_{i=1}^{n}\frac{\left(L_i + A - n\right)!}{\left(L_i + n\right)!\left(A-2i\right)!}\prod_{j=1}^{n}\prod_{i=1}^{j-1}\left(L_i - L_j\right)\left(L_i + L_j + A + 1\right). \qquad (40)$$

In order to apply this to $\det\left(\binom{i+j+1}{i-j+1}\right)_{i,j=0}^{n-1}$ we first change $i \to i-1$ and $j \to j-1$ and then reverse the order of the rows and columns in the matrix $\left(\binom{i+j-1}{i-j+1}\right)_{i,j=1}^{n}$. This means we change $i \to n+1-i$ and $j \to n+1-j$.

We then get the matrix $\left(\binom{2n+1-i-j}{j-i+1}\right)_{i,j=1}^{n}$.

Since this operation lets the determinant unchanged we get

$\det A_n = \det\left(\binom{2n+1-i-j}{j-i+1}\right)_{i,j=1}^{n}$. Choosing $L_i = 1-i$ and $A = 2n$ in (40) we get

$$\det A_n = \prod_{i=1}^{n}\frac{\left(1-i+n\right)!}{\left(1-i+n\right)!\left(2n-2i\right)!}\prod_{j=1}^{n}\prod_{i=1}^{j-1}\left(j-i\right)\left(2-i-j+2n+1\right)$$

$$= \prod_{i=1}^{n}\frac{(i-1)!}{\left(2n-2i\right)!}\prod_{j=1}^{n}\prod_{i=1}^{j-1}\left(3-i-j+2n\right).$$

In order to simplify this let $f(n) = \prod_{i=1}^{n}\frac{(i-1)!}{\left(2n-2i\right)!} = \prod_{i=1}^{n}\frac{(i-1)!}{\left(2i-2\right)!} = \frac{(n-1)!}{(2n-2)!}f(n-1)$

and

$$g(n)=\prod_{j=1}^{n}\prod_{i=1}^{j-1}(3-i-j+2n)=\prod_{j=1}^{n}\frac{(2n+2-j)!}{(2n+3-2j)!}$$
$$=\frac{(2n)!}{(2n-1)!}\frac{(2n-1)!}{(2n-3)!}\cdots\frac{(n+2)!}{(3)!}=\frac{(2n)!}{(n+1)!}g(n-1).$$

This gives

$$f(n)g(n)=\frac{(n-1)!}{(2n-2)!}\frac{(2n)!}{(n+1)!}f(n-1)g(n-1)=\frac{(2n)(2n-1)}{n(n+1)}f(n-1)g(n-1)=\frac{1}{n+1}\binom{2n}{n}.$$

The same method also leads to

Theorem 4

For $a(i,j)=\binom{i+j}{i-j}$ *we get*

$$\det\left(a(i+m,j)\right)_{i,j=0}^{n-1}=\det\left(\binom{i+j+m}{i-j+m}\right)_{i,j=0}^{n-1}=\det\left(C_{n+i+j}\right)_{i,j=0}^{m-1}. \tag{41}$$

Proof

First we write $\det\left(\binom{i+j+m}{i-j+m}\right)_{i,j=0}^{n-1}=\det\left(\binom{i+j+m-2}{i-j+m}\right)_{i,j=1}^{n}$ and then we again reverse the order of the rows and columns. Comparing with (40) we choose $L_i=m-i$ and $A=2n$ and get

$$\det\left(\binom{i+j+m}{i-j+m}\right)_{i,j=0}^{n-1}=\det\left(\binom{2n+m-i-j}{j-i+m}\right)_{i,j=1}^{n}$$
$$=\prod_{i=1}^{n}\frac{(m-i+n)!}{(m-i+n)!(2n-2i)!}\prod_{j=1}^{n}\prod_{i=1}^{j-1}(j-i)(2m+1-i-j+2n)$$
$$=\prod_{i=1}^{n}\frac{(i-1)!}{(2n-2i)!}\prod_{j=1}^{n}\frac{(2m-j+2n)!}{(2m+1+2n-2j)!}=f(n)g(n,m)$$

with $f(n)=\prod_{i=1}^{n}\frac{(i-1)!}{(2n-2i)!}$ and

$$g(n,m)=\prod_{j=1}^{n}\frac{(2m-j+2n)!}{(2m+1+2n-2j)!}=\frac{(2m+2n-1)!(2m+2n-2)!\cdots(2m+n)!}{(2m+2n-1)!(2m+2n-3)!\cdots(2m+1)!}.$$

This gives $\dfrac{f(n)}{f(n-1)}=\dfrac{(n-1)!}{(2n-2)!}$ and $\dfrac{g(n,m)}{g(n-1,m)}=\dfrac{(2m+2n-2)!}{(2m+n-1)!}$

because

$$g(n-1,m)=\frac{(2m+2n-3)!(2m+2n-4)!\cdots(2m+n-1)!}{(2m+2n-3)!(2m+2n-5)!\cdots(2m+1)!}.$$

Therefore we have

$$\frac{f(n)g(n,m)}{f(n-1)g(n-1,m)}=\frac{(n-1)!}{(2n-2)!}\frac{(2m+2n-2)!}{(2m+n-1)!} \tag{42}$$

On the other hand it is well known that (cf.e.g. [13],Theorem 33)

$$\det\left(C_{n+i+j}\right)_{i,j=0}^{m-1}=\prod_{j=1}^{n-1}\prod_{i=1}^{j}\frac{2m+i+j}{i+j}=\prod_{j=1}^{n-1}\frac{j!}{(2j)!}\prod_{j=1}^{n-1}\frac{(2m+2j)!}{(2m+j)!} \tag{43}$$

Comparing (42) with (43) we get (41).

Remark

As already mentioned the appearance of the Hankel determinants $\det\left(C_{n+i+j}\right)_{i,j=0}^{m-1}$ is somewhat mysterious, but some experimentation leads to

Conjecture 5

Let $r_n(x)=\sum_{j=0}^{n}(-1)^{n-j}r(n,j)x^j$ be monic orthogonal polynomials with moments M_n. Then

$$\det\left(r(i+m,j)\right)_{i,j=0}^{n-1}=\frac{\det\left(M_{n+i+j}\right)_{i,j=0}^{m-1}}{\det\left(M_{i+j}\right)_{i,j=0}^{m-1}}. \tag{44}$$

This formula can also be interpreted in the reverse direction. Then it gives a formula for the Hankel determinants $\det\left(M_{n+i+j}\right)_{i,j=0}^{m-1}$ in terms of the coefficients of the corresponding orthogonal polynomials.

For example for $a(2n)=C_n$ and $a(2n+1)=0$ and $r_n(x)=F_n(x)$ this gives

$$\det\left(a(i+j+2)\right)_{i,j=0}^{2m-1}=\det\left(r(2m+i,j)\right)_{i,j=0}^{1}=\det\begin{pmatrix}1 & 0\\ 0 & m+1\end{pmatrix}=m+1,$$

$$\det\left(a(i+j+2)\right)_{i,j=0}^{2m}=\det\left(r(2m+1+i,j)\right)_{i,j=0}^{1}=\det\begin{pmatrix}0 & (-1)^{m+1}(m+1)\\ (-1)^m & 0\end{pmatrix}=m+1$$

and therefore we get the well-known result $\det\left(a(i+j+2)\right)_{i,j=0}^{m-1} = \left\lfloor \frac{m+2}{2} \right\rfloor.$

For another illustration consider the sequence $a(n) = \frac{1}{n+1},$ which gives the famous Hilbert matrix.

As shown in [8] the corresponding orthogonal polynomials are

$$r_n(x) = \sum_{j=0}^{n} (-1)^{n-j} \frac{\binom{n}{j}\binom{n+j}{j}}{\binom{2n}{n}} x^j.$$

Thus (44) reduces to

$$\det\left(\frac{\binom{i+m}{j}\binom{i+m+j}{j}}{\binom{2i+2m}{i+m}}\right)_{i,j=0}^{n-1} = \frac{\det\left(\frac{1}{n+i+j+1}\right)_{i,j=0}^{m-1}}{\det\left(\frac{1}{i+j+1}\right)_{i,j=0}^{m-1}}. \tag{45}$$

To compute the Hankel determinants $\left(\frac{1}{i+j+n+1}\right)_{i,j=0}^{m-1} = \left(\frac{1}{i+j+n-1}\right)_{i,j=1}^{m}$ we use Cauchy's formula (cf. [12],(2.7))

$$\det\left(\frac{1}{x_i + y_j}\right)_{i,j=1}^{m} = \frac{\prod_{1\le i<j\le m} (x_i - x_j)(y_i - y_j)}{\prod_{1\le i,j\le m} (x_i + y_j)}$$

with $x_i = i+n-1, y_j = j.$ This gives

$$\det\left(\frac{1}{n+i+j+1}\right)_{i,j=0}^{m-1} = \prod_{j=0}^{m-1} \frac{j!\,j!\,(n+j)!}{(n+m+j)!} \tag{46}$$

and thus

$$\frac{\det\left(\frac{1}{n+i+j+1}\right)_{i,j=0}^{m-1}}{\det\left(\frac{1}{i+j+1}\right)_{i,j=0}^{m-1}} = \frac{\prod_{j=0}^{m-1} \frac{j!\,j!\,(n+j)!}{(n+m+j)!}}{\prod_{j=0}^{m-1} \frac{j!\,j!\,j!}{(m+j)!}} = \prod_{j=0}^{m-1} \frac{(n+j)!\,(m+j)!}{(n+m+j)!\,j!}.$$

For the left-hand side of (45) we get

$$\det\left(\frac{\binom{i+m}{j}\binom{i+m+j}{j}}{\binom{2i+2m}{i+m}}\right)_{i,j=0}^{n-1}=\det\left(\frac{(i+m)!^2(2j)!}{j!\,j!(2i+2m)!}\binom{i+j+m}{i-j+m}\right)_{i,j=0}^{n-1}$$

$$=\det\left(\binom{i+j+m}{i-j+m}\right)_{i,j=0}^{n-1}\prod_{j=0}^{n-1}\frac{(i+m)!^2(2j)!}{j!\,j!(2i+2m)!}$$

$$=\prod_{j=1}^{n-1}\frac{j!}{(2j)!}\frac{\left(2m+2j\right)!}{\left(2m+j\right)!}\frac{(2j)!}{j!\,j!}\frac{(j+m)!(j+m)!}{(2j+2m)!}=\prod_{j=1}^{n-1}\frac{(j+m)!(j+m)!}{j!\left(2m+j\right)!}.$$

To verify (45) we must show that

$$u(n,m)=\prod_{j=1}^{n-1}\frac{(j+m)!(j+m)!}{j!\left(2m+j\right)!}=v(n,m)=\prod_{j=0}^{m-1}\frac{(n+j)!(m+j)!}{j!(n+m+j)!}.$$

This follows from $u(0,m)=v(0,m)$ and $\dfrac{u(n,m)}{u(n-1,m)}=\dfrac{(n+m-1)!(n+m-1)!}{(n-1)!(2m+n-1)!}$ and $\dfrac{v(n,m)}{v(n-1,m)}=\prod_{j=0}^{m-1}\dfrac{(n+j)}{(n+m+j)}=\dfrac{(n+m-1)!}{(n-1)!}\dfrac{(n+m-1)!}{(n+2m-1)!}$.

As a further generalization we get

Theorem 6

$$\det\left(\binom{i+j+k+m}{i-j+m}\right)_{i,j=0}^{n-1}=\det\left(\binom{2n+m+k-i-j+2}{j-i+m}\right)_{i,j=1}^{n}=\det\left(C_{n-i+j}^{(2i+k+1)}\right)_{i,j=0}^{m-1}. \qquad (47)$$

Let us first show that this indeed generalizes Theorem 5.

First we see that

$$\det\begin{pmatrix}C_n & C_{n+1}\\ C_{n+1} & C_{n+2}\end{pmatrix}=\det\begin{pmatrix}C_n^{(1)} & C_{n+1}^{(1)}\\ C_{n-1}^{(3)} & C_n^{(3)}\end{pmatrix}$$

From $c(z)^2=c(z)+zc(z)^3$ and $C_n^{(2)}=C_{n+1}^{(1)}=C_{n+1}$ we get $C_{n+1}=C_n+C_{n-1}^{(3)}$.

Thus $\det\begin{pmatrix}C_n & C_{n+1}\\ C_{n+1} & C_{n+2}\end{pmatrix}=\det\begin{pmatrix}C_n & C_{n+1}\\ C_{n+1}-C_n & C_{n+2}-C_{n+1}\end{pmatrix}=\det\begin{pmatrix}C_n^{(1)} & C_{n+1}^{(1)}\\ C_{n-1}^{(3)} & C_n^{(3)}\end{pmatrix}$.

In the same way we get

$$\det\begin{pmatrix} C_n & C_{n+1} & C_{n+2} \\ C_{n+1} & C_{n+2} & C_{n+3} \\ C_{n+2} & C_{n+3} & C_{n+4} \end{pmatrix} = \det\begin{pmatrix} C_n & C_{n+1} & C_{n+2} \\ C_{n+1} & C_{n+2} & C_{n+3} \\ C_n^{(3)} & C_{n+1}^{(3)} & C_{n+2}^{(3)} \end{pmatrix} = \det\begin{pmatrix} C_n & C_{n+1} & C_{n+2} \\ C_{n-1}^{(3)} & C_n^{(3)} & C_{n+1}^{(3)} \\ C_n^{(3)} & C_{n+1}^{(3)} & C_{n+2}^{(3)} \end{pmatrix}$$
$$= \det\begin{pmatrix} C_n & C_{n+1} & C_{n+2} \\ C_{n-1}^{(3)} & C_n^{(3)} & C_{n+1}^{(3)} \\ C_{n-2}^{(5)} & C_{n-1}^{(5)} & C_n^{(5)} \end{pmatrix} = \det\begin{pmatrix} C_n^{(1)} & C_{n+1}^{(1)} & C_{n+2}^{(1)} \\ C_{n-1}^{(3)} & C_n^{(3)} & C_{n+1}^{(3)} \\ C_{n-2}^{(5)} & C_{n-1}^{(5)} & C_n^{(5)} \end{pmatrix}.$$

Analogously in the general case.

Proof of Theorem 6

Comparing with (40) we choose $L_i = -i + m$ and $A = 2n + k$. Then we get

$$w(n,m,k) = \det\left(\binom{2n+m+2+k-i-j}{j-i+m}\right)_{i,j=1}^{n}$$
$$= \prod_{i=1}^{n} \frac{(k+m+2-i+n)!}{(m-i+n)!(2n+k+2-2i)!} \prod_{j=1}^{n}\prod_{i=1}^{j-1}(j-i)(2m+3-i-j+2n+k)$$
$$= \prod_{i=1}^{n} \frac{(i-1)!(k+m+2-i+n)!}{(2n+k+2-2i)!(m-i+n)!} \prod_{j=1}^{n} \frac{(2m-j+2n+k+2)!}{(2m+3+2n-2j+k)!}.$$

This implies

$$v(n,m,k) = \frac{w(n,m,k)}{w(n-1,m,k)} = \frac{(n-1)!(k+m+1+n)!(2m+2n+k)!}{(2n+k)!(m+n-1)!(n+2m+k+1)!}$$
$$= \frac{\prod_{\ell=0}^{2m-1}(2n+1+k+\ell)}{\prod_{\ell=0}^{m-1}(n+\ell)(n+k+\ell+m+2)}. \tag{48}$$

for

$$\frac{w(n,m,k)}{w(n-1,m,k)} = \frac{\prod_{i=1}^{n} \frac{(i-1)!(k+m+2-i+n)!}{(2n+k+2-2i)!(m-i+n)!} \prod_{j=1}^{n} \frac{(2m-j+2n+k+2)!}{(2m+3+2n-2j+k)!}}{\prod_{i=1}^{n-1} \frac{(i-1)!(k+m-i+n+1)!}{(2n+k-2i)!(m-i+n-1)!} \prod_{j=1}^{n-1} \frac{(2m-j+2n+k)!}{(2m+1+2n-2j+k)!}}$$
$$= \frac{(n-1)!(k+m+1+n)!(2m+2n+k)!}{(2n+k)!(m+n-1)!(n+2m+k+1)!} = \frac{1}{\prod_{\ell=0}^{m-1}(n+\ell)} \frac{\prod_{\ell=0}^{2m-1}(2n+1+k+\ell)}{\prod_{\ell=0}^{m-1}(n+m+\ell+k+2)}$$

Thus

$$\det\left(\binom{2n+m+2+k-i-j}{j-i+m}\right)_{i,j=1}^{n}=\prod_{j=1}^{n}v(j,m,k).$$

Consider now the right-hand side $M(m,n,k)=\det\left(C_{n-i+j}^{(k+1+2i)}\right)_{i,j=0}^{m-1}$.

This determinant can be computed with the Condensation method (cf. [12], Prop.10). This gives

$$\begin{aligned}&M(m,n,k)M(m-2,n,k+2)\\&=M(m-1,n,k+2)M(m-1,n,k)-M(m-1,n+1,k)M(m-1,n-1,k+2).\end{aligned} \tag{49}$$

Since no determinant vanishes this determines $M(m,n,k)$ by induction with m.

Let us verify the first terms. We have $M(0,n,k)=1=\prod_{j=1}^{n}v(j,0,k)$,

$$\prod_{j=1}^{n}v(j,1,k)=\prod_{j=1}^{n}\frac{(2j+1+k)(2j+k+2)}{j(j+3+k)}=\frac{3+k}{3+n+k}\binom{2n+k+2}{n}=C_n^{(k+1)}$$

Since

$$\frac{v(j,m,k)v(j,m-2,k+2)}{v(j,m-1,k)v(j,m-1,k+2)}=\frac{(k+2+j+m)(j+m-1)}{(k+j+m+1)(j+m-2)}$$

we get

$$\prod_{j=1}^{n-1}\frac{v(j,m,k)v(j,m-2,k+2)}{v(j,m-1,k)v(j,m-1,k+2)}=\frac{(m-1)(k+2+m)}{(n+m-2)(m+k+n+1)}$$

Therefore (49) reduces to

$$\begin{aligned}&\frac{(m-1)(m+2+k)}{(n+m-2)(m+k+n+1)}v(n,m,k)v(n,m-2,k+2)\\&=v(n,m-1,k+2)v(n,m-1,k)-v(n+1,m-1,k)v(n,m-1,k)\end{aligned}$$

which is easily verified.

3. q-analogues

3.1. The Carlitz q-Catalan numbers

As usual we write $[n]=\frac{1-q^n}{1-q}$, $[n]!=[1][2]\cdots[n]$ and $\begin{bmatrix}n\\k\end{bmatrix}=\frac{[n]!}{[k]![n-k]!}$ for $0\le k\le n$.

Further we let $(x;q)_n=(1-x)(1-qx)\cdots(1-q^{n-1}x)$.

For the Carlitz $q-$Catalan numbers $c_n(q)$ which satisfy $c_n(q)=\sum_{k=0}^{n-1} q^k c_k(q) c_{n-1-k}(q)$ with $c_0(q)=1$ the corresponding orthogonal polynomials are given by

$f_n(x,q)=\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k q^{2\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix} x^{n-2k}$. Here we have $s(n)=0$ and $t(n)=q^n$ (cf. e.g.[4]). In this case the method of orthogonal polynomials gives nice $q-$analogues, but there are no closed formulas for $c_n(q)$ and the corresponding $c_n^{(k)}(q)$.

Let me only mention the analogue of (1).

$$\det\left(q^{2\binom{i-j}{2}} \begin{bmatrix} i+1+j \\ i+1-j \end{bmatrix}\right)_{i,j=0}^{n-1} = c_n(q). \tag{50}$$

The first terms are

$$\det(1)=1,\ \det\begin{pmatrix} 1 & q^2 \\ 1 & [3] \end{pmatrix} = 1+q = c_2(q),$$

$$\det\begin{pmatrix} 1 & q^2 & 0 \\ 1 & [3] & q^2 \\ q^2 & \left(1+q^2\right)[3] & [5] \end{pmatrix} = 1+2q+q^2+q^3 = c_3(q),\cdots.$$

In [3] we have introduced a $q-$analogue $g_n(r,q)$ of $\frac{1}{rn+1}\binom{rn+1}{n}$ defined by

$g_n(r,q)=\sum_{k_1+\cdots+k_r=n-1} \prod_{j=1}^{r} \left(q^{(r-j)k_j} g_{k_j}(r,q)\right)$ with initial value $g_0(r,q)=1$.

[3],(10), gives a combinatorial proof of the identity

$$\sum_{j=0}^{n} (-1)^{n-j} q^{r\binom{n-j}{2}} \begin{bmatrix} (r-1)j+1 \\ n-j \end{bmatrix} g_j(r,q) = [n=0]. \tag{51}$$

This implies

$$\det\left(q^{\binom{i-j+1}{2}} \begin{bmatrix} (r-1)j+1 \\ i-j+1 \end{bmatrix}\right)_{i,j=0}^{n-1} = g_n(r,q), \tag{52}$$

which generalizes (50) and (37) for $k=1$.

3.2. The q-Catalan numbers $C_n(q)=\frac{1}{[n+1]}\begin{bmatrix}2n\\n\end{bmatrix}$.

The orthogonal polynomials whose moments are $C_n(q)=\frac{1}{[n+1]}\begin{bmatrix}2n\\n\end{bmatrix}$ have no closed formulas. But there are other polynomials with these moments. As shown in [7] the linear functional Λ defined by $\Lambda\left(f(n,x,q)\right)=[n=0]$ for the polynomials $f(n,x,q)=\sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}n-j\\j\end{bmatrix}x^{n-2j}$ satisfies $\Lambda\left(x^{2n}\right)=C_n(q)=\frac{1}{[n+1]}\begin{bmatrix}2n\\n\end{bmatrix}$ and $\Lambda\left(x^{2n+1}\right)=0.$

Therefore

$$\Lambda\left(f(2n,x,q)\right)=\Lambda\left(\sum_{j=0}^{n}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}2n-j\\j\end{bmatrix}x^{2n-2j}\right)=\Lambda\left(\sum_{j=0}^{n}(-1)^{n-j} q^{\binom{n-j}{2}}\begin{bmatrix}n+j\\n-j\end{bmatrix}x^{2j}\right)$$
$$=\sum_{j=0}^{n}(-1)^{n-j} q^{\binom{n-j}{2}}\begin{bmatrix}n+j\\n-j\end{bmatrix}C_j(q)=[n=0]$$

which implies

$$\det\left(q^{\binom{i-j+1}{2}}\begin{bmatrix}i+j+1\\i-j+1\end{bmatrix}\right)_{i,j=0}^{n-1}=C_n(q)=\frac{1}{[n+1]}\begin{bmatrix}2n\\n\end{bmatrix}. \tag{53}$$

The first terms are

$$\det\begin{pmatrix}1\end{pmatrix}=1=C_1(q),\quad \det\begin{pmatrix}1 & 1\\ q & [3]\end{pmatrix}=1+q^2=C_2(q),$$
$$\det\begin{pmatrix}1 & 1 & 0\\ q & [3] & 1\\ q^3 & q+q^2+2q^3+q^4+q^5 & [5]\end{pmatrix}=(1-q+q^2)[5]=C_3(q),\cdots.$$

Since $\binom{i-j+1}{2}-\binom{i-j}{2}=i-j$

we also get

$$\det\left(q^{\binom{i-j}{2}}\begin{bmatrix}i+j+1\\i-j+1\end{bmatrix}\right)_{i,j=0}^{n-1}=C_n(q). \tag{54}$$

As $q-$analogue of (32) we prove

Theorem 7

$$\det\left(q^{\binom{i-j}{2}}\begin{bmatrix} i+j+k \\ i-j+1 \end{bmatrix}\right)_{i,j=0}^{n-1} = \det\left(q^{\binom{i-j+1}{2}}\begin{bmatrix} i+j+k \\ i-j+1 \end{bmatrix}\right)_{i,j=0}^{n-1} = \frac{[k]}{[2n+k]}\begin{bmatrix} 2n+k \\ n \end{bmatrix}. \qquad (55)$$

Proof

It suffices to prove

$$\sum_{j=0}^{n}(-1)^{n-j}q^{\binom{n-j}{2}}\begin{bmatrix} n+j+k-1 \\ n-j \end{bmatrix}\frac{[k]}{[2j+k]}\begin{bmatrix} 2j+k \\ j \end{bmatrix} = [n=0]. \qquad (56)$$

for $n,k \in \mathbb{N}.$

Using $(q;q)_{n-k} = \frac{(q;q)_n}{(q^{-n};q)_k}(-1)^k q^{\binom{k}{2}-nk}$ and $q-$Vandermonde's formula (cf. [10], II(6))

$${}_2\varphi_1\left[\begin{matrix} a, q^{-n} \\ c \end{matrix};q,q\right] = \sum_{k=0}^{n}\frac{(a;q)_k (q^{-n};q)_k}{(q;q)_k (c;q)_k}q^k = \frac{(c/a;q)_n}{(c;q)_n}a^n$$

we get

$$\sum_{j=0}^{n}(-1)^j q^{\binom{j}{2}}\begin{bmatrix} 2n-j+k-1 \\ j \end{bmatrix}\frac{[k]}{[2n-2j+k]}\begin{bmatrix} 2n-2j+k \\ n-j \end{bmatrix}$$
$$= \sum_{j=0}^{n}(-1)^j q^{\binom{j}{2}}\frac{(q;q)_{2n+k-1-j}[k](q;q)_{2n-2j+k-1}}{(q;q)_j (q;q)_{2n+k-1-2j}(q;q)_{n-j}(q;q)_{n+k-j}}$$
$$= \sum_{j=0}^{n}(-1)^j q^{\binom{j}{2}}\frac{(q;q)_{2n+k-1-j}[k]}{(q;q)_j (q;q)_{n-j}(q;q)_{n+k-j}}$$
$$= \sum_{j=0}^{n}(-1)^j q^{\binom{j}{2}}\frac{(q;q)_{2n+k-1-j}[k]}{(q;q)_j (q;q)_{n-j}(q;q)_{n+k-j}}$$
$$= \frac{[k]}{[2n+k]}\begin{bmatrix} 2n+k \\ n \end{bmatrix}\sum_{j=0}^{n}q^j\frac{(q^{-n};q)_j (q^{-n-k};q)_j}{(q;q)_j (q^{-2n-k+1};q)_j} = \frac{[k]}{[2n+k]}\begin{bmatrix} 2n+k \\ n \end{bmatrix}\ {}_2\varphi_1\left[\begin{matrix} q^{-n}, q^{-n-k} \\ q^{-2n-k+1} \end{matrix};q,q\right] = [n=0].$$

As $q-$analog of (33) we get

Theorem 8

Let $c(n,j)=q^{2\binom{n-j}{2}}\frac{\left(-q^{j};q\right)_{n-j}}{\left(-q;q\right)_{n-j}}\begin{bmatrix}j\\n-j\end{bmatrix}$. *Then we get*

$$\det\left(c(i+k+1,j+k)\right)_{i,j=0}^{n-1}=C(n,k,q)=\frac{[k]}{[2n+k]}\begin{bmatrix}2n+k\\n\end{bmatrix}\frac{(-q^{n+1};q)_{k-1}}{(-q;q)_{k-1}}. \qquad (57)$$

It suffices to prove

Lemma 9

$$\sum_{j=0}^{n}(-1)^{n-j}c(n+k,j+k)\frac{[k]}{[2j+k]}\begin{bmatrix}2j+k\\j\end{bmatrix}\frac{(-q^{j+1};q)_{k-1}}{(-q;q)_{k-1}}=[n=0]. \qquad (58)$$

For $k=1$ this has been found by George Andrews [2] and later a combinatorial proof has been given in [11].

To prove Lemma 9 we use the $q-$Pfaff-Saalschütz summation (cf. [10],II.12)

$$ {}_3\varphi_2\left[\begin{matrix}a,b,q^{-n}\\c,abc^{-1}q^{1-n}\end{matrix},q;q\right]=\sum_{k=0}^{n}\frac{\left(a;q\right)_k\left(b;q\right)_k\left(q^{-n};q\right)_k}{\left(q;q\right)_k\left(c;q\right)_k\left(abc^{-1}q^{1-n};q\right)_k}q^k=\frac{\left(c/a;q\right)_n\left(c/b;q\right)_n}{\left(c;q\right)_n\left(c/ab;q\right)_n}$$

and the following elementary identities:

$$\left(-q^{n-k+1};q\right)_k=\left(q^{-n};q\right)_k q^{nk-\binom{k}{2}},\left(q;q\right)_{n-k}=\frac{\left(q;q\right)_n}{\left(q^{-n};q\right)_k}(-1)^k q^{\binom{k}{2}-nk},$$
$$\left(a;q\right)_k\left(-a;q\right)_k=\left(a^2;q^2\right)_k,\left(a;q\right)_{2k}=\left(a;q^2\right)_k\left(aq;q^2\right)_k.$$

First we change the order of summation

$$\sum_{j=0}^{n}(-1)^{n-j}q^{2\binom{n-j}{2}}\frac{\left(-q^{j+k};q\right)_{n-j}}{\left(-q;q\right)_{n-j}}\begin{bmatrix}j+k\\n-j\end{bmatrix}\frac{[k]}{[2j+k]}\begin{bmatrix}2j+k\\j\end{bmatrix}\frac{(-q^{j+1};q)_{k-1}}{(-q;q)_{k-1}}$$
$$=\sum_{j=0}^{n}(-1)^{j}q^{2\binom{j}{2}}\frac{\left(-q^{n-j+k};q\right)_{j}}{\left(-q;q\right)_{j}}\begin{bmatrix}n-j+k\\j\end{bmatrix}\frac{[k]}{[2n-2j+k]}\begin{bmatrix}2n-2j+k\\n-j\end{bmatrix}\frac{(-q^{n-j+1};q)_{k-1}}{(-q;q)_{k-1}}.$$

Then we make the following changes:

$$\left(-q^{n-j+k};q\right)_j=\left(-q^{-n-k+1};q\right)_j q^{(n+k-1)j-\binom{j}{2}},\quad \frac{1}{(q;q)_j(q;q)_{n+k-2j}}=\frac{\left(q^{-n-k};q\right)_{2j}}{(q;q)_j(q;q)_{n+k}q^{\binom{2j}{2}-2j(n+k)}},$$

$$(q;q)_{2n+k-1-2j}=\frac{(q;q)_{2n+k-1}}{\left(q^{-2n-k+1};q\right)_{2j}}q^{\binom{2j}{2}-2j(2n+k-1)},\quad \frac{1}{(q;q)_{n-j}}=\frac{\left(q^{-n};q\right)_j}{(q;q)_n}q^{nj-\binom{j}{2}},$$

$$\frac{1}{(q;q)_{n-j+k}}=\frac{\left(q^{-n-k};q\right)_j}{(q;q)_{n+k}}q^{(n+k)j-\binom{j}{2}}.$$

Observing that

$$\frac{\left(-q^{-n-k+1};q\right)_j\left(q^{-n};q\right)_j(-q^{n-j+1};q)_{k-1}}{(-q^{n+1};q)_{k-1}}=\frac{\left(q^{-2n};q^2\right)_j}{q^{(k-1)j}}$$

we get

$$\sum_{j=0}^{n}(-1)^j q^{2\binom{j}{2}}\frac{\left(-q^{n-j+k};q\right)_j}{\left(-q;q\right)_j}\begin{bmatrix}n-j+k\\ j\end{bmatrix}\frac{[k]}{[2n-2j+k]}\begin{bmatrix}2n-2j+k\\ n-j\end{bmatrix}\frac{(-q^{n-j+1};q)_{k-1}}{(-q;q)_{k-1}}$$

$$=\frac{[k]}{[2n+k]}\begin{bmatrix}2n+k\\ n\end{bmatrix}\frac{(-q^{n+1};q)_{k-1}}{(-q;q)_{k-1}}q^{(k+1)j}\frac{\left(-q^{-n-k+1};q\right)_j\left(q^{-n-k};q\right)_{2j}\left(q^{-n};q\right)_j}{\left(-q;q\right)_j(q;q)_j\left(q^{-2n-k+1};q\right)_{2j}}\frac{(-q^{n-j+1};q)_{k-1}}{(-q^{n+1};q)_{k-1}}$$

$$=\frac{[k]}{[2n+k]}\begin{bmatrix}2n+k\\ n\end{bmatrix}\frac{(-q^{n+1};q)_{k-1}}{(-q;q)_{k-1}}\frac{\left(q^{-2n};q^2\right)_j\left(q^{-n-k};q^2\right)_j\left(q^{-n-k+1};q^2\right)_j}{(q^2;q^2)_j\left(q^{-2n-k+1};q^2\right)_j\left(q^{-2n-k+2};q^2\right)_j}q^{2j}$$

$$=C(n,k,q)\,{}_3\varphi_2\left[\begin{matrix}q^{-n-k},q^{-n-k+1},q^{-2n}\\ q^{-2n-k+1},q^{-2n-k+2}\end{matrix};q^2;q^2\right]=\frac{\left(q^{-n+1};q^2\right)_n\left(q^{-n};q^2\right)_n}{\left(q^{-2n-k+1};q^2\right)_n\left(q^k;q^2\right)_n}=[n=0].$$

3.3. q-Chebyshev polynomials and the q-Catalan numbers of George Andrews

For the monic Chebyshev polynomials of the second kind

$u_n(x)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\left(-\frac{1}{4}\right)^k\binom{n-k}{k}x^{n-2k}=\frac{F_n(2x)}{2^n}$ there exist $q-$analogues with nice formulas of both the orthogonal polynomials and its moments.

As analogues of $u_n(x)$ we get the monic $q-$Chebyshev polynomials of the second kind (cf.[6])

$$u_n(x,q)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^k}{\left(-q;q\right)_k\left(-q^{n+1-k};q\right)_k}q^{k^2}\begin{bmatrix}n-k\\ k\end{bmatrix}x^{n-2k}\tag{59}$$

which satisfy $u_n(x,q) = xu_{n-1}(x,q) - \frac{q^{n-1}}{\left(1+q^{n-1}\right)\left(1+q^n\right)} u_{n-2}(x,q)$ with initial values $u_0(x,q) = 1$ and $u_1(x,q) = x.$ Their moments are Andrews' $q-$Catalan numbers

$$M_n = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix} \frac{1+q}{1+q^{n+1}} \frac{q^n}{\left(-q;q\right)_n^2}. \tag{60}$$

Here we get as analogue of (32)

$$\det\left(\frac{q^{(i+1-j)^2}}{\left(-q;q\right)_{i+1-j} \left(-q^{i+j+k+1};q\right)_{i+1-j}} \begin{bmatrix} i+j+k \\ i+1-j \end{bmatrix} \right)_{i,j=0}^{n-1} = q^n \frac{1+q^k}{1+q^{n+k}} \frac{[k]}{[2n+k]} \begin{bmatrix} 2n+k \\ n \end{bmatrix} \frac{1}{\left(-q;q\right)_n \left(-q^k;q\right)_n}.$$

The proofs use orthogonality and will be omitted.

3.4. Final Remarks

We could not find $q-$analogues of all results. For example with Lemma 3 we get

$$\det\left(q^{\binom{i-j}{2}} \begin{bmatrix} x+i+j+m \\ i-j+m \end{bmatrix} \right)_{i,j=0}^{n-1} = \prod_{j=1}^{n} \frac{\prod_{\ell=0}^{2k-1} [2j-1+x+\ell]}{\prod_{\ell=0}^{k-1} [j+\ell][j+k+x+\ell]},$$

but we do not know if there is an analogue of the Hankel determinant in (41).

The formula

$$\det\left(q^{\binom{i-j}{2}} \frac{[2i+k+1]}{[i+j+k]} \begin{bmatrix} i+j+k \\ i-j+1 \end{bmatrix} \right)_{i,j=0}^{n-1} = \begin{bmatrix} 2n+k-1 \\ n \end{bmatrix} \tag{61}$$

is a nice $q-$analogue of (28), which follows from

$$\sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} \frac{[2n+k-1]}{[2n-j+k-1]} \begin{bmatrix} 2n-j+k-1 \\ j \end{bmatrix} \begin{bmatrix} 2n-2j+k-1 \\ n-j \end{bmatrix} =$$
$$\begin{bmatrix} 2n+k-1 \\ n \end{bmatrix} {}_2\varphi_1 \begin{bmatrix} q^{-n}, q^{-n-k+1} \\ q^{-2n-k+2} \end{bmatrix} ;q,q \Bigg] = \frac{\left(q^{-n+1};q\right)_n}{\left(q^{-2n-k+2};q\right)_n} = [n=0].$$

It would be interesting if there is also a nice $q-$analogue of (29).

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