Some remarks and conjectures about Hankel determinants

Johann Cigler

1. Introduction

I give a survey of some known results about Hankel determinants of Catalan-like sequences and present some conjectures about related themes which have been obtained by using Martin Rubey's computer program Guess ([4]).

Define numbers $a(n,k,c,x)$ by the Stieltjes tableau

$$\begin{aligned} &a(0,k,c,x)=[k=0] \\ &a(n,0,c,x)=(c+x)a(n-1,0,c,x)+a(n-1,1,c,x) \\ &a(n,k,c,x)=a(n-1,k-1,c,x)+ca(n-1,k,c,x)+a(n-1,k+1,c,x) \end{aligned} \tag{1.1}$$

for real numbers c and x.

We are interested in computing the Hankel determinants

$$d_p(n,k,c,x)=\det\left(a(i+j+p,k,c,x)\right)_{i,j=0}^{n-1} \tag{1.2}$$

for $0 \le p \le 2$ and non-negative integers k.
The determinants are defined for $n>0$. It is convenient to set $d_p(0,k,c,x)=1$.

For $k=0$ such determinants have been extensively studied (cf. e.g. [1]). For $k>0$ the first results have been obtained in [2].

The simplest case occurs for $(c,x)=(2,-1)$. Here we get the Catalan numbers $a(n,0,2,-1)=C_n=\frac{1}{n+1}\binom{2n}{n}$.
For general k we get

$$a(n,k,2,-1)=\frac{2k+1}{n+k+1}\binom{2n}{n+k}. \tag{1.3}$$

In [2] the following Hankel determinants have been computed:

[2] Corollary 12:
For all non-negative integers n,k *we have*

$$\begin{aligned} &d_0((2k+1)n,k,2,-1)=(-1)^{nk}, \\ &d_0((2k+1)n+k+1,k,2,-1)=(-1)^{(n+1)k+\binom{k}{2}} \end{aligned} \tag{1.4}$$

and $d_0(n,k,2,-1)=0$ *else.*

[2] Corollary 13:

$$\begin{aligned} &d_1((2k+1)n,k,2,-1)=(-1)^{nk}, \\ &d_1((2k+1)n+k,k,2,-1)=(-1)^{(n+1)k+\binom{k+1}{2}} \end{aligned} \tag{1.5}$$

and $d_1(n,k,2,-1)=0$ *else.*

[2] Corollary 16:

$$\begin{aligned} &d_2(n,0,2,-1)=n+1, \\ &d_2(3n,1,2,-1)=(-1)^n, \\ &d_2(3n+1,1,2,-1)=(-1)^n 3(n+1), \\ &d_2(3n+2,1,2,-1)=(-1)^n 3(n+1). \end{aligned} \tag{1.6}$$

For $k \geq 2$ *we have*

$$\begin{aligned} &d_2((2k+1)n,k,2,-1)=(-1)^{nk}, \\ &d_2((2k+1)n+k-1,k,2,-1)=(-1)^{(n+1)k+\binom{k}{2}+1}, \\ &d_2((2k+1)n+k,k,2,-1)=(-1)^{(n+1)k+\binom{k+1}{2}}(2k+1)(n+1), \\ &d_2((2k+1)n+2k,k,2,-1)=(-1)^{nk}(2k+1)(n+1) \end{aligned} \tag{1.7}$$

and $d_2(n,k,2,-1)=0$ *else.*

A concise formulation of these results can be obtained by using generating functions.
For $k \geq 1$ we get the generating functions

$$\sum_{n\geq 0} d_0(n,k,2,-1)z^n = \frac{1+(-1)^{\binom{k+1}{2}} z^{k+1}}{1+(-1)^{k+1} z^{2k+1}} \tag{1.8}$$

and

$$\sum_{n\geq 0} d_1(n,k,2,-1)z^n = \frac{1+(-1)^{\binom{k}{2}} z^{k}}{1+(-1)^{k+1} z^{2k+1}} \tag{1.9}$$

and for $k \geq 2$

$$\sum_{n\geq 0} d_2(n,k,2,-1)z^n$$
$$= \frac{1+(-1)^{\binom{k-1}{2}} z^{k-1} + (2k+1)(-1)^{\binom{k}{2}} z^k + (2k+1)z^{2k} + (-1)^{k+1} z^{2k+1} + (-1)^{\binom{k}{2}} z^{3k}}{\left(1+(-1)^{k+1} z^{2k+1}\right)^2}. \tag{1.10}$$

It should be noted that for $k=0$ the numerator in (1.8) and (1.9) must be replaced by 1.
In (1.10) the numerator for $k=0$ is 1 and for $k=1$ it is $(1+z)^3$.

Remark
As is well known Hankel determinants and Catalan numbers are intimately related. Catalan numbers are characterized by the fact that $d_0(n,0,2,-1) = d_1(n,0,2,-1) = 1$ for all n. This is of course an easy consequence of (1.4) and (1.5). So it is perhaps not very surprising that also for $k>0$ simple formulas hold.
Our purpose is to find other instances where simple or beautiful formulas hold.

2. The case x=0.

Let $P_n^+(i,k)$ be the generating function $\sum_P w(P)$, where P runs over all paths from $(0,i)$ to (n,k) consisting of steps from $\{1,0),(1,1),(1,-1)\}$, which never run below the $x-$axis. Here $w(P)$ is the product of all weights of the steps of P, where the weights of the steps are defined by $w((1,0)) = x+y, w((1,1)) = 1,$ and $w((1,-1)) = xy.$
It has been proved in [2], that

$$\det\left(P_{i+j}^+(0,k)\right)_{i,j=0}^{n-1} = \det\left(P_j^+(i,k)\right)_{i,j=0}^{n-1} = \begin{cases} (-1)^{m\binom{k+1}{2}} (xy)^{(k+1)^2\binom{m}{2}} & n=(k+1)m \\ 0 & n \not\equiv 0 (\operatorname{mod}(k+1)) \end{cases} \tag{2.1}$$

In the following I shall consider only the special case $xy=1.$ Setting $x+y=c$ we get

$$x = \frac{c-\sqrt{c^2-4}}{2}, y = \frac{c+\sqrt{c^2-4}}{2}. \tag{2.2}$$

It is clear that $P_n^+(0,k) = a(n,k,c,0)$ for this choice of weights.
To make formulations simpler we write in this case $a(n,k,c)$ instead of $a(n,k,c,0).$

Then the above result implies that

$$d_0((k+1)n,k,c)=(-1)^{n\binom{k+1}{2}} \tag{2.3}$$

and $d_0(n,k,c)=0$ if $n\not\equiv 0\left(\operatorname{mod}(k+1)\right)$.

For the Hankel determinants d_1 it has been proved in [2] that

$$\det\left(P^+_{i+j+1}(0,k)\right)_{i,j=0}^{n-1}=\det\left(P^+_{j+1}(i,k)\right)_{i,j=0}^{n-1}$$
$$=\begin{cases}(-1)^{m\binom{k+1}{2}}(xy)^{(k+1)^2\binom{m}{2}}\dfrac{y^{(k+1)(m+1)}-x^{(k+1)(m+1)}}{y^{k+1}-x^{k+1}} & n=(k+1)m\\ (-1)^{m\binom{k+1}{2}+\binom{k}{2}}(xy)^{(k+1)^2\binom{m}{2}+mk(k+1)}\dfrac{y^{(k+1)(m+1)}-x^{(k+1)(m+1)}}{y^{k+1}-x^{k+1}} & n=(k+1)m+k\\ 0 & n\not\equiv 0,k\left(\operatorname{mod}(k+1)\right)\end{cases} \tag{2.4}$$

In order to give a concise formulation of this result in the special case $xy=1$ we consider a variant of the Fibonacci and Lucas polynomials.

Define Fibonacci polynomials $F_n(z)$ by

$$F_n(z)=zF_{n-1}(z)-F_{n-2}(z) \tag{2.5}$$

with initial values $F_0(z)=0, F_1(z)=1$ and Lucas polynomials $L_n(z)$ by

$$L_n(z)=zL_{n-1}(z)-L_{n-2}(z) \tag{2.6}$$

with initial values $L_0(z)=2, L_1(z)=z.$

The first terms are
$\left(F_n(z)\right)=\left(0,1,z,z^2-1,z^3-2z,z^4-3z^2+1,\cdots\right)$
and
$\left(L_n(z)\right)=\left(2,z,z^2-2,z^3-3z,z^4-4z^2+2,z^5-5z^3+5z,\cdots\right).$
For $n\geq 1$ we have $L_n(z)=F_{n+1}(z)-F_{n-1}(z).$

It is clear that by defining x,y as in (2.2) we have

$$F_n(c)=\frac{y^n-x^n}{y-x} \tag{2.7}$$

and

$$L_n(c)=x^n+y^n. \tag{2.8}$$

Now observe that $F_n(x^k + y^k) = \dfrac{y^{kn} - x^{kn}}{y^k - x^k}.$

This follows from

$$\frac{y^{k(n+2)} - x^{k(n+2)}}{y^k - x^k} = (x^k + y^k)\frac{y^{k(n+1)} - x^{k(n+1)}}{y^k - x^k} - \frac{y^{kn} - x^{kn}}{y^k - x^k}$$

by using $xy = 1.$

Therefore $\dfrac{y^{(k+1)(n+1)} - x^{(k+1)(n+1)}}{y^{k+1} - x^{k+1}} = F_{n+1}\left(x^{k+1} + y^{k+1}\right) = F_{n+1}\left(L_{k+1}(c)\right).$

This implies

2.1 Theorem

$$\begin{aligned} &d_1\left((k+1)n, k, c\right) = (-1)^{n\binom{k+1}{2}} F_{n+1}\left(L_{k+1}(c)\right) \\ &d_1\left((k+1)n + k, k, c\right) = (-1)^{\binom{k}{2} + n\binom{k+1}{2}} F_{n+1}\left(L_{k+1}(c)\right) \\ &d_1(n, k, c) = 0 \qquad\qquad n \not\equiv 0, k(\operatorname{mod}(k+1)) \end{aligned} \tag{2.9}$$

In terms of generating functions we get

$$\sum_{n\geq 0} d_1(n,0,c) z^n = \frac{1}{1 - cz + z^2} \tag{2.10}$$

and for $k > 0$

$$\sum_{n\geq 0} d_1(n,k,c) z^n = \frac{1 + (-1)^{\binom{k}{2}} z^k}{1 + (-1)^{\binom{k-1}{2}} L_{k+1}(c) z^{k+1} + z^{2k+2}}. \tag{2.11}$$

Remark

Let Λ be the linear functional on the polynomials in z defined by $\Lambda\left(F_{n+1}(z-c)\right) = [n=0],$ then the moments are given by $\Lambda\left(z^n\right) = a(n,0,c).$ More generally

$$\sum_{k=0}^{n} a(n,k,c) F_{k+1}(z-c) = z^n. \tag{2.12}$$

The polynomials $F_{n+1}(z-c)$ are orthogonal with respect to $\Lambda.$

Most approaches to Hankel determinants for sequences $a(n,0,c)$ are using orthogonal polynomials (see e.g. [3]). But for sequences $a(n,k,c)$ with $k>0$ other methods are needed.

Observing Cassini's identity

$$F_{n+1}(x)F_{n-1}(x)=F_n(x)^2-1 \tag{2.13}$$

we see that $d_1\left((k+1)n,k,c\right)$ satisfies

$$d_1\left((k+1)(n+1),k,c\right)d_1\left((k+1)(n-1),k,c\right)-d_1\left((k+1)n,k,c\right)^2+1=0. \tag{2.14}$$

For a matrix A the condensation method (cf. [3]) states that

$$\det A \det A_{1,n}^{1,n}=\det A_1^1 \det A_n^n-\det A_1^n \det A_n^1. \tag{2.15}$$

Here A_I^J denotes the matrix where the rows with indices in I and the columns in J are removed.
In the case of Hankel determinants this means

$$d_0(n)d_2(n-2)=d_2(n-1)d_0(n-1)-d_1(n-1)^2. \tag{2.16}$$

Since $d_0(n,0,c)=1$ we get $d_2(n,0,c)-d_2(n-1,0,c)=d_1(n,0,c)^2$ and thus

$$d_2(n,0,c)=d_1(n,0,c)^2+d_1(n-1,0,c)^2+\cdots+d_1(1,0,c)^2+d_1(0,0,c)^2$$

$$=\sum_{j=0}^{n}\left(F_{j+1}(c)\right)^2.$$

A further consequence is

$$d_0((k+1)n+i,k,c)d_2((k+1)n+i-2,k,c)=d_2((k+1)n+i-1,k,c)d_0((k+1)n+i-1)$$
$$-d_1((k+1)n+i-1,k,c)^2$$

Choosing $i=k+1$ we see that

$$d_0((k+1)(n+1),k,c)d_2((k+1)n+k-1,k,c)=d_2((k+1)n+k,k,c)d_0((k+1)n+k)$$
$$-d_1((k+1)n+k,k,c)^2$$

or

$$(-1)^{(n+1)\binom{k+1}{2}}d_2((k+1)n+k-1,k,c)=-d_1((k+1)n+k,k,c)^2=-\left(F_{n+1}\left(L_{k+1}(c)\right)\right)^2.$$

By choosing $i=1$ we get

$$d_2((k+1)n,k,c)(-1)^{n\binom{k+1}{2}}=d_1((k+1)n,k,c)^2.$$

This proves part of the following

2.2 Conjecture

For $k \geq 1$ *the following formulae hold:*

$$d_2((k+1)n,k,c) = (-1)^{n\binom{k+1}{2}} \left(F_{n+1}\left(L_{k+1}(c)\right)\right)^2 \tag{2.17}$$

$$d_2((k+1)n+k-1,k,c) = (-1)^{\binom{k-1}{2}+n\binom{k+1}{2}} \left(F_{n+1}\left(L_{k+1}(c)\right)\right)^2 \tag{2.18}$$

For $k \geq 0$

$$d_2\left((k+1)n+k,k,c\right) = (-1)^{\binom{k}{2}+n\binom{k+1}{2}} (k+1)F_{k+1}(c)\sum_{j=0}^{n}\left(F_{j+1}\left(L_{k+1}(c)\right)\right)^2. \tag{2.19}$$

All other values $d_2(n,k,c)$ *vanish.*

As a consequence we see as in (2.14) that $d_2\left((k+1)n,k,c\right)$ satisfies the recurrence

$$d_2\left((k+1)(n+1),k,c\right)d_2\left((k+1)(n-1),k,c\right) = \left((-1)^{n\binom{k+1}{2}} d_2\left((k+1)n,k,c\right)-1\right)^2 \tag{2.20}$$

with initial values $d_2\left(0,k,c\right)=1$ and $d_2\left((k+1),k,c\right) = (-1)^{\binom{k+1}{2}}\left(L_{k+1}(c)\right)^2$.

Linear recurrences for the numbers (2.17) - (2.19) are easily obtained from the generating functions

$$\sum_{n\geq 0} F_{n+1}(c)^2 z^n = \frac{1+z}{1-(c^2-1)z+(c^2-1)z^2-z^3} = \frac{1+z}{(1-z)\left(z^2-(c^2-2)z+1\right)} = \frac{1+z}{(1-z)\left(z^2-L_2(c)z+1\right)} \tag{2.21}$$

and with

$$g_n(c) = \sum_{j=0}^{n} F_{j+1}(c)^2 \tag{2.22}$$

$$\sum_{n\geq 0} g_n(c)z^n = \frac{1+z}{1-c^2z+2(c^2-1)z^2-c^2z^3+z^4} = \frac{1+z}{(1-z)^2\left(z^2-L_2(c)z+1\right)}. \tag{2.23}$$

Observe that

$$L_2(L_k(c)) = \left(L_k(c)\right)^2 - 2 = L_{2k}(c). \tag{2.24}$$

We can again state these results using generating functions for $d_2(n,k,c)$:

2.3 Conjecture

$$\sum_{n\geq 0} d_2(n,0,c)z^n = \frac{1+z}{(1-z)^2\left((1+z)^2-c^2z\right)} = \frac{1+z}{(1-z)^2\left(1-L_2(c)z+z^2\right)} \tag{2.25}$$

$$\sum_{n\geq 0} d_2(n,1,c)(-z)^n = \frac{1-2cz+2cz^3-z^4}{\left(1+z^2\right)^2\left(\left(1-z^2\right)^2+(c^2-2)^2z^2\right)} = \frac{(1-z^2)\left(1-2cz+z^2\right)}{\left(1+z^2\right)^2\left(1+L_4(c)z^2+z^4\right)} \tag{2.26}$$

and for $k \geq 2$

$$\sum_{n\geq 0}(-1)^{nk} d_2(n,k,c)z^n$$

$$= \frac{1+(-1)^{\binom{k-1}{2}}z^{k-1}-(-1)^{\binom{k-1}{2}}(k+1)F_{k+1}(c)z^k+(k+1)F_{k+1}(c)z^{2k+1}-z^{2k+2}-(-1)^{\binom{k-1}{2}}z^{3k+1}}{\left(1+(-1)^{\binom{k-1}{2}}z^{k+1}\right)^2\left(\left(1-(-1)^{\binom{k-1}{2}}z^{k+1}\right)^2+(-1)^{\binom{k-1}{2}}z^{k+1}\left(L_{k+1}(c)\right)^2\right)}$$

$$= \frac{\left(1-(-1)^{\binom{k-1}{2}}z^{k+1}\right)\left(1+(-1)^{\binom{k-1}{2}}z^{k-1}-(-1)^{\binom{k-1}{2}}(k+1)F_{k+1}(c)z^k+(-1)^{\binom{k-1}{2}}z^{k+1}+z^{2k}\right)}{\left(1+(-1)^{\binom{k-1}{2}}z^{k+1}\right)^2\left(1+(-1)^{\binom{k-1}{2}}L_{2k+2}(z)z^{k+1}+z^{2k+2}\right)} \tag{2.27}$$

2.4 Remark

The simplest special cases occur for $c \in \{-2,-1,0,1,2\}$.
Since $a(n,k,-c) = (-1)^{n-k}a(n,k,c)$ we get

$d_p\left(n,k,-c\right) = (-1)^{n(n+p-k-1)}d_p\left(n,k,c\right)$. Therefore we need only consider $c \in \{0,1,2\}$.

This corresponds to the following $a(n,k,c)$:
For $k=0$ we get the Catalan numbers $a(2n,0,0) = C_n = \frac{1}{n+1}\binom{2n}{n}$, the Motzkin numbers $a(n,0,1) = M_n$ and the Catalan numbers $a(n,0,2) = C_{n+1}$.

For general k the numbers $a(n,k,c)$ are given by

$$a(2n,2k,0)=\binom{2n}{n+k}-\binom{2n}{n+k+1}, a(2n+1,2k+1,0)=\binom{2n+1}{n+k+1}-\binom{2n+1}{n+k+2},$$

$$a(n,k,1)=\sum_{j=0}^{n}\binom{n}{j}\left(\binom{n-j}{k+j}-\binom{n-j}{k+j+2}\right)$$

$$a(n,k,2)=\frac{2(k+1)}{n+k+2}\binom{2n+1}{n+k+1}$$

In these cases the sequence $(L_k(c))$ is periodic:

$(L_k(0))_{k\geq 0}=(\underline{2},0,-2,\underline{0},\cdots)$

$(L_k(1))_{k\geq 0}=(\underline{2},1,-1,-2,-1,\underline{1},\cdots)$

$(L_k(2))_{k\geq 0}=(\underline{2},\cdots)$

This implies that the sequences $w(k,c)=(F_{n+1}(L_{k+1}(c)))_{n\geq 0}$ are periodic with respect to k.

For $c=0$ we have

$w(0,0)=w(2,0)=(\underline{1},0,-1,\underline{0},\cdots)$

$w(1,0)=(1,-2,3,-4,\cdots)$

$w(3,0)=(1,2,3,4,\cdots)$

For $c=1$ we get

$w(0,1)=w(4,1)=(\underline{1},1,0,-1,-1,\underline{0},\cdots)$

$w(1,1)=w(3,1)=(\underline{1},-1,\underline{0},\cdots)$

$w(2,1)=(1,-2,3,-4,\cdots)$

$w(5,1)=(1,2,3,4,\cdots)$

and for $c=2$ we have

$w(0,2)=(\underline{2},\cdots).$

Furthermore we have

$(F_k(0))=(\underline{0},1,0,-\underline{1},\cdots)$

$(F_k(1))=(\underline{0},1,1,0,-1,-\underline{1},\cdots)$

$(F_k(2))=(0,1,2,3,\cdots)$

Example

Let us compute $d_2(5,2,c)$.

Now

$$d_2(5,2,c)=d_2(3\cdot 1+2,2,c)=(-1)^{\binom{2}{2}+1\cdot\binom{3}{2}}3F_3(c)\left(F_1(L_3(c))^2+F_2(L_3(c))^2\right)$$
$$=3(c^2-1)\left(1+(c^3-3c)^2\right)=3c^8-21c^6+45c^4-24c^2-3$$

As special cases we get $d_2(5,2,0)=-3, d_2(5,2,1)=0, d_2(5,2,1)=45.$

2.5 Remark

If we extend the sequence $\left(a(n,k,c)\right)$ to all $n\in\mathbb{Z}$ by $a(n,k,c)=0$ for $n<0,$ then the Hankel determinants $d_p(n,k,c)$ are defined for all $p\in\mathbb{Z}.$

We then get

2.6 Conjecture

$$d_{-p}\left(n+p+k+1,k,c\right)=(-1)^{\binom{k+p+1}{2}}d_p(n,k,c) \tag{2.28}$$

for $n,k\in\mathbb{N}.$

The condensation formula (2.16) gives

$$d_p(n,k,c)d_{p+2}(n-2,k,c)=d_{p+2}(n-1,k,c)d_p(n-1,k,c)-d_{p+1}(n-1,k,c)^2. \tag{2.29}$$

Since $d_0(n,0,c)=1$ and $d_1(n,0,c)=F_{n+1}(c)=c^n+\cdots$ formula (2.29) gives by induction that $d_p(n,0,c)=c^{np}+\cdots$ is a monic polynomial with highest term $c^{pn}.$ Therefore no $d_p(n,0,c)$ vanishes.

I shall prove this conjecture for the case $k=0.$

For $p=0$ (2.28) reduces to $d_0\left(n+1,0,c\right)=d_0(n,0,c)=1.$

For $p=-1$ (2.29) gives

$d_{-1}(n+2,0,c)d_1(n,0,c)-d_1(n+1,0,c)d_{-1}(n+1,0,c)=-1.$

Now (2.9) gives

$d_1(n,0,c)=F_{n+1}(c).$

Therefore we get

$$d_{-1}(n+2,0,c)F_{n+1}(c)-d_{-1}(n+1,0,c)F_{n+2}(c)=-1. \tag{2.30}$$

Since $d_{-1}(1,0,c)=0$ this gives $d_{-1}(2,0,c)=-1=-F_1(c).$

Comparing with Cassini's identity (2.13) we see that

$d_{-1}\left(n+2,0,c\right)=-d_1(n,0,c).$

Therefore (2.28) holds for $p=0$ and $p=1.$

The condensation formula gives

$$d_{-p}(n,0,c)d_{-p+2}(n-2,0,c)=d_{-p+2}(n-1,0,c)d_{-p}(n-1,0,c)-d_{-p+1}(n-1,0,c)^2.$$

If (2.28) holds for all $i<p$ then we get

$$d_{-p}(n+p+1,0,c)d_{p-2}(n,0,c)=d_{p-2}(n+1,0,c)d_{-p}(n+p,0,c)-(-1)^{\binom{p-1}{2}}d_{p-1}(n+1,0,c)^2. \tag{2.31}$$

Now we have $d_{-p}(n,0,c)=0$ for $n\le p$ and $d_{-p}(p+1,0,c)=(-1)^{\binom{p+1}{2}}$.

Since $d_{p-2}(n-p-3,0,c)\neq 0$ we see that $d_{-p}(n,0,c)$ is uniquely determined by (2.31).

On the other hand we have

$$d_p(n,0,c)d_{p-2}(n,0,c)=d_{p-2}(n+1,0,c)d_p(n-1,0,c)+d_{p-1}(n,0,c)^2. \tag{2.32}$$

Comparing (2.31) and (2.32) we see that (2.28) holds for $k=0$.

For $k>0$ the same argument can be applied. But I have no proof that always $d_p(n,0,c)\neq 0$.

As a simple example consider

$$d_{-1}(3,0,c)=\det\begin{pmatrix}0 & 1 & c\\ 1 & c & c^2+1\\ c & c^2+1 & c^3+3c\end{pmatrix}=-d_1(1,0,c)=-\det(c)=-c.$$

3. Associated matrices

To each of the above Hankel matrices we associate another matrix hoping that these matrices will give more insight into the situation.

3.1 Theorem

If we choose the weights (2.2) *then*

$$P_n^+(k,j)=\sum_{\ell=\max(k-j,0)}^{k}a(n,j-k+2\ell). \tag{3.1}$$

Proof

The weight of all paths which do not run below the axis $y=k$ is $a(n,j-k)$. If the absolute minimum of the path is $k-1$ then we change the first down-step $(i,k)\to(i+1,k-1)$ into an up-step $(i,k)\to(i+1,k+1)$ and attach the remnant path which gives a path from $(0,k)\to(n,j+2)$. Each such path which does not run below the axis $y=k$ can be obtained in this way. We have only to change the up-step after the last minimum into a down-step. The weight of all such paths is $a(n,j-k+2)$. In this way we continue and get the above result.

Equation (3.1) can be written in matrix notation as

$$\left(P_i^+(j,k)\right)_{i,j=0}^{n-1}=\left(a(i,m,c)\right)_{i,m=0}^{n-1}H_{n,k} \tag{3.2}$$

where

$$H_{n,k} = \left(h(i,j,k)\right)_{i,j=0}^{n-1} \tag{3.3}$$

is the matrix with entries

$h(i,j,k) = 1$ for $i = k - j + 2\ell$ with $\max(j-k,0) \le \ell \le j,$ and $h(i,j,k) = 0$ else.

This means that
$h(i,i+2j,2k) = 1$ if $i + j \ge 2k$ and $|j| \le k$
and
$h(i,i+2j+1,2k+1) = 1$ if $i + j \ge 2k+1$ and $|j| \le k$

We call $H_{n,k}$ the associated matrix to $d_0(n,k,c)$.

For example $H_{n,0} = I_n$ is the identity matrix and $H_{8,3}$ is given by

$$H_{8,3} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

The determinant of such matrices $H_{n,k}$ can be easily computed. Consider the submatrix consisting of the first $k+1$ rows and columns. By elementary row manipulations this can be reduced to the matrix J_{k+1} where the entries with $i + j = k$ are 1 and all other entries vanish. Then by elementary column operations $H_{n,k}$ can be reduced to a matrix of the form $\begin{pmatrix} J_{k+1} & 0 \\ K & H_{n-k-1,k} \end{pmatrix}$ and by elementary row operations to the matrix $\begin{pmatrix} J_{k+1} & 0 \\ 0 & H_{n-k-1,k} \end{pmatrix}$.

This implies $\det H_{(k+1)n,k} = (-1)^{n\binom{k+1}{2}}$ and $\det\left(H_{n,k}\right) = 0$ else.

Since $\left(a(i,k,c)\right)_{i,k=0}^{n-1}$ is a triangular matrix with determinant 1 we get

3.2 Corollary

$$\det\left(H_{(k+1)n,k}\right)=(-1)^{n\binom{k+1}{2}} \qquad (3.4)$$

and

$$\det\left(H_{n,k}\right)=0 \qquad (3.5)$$

if $n\not\equiv 0(\mathrm{mod}(k+1))$.

Now observe that $P_{n+1}^{+}(j,k)=P_{n}^{+}(j,k-1)+cP_{n}^{+}(j,k)+P_{n}^{+}(j,k+1)$.
This implies

$$\left(P_{i+1}^{+}(j,k)\right)_{i,j=0}^{n-1}=\left(a(i,m,c)\right)_{i,m=0}^{n-1}W_{n,k}^{1}(c) \qquad (3.6)$$

with

$$W_{n,k}^{1}(c)=H_{n,k-1}+cH_{n,k}+H_{n,k+1}, \qquad (3.7)$$

where $H_{n,-1}=0$.
E.g we have

$$\det\left(W_{n,0}^{1}(c)\right)=\det\begin{pmatrix} c & 1 & 0 & 0 & \cdots & & \\ 1 & c & 1 & 0 & \cdots & & \\ 0 & 1 & c & 1 & \cdots & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & c & 1 \\ & & & & & 1 & c \end{pmatrix}=F_{n+1}(c). \qquad (3.8)$$

This can immediately be seen by expanding the determinant with respect to the first row.

From (3.6) we deduce

3.3 Corollary

$$\begin{aligned}&\det\left(W_{(k+1)n,k}^{1}(c)\right)=(-1)^{n\binom{k+1}{2}}F_{n+1}\left(L_{k+1}(c)\right)\\&\det\left(W_{(k+1)n+k,k}^{1}(c)\right)=(-1)^{n\binom{k+1}{2}+\binom{k}{2}}F_{n+1}\left(L_{k+1}(c)\right)\\&\det\left(W_{n,k}^{1}(c)\right)=0 \qquad n\not\equiv 0,k(\mathrm{mod}(k+1))\end{aligned} \qquad (3.9)$$

3.4 Remark

To compute $\det\left(W_{n,k}^{1}\right)$ without recourse to (2.9) we can use the same method as in [2]. First we replace row $\left((2k+2)h+b\right)$ by $\sum_{\ell=0}^{h} row\left((2k+2)\ell+b\right)-\sum_{\ell=1}^{h} row\left((2k+2)\ell-b-2\right)$ if $0 \le b \le k-1,$ and by

$\sum_{\ell=0}^{h} row\left((2k+2)\ell+b\right)-\sum_{\ell=1}^{h+1} row\left((2k+2)\ell-b-2\right)$

if $k+1 \le b \le 2k.$

In this way we get a matrix $W_{n,k}^{*}$.

If $n \not\equiv 0, k(\mathrm{mod}(k+1))$ one row vanishes and therefore the determinant vanishes too.

By a reordering of the rows the matrix $W_{(k+1)n,k}^{*}$ we get a matrix $V_{n(k+1),k}^{1}=\left(v_k(i,j)\right)$ which is almost triangular:

$v_k(i,i-1)=1,$

$v_k((k+1)i+m,i+2j)=c$ for $0 \le j \le k-m$

$v_k((k+1)i+m,i+2j+1)=2$ for $0 \le j \le k-m-1$

$v_k((k+1)i+m,i+2k-2m+1)=1$

All other entries vanish.

For example we have

$$W_{6,2}^{1}=\begin{pmatrix} 0 & 1 & c & 1 & 0 & 0 \\ 1 & c & 2 & c & 1 & 0 \\ c & 2 & c & 2 & c & 1 \\ 1 & c & 2 & c & 2 & c \\ 0 & 1 & c & 2 & c & 2 \\ 0 & 0 & 1 & c & 2 & c \end{pmatrix}, V_{6,2}^{1}=\begin{pmatrix} c & 2 & c & 2 & c & 1 \\ 1 & c & 2 & c & 1 & 0 \\ 0 & 1 & c & 1 & 0 & 0 \\ 0 & 0 & 1 & c & 2 & c \\ 0 & 0 & 0 & 1 & c & 2 \\ 0 & 0 & 0 & 0 & 1 & c \end{pmatrix} \qquad (3.10)$$

Now we choose the uniquely determined numbers $\lambda_i, 0 \le i \le (k+1)n-1,$ with $\lambda_0=1$ such that $\sum_{i=0}^{(k+1)n-1} \lambda_i v_k(i,j)=0$ for $0 \le j < (k+1)n-1.$ Then

$\det\left(W_{(k+1)n,k}^{1}\right)=(-1)^{(k+1)n-1}\sum_{i=0}^{(k+1)n-1} \lambda_i v_k(i,(k+1)n-1).$

Formula (5.20) in [2] reduces in our special case to

$$\lambda_i=(-1)^i L_b^{*}(c) F_{h+1}\left(L_{k+1}(c)\right) \qquad (3.11)$$

if $i=(k+1)h+b, 0 \le b \le k,$ and $L_n^{*}(c)=L_n(c)$ for $n>0$ and $L_0^{*}(c)=1.$

This gives (3.9). For by induction we get
$cL_k^*(c)-2L_{k-1}^*(c)+cL_{k-2}^*(c)-2L_{k-3}^*(c)+\cdots+(-1)^{k-1}\left([k\equiv 0 \bmod 2]+c[k\equiv 1 \bmod 2]\right)L_0^*(c)$
$=L_{k+1}(c)$
and $L_{k+1}(c)F_n\left(L_{k+1}(c)\right)-F_{n-1}\left(L_{k+1}(c)\right)=F_{n+1}\left(L_{k+1}(c)\right).$

In our example (3.10) we see that

$$\det\left(V_{6,2}^1\right)=\left(cL_2(c)-2L_1(c)+c\right)F_2(L_3(c))-1=\left(c^3-3c\right)^2-1=c^6-6c^3+9c^2-1.$$

The determinants $W_{(k+1)n+k,k}^1(c)$ can be treated in the same way. In this case $W_{(k+1)n+k,k}^*=\begin{pmatrix} W_{(k+1)n,k}^* & B \\ 0 & C \end{pmatrix}$, where C is a matrix where all elements of the anti-diagonal are 1 and the elements above the antidiagonal vanish. Details can be found in [2].

Another interesting fact is that $W_{n,k}^1(c)-W_{n,0}^1(c)H_{n,k}$ has all entries 0 except in the last row r_{n-1} where $r_{n-1}(j)=1$ for $j=n-k,n-k+2,n-k+4,\cdots$.

As an example consider

$$W_{6,3}^1(c)=\begin{pmatrix} 0 & 0 & 1 & c & 1 & 0 \\ 0 & 1 & c & 2 & c & 1 \\ 1 & c & 2 & c & 2 & c \\ c & 2 & c & 2 & c & 2 \\ 1 & c & 2 & c & 2 & c \\ 0 & 1 & c & 2 & c & 2 \end{pmatrix}.$$

Comparing this with

$$W_{6,0}^1(c)H_{6,3}=\begin{pmatrix} c & 1 & 0 & 0 & 0 & 0 \\ 1 & c & 1 & 0 & 0 & 0 \\ 0 & 1 & c & 1 & 0 & 0 \\ 0 & 0 & 1 & c & 1 & 0 \\ 0 & 0 & 0 & 1 & c & 1 \\ 0 & 0 & 0 & 0 & 1 & c \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 1 & c & 1 & 0 \\ 0 & 1 & c & 2 & c & 1 \\ 1 & c & 2 & c & 2 & c \\ c & 2 & c & 2 & c & 2 \\ 1 & c & 2 & c & 2 & c \\ 0 & 1 & c & 1 & c & 1 \end{pmatrix}$$

we see that

$$W_{6,3}^1(c) - W_{6,0}^1(c)H_{6,3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

This observation can be used to give another computation of $\det\left(W_{n,1}^1(c)\right)$.

We get

$\det\left(W_{n,1}^1(c)\right) = \det\left(W_{n,0}^1(c)\right)\det\left(H_{n,1}\right) + \det\left(W_{n-1,1}^1(c)\right).$

Since $\det\left(H_{2n+1,1}\right) = 0$ we get $\det\left(W_{2n+1,1}^1(c)\right) = \det\left(W_{2n,1}^1(c)\right)$.

$\det\left(W_{2n,1}^1(c)\right) = \det\left(W_{2n,0}^1(c)\right)\det\left(H_{2n,1}\right) + \det\left(W_{2n-1,1}^1(c)\right) = (-1)^n F_{2n+1}(c) + \det\left(W_{2n-1,1}^1(c)\right)$

$= (-1)^n F_{2n+1}(c) + \det\left(W_{2n-2,1}^1(c)\right).$

Therefore we get

$$\det\left(W_{2n,1}^1(c)\right) = F_1(c) - F_3(c) + - \cdots + (-1)^n F_{2n+1}(c) = (-1)^n F_{n+1}(c^2 - 2) = (-1)^n F_{n+1}\left(L_3(c)\right). \quad (3.12)$$

Here we used the fact that $F_n(c^2-2) + F_{n+1}(c^2-2) = F_{2n+1}(c)$, which is a consequence of

$F_{2n+1}(c) = cF_{2n}(c) - F_{2n-1}(c) = c^2 F_{2n-1}(c) - cF_{2n-2}(c) - F_{2n-1}(c) = \left(c^2-2\right)F_{2n-1}(c) + \left(F_{2n-1}(c) - cF_{2n-2}(c)\right)$

$= \left(c^2-2\right)F_{2n-1}(c) - F_{2n-3}(c).$

It would be interesting to find simpler methods for obtaining $\det\left(W_{n,k}^1\right)$.

In the same way as above we get

$$\det\left(P_{i+j+2}^+(0,k)\right) = \det W_{n,k}^2(c) \quad (3.13)$$

with

$$W_{n,k}^2(c) = W_{n,k-1}^1(c) + cW_{n,k}^1(c) + W_{n,k+1}^1(c). \quad (3.14)$$

E.g.

$$\det W_{n,0}^2(c)=\det\begin{pmatrix} 1+c^2 & 2c & 1 & \cdots & 0 & 0 \\ 2c & 2+c^2 & 2c & \cdots & 0 & 0 \\ 1 & 2c & 2+c^2 & \cdots & 1 & 0 \\ \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & & 2+c^2 & 2c \\ 0 & 0 & 0 & \cdots & 2c & 2+c^2 \end{pmatrix}=\sum_{j=0}^{n}F_{j+1}(c)^2.$$

Or

$$W_{5,2,}^2(c)=\begin{pmatrix} 1 & 2c & c^2+2 & 2c & 1 \\ 2c & c^2+3 & 4c & c^2+3 & 2c \\ c^2+2 & 4c & c^2+4 & 4c & c^2+3 \\ 2c & c^2+3 & 4c & c^2+4 & 4c \\ 1 & 2c & c^2+3 & 4c & c^2+4 \end{pmatrix}$$

and therefore

$$\det\left(W_{5,2}^2(c)\right)=3F_3(c)\left(1+L_3(c)^2\right)=3c^8-21c^6+45c^4-24c^2-3.$$

Remark

These matrices have a particular nice form. I wonder if they occur also in other parts of mathematics. Are there simple methods to compute these determinants?

4. The general case

Now we consider the slightly more general situation $a(n,k,c,x)$ for $x\neq 0.$.

In this case we get

$$P_n^+(k,j)=\sum_{\ell=\max(k-j,0)}^{k}a(n,j-k+2\ell,c,x)+\sum_{\ell=1}^{n}x^{\ell}a(n,j+k+\ell,c,x). \tag{4.1}$$

Equation (4.1) can be written in matrix notation as

$$\left(P_i^+(j,k)\right)_{i,j=0}^{n-1}=\left(a(i,m,c,x)\right)_{i,m=0}^{n-1}H_{n,k}(x) \tag{4.2}$$

where

$$H_{n,k}(x)=\left(h(i,j,k,x)\right)_{i,j=0}^{n-1} \tag{4.3}$$

is the matrix with entries

$h(i,j,k,x)=1$ if $k+i+j\equiv 0(\mathrm{mod}\,2), i+j\ge k$ and $|i-j|\le k$,
$h(i,j,k,x)=x^{\ell}$ if $j-i=k+\ell,\ \ell\ge 1$,
and $h(i,j,k,x)=0$ else.

E.g. we have

$$H_{5,2}(x)=\begin{pmatrix}0&0&1&x&x^2\\0&1&0&1&x\\1&0&1&0&1\\0&1&0&1&0\\0&0&1&0&1\end{pmatrix}.$$

From this it is obvious that $d_0(n,k,c,x)$ does not depend on c.

First we consider the case $k=1$.
Here we get the recurrence relation
$d_0(n,1,c,x)=-xd_0(n-1,1,c,x)-d_0(n-2,1,c,x)$ with initial values
$d_0(1,1,c,x)=0, d_0(2,1,c,x)=-1.$
This gives

$$d_0(n,1,c,x)=(-1)^{n-1}F_{n-1}(x). \qquad (4.4)$$

This is equivalent with $\det\left(H_{n+1,1}(x)\right)=(-1)^n F_n(x).$
For example

$$\det\left(H_{5,1}(x)\right)=\det\begin{pmatrix}0&1&x&x^2&x^3\\1&0&1&x&x^2\\0&1&0&1&x\\0&0&1&0&1\\0&0&0&1&0\end{pmatrix}=F_4(x)=x^3-2x.$$

The determinants $\det\left(H_{n,1}(x)\right)$ are easily computed. We need only find linear combinations of the rows such that the first $n-1$ entries of the first row vanish. Here it suffices to observe that
$F_{n+1}(x)+\sum_{j=0}^{n-1}F_{n-1-j}(x)x^j=x^n.$

For $k>1$ the situation becomes more involved.

By Rubey's program Guess we get the interesting relation

4.1 Conjecture

$$\begin{aligned}&d_0(n,k,c,x)d_0(n-2k-2,k,c,x)-x^2d_0(n-2k-1,k,c,x)d_0(n-1,k,c,x)\\&+(x^2-1)d_0(n-k-1,k,c,x)^2=0.\end{aligned} \tag{4.5}$$

For $k=1$ this reduces to
$F_n(x)F_{n-4}(x)-x^2F_{n-1}(x)F_{n-3}(x)+(x^2-1)F_{n-2}(x)^2=0.$
This is a consequence of Cassini's formula and the formula
$F_{n-2}(x)^2-F_n(x)F_{n-4}(x)=x^2.$

It is interesting to note that the recurrence $f(n)+xf(n-1)+f(n-2)=0$ with arbitrary initial values $f(0)=a,\ f(1)=b$ satisfies the quadratic recursion

$$f(n)f(n-4)-x^2f(n-1)f(n-3)+(x^2-1)f(n-2)^2=0. \tag{4.6}$$

Equation (4.5) can be used to compute $d_0(n,k,c,x)$ only for $n>(k+1)^2+1,$ because for each k the determinant $d_0(n,k,c,x)$ vanishes for the $\binom{k+1}{2}$ values
$n\in\{1,\cdots k,k+2\cdots 2k,2k+3\cdots 3k,\cdots,(k-1)k+k=k^2\}.$
The first values of the sequence $(d_0(n,k,c,x))$ can be easily guessed:

Let $w(i,k)=(d_0(n,k,c,x))_{n=ik+1}^{i(k+1)}.$

$w(1,k)=\left((-1)^{\binom{k+1}{2}}\right)$

$w(2,k)=\left(x^k,-(x^2-1)\right)$ for $k\geq 1,$

$w(3,k)=\left((-1)^{\binom{k+1}{2}}x^{2(k-1)}(x^2-1),x^k\left(kx^2-(k+1)\right),-(x^2-1)^6\right)$ for $k\geq 2,$

$$w(4,k)=\begin{pmatrix}x^{3(k-2)}\left(x^2-1\right)^3,-x^{2(k-1)}\left(\binom{k+1}{2}\left(x^2-1\right)^2-x^2\right),\\-x^k\left(x^2-1\right)\left(\binom{k}{2}x^4-(k^2+k-1)x^2+\binom{k+2}{2}\right),\left(x^2-1\right)^6\end{pmatrix}$$ for $k\geq 3.$

Some more terms could be computed. But these become increasingly complicated.

For small values of k more can be said. Consider first the special case $k=2$: Here we get

4.2 Conjecture

$$f(n,2) = d_0(n,2,c,x) \tag{4.7}$$

satisfies

$$f(n,2) + f(n-1,2) + x^2 f(n-2,2) + f(n-3,2) + f(n-4,2) = 0 \tag{4.8}$$

with initial values

$$f(0,2) = 1,\ f(1,2) = 0,\ f(2,2) = 0,\ f(3,2) = -1. \tag{4.9}$$

The first values are

$$1, 0, 0, -1, 0, x^2, 1-x^2, x^2-x^4, -3x^2+2x^4, \cdots. \tag{4.10}$$

In order to study these polynomials we define numbers $c(k,n)$ for $k \in \mathbb{Z}, n \in \mathbb{N}$ by the generating function

$$\sum_{n\geq 0} c(k,n) z^n = \frac{1}{(1-z)^k \left(1-z^3\right)^{k+1}}. \tag{4.11}$$

Then it is easily verified that

$$nc(k,n) = kc(k,n-1) + kc(k,n-2) + (n+4k)c(k,n-3) \tag{4.12}$$

with initial values

$$c(k,0) = 1, c(k,1) = k, c(k,2) = \binom{k+1}{2}. \tag{4.13}$$

Define polynomials

$$A_n(x,s) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} c(k, n-2k) s^{n-2k} x^k. \tag{4.14}$$

The first values are

$1, 0, x, s^3+sx, s^2x+x^2, 3s^3x+2sx^2, s^6+3s^4x+3s^2x^2+x^3, \cdots.$

Then

$$\sum_{n\geq 0} A_n(x,s) z^n = \sum_{n\geq 0} z^n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} c(k,n-2k) s^{n-2k} x^k = \sum_k z^{2k} x^k \sum_n c(k,n-2k)(sz)^{n-2k}$$

$$= \sum_k z^{2k} x^k \frac{1}{(1-sz)^k \left(1-s^3z^3\right)^{k+1}} = \frac{1}{\left(1-s^3z^3\right)} \frac{1}{1-\dfrac{z^2 x}{(1-sz)\left(1-s^3z^3\right)}} = \frac{1-sz}{(1-sz)\left(1-s^3z^3\right) - xz^2}$$

Comparing this with (4.8) and (4.10) we see that

$$d_0(n+3,2,c,x) = f(n+3,2) = -A_n(-x^2,-1). \tag{4.15}$$

It should be noted that

$$s^2 A_n(x,s)A_{n+6}(x,s) + xA_{n+1}(x,s)A_{n+5}(x,s) - (x+s^2)\left(A_n(x,s)\right)^2 = 0. \tag{4.16}$$

It seems that for each k the sequence

$$f(n,k) = d_0(n,k,c,x) \tag{4.17}$$

also satisfies a linear recurrence with coefficients in $\mathbb{Z}[x]$.

For $k = 3$ we have

4.3 Conjecture

$$\begin{aligned} &-f(n,3) + xf(n-1,3) + (x^3 - x)f(n-3,3) + (-x^4 - x^2 + 2)f(n-4,3) \\ &+(x^3 - x)f(n-5,3) + xf(n-7,3) - f(n-8,3) = 0 \end{aligned} \tag{4.18}$$

For $k = 4$ we get

4.4 Conjecture

$$\begin{aligned} &f(n,4) - f(n-1,4) + x^2 f(n-3,4) + (-2x^4 + x^2)f(n-4,4) \\ &+(x^4 + 2x^2 - 3)f(n-5,4) + (-2x^2 + 3)f(n-6,4) + (-x^6 + x^4 - x^2)f(n-7,4) \\ &+(x^8 + x^4 - 2x^2)f(n-8,4) + (-x^6 + x^4 - x^2)f(n-9,4) \\ &+(-2x^2 + 3)f(n-10,4) + (x^4 + 2x^2 - 3)f(n-11,4) + (-2x^4 + x^2)f(n-12,4) \\ &+x^2 f(n-13,4) - f(n-15,4) + f(n-16,4) = 0. \end{aligned} \tag{4.19}$$

I could not find a general formula which holds for all k.

Next we consider determinants $d_1(n,k,c,x)$.
Let now

$$W^1_{n,k}(c,x) = H_{n,k+1}(x) + cH_{n,k}(x) + H_{n,k-1}(x) \tag{4.20}$$

for $k > 0$ and

$$W^1_{n,0}(c,x) = H_{n,1}(x) + (c+x)H_{n,0}(x). \tag{4.21}$$

Then we have

$$d_1(n,k,c,x) = \det\left(W^1_{n,k}(c,x)\right). \tag{4.22}$$

For example we have

$$d_1(4,1,c,x) = \det\begin{pmatrix} 1 & c+x & 1+cx+x^2 & x(1+cx+x^2) \\ c & 2 & c+x & 1+cx+x^2 \\ 1 & c & 2 & c+x \\ 0 & 1 & c & 2 \end{pmatrix}.$$

These determinants depend on c and x.
For $k=0$ we get that

$$d_1(n,0,c,x) = \det\begin{pmatrix} c+x & 1+cx+x^2 & x(1+cx+x^2) & x^2(1+cx+x^2) & \cdots \\ 1 & c+x & 1+cx+x^2 & x(1+cx+x^2) & \cdots \\ 0 & 1 & c+x & 1+cx+x^2 & \cdots \\ 0 & 0 & 1 & c+x & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = F_{n+1}(c) + xF_n(c). \tag{4.23}$$

For $k=2$ we get for example

$$d_0(5,2,c,x) = \det\begin{pmatrix} 0 & 0 & 1 & x & x^2 \\ 0 & 1 & 0 & 1 & x \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} = x^2$$

In the same way we set

$$W_{n,k}^2(c,x) = W_{n,k+1}^1(c,x) + cW_{n,k}^1(c,x) + W_{n,k-1}^1(c,x) \tag{4.24}$$

for $k>0$ and

$$W_{n,0}^2(c,x) = W_{n,1}^1(c,x) + (c+x)W_{n,0}^1(c,x). \tag{4.25}$$

Then we have

$$d_2(n,k,c,x) = \det\left(W_{n,k}^2(c,x)\right). \tag{4.26}$$

For $k=0$ we get an explicit formula

$$\begin{pmatrix} 1+c^2+2cx+x^2 & (c+x)(2+cx+x^2) & (1+cx+x^2)^2 & x(1+cx+x^2)^2 & \cdots \\ 2c+x & 2+c^2+2cx+x^2 & (c+x)(2+cx+x^2) & (1+cx+x^2)^2 & \cdots \\ 1 & 2c+x & 2+c^2+2cx+x^2 & (c+x)(2+cx+x^2) & \cdots \\ 0 & 1 & 2c+x & 2+c^2+2cx+x^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (4.27)$$

$$= d_2(n,0,c,x) = \sum_{j=0}^{n+1} F_j(c)^2 + 2cx\sum_{j=0}^{n/2} F_{n-2j}(c)^2 + x^2\sum_{j=0}^{n} F_j(c)^2 = \sum_{j=0}^{n}\left(F_{j+1}(c)+x^2F_j(c)\right)^2$$

For $k>0$ formulas become more complicated.
For $c=0$ we have

$$a(n,0,0,x)=\sum_{2k\le n}\left(\binom{2n}{k}-\binom{2n}{k-1}\right)x^{n-2k}.$$

Here we get

$$d_1(n,0,0,x)=\det\begin{pmatrix} x & 1+x^2 & x(1+x^2) & x^2(1+x^2) & \cdots \\ 1 & x & 1+x^2 & x(1+x^2) & \cdots \\ 0 & 1 & x & 1+x^2 & \cdots \\ 0 & 0 & 1 & x & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{cases} (-1)^m & \text{n=2m} \\ (-1)^m x & \text{n=2m+1} \end{cases}$$

and

$$d_2(n,0,0,x)=\begin{pmatrix} 1+x^2 & x(2+x^2) & (1+x^2)^2 & x(1+x^2)^2 & \cdots \\ x & 2+x^2 & x(2+x^2) & (1+x^2)^2 & \cdots \\ 1 & x & 2+x^2 & x(2+x^2) & \cdots \\ 0 & 1 & x & 2+x^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{cases} nx^2+n+1 & \text{n=2m} \\ (n+1)(x^2+1) & \text{n=2m+1} \end{cases}$$

5. The case x=1 and x=-1.

Finally I want to state some results for $a(n,k,c,\pm 1)$.
Observing that $a(n,k,c,-1)=(-1)^{n-k}a(n,k,-c,1)$ we need only consider the case $x=1$.
The determinants $d_0(n,k,c,1)$ do not depend on c.
We get

$$d_0((2k+1)n,k,c,1)=1$$
$$d_0((2k+1)n+k+1,k,c,1)=(-1)^{\binom{k+1}{2}}. \qquad (5.1)$$

For the determinants $d_1(n,k,c,1)$ we get

5.1 Conjecture

$$d_1(2k+1)n,k,c,1) = F_n\left(L_{2k+1}(c)\right) + F_{n+1}\left(L_{2k+1}(c)\right) \tag{5.2}$$

For $k>0$ *we have*

$$d_1(2k+1)n+k,k,c,1) = (-1)^{\binom{k}{2}} d_1(2k+1)n,k,c,1) \tag{5.3}$$

$$d_1(2k+1)n+k+1,k,c,1) = d_1(k+1,k,c,1)F_{n+1}\left(L_{2k+1}(c)\right) \tag{5.4}$$

$$d_1(2k+1)n+2k,k,c,1) = (-1)^{\binom{k}{2}} d_1(2k+1)n+k+1,k,c,1) \tag{5.5}$$

Here

$$d_1(k+1,k,c,1) = (-1)^{\binom{k+1}{2}}\left(L_k(c) + L_{k+1}(c)\right) \tag{5.6}$$

for $k>0.$

For the generating functions we get

5.2 Conjecture

For $k=0$

$$\sum_{n\ge 0} d_1(n,0,c,1)z^n = \frac{1+z}{1-cz+z^2}. \tag{5.7}$$

For $k=1$ *we have*

$$\sum_{n\ge 0} d_1(n,1,c,1)z^n = \frac{1+z-\left(L_1(c)+L_2(c)\right)z^2+z^3+z^4}{1-L_3(c)z^3+z^6} \tag{5.8}$$

For $k\ge 2$ *we get*

$$\sum_{n\ge 0} d_1(n,k,c,1)z^n$$

$$= \frac{1+(-1)^{\binom{k}{2}}z^k + (-1)^{\binom{k+1}{2}}\left(L_k(c)+L_{k+1}(c)\right)z^{k+1} + (-1)^k\left(L_k(c)+L_{k+1}(c)\right)z^{2k} + z^{2k+1} + (-1)^{\binom{k}{2}}z^{3k+1}}{1-L_{2k+1}(c)z^{2k+1}+z^{4k+2}} \tag{5.9}$$

For the determinants $d_2(n,k,c,1)$ we get

5.3 Conjecture

$$d_2(n,0,c,1)=\sum_{j=0}^{n}\left(F_j(c)+F_{j+1}(c)\right)^2. \tag{5.10}$$

$$d_2((2k+1)n,k,c,1)=\left(F_n(L_{2k+1}(c))+F_{n+1}(L_{2k+1}(c))\right)^2 \tag{5.11}$$

$$d_2((2k+1)n+k-1,k,c,1)=(-1)^{\binom{k-1}{2}}\left(F_n(L_{2k+1}(c))+F_{n+1}(L_{2k+1}(c))\right)^2 \tag{5.12}$$

$$d_2((2k+1)n+k+1,k,c,1)=(-1)^{\binom{k+1}{2}}d_1((2k+1)n+k+1,k,c,1)^2 \tag{5.13}$$

$$d_2((2k+1)n+2k-1,k,c,1)=-d_1((2k+1)n+k+1,k,c,1)^2 \tag{5.14}$$

It should be noted that (5.11) - (5.14) can be derived from 5.1 with the help of the condensation formula.

For the determinants $d_2((2k+1)n+k,k,c,1)$ and $d_2((2k+1)n+2k,k,c,1)$ the formulas are more complicated. For $k\ge 1$ we have

$$\begin{aligned}d_2((2k+1)n+k,k,c,1)&=d_2(k,k,c,1)\sum_{j=0}^{n}\left(F_j(L_{2k+1}(c))+F_{j+1}(L_{2k+1}(c))\right)^2\\&-\left(d_2(k,k,c,1)+(-1)^{\binom{k-1}{2}}d_2(2k,k,c,1)\right)\sum_{j=0}^{n-1}\left(F_{j+1}(L_{2k+1}(c))\right)^2\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}d_2((2k+1)n+2k,k,c,1)&=(-1)^{\binom{k-1}{2}+1}d_2(k,k,c,1)\sum_{j=0}^{n}\left(F_j(L_{2k+1}(c))+F_{j+1}(L_{2k+1}(c))\right)^2\\&+\left(d_2(2k,k,c,1)+(-1)^{\binom{k-1}{2}}d_2(k,k,c,1)\right)\sum_{j=0}^{n}\left(F_{j+1}(L_{2k+1}(c))\right)^2\end{aligned} \tag{5.16}$$

The values $d_2(k,k,c,1)$ and $d_2(2k,k,c,1)$ are given by

$$d_2(k,k,c,1)=(-1)^{\binom{k}{2}}\left(kF_k(c)+(k+1)F_{k+1}(c)\right) \tag{5.17}$$

and

$$d_2(2k,k,c,1)=(-1)^k\left(kF_{3k+2}(c)+(k+1)F_{3k+1}(c)+2kF_{k+1}(c)+2(k+1)F_k(c)\right). \tag{5.18}$$

Here

$$\left(kF_k(c)+(k+1)F_{k+1}(c)\right)_{k\geq 0}=\left(1,2c+1,3c^2+2c-3,4c^3+3c^2-8c-3,\cdots\right) \tag{5.19}$$

and

$$\begin{aligned}&\left(kF_{3k+2}(c)+(k+1)F_{3k+1}(c)+2kF_{k+1}(c)+2(k+1)F_k(c)\right)_{k\geq 0}\\&=\left(1,c^4+2c^3-3c^2-2c+5,2c^7+3c^6-12c^5-15c^4+20c^3+22c^2-2c-7,\cdots\right).\end{aligned} \tag{5.20}$$

If we extend $a(n,k,c,x)$ to negative n by $a(n,k,c,x)=0$ for $n<0,$ then we obtain in the same way as in 2.6

5.4 Theorem

$$d_{-p}(n+p+1,0,c,\pm 1)=(-1)^{\binom{p+1}{2}}d_p(n,0,c,0). \tag{5.21}$$

For example we have

$$d_{-1}(5,0,2,-1)=\det\begin{pmatrix}0&1&1&2&5\\1&1&2&5&14\\1&2&5&14&42\\2&5&14&42&132\\5&14&42&132&429\end{pmatrix}=-d_1(3,0,2,0)=-\det\begin{pmatrix}2&5&14\\4&14&42\\14&42&132\end{pmatrix}=-4.$$

References

[1] M. Aigner, Catalan-like numbers and determinants, J. Comb. Th. A 87 (1998), 33-51

[2] J. Cigler & C. Krattenthaler, Some determinants of path generating functions

[3] C. Krattenthaler, Advanced determinant calculus, Sém. Loth. 42 (1966)

[4] M. Rubey & W. Hebisch, Extended Rate, more GFun